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AN EFFICIENT AND ACCURATE METHOD FOR GRADIENT COMPUTATION OF NONLINEAR SOIL-STRUCTURE INTERACTION (SSI) SYSTEMS

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ABSTRACT

In the context of nonlinear finite element (FE) method, this paper presents a computational framework for ‘exact’ response sensitivity analysis of soil-structural interaction (SSI) systems based on the direct differentiation method (DDM). The DDM requires analytically differentiating the FE algorithm for the response computation with respect to material, geometry or loading parameters, involving the differentiation of various hierarchical layers of FE response, namely: (1) structure level, (2) element level, (3) section level, and (4) material level. To achieve this analysis framework, the DDM-based response sensitivity analysis algorithm is extended to various element and material models used in Reinforced Concrete (RC) frame, as well as a multi-yield surface J_2 plasticity material model used to represent cohesive soil behavior under cyclic loading conditions. Furthermore, the DDM algorithm is extended to accommodate the multi-point constraints (MPC) conditions used in FE models (e.g., Transformation method, penalty method). The framework is implemented in general purpose FE software, OpenSees, an open system for earthquake engineering simulation.

A three dimensional frame-soil interaction example is used to demonstrate the implementation and usage of the DDM-based response sensitivity analysis algorithm for SSI systems subjected to two orthogonal horizontal earthquake excitations. The frame is modeled with beam-column element, fiber section and nonlinear concrete and steel materials, while the soil by multi-yield surface J_2 plasticity material model. The sensitivity parameters are taken as the material parameters used to represent the elastoplastic behaviors of the various soil and structural materials. The response sensitivity results obtained by using DDM are verified by their counterparts obtained by using forward finite difference (FFD) method. The asymptotic convergence of FFD results towards DDM results verifies the presented DDM framework. As the application, the sensitivity analysis results provide the relative importance of the various material parameters for selected global and/or local response parameters of the SSI systems.

Keywords: soil-structure interaction, direct differentiation method, nonlinear finite element analysis, OpenSees, sensitivity analysis.

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1. INTRODUCTION

Finite element response sensitivities represent an essential ingredient for gradient-based optimization methods needed in various subfields of structural and geotechnical engineering such as structural reliability analysis, structural/geotechnical system identification, and FE model updating (e.g., Ditlevsen and Madsen 1996, Kleiber 1997). In addition, FE response sensitivities are invaluable for gaining insight into the effect and relative importance of system and loading parameters in regards to structural response behavior.

Several methods are available for response sensitivity computation, such as the Finite Difference Method (FDM), the Adjoint Method (AM), the Perturbation Method (PM), and the Direct Differentiation Method (DDM). These methods are described by Zhang and Der Kiureghian (1993), Kleiber et al. (1997), Conte et al. (2003, 2004), and Gu et al (2009, 2013). The FDM is the simplest method for response sensitivity computation, but is computationally expensive and can be negatively affected by numerical noise (i.e., truncation and round-off errors). The AM is extremely efficient for linear and non-linear elastic systems, but is not a competitive method for path-dependent problems. The PM is computationally efficient, but generally not very accurate. The DDM, on the other hand, is very general, accurate and efficient and is applicable to any material constitutive model (both path-independent and path-dependent). The computation of FE response sensitivities to system and loading parameters based on the DDM requires extension of the FE algorithms for response-only computation (Conte et al. 2003). This paper focuses on finite element response sensitivity analysis of SSI systems based on the DDM.

In the past decade, DDM framework has been implemented in a general purpose FE software, OpenSees (Open System for Earthquake Engineering Simulation, <http://OpenSees.berkeley.edu>). To extend the DDM framework to SSI system, the sensitivities of various elements, sections and materials are derived and implemented. Furthermore, the DDM algorithm is extended to accommodate the multi-point constraints (MPC) conditions used in FE models (e.g., Transformation method, penalty method, Gu et al. 2009). It is worth mentioning that among these materials, two kind of 3D soil material model has been re-derived to accommodate the DDM sensitivity algorithms, i.e., multi-yield surface soil model and bounding surface soil model. These are two of the most popular soil models and are widely used in geotechnical engineering for simulating elastoplastic nonlinear soil behaviors during the earthquake (Gu et al, 2009, 2013). In this paper, only multi-yield surface soil model is used for sensitivity of SSI systems.

2. DDM BASED FE RESPONSE SENSITIVITY ANALYSIS FRAMEWORK

In the context of nonlinear FE analysis, the consistent FE response sensitivities based on DDM are computed at each time step, after convergence is achieved for the response computation. This requires differentiation of the FE algorithm for the response computation with respect to each sensitivity parameter q . Consequently, the response sensitivity computation algorithm involves the various hierarchical layers of FE response analysis, namely: (1) structure level, (2) element level, (3)

Gauss point level (or section level), and (4) material level. Details on the derivation of the DDM-based sensitivity equations for classical displacement-based, force-based and mixed finite elements can be found in a number of references (Zhang and Der Kiureghian 1993, Kleiber et al. 1997, Conte et al. 2003, Gu et al. 2013).

2.1. General Response Sensitivity Analysis Based on DDM

After spatial discretization using the FEM, and integrate numerically the equation of motion in time using the Newmark- β method of structural dynamics, the dynamic residual $\Psi(\mathbf{u}_{n+1})$ expressed at discrete time $t = t_{n+1}$ is given by

$$\Psi(\mathbf{u}_{n+1}) = \tilde{\mathbf{F}}_{n+1} - \left[\frac{1}{\beta(\Delta t)^2} \mathbf{M} \mathbf{u}_{n+1} + \frac{\alpha}{\beta(\Delta t)} \mathbf{C} \mathbf{u}_{n+1} + \mathbf{R}(\mathbf{u}_{n+1}) \right] = \mathbf{0} \quad (1)$$

Where t = time, θ = sensitivity parameter, $\mathbf{u}(t)$ = vector of nodal displacements, $\mathbf{M}$ = mass matrix, $\mathbf{C}$ = damping matrix, $\mathbf{R}(\mathbf{u}, t)$ = history dependent internal (inelastic) resisting force vector, $\tilde{\mathbf{F}}_{n+1}$ = equivalent applied dynamic load vector. Differentiating Eq. (2) with respect to θ , recognizing that $\mathbf{R} = \mathbf{R}(\mathbf{u}(t, \theta), \theta)$,

$$\left[\frac{1}{\beta(\Delta t)^2} \mathbf{M} + \frac{\alpha}{\beta(\Delta t)} \mathbf{C} + (\mathbf{K}_T^{stat})_{n+1} \right] \frac{\partial \mathbf{u}_{n+1}}{\partial \theta} = - \left(\frac{1}{\beta(\Delta t)^2} \frac{\partial \mathbf{M}}{\partial \theta} + \frac{\alpha}{\beta(\Delta t)} \frac{\partial \mathbf{C}}{\partial \theta} \right) \mathbf{u}_{n+1} - \left. \frac{\partial \mathbf{R}(\mathbf{u}_{n+1}(\theta), \theta)}{\partial \theta} \right|_{\mathbf{u}_{n+1}} + \frac{\partial \tilde{\mathbf{F}}_{n+1}}{\partial \theta} \quad (2)$$

Where $\frac{\partial \tilde{\mathbf{F}}_{n+1}}{\partial \theta}$ can be computed easily based on the external loading and variables in the previous

time step. In Eq. (2), the term $\left. \partial \mathbf{R}(\mathbf{u}_{n+1}(\theta), \theta) / \partial \theta \right|_{\mathbf{u}_{n+1}}$ represents the partial derivative of the internal resisting force vector $\mathbf{R}(\mathbf{u}_{n+1})$ with respect to sensitivity parameter θ under the condition that the displacement vector $\mathbf{u}_{n+1}$ remains fixed. This conditional derivative term is expressed as an assembly of contributions from all elements, and thus from all integration/material points.

In Eq. (2), the term $\mathbf{K}_T^{dyn} = \left[\frac{1}{\beta(\Delta t)^2} \mathbf{M} + \frac{\alpha}{\beta(\Delta t)} \mathbf{C} + (\mathbf{K}_T^{stat})_{n+1} \right]$ denotes the tangent dynamic stiffness matrix, in

which $\mathbf{K}_T^{stat}$ denotes the consistent tangent stiffness matrix defined by $(\mathbf{K}_T^{stat})_{n+1} = \frac{\partial \mathbf{R}_{n+1}}{\partial \mathbf{u}_{n+1}}$. The attribute

‘consistent’ emphasizes that the tangent operator is obtained through consistent linearization of the the constitutive law integration scheme, which guarantees the quadratic rate of asymptotic convergence of iterative solution strategies based on Newton’s method (Simo and Taylor 1985).

For material sensitivity parameters, the term $\partial \mathbf{R}(\mathbf{u}_{n+1}(\theta), \theta) / \partial \theta \big|_{\mathbf{u}_{n+1}}$ in Eq. (2) simplifies to

$$\frac{\partial \mathbf{R}(\mathbf{u}_{n+1}(\theta), \theta)}{\partial \theta} \bigg|_{\mathbf{u}_{n+1}} = \sum_{n=1}^{Nel} \left[\int_{\Omega_e} \mathbf{B}^T(\mathbf{x}) \cdot \frac{\partial \boldsymbol{\sigma}(\mathbf{x})}{\partial \theta} \bigg|_{\mathbf{x}_{n+1}} \cdot d\Omega_e \right] \quad (3)$$

where $\mathbf{B}$ is the strain-displacement transformation matrix. The conditional stress sensitivity $\frac{\partial \boldsymbol{\sigma}(\mathbf{x})}{\partial \theta} \big|_{\mathbf{x}_{n+1}}$ is computed by following the same elastic predictor multi-plastic corrector stress computation scheme as mentioned above.

2.2. Response Sensitivity Algorithm for Elements, sections and Materials used to Model RC Frame

RC frame can be modeled by displacement based Euler-Bernoulli frame beam column elements with distributed plasticity. Section stress resultants at the integration points may be computed by using fiber sections with concrete and reinforcing steel material layers. Currently the concrete material may be modeled using Kent-Scott-Park model with no tension stiffening, or smoothed Popovics-Saenz model (Gu 2008); while the reinforced steel may be modeled by using uniaxial J_2 plasticity model with kinematic and isotropic hardening, or Menegotto-Pinto model. The DDM-based response sensitivity analysis is extended to the elements, sections, materials mentioned above in order to perform the sensitivity computation.

2.3. Extension of the DDM to accomodate the Multi-Point Constrain (MPC) method

The DDM for FE response sensitivity analysis is extended to linear and nonlinear FE models with multi-point constraints (MPCs). The analytical developments are provided for three different constraint handling methods, namely: the transformation equation method, the Lagrange multipliers method and the penalty function method. This extension allows user to apply MPCs including: (1) “equal DOF”, which enforces equal displacements/rotations at different DOFs of the FE model. This may be used to model the simple shear condition of soil by tying the corresponding boundary nodes (usually at the same depth) together along some directions. (2) “rigid link”, which imposes a rigid connection between different DOFs, and (3) “rigid diaphragm”, which imposes a rigid behavior for the in-plane motion of nodes belonging to the same plane (Gu et al. 2009).

2.4. Response Sensitivity Algorithm for Multi-Yield Surface J_2 Plasticity Model and bounding surface soil model

The multi-yield surface J_2 plasticity model was first developed by Iwan and Mroz (1967), then further developed and applied to soil mechanics by Prevost (1977), and recently implemented in OpenSees by Elgamal et al. (2003). In contrast to the classical J_2 plasticity model with a single yield

surface, the multi-yield surface J_2 plasticity model employs the concept of a field of plastic moduli to achieve a more realistic representation of the material plastic behavior under cyclic loading conditions. This field is defined by a collection of nested yield surfaces each of constant size (i.e., no isotropic hardening) in the stress space, which define the regions of constant plastic shear moduli. At each time step, it is not possible to know a priori which and how many yield surfaces will be reached until global equilibrium is achieved at the end of the step. Hence, the expressions for the response sensitivities at the current stress point (converged time step) depends on those yield surfaces that have contributed to the change of stress state from the last converged time step. The DDM based sensitivity algorithm has been extended to this soil model recently (Gu 2009).

The bounding surface model was developed for simulating the pressure-dependent behaviors of sandy soils under complex loading conditions (Gu 2013). Compared with the classical plastic theory using yield surfaces, flow rules and hardening laws to characterize the plastic behavior of a material, this model generalizes the yield-surface-based plasticity theory by defining a bounding surface or a failure surface. The plastic deformation within the bounding surface is determined by a varying plastic modulus, which is defined as a continuous function of the distance from the current stress to a properly defined ‘image’ stress on the bounding surface. The model was further improved to incorporate the basic premises of critical-state soil mechanisms to allow for the realistic modeling of the shear-induced volumetric changes (i.e., contraction or dilation) in either a loose or a dense state, and the phase transition from one state to another, which is the basis for modeling the liquefaction behavior of sandy soils. The DDM sensitivity algorithm was recently developed for this model (Gu 2013).

3. APPLICATION EXAMPLES

In this section, a soil-foundation-structure-interaction system subjected to earthquake excitation is studied. The application example considered herein consists of a three-dimensional reinforced concrete (RC) frame with concrete slabs at each floor as shown in Figure 1. The frame consists of three stories each of height $h = 3.66\text{m}$ (12ft) and one bay of span $L = 6.10\text{m}$ (20ft) in each direction. Beam and column cross-sections are also shown in Figure 2. Beams and columns are modeled using displacement-based Euler-Bernoulli frame elements, each with four Gauss-Legendre integration points. Each column and beam is discretized into two and three finite elements, respectively. Beam and column cross-sections are discretized into fibers of confined concrete, unconfined concrete and steel reinforcement. The reinforcement steel is modeled through a bilinear hysteretic model, while the concrete is represented by the Kent-Scott-Park model with zero tension stiffening (Scott et al., 1982) as shown in Figure 1. Different material parameters are used for the confined (core) and unconfined (cover) concrete in the columns and beams. The concrete slabs are modeled through a diaphragm constraint at each floor to enforce rigid in-plane behavior. In the foundation, the 4m long RC piles are modeled in the same manner as the columns in the upper structure. The same beam/column elements, fiber sections, and material properties are used for the piles and columns. The material properties of the piles and columns have the same nominal values, but are treated as

different sensitivity parameters for the response sensitivity analysis. The foundation soil is discretized into four layers and the soil in each layer is modeled using the multi-yield surface J_2 plasticity model (Elgamal et al. 2003) with material parameters varying across layers. The connection between the piles and the soil is achieved by tying the three translational DOFs of the corresponding pile nodes (6-DOFs) and soil nodes (3-DOFs). A simple shear condition in the soil is modeled by tying together (in the x or y directions) the corresponding horizontal DOFs of the boundary nodes at the same depth.

Figure 1. Geometry and column section properties of a 3D SFSI system

Twenty three model parameters are modeled as random variables, taken from the material parameters of the soil, foundation and frame. Their relative importance in terms of specific system response parameters will be presented and discussed later. The parameters used to model the frame and soil materials are listed in Table 1. Poisson's ratio is approximately 0.35 for all four soil layers.

Table 1: Material properties

	Frame and pile foundations				Soil			
	Concrete		Steel					
Mat par.	Core	Cover	Mat		Layer	G	$\tau_{\max}$	K
f_c	34473.8	27579.04	E	2.1e8	#1	54450	33.	1.6e5
f_{cu}	24131.66	0	σ_Y	248200	#2	33800	26.	1.0e5
ϵ_c	0.005	0.002	b	0.02	#3	61250	35.	1.8e5
ϵ_{cu}	0.02	0.006			#4	96800	44.	2.9e5

After static application of the gravity loads, the structure is subjected to a bi-directional base excitation taken as the fault normal and fault parallel components of the 1978 Tabas earthquake. The maximum accelerations in both directions are approximately 1g as shown in Figure 2 (9.81 m/s^2). Both response and DDM-based response sensitivity analyses are performed using a constant integration time step of 0.01 sec. Part of the response of the frame and different soil layers are shown in Figure 2. The local moment-curvature response at point A in the frame and shear stress-strain response at point B in the soil are also plotted in Figure 2. From these simulation

results, it is clear that the system yields significantly under the earthquake excitation considered. Response sensitivity analysis is performed using the DDM. Response sensitivity results obtained using the DDM are verified by the Forward Finite Difference (FFD) method as shown in Figure 3. It is observed that the FFD results approach the DDM results as the perturbation is increased. This is due to the fact that the convergence threshold for the response is set to a relatively large value (1.0e-3 [m]) such that a too small parameter perturbation causes the FFD results to be inaccurate due to computational round off errors. The normalized sensitivity results for the first interstory drift are analyzed for all the material (structural and soil) parameters taken as sensitivity parameters. Some of these results are shown in Figure 3. Based on all the analysis results, it is found that the relative importance of the few most sensitive material parameters for the first interstory drift response in the x-direction are: $\tau_{\max,4} > E_{\text{steel}} > \tau_{\max,3} > \sigma_{y,\text{steel}} > f_{c,\text{cover}} > f_{c,\text{core}} > G_2 > \tau_{\max,1} > G_1 > E_{\text{foundation}} > \sigma_{y,\text{foundation}}$. Similarly, for the first interstory drift in the y-direction, the relative importance of the first few most sensitive material parameters are: $\sigma_{y,\text{steel}} > \tau_{\max,4} > E_{\text{steel}} > \tau_{\max,3} > f_{c,\text{cover}} > f_{c,\text{core}} > \tau_{\max,1} > \sigma_{y,\text{foundation}} > G_2 > \tau_{\max,2} > G_3$. The normalized sensitivity results are analyzed and the relative importance of the few most sensitive material parameters in determining the second floor drift in x direction are: $\tau_{\max,4} > \tau_{\max,3} > E_{\text{steel}} > \sigma_{y,\text{steel}} > f_{c,\text{cover}} > G_2 > G_1 > G_3 > \tau_{\max,1} > f_{c,\text{core}} > G_4 > E_{\text{foundation}} > \sigma_{y,\text{foundation}}$, and in y direction: $\tau_{\max,3} > E_{\text{steel}} > \sigma_{y,\text{steel}} > \tau_{\max,4} > f_{c,\text{cover}} > \tau_{\max,1} > G_2 > \varepsilon_{c,\text{cover}} > G_3 > E_{\text{foundation}} > G_4 > f_{c,\text{core}} > \sigma_{y,\text{foundation}}$. From these results, it is observed that the importance sequence for x and y direction are almost same. Parameters $\sigma_{y,\text{steel}}, E_{\text{steel}}, \tau_{\max,3}, \tau_{\max,4}$ are always most important parameters; while the other few parameters $f_{c,\text{cover}}, f_{c,\text{core}}, G_2, G_1, \tau_{\max,1}, E_{\text{foundation}}, \sigma_{y,\text{foundation}}$ are sensitive as well. The relative importance to second floor drift is similar with that to the first drift and is not shown in this paper.

The reason for the sensitivity ranking described herein is that the Tabas earthquake is strong enough ($PGA \approx 1.0g$) such that the building and the soil in the third and forth layer (i.e., the bottom 2 layers, refer to Figure 1) yield significantly. Thus the parameters related to the strength $\sigma_{y,\text{steel}}, \tau_{\max,3}, \tau_{\max,4}$ become as important as or more important than the stiffness parameters (e.g. E_{steel}), which are observed to be most sensitive when system is subjected to small or moderate earthquake. Parameters $f_{c,\text{cover}}, f_{c,\text{core}}, G_2, G_1$ are closely related to the stiffness of the system, and are very sensitive for small earthquakes, however become less sensitive for strong earthquake. Foundation is not yield so much and thus its parameters are not so sensitive. Same sensitivity analysis is performed for local strain at point B in soil (see Figure 1). Part of the normalized sensitivity results are shown in Figure 4. From these analysis results, it is observed that, the relative importance to strain a point B is: $\tau_{\max,1} > \tau_{\max,4} > \tau_{\max,2} > \sigma_{y,\text{steel}} > \tau_{\max,3} > G_1 > f_{c,\text{cover}} > G_4 > E_{\text{steel}} > G_2 > f_{c,\text{core}}$, which is slightly different than the sensitivity ranking of parameters to the global responses (first and second interstory drift), in the sense that the local soil parameters close to point B becomes remarkable (e.g., $\tau_{\max,1}, \tau_{\max,2}$), while those far from point B becomes less important (e.g., E_{steel}). The sensitivity analysis is performed for the moment at point A in structure (Figure 2) as well. Part of the results are shown in Figure 3. The normalized sensitivity shows the relative importance of the material parameters are: $G_3 > \tau_{\max,4} > E_{\text{steel}} > \tau_{\max,3} > f_{c,\text{cover}} > \tau_{\max,1} > \tau_{\max,2} > f_{c,\text{core}} > \varepsilon_{c,\text{core}} > \varepsilon_{c,\text{cover}}$. This ranking is slight different from the

ranking of parameter sensitive for global response, and the parameters closely related to the structure (e.g., $f_{c,cover}$, $f_{c,core}$) becomes more important.

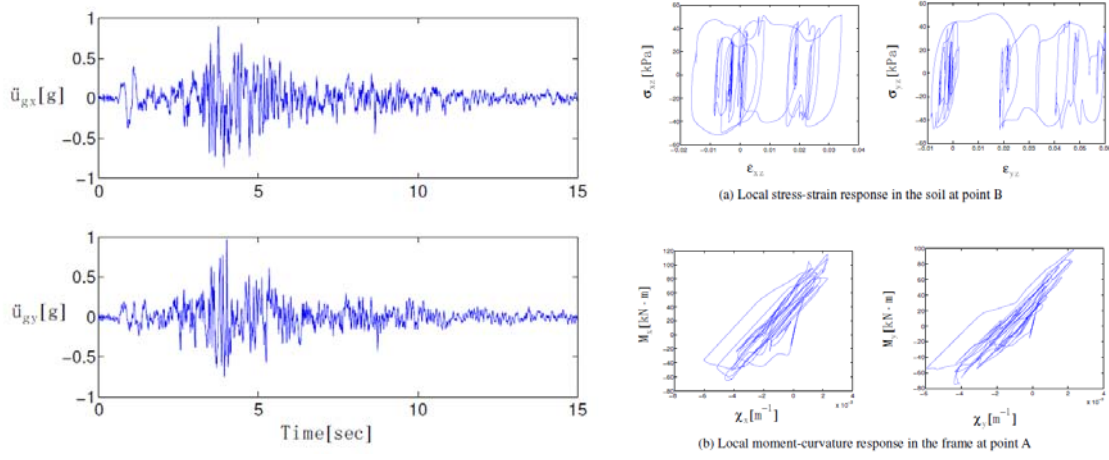


Figure 2: Base excitation and local responses

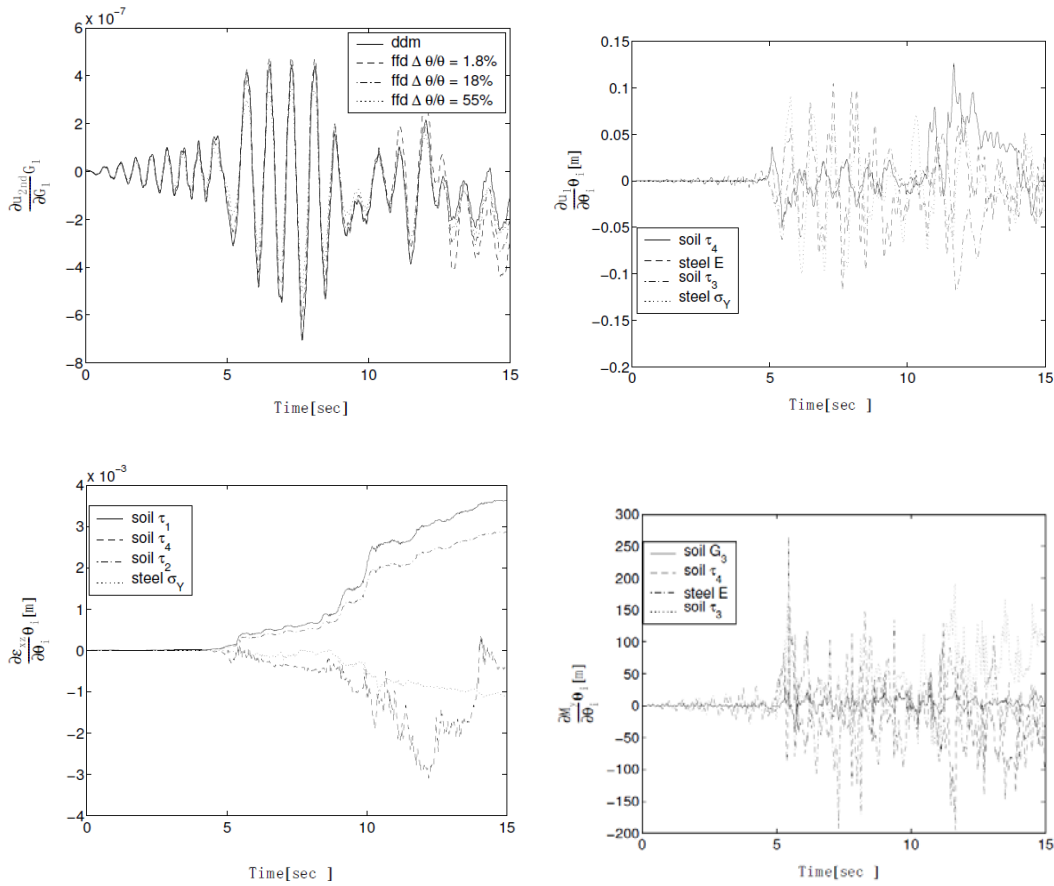


Figure 3: Verification and normalization of DDM based response sensitivities

4. CONCLUSIONS

This paper presents a framework for FE response sensitivity analysis of soil structure interaction (SSI) systems based on the direct differentiation method (DDM). A 3D frame-soil system is taken as application example, when subjected to dynamic earthquake base excitation and static push over loading conditions. The response sensitivity obtained by using DDM is verified by using Finite Difference method. The response sensitivities are also used to evaluate the relative importance of material parameters on the response quantities of interest.

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