第１４回偏微分方程式論
札幌シンポジウム

（代表者　上見　練太郎）

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代表者　上見　練太郎

記

１．　日　時　　　１９８９年８月３日（木）　～　８月５日（土）

２．　場　所　　　北海道大学数学教室　４－５０８室

３．　講　演

８月４日（金）

９：３０～１０：３０　服 部 晴 海（West Virginia Univ.）
Harumi Hattori
A phase transition problem - Existence and asymptotic behavior for an initial boundary value problem　・・・ ２２

１１：００～１２：００　Enrico Bernardi(Univ. of Bologna)
Hyperbolic operators with triple characteristic roots

１３：３０～１４：１０　＊

１４：１５～１４：４５　平 良 和 昭（筑波大数学系）
Kazuaki Taira (Tsukuba Univ.)
On initial boundary value problems for semilinear parabolic differential equations　・・・ ２３

１５：００～１５：３０　林 仲 夫、斎 藤 三 郎、相 川 弘 明（群馬大工）
Nakao Hayashi, Saburou Saitoh
Hiroaki Aikawa(Gunma Univ.)
Analyticity of solutions for semilinear heat equations in one space dimension　・・・ ３３

１５：３０～１６：３０　＊

８月５日（土）

９：３０～１０：３０　岩 塚 明（京大理）
Akira Iwatsuka(Kyoto Univ.)
Schrödinger operators with periodic magnetic fields　・・・ ３９

１１：００～１２：００　川 島 秀 一（九大工）
Shuichi Kawashima(Kyushu Univ.)
On initial boundary problems for descrete Boltzmann equations　・・・ ４５

１２：００～１３：００　＊

＊　この時間は講演者を囲んで自由な質問の時間とする予定です。

連絡先　北海道大学理学部数学教室
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Boundary behavior of harmonic maps on non-smooth domains and complete negatively curved manifolds

Patricio Aviles

Abstract: In this talk I shall discuss two classical topics of harmonic analysis; the solvability of the Dirichlet problem in regular domains and the Fatou's theorem in the case of the harmonic map system of equations. We shall also mention some ongoing work. Since every bounded harmonic function can be considered as a harmonic map into $\mathbb{R}$ our analysis below (and the one of the ongoing work) can be considered as a natural and important extension of part of the classical function theory to harmonic maps. There are several new difficulties that we have overcome in the process of doing harmonic analysis for harmonic mappings which can be briefly explained as follows. From the point of view of extending the classical known results for harmonic functions, cf. Hunt and Wheeden [HW], Dahlberg [D], Jerison and Kenig [JK] Anderson [A], Sullivan [S], Anderson and Schoen [AS] and Ancona [An], to harmonic maps the main difficulties come from the fact that we are dealing with a non-linear elliptic system as opposed to a linear equation which furthermore is a degenerated non-linear elliptic system when in a complete manifold. From the point of view of extending the by now classical existence results of harmonic maps due to Eells and Sampson [ES], and Hildebrandt Kaul and Widman [HKW] the difficulties in the situation at hand lies in the important fact that we are

dealing with possible unrectifiable boundaries and therefore the classical methods of the calculus of variations used by the authors mentioned above are not applicable.

We shall now describe our results, see [A], [ACM]. Let Ω be an open connected set in a complete Riemannian manifold. If Ω is a Greenian domain, that is, the Green function with pole at $x \in \Omega$ exists for all $x \in \Omega$, i.e. $\Delta_g G = \delta(x)$, $G|\partial\Omega = 0$, then the Martin boundary, $\mathcal{M}(\Omega)$, can be defined, see [A]. Furthermore, if we let $\hat{\Omega} = \Omega \cup \mathcal{M}(\Omega)$, $\hat{\Omega}$ is complete and compact with boundary $\mathcal{M}(\Omega)$ with respect to the so-called Martin metric, which defines a topology in Ω which agrees with its topology as a Riemannian manifold, see [Db] for further details. We shall now make the following

<u>Definition</u>: Ω is a <u>regular</u> domain if for every point $\xi \in \mathcal{M}(\Omega)$ there exists $\omega_\xi \in C^0(\hat{\Omega})$ so that

(i) ω_ξ is super-harmonic (with respect to the Laplace-Beltrami operator) in Ω;

(ii) $\omega_\xi > 0$ in $\overline{\Omega} - \{\xi\}$, $\omega_\xi(\xi) = 0$.

Examples of regular domains which satisfy the above definition are those considered in [HW], [D], [JK], [Wu], [Ad], [S], [Ad-Sc], [An].

The image of the mapping in consideration will be required to be in a convex ball $\overline{B}_\tau(p)$ of a complete C^∞ Riemannian manifold N. Consequently $\overline{B}_\tau(p)$ shall denote the closed geodesic ball of radius τ and center at p in N with

$$r < \min\{\frac{\pi}{2\sqrt{K}}, \text{ injectively radius of } N \text{ at } p\}$$

where $K \geq 0$ is an upper bound for the sectional curvatures of N. We remark that, in general, there are examples that show that the theorems below do not hold without the hypothesis that the image of the mapping is in a ball as described above.

We also recall that for $f \in C^1(M,N)$, the energy density $e(f)$ of f is defined by

$$e(f)(x) = \frac{1}{2} g^{ij}(x)h_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}$$

where $x = (x^1,\dots,x^m)$, $y = (y^1,\dots,y^n)$ are local co-ordinates on M and N respectively, $f^\alpha(x) = y^\alpha(f(x))$, $g_{ij}dx^i dx^j$ and $h_{\alpha\beta}dy^\alpha dy^\beta$ define the Riemannian metrics on M and N respectively and the matrix (g^{ij}) is the inverse of (g_{ij}). The energy $E(f)$ of f is defined by $E(f) = \int_M e(f)dv$ where dv is the volume element of M. The map $u \in C^1(M,N)$ is said to be harmonic, if it is a critical point of the functional $E : C^1(M,N) \to \mathbb{R}$ with respect to compactly supported variations. A simple calculation shows that the Euler–Lagrange system of equations have to be satisfied by a harmonic map are

$$\Delta u^\alpha(x) + g^{ij}(x)\, \Gamma^\alpha_{\beta\gamma}(u(x)) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} = 0$$

where Δ is the Laplace–Beltrami operator on M and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffe symbols of the metric on N.

Theorem A (Dirichlet Problem). Let Ω be a regular domain for the Dirichlet problem for the Laplace–Beltrami operator. For each $\phi \in C^0(\mathcal{M}(\Omega), \mathbb{B}_\tau(p))$ there exists a unique $u \in C^0(\hat{\Omega}, \mathbb{B}_\tau(p)) \cap C^\infty(\Omega, \mathbb{B}_\tau(p))$ which is a harmonic map on Ω and which equals ϕ on $\mathcal{M}(\Omega)$.

A classical problem in linear harmonic analysis is to study the boundary regularity of harmonic functions in arbitrary bounded domain. We recall the Wiener criterion. We shall assume that the dimension of the domains is greater or equal to three.

Wiener Condition. Let Ω be a bounded domain which is contained in the interior of a complete Riemannian manifold M. Given a point $x_0 \in \partial\Omega$, let $X(\sigma) = \sigma^{2-n}\mathrm{Cap}\{x \in B_\sigma(x_0) | x \notin \Omega\}$ where the capacity is measured with respect to a fixed smooth compact subset R of M which contains $B_\sigma(x_0)$ in its interior. x_0 is said to satisfy the Wiener condition if $\sum_{j=1}^{\infty} X(\sigma^j)$ diverges for $\sigma \in (0,1)$.

Definition. Let Ω be a bounded domain which is contained in the interior of a complete Riemannian manifold M. We say $x_0 \in \partial\Omega$ is a regular point for the Laplace–Beltrami operator, if for every $\phi \in L^\infty(\partial\Omega, \mathbb{R})$ which is continuous at x_0, the solution of the Dirichlet problem with boundary values ϕ is continuous at x_0.

Theorem (Wiener criterion [Wn], [LSW]). x_0 is regular for the Laplace–Beltrami operator iff x_0 satisfies the Wiener condition.

We now make the following

Definition. Let Ω be a bounded domain which is contained in the interior of a complete Riemannian manifold M, $x_0 \in \partial\Omega$ is regular for the harmonic map system if for all $\phi \in L^\infty(\partial\Omega, \bar{B}_\tau(p))$ which is continuous at x_0, the solution of the Dirichlet problem for the harmonic map system with boundary values ϕ is continuous at x_0. We then have

Theorem B. x_0 is a regular point for the harmonic map system iff x_0 satisfies the Wiener Condition.

Using different methods and ideas a version of the Wiener text for harmonic mappings of finite energy was established by Paulik [P]).

It is of interest to study the solution of the Dirichlet problem of the harmonic map system with further details. We shall next state a result in which we discuss sharp bounds for such solutions in the important case of simply connected negatively curved manifolds.

If M denote a complete, simply connected Riemannian manifold of dimension m, with sectional curvature K_M, $-b^2 \le K_M \le -a^2 < 0$, the sphere at infinity $S(\infty)$ of M is defined to be the set of asymptotic classes of geodesic rays in M: two rays γ_1 and γ_2 are asymptotic if $\text{dist}(\gamma_1(t), \gamma_2(t))$ is bounded for $t \ge 0$. The cone topology on $\bar{M} = M \cup S(\infty)$ is defined by: let q be a fixed point in M and let γ be a geodesic ray passing through q with tangent vector v at q. The cone $C_q(\gamma,\theta)$ of angle θ about γ is defined by $C_q(\gamma,\theta) = \{x \in M | \text{angle between } v \text{ and tangent vector at } q \text{ of geodesic joining } q \text{ to } x \text{ is less than } \theta\}$. Let $T_q(\gamma,\theta,R) = C_q(\gamma,\theta) \mid \bar{B}_R(q)$ denote a truncated cone, then the domains $T_q(\gamma,\theta,R)$ together with the open geodesic balls $B_\delta(x), x \in M$ form a local basis for the cone topology. Let $\xi : [0,1] \to [0,\infty]$ be a fixed homeomorphism which is diffeomorphism on $[0,1)$. The map $E(v) = \exp_q(\xi(|v|)v)$ is a diffeomorphism of the open unit ball $B_1(0)$ in $T_q(M)$ onto M; moreover E_ξ extends to a homeomorphism of the sphere $S_1 = \partial B_1(0)$ into $S(\infty)$. We identify $\bar{M}$, M and $S(\infty)$ with $\bar{B}_1(0)$, $B_1(0)$ and S_1 respectively.

It was proved by Anderson [Ad], Sullivan [S] and later by Schoen ([Ad–Sc] pp. 435–438) that every point of $S(\infty)$ is a regular point for the Dirichlet problem with respect to the Laplace–Beltrami operator on M. Furthermore Anderson and Schoen [Ad–Sc] showed that the Martin boundary of M is homeomorphic to $S(\infty)$. Hence the solution to the Dirichlet problem with data $\phi \in C^0(S(\infty), \bar{B}_\tau(p))$ exists.

<u>Theorem</u> C (Bounds for the solution of the Dirichlet problem). Let M be a complete, simply connected Riemannian manifold of dimension m, with sectional curvatures $-b^2 \le K_M \le -a^2 < 0$ and let $\bar{B}_\tau(p)$ as described above. Given $\phi \in C^\alpha(S(\infty), \bar{B}_\tau(p))$, $\alpha \in (0,1]$, the harmonic map $u : \bar{M} \to \bar{B}_\tau(p)$, $u|_{S(\infty)} = \phi$ satisfies the following decay estimates;

(i) $\rho(u(x), \phi(x)) \le C_1 \, e^{-1/2 \; \delta r(x)}$

(ii) for any $\beta \in [0,1)$, $|Du(x)|_{C^{0,\beta}} \le C_2 e^{-1/2 \; \delta r(x)}$

where $r(x)$ = distance of x from some fixed point $q \in M$, $\delta > 0$ is

$$\delta = \begin{cases} \alpha a & \text{if } \alpha < 1 \text{ or } \alpha = 1 \text{ and } m \ge 3 \\ \text{any positive number strictly less than } a & \text{if } \alpha = 1 \text{ and } m = 2, \end{cases}$$

C_1, C_2 depend on the geometry, $(\bar{B}_\tau(p), a, b, \phi, m)$ but u itself. If only $\phi \in C^0(S(\infty), \bar{B}_\tau(p))$ then the energy density $e(u)(x) \to 0$ as $x \to S(\infty)$.

Finally, I shall state the Fatou's theorem for harmonic mappings. This is a quite interesting result. Technically speaking, it is interesting because the basic tools that there are used to prove it for bounded harmonic functions, that is, boundary representation, the correct inequality between the non-tangential maximal function and the Hardy-Littlewood maximal function associated to the harmonic function (see for instance Stein's book [St]) are not available or they are not true. Geometrically it is interesting because if we combine it with the solution of the Dirichlet problem for L^∞-data we obtain the following.

<u>Theorem</u> D. There is a one to one correspondence between $\phi \in L^\infty(\partial\Omega, \bar{B}_\tau(p))$ and harmonic maps $u \in L^\infty(\bar{\Omega}, \bar{B}_\tau(p)) \cap C^\infty(\Omega, \bar{B}_\tau(p))$ so that $\lim u(x) = \phi(Q)$ almost everywhere $Q \in \partial\Omega$ as $x \to Q$, where almost everywhere is with respect to the natural measure associated to $\partial\Omega$ (harmonic measure) and where $x \to Q$ non-tangentially.

<u>Theorem</u> E (Fatou's theorem). Let $u : \Omega \to \bar{B}_\tau(p)$ be a harmonic map where $\bar{B}_\tau(p)$ is as defined above and Ω is either (i) a bounded Lipschitz domain contained in the interior of a complete Riemannian manifold or (ii) a complete, simply connected Riemannian manifold with sectional curvatures K_Ω, $-b^2 \le K_\Omega \le -a^2 < 0$. Then u has the Fatou property, that is, for almost every $Q \in \partial\Omega$ with respect to the harmonic measure in $\partial\Omega$, $\lim_{x\to Q} u(x)$ exists whenever $x \to Q$ non-tangentially.

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Uniqueness and Existence of Viscosity Solutions of Generalized Mean Curvature Flow Equations

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1. Introduction. We construct unique global continuous viscosity solutions of the initial value problem in $\mathbb{R}^n$ for a class of degenerate parabolic equations that we shall call *geometric*. A typical example is

$$(1) \qquad u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \nu|\nabla u| = 0 \quad \left(u_t = \frac{\partial u}{\partial t},\ \nabla u = \text{grad } u, \quad \nu \in \mathbb{R}\right).$$

Our method is based on the comparison principle of viscosity solutions developed recently by Jensen [8] and Ishii [6]. However, as is observed from (1), our equation is singular at $\nabla u = 0$ so we are forced to extend their theory to our situation.

The equation (1) has a geometric significance because γ-level surface $\Gamma(t)$ of u moves by its mean curvature when $\nu = 0$ provided that ∇u does not vanish on $\Gamma(t)$. Such a motion of surfaces has been studied by many authors [1,5]. However, so far whole *unique* evolution families of surfaces were only constructed under geometric restrictions on initial surfaces such as convexity [3,5] except $n = 2$ [1,4]. When $n = 2$, Grayson [4] has shown that any embedded curve moved by its curvature never becomes singular unless it shrinks to a point. However when $n \geq 3$ even embedded surfaces may become singular before it shrinks to a point.

Our goal is to construct whole evolution family of surfaces even after the time when there appear singularities. This program is carried out by Angenent [1] when $n = 2$. Contrary to [1] we avoid parametrization and rather understand surfaces as level set of

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viscosity solutions of (1). Let $D(t)$ denote the open set of $x \in \mathbb{R}^n$ such that $u(x,t) > \gamma$. When the equation is geometric, it turns out that the family $(\Gamma(t), D(t))$ $(t \geq 0)$ is uniquely determined by $(\Gamma(0), D(0))$ and is independent of u and γ. By unique existence of viscosity solution of (1) we have a unique family of $(\Gamma(t), D(t))$ for all $t \geq 0$ provided $D(0)$ is bounded open and that $\Gamma(0)$ $(\subset \mathbb{R}^n \setminus D(0))$ is compact. As is expected, we conclude that $(\Gamma(t), D(t))$ becomes empty in a finite time provided $\nu \leq 0$. This extends a result of Huisken [5] where he proved that $\Gamma(t)$ disappears in a finite time provided $\Gamma(0)$ is a uniformly convex C^2 hypersurface.

In this note we state our main results almost without proofs; the details will be published elsewhere.

2. A parabolic comparison principle. For $h : L \to \mathbb{R}$ $(L \subset \mathbb{R}^d)$ we associate its *lower (upper) semicontinuous relaxation* $h_*(h^*) : \bar{L} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ defined by

$$h_*(z) = \lim_{\varepsilon \downarrow 0} \inf_{|z-y|<\varepsilon} h(y), \quad h^*(z) = -(-h)_*(z), \quad z \in \bar{L}.$$

Let Ω be an open subset of $\mathbb{R}^n$. We write $J(\Omega) = \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$ and $W = \Omega \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \times S^{n\times n}$ where $S^{n\times n}$ denotes the space of $n \times n$ real symmetric matrices. Let $F = F(t, x, s, p, X)$ be a real valued function defined in $(0, T] \times W$ for $T < \infty$. Since W is dense in $J(\Omega)$, we see $F^*, F_* : [0,T] \times J(\Omega) \to \bar{\mathbb{R}}$. Any function $u : \Omega_T \to \mathbb{R}$ is called a *viscosity sub-(super) solution* of

$$u_t + F(t, x, u, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad \Omega_T = (0,T] \times \Omega \tag{2}$$

if $u^* < \infty (-\infty < u_*)$ on $\bar{\Omega}_T$ and if, whenever $\phi \in C^2(\Omega_T)$, $(t,y) \in \Omega_T$ and $(u^* - \phi)(t,y) = \max_{\Omega_T}(u^* - \phi)$ $((u_* - \phi)(t,y) = \min_{\Omega_T}(u_* - \phi))$

$$\phi_t(t,y) + (F_*(t, y, u^*(t,y), \nabla\phi(t,y), \nabla^2\phi(t,y)) \leq 0$$

$$(\phi_t(t,y) + (F^*(t, y, u_*(t,y), \nabla\phi(t,y), \nabla^2\phi(t,y)) \geq 0).$$

If $u : \Omega_T \to \mathbb{R}$ is both a viscosity sub- and supersolution of (2), u is called a *viscosity solution* of (2). We say F is *degenerate elliptic* if

$$F(t, x, s, p, X + Y) \leq F(t, x, s, p, X)$$

for every $(t,x,s,p,X) \in W$ and $Y \geq O$. We say (2) is a *degenerate parabolic* equation if F is degenerate elliptic.

Example 1: The equation (1) is degenerate parablic since (1) is expressed in the form (2) by taking

$$F(t,x,s,p,X) = -\operatorname{trace}((I - \bar{p}^t\bar{p})X) - \nu|p|, \quad \bar{p} = p/|p|.$$

Example 2: For $\omega \geq 0$ we set

$$\psi^{\pm}(t,x) = \mp(|x| - \omega t)^4 \quad \text{if} \quad |x| > \omega t \quad \text{otherwise} \quad \psi^{\pm}(t,x) = 0. \tag{3}$$

Suppose that F is elliptic and satisfies

$$-\nu|p| \leq F(t,x,s,p,O)(\leq \mu|p|) \quad \text{in} \quad W \tag{4}$$

for some constant $\nu(\mu)$. Then $\psi^+(\psi^-)$ is a viscosity super-(sub) solution of (2) with $\Omega = \mathbb{R}^n$ provided $\omega \geq \nu$ $(\omega \geq \mu)$.

Example 3: The function $U_{\xi h}(t,x) = h(2(n-1)t + |x-\xi|^2)$ is a viscosity solution of (1) in $\mathbb{R}^n_T$ for every T when $\nu = 0$ provided that h is a continuous monotone function on $\mathbb{R}$.

We now state our main comparison result.

Theorem 4. *Let Ω be bounded and let $F : (0,T] \times W \to \mathbb{R}$ is continuous, degenerate elliptic and independent of $x \in \Omega$. Assume that there is a constant $c = c(\Omega, T, M, n)$ such that the function $s \mapsto F(t,s,p,X) + cs$ is nondecreasing in $s \in \mathbb{R}$ for all $t \in (0,T]$, $|s| \leq M$, $p \in \mathbb{R}^n \setminus \{0\}$, $X \in S^{n\times n}$. Suppose furthermore that*

$$-\infty < F_*(t,s,0,O) = F^*(t,s,0,O) < \infty, \quad t \in (0,T], \quad s \in \mathbb{R}. \tag{5}$$

Let u and v be, respectively, viscosity sub- and supersolutions of (2) in Ω_T. If $u^* \leq v_*$ on $\partial_p\Omega_T = \{0\} \times \Omega \cup [0,T] \times \partial\Omega$, then $u^* \leq v_*$ on Ω_T.

REMARK 5: If two inequalities in (4) hold for F and $(t,s,X) \mapsto F(t,s,p,X)$ is equicontinuous for small p, then (5) holds. In particular Theorem 4 is applicable to the equation (1). Although our proof is based on a parabolic version of Ishii's Proposition 5.1 in [6] (cf.[7]), new idea is necessary to prove Theorem 4 since F is not continuous at $p=0$ even if we consider its elliptic version. We note that Theorem 3.1 in [6] can be extended even if F is not continuous at $p=0$ provided (5) holds. Using Perron's method as in [6] we obtain an existence result.

THEOREM 6. *Let Ω and F be as above. Suppose that there is a viscosity subsolution f and a viscosity supersolution g of (2) such that f,g are locally bounded in $\bar{\Omega}_T$, $f \leq g$ in Ω_T and $f_* = g^*$ on $\partial_p\Omega_T$. Then there is a viscosity solution u of (2) satisfying $u \in C(\bar{\Omega}_T)$ and $f \leq u \leq g$ on $\bar{\Omega}_T$, where $\bar{\Omega}_T = \partial_p\Omega_T \cup \Omega_T$.*

3. Geometric equations. We consider a special class of degenerate parabolic equations including (1).

DEFINITION 7: A function $F : (0,T] \times W \to \mathbb{R}$ is called *geometric* if F does not depend on $s \in \mathbb{R}$ i.e.

$$F(t,x,s,p,X) = F(t,x,p,X)$$

and for every $\lambda > 0$ and $\sigma \in \mathbb{R}$ it holds

$$F(t,x,\lambda p, \lambda X + \sigma p\, {}^t p) = \lambda F(t,x,p,X).$$

THEOREM 8. *Suppose that F is degenerate elliptic and geometric in $(0,T] \times W$. If u is a locally bounded viscosity sub-(super) solution of (2) in Ω_T, so is $\theta(u)$ whenever $\theta : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function.*

The proof depends on approximation of u by semiconvex Lipschitz functions. Example 3 follows from Theorem 8.

4. Evolutions of level surfaces Suppose that $a \in C(\mathbb{R}^n)$ and $a - \alpha$ is compactly supported for some $\alpha \in \mathbb{R}$. Let u_a denote a viscosity solution of (2) in Ω_T such that

$u_a \in C(\bar{\Omega}_T)$ with $u(0,x) = a(x)$ and that $u - \alpha$ has a compact support in $\bar{\Omega}_T$. We state our uniqueness result when $\Omega = \mathbb{R}^n$.

THEOREM 9. *For $\Omega = \mathbb{R}^n$ we assume F is continuous, geometric, degenerate elliptic and is independent of x in $(0,T] \times W$. Suppose that F satisfies (4) and (5). Then there is at most one viscosity solution u_a of (2) in Ω_T with initial data a. Moreover, if $b \geq a$ then $u_b \geq u_a$ on $\bar{\Omega}_T$.*

PROOF: We may assume $\alpha = 0$. For $\psi^{\pm}$ in (3) we set

$$f_R = \min(\psi^- - R^4, 0), g_R = \max(\psi^+ + R^4, 0)$$

where $\omega \geq \max(\nu, \mu)$ and $R > 0$. We take R large enough so that $f_R \leq a(x) \leq b(x) \leq g_R$ holds at $t = 0$. Example 2 and Theorem 8 imply that f_R and g_R are, respectively, viscosity sub- and supersolutions of (2) in $\mathbb{R}^n_T$. Take R_1 such that u_a, u_b, f_R, g_R are supported in $[0,T] \times B(R_1)$ where $B(R_1)$ denotes the open ball of radius R_1 centered at the origin. Applying comparison Theorem 4 with $\Omega = B(R_1)$ yields $u_b \geq u_a$. This implies uniqueness of u_a.

Theorems 8 and 9 yield

THEOREM 10. *Suppose F and u_a are as in Theorem 9. Let $\Gamma(t)$ be γ-level set of $u_a(t,\cdot)$ and $D(t)$ be a set of $x \in \mathbb{R}^n$ such that $u_a(t,x) > \gamma$. If $\gamma > \alpha$ then $(\Gamma(t), D(t))$ $(t \geq 0)$ is uniquely determined by $(\Gamma(0), D(0))$ and is independent of a, α and γ. We call $(\Gamma(t), D(t))$ is a solution family of (2) with intial data $(\Gamma(0), D(0))$.*

When (2) is quasilinear, one can construct a global viscosity solution u_a for a given initial data a.

THEOREM 11. *Suppose that F and a are as in Theorem 9 and that F is linear in X. Then the viscosity solution u_a of (2) in Theorem 9 (uniquely) exists for every $T > 0$.*

For general F we approximate (2) by uniformly parabolic equations and prove convergence of approximate solutions at least for $a \in C^2$. Here f_R and g_R in (6) play a role of "barriers" to get uniform estimates for first derivatives of approximate solutions. As their limit we obtain the viscosity solution u_a. For continuous a, we can approximate it by regular functions and find that Theorem 6 is applicable to get the solution .

For the equation (1) with $\nu = 0$ one can construct u_a via Theorem 6 without using approximate equations. There are viscosity sub- and supersolutions f, g satisfing assumptions of Theorem 6 with $f = g = a$ at $t = 0$. Indeed, for $\xi \in \mathbb{R}^n$ there is a decreasing continuous function h such that $U_{\xi h}(0, x) \leq a(x)$ and $U_{\xi h}(0, \xi) = a(\xi)$ where $U_{\xi h}$ is in Example 3. We define f as the supremum of such $U_{\xi h}$ and find that $f = a$ at $t = 0$ and f is a viscosity subsolution of (2) in $\mathbb{R}^n_T$. The function g can be constructed similarly. By comparison with $f_R + \alpha, g_R + \alpha$ in (6), we see $f = g = \alpha$ outside $[0, T] \times B(R)$ if R is sufficiently large. Theorem 6 with $\Omega = B(R)$ yields the desired solution u_a by defining its value as α outside $B(R)$.

Corollary 12. *(i) Suppose F is as in Theorem 11. Suppose that $D(0)$ is a bounded open set and $\Gamma(0) \subset \mathbb{R}^n \setminus D(0)$ is a compact set. Then there is a unique solution family $(\Gamma(t), D(t))$ for all $t \geq 0$ with initial data $(\Gamma(0), D(0))$.*

(ii) Let $(\Gamma(t), D(t))$ be a solution family of (1) with $\nu \leq 0$ such that $D(0) \cup \Gamma(0)$ is bound. Then $(\Gamma(t), D(t))$ becomes empty in finite time.

We note (i) follows from Theorems 10 and 11 with a suitable choice of a. For mean curvature flow equation (1), Example 3 yields (ii) by a comparison.

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Homoclinic orbits for a singular second order Hamiltonian system

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1. Introduction and statement of result

There is a large literature on the use of variational methods to prove the existence of periodic solutions of Hamiltonian systems. However it is only recently that these methods have been applied to the existence of homoclinic or heteroclinic orbits of Hamiltonian system. See [**3,4,7,9,10,11**].

Our purpose of this talk is to consider the existence of homoclinic orbits for the second order *singular* Hamiltonian system:

$$\ddot{q} + V'(q) = 0. \tag{HS}$$

Here $q = (q_1, q_2, \cdots, q_N)$, $N \geq 3$ and $V(q)$ satisfies

$(V1)$ There is an $e \in \mathbf{R}^N$, $e \neq 0$ and $V \in C^2(\mathbf{R}^N \setminus \{e\}, \mathbf{R})$;

$(V2)$ $V(q) \leq 0$ for all $q \in \mathbf{R}^N \setminus \{e\}$ and $V(q) = 0$ if and only if $q = 0$, and $\limsup_{|q| \to \infty} V(q) \equiv \overline{V} < 0$;

$(V3)$ There is a constant $\delta \in (0, \frac{1}{2} \mid e \mid)$ such that $V(q) + \frac{1}{2}(V'(q), q) \leq 0$ for all $q \in B_\delta(0)$, where $B_\delta(0) = \{x \in \mathbf{R}^N; \mid x \mid < \delta\}$;

$(V4)$ $-V(q) \to \infty$ as $q \to e$;

$(V5)$ There is a neighbourhood W of e in $\mathbf{R}^N$ and a function $U \in C^1(\mathbf{R}^N \setminus \{e\}, \mathbf{R})$ such that $U(q) \to \infty$ as $q \to e$ and $-V(q) \geq \mid U'(q) \mid^2$ for $q \in W \setminus \{e\}$.

Our main result is as follows:

Theorem ([**11**]). *If V satisfies $(V1) - (V5)$, then (HS) possesses at least one (nontrivial) homoclinic orbit which begins and ends at* 0.

Remark. The assumption $(V5)$ is the so-called *strong force* condition (c.f. Gordon [**5**]) and it will be used to verify the Palais-Smale compactness condition for the functional corresponding to the approximate problem $(HS:T)$ (see below). For example, $(V5)$ is satisfied when $V(q) = -\mid q-e \mid^{-\alpha}$ $(\alpha \geq 2)$ in a neighbourhood of e. The assumption $(V3)$ is a kind of concavity condition for $V(q)$ near 0. In particular $(V3)$ holds for small $\delta > 0$ when $V''(0)$ is negative definite.

This work is largely motivated by the work of Rabinowitz [**9**] and the works [**1**,**2**,**6**]. [**9**] studied via a variational method the existence and the multiplicity of *heteroclinic* orbits joining global maxima of $V(q)$ for a periodic Hamiltonian system. On the other hand [**1**,**2**,**6**] studied the existence of time periodic solutions of prescribed period for the second order singular Hamiltonian system (HS).

2. Outline of the proof of Theorem

First we consider the approximate problem:

$$\begin{aligned} &\ddot{q} + V'(q) = 0, \quad \text{in } (0,T), \\ &q(0) = q(T) = 0. \end{aligned} \tag{$HS:T$}$$

Solutions of this approximate problem will be obtained as critical points of the functional $I_T(q)$. We show the existence of critical points of $I_T(q)$ via minimax argument, which is essentially due to Bahri and Rabinowitz [**2**] (see also Lyusternik and Fet [**8**]). We also get some estimates, which are uniform with respect to $T \geq 1$, for minimax values and corresponding critical points $q(t;T)$. These uniform estimates permit us to let $T \to \infty$; for a suitable sequence $(\tau_k)_{k=1}^{\infty}$ and a subsequence $T_k \to \infty$, we see $q(t+\tau_k;T_k)$ converges weakly to a homoclinic orbit of (HS) as $k \to \infty$.

Let $H_0^1(0,T;\mathbf{R}^N)$ denote the usual Sobolev space on $(0,T)$ with values in $\mathbf{R}^N$ under the norm $\|q\| = (\int_0^T \mid \dot{q} \mid^2 dt)^{1/2}$. Let

$$\Lambda_T = \{q \in H_0^1(0,T;\mathbf{R}^N)\,;\, q(t) \neq e \text{ for all } t \in [0,T]\,\}.$$

Clearly Λ_T is an open subset of $H_0^1(0,T;\mathbf{R}^N)$. Consider

$$I_T(q) = \frac{1}{2}\int_0^T \mid \dot{q} \mid^2 dt - \int_0^T V(q)dt \in C^1(\Lambda_T, \mathbf{R}).$$

Then there is an one-to-one correspondence between critical points of $I_T(q)$ and classical solutions of $(HS:T)$.

We can see that $I_T(q)$ satisfies the Palais-Smale compactness condition on Λ_T, that is,

$(P.S.)$: if $(q_m)_{m=1}^{\infty} \subset \Lambda_T$ is a sequence such that $I_T(q_m)$ is bounded and $I'_T(q_m) \to 0$, then (q_m) possesses a subsequence converging to some $q \in \Lambda_T$,

and we can apply the minimax argument to $I_T(q)$.

Now we introduce a minimax procedure for $I_T(q)$. Let

$$D^{N-2} = \{x \in \mathbf{R}^{N-2};\ | x | \leq 1\},$$

$$\Gamma_T = \{\gamma \in C(D^{N-2}, \Lambda_T);\ \gamma(x)(t) = 0 \text{ for all } x \in \partial D^{N-2} \text{ and } t \in [0,T]\}.$$

For $\gamma \in \Gamma_T$ we observe $\gamma(x)(t) = 0$ for all $(x,t) \in (\partial D^{N-2} \times [0,T]) \cup (D^{N-2} \times \{0, T\}) \equiv \partial(D^{N-2} \times [0,T])$. Since $D^{N-2} \times [0,T]/\partial(D^{N-2} \times [0,T]) \simeq S^{N-1}$, we can associate for each $\gamma \in \Gamma_T$ a map $\widetilde{\gamma} : S^{N-1} \to S^{N-1}$ defined by

$$\widetilde{\gamma}(x,t) = \frac{\gamma(x)(t) - e}{| \gamma(x)(t) - e |}.$$

We denote by $\deg \widetilde{\gamma}$ the Brouwer degree of a map $\widetilde{\gamma} : S^{N-1} \to S^{N-1}$. Let

$$\Gamma_T^* = \{\gamma \in \Gamma_T;\ \deg \widetilde{\gamma} \neq 0\}.$$

It is clear that $\Gamma_T^* \neq \emptyset$. We define a minimax value of $I_T(q)$ by

$$c(T) = \inf_{\gamma \in \Gamma_T^*} \sup_{x \in D^{N-2}} I_T(\gamma(x)).$$

Using the standard deformation argument, we have

Proposition 1. *$c(T) > 0$ is a critical value of $I_T(q)$, that is, the problem $(HS : T)$ has a solution $q(t;T)$ such that $I_T(q(\,\circ\,;T)) = c(T)$. Moreover, there are constants c_0, $c_1 > 0$ which are independent of $T \geq 1$ such that*

$$0 < c_0 \leq c(T) = I_T(q(\,\circ\,;T)) \leq c_1 < \infty \quad \textit{for } T \geq 1. \tag{1}$$

Here the estimate (1) is obtained from the minimax characterization of $c(T)$. We can use (1) to get uniform estimates for $q(t;T)$. We get directly from (1)

$$\| \dot{q}(\,\circ\,;T)\|_{L^2(0,T)}, \quad \int_0^T -V(q(t;T))\, dt \leq C \quad \text{for all } T \geq 1. \tag{2}$$

Moreover we can deduce from (2) and $(V2)$ that

$$\|q(\,\circ\,;T)\|_{L^\infty(0,T)} \leq C, \tag{3}$$

$$E_T \to 0 \quad \text{as } T \to \infty, \tag{4}$$

where

$$E_T \equiv \frac{1}{2} | \dot{q}(t;T) |^2 + V(q(t;T)).$$

On the other hand, the following proposition gives us an L^∞-bound from below for $q(t;T)$. Here the condition $(V3)$ plays an role.

Proposition 2. $\|q(\circ\,;T)\|_{L^\infty(0,T)} \geq \delta$ *for all* $T \geq 1$.

By the above proposition, we can find a number $\tau_T \in (0,T)$ such that

$$q(\tau_T;T) \in \partial B_\delta(0). \tag{5}$$

Now we construct a homoclinic orbit of (HS) as a limit of $q(t;T)$ as $T \to \infty$ and we complete the proof of Theorem.

For each $T \geq 1$, we define $\widetilde{q}(t;T) \in H^1(\mathbf{R},\mathbf{R}^N)$ by

$$\widetilde{q}(t;T) = \begin{cases} q(t+\tau_T;T), & \text{if } t \in [-\tau_T, T-\tau_T]; \\ 0, & \text{otherwise.} \end{cases}$$

By (2)-(3), we can extract a subsequence $T_k \to \infty$ such that $\widetilde{q}(t;T_k)$ converges to some $y(t) \in C(\mathbf{R},\mathbf{R}^N) \cap L^\infty(\mathbf{R},\mathbf{R}^N)$ with $\mathring{y}(t) \in L^2(\mathbf{R},\mathbf{R}^N)$ in the following sense:

$$\begin{aligned} \widetilde{q}(t;T_k) &\to y(t) \qquad \text{in } L^\infty_{loc}(\mathbf{R},\mathbf{R}^N), \\ \mathring{\widetilde{q}}(t;T_k) &\rightharpoonup \mathring{y}(t) \qquad \text{weakly in } L^2(\mathbf{R},\mathbf{R}^N), \end{aligned}$$

$$\int_{-\infty}^{\infty} -V(y(t))\,dt \leq \liminf_{T\to\infty} \int_{-\infty}^{\infty} -V(\widetilde{q}(t;T))\,dt \leq C < \infty,$$

By (5), $y(t) \not\equiv 0$ and we can see that $y(t)$ is a nontrivial solution of (HS) on $\mathbf{R}$ such that

$$y(t),\ \mathring{y}(t) \to 0 \quad \text{as } t \to \pm\infty. \qquad \blacksquare$$

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Title

Existence and asymptotic behavior for an equation related to a phase transition problem.

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Abstract

I would like to discuss the following initial boundary value problem:

$$u_{tt} = \sigma(u_x)_x + \nu u_{xxt} - \eta u_{xxx} \qquad 0 < x < 1,\ t > 0, \qquad (1)$$

$$\text{B.C.}\quad u(0,t) = 0, \quad \sigma(u_x(1,t)) + \nu u_{xt}(1,t) - \eta u_{xxx}(1,t) = P, \qquad (2)$$

$$u_{xx}(0,t) = 0, \quad u_{xx}(1,t) = 0, \qquad (3)$$

$$\text{I.C.}\quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad (4)$$

where σ is a nonmonotone function of its argument. The boundary conditions (2) corresponds to a bar in a soft loading device and (3) are the natural boundary conditions for the variational problem corresponding to the static problem for (1).

The above problem is related to a phase transition problem. First, I shall discuss the various results on the phase transition problem related to the above equation. Then, I would like to show the existence and asymptotic behavior of solutions to the above problem and discuss the connecting orbit problem when there are more than one steady state solutions.

On Initial-Boundary Value Problems For Semilinear Parabolic Differential Equations

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ABSTRACT

This Note is devoted to an L^p approach to a class of *degenerate* boundary value problems for second-order elliptic differential operators which includes as particular cases the Dirichlet and Neumann problems. By using the L^p theory of pseudo-differential operators, we show that this class of boundary value problems provides a new example of analytic semigroups. Furthermore, via the theory of analytic semigroups, one can apply this result to a class of initial-boundary value problems for *semilinear* parabolic differential equations. Our semigroup approach can be traced back to the pioneering work of Fujita-Kato on the Navier-Stokes equation.

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Key words and phrases : Elliptic boundary value problems, L^p approach, analytic semigroups, initial-boundary value problems, semilinear parabolic equations.

§1. Introduction and Results

Let Ω be a bounded domain of Euclidian space $\mathbb{R}^n$, with C^∞ boundary Γ; its closure $\overline{\Omega} = \Omega \cup \Gamma$ is an n-dimensional, compact C^∞ manifold with boundary Γ. We let

$$A = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} + c(x)$$

be a second-order *elliptic* differential operator with real C^∞ coefficients on $\overline{\Omega}$ such that :

1) $a^{ij}(x) = a^{ji}(x)$, $x \in \overline{\Omega}$, $1 \leqq i, j \leqq n$.

2) $\displaystyle\sum_{i,j=1}^{n} a^{ij}(x)\, \xi_i \xi_j \geqq c_0 |\xi|^2$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$,

   with a constant $c_0 > 0$.

We consider the following boundary value problem : Given functions f and ϕ defined in Ω and on Γ respectively, find a function u in Ω such that

$$(*) \begin{cases} (A - \lambda)u = f & \text{in } \Omega, \\ Bu \equiv a \dfrac{\partial u}{\partial \nu} + bu|_\Gamma = \phi & \text{on } \Gamma. \end{cases}$$

Here :

1° λ is a complex parameter.

2° a and b are real-valued C^∞ functions on Γ.

3° $\partial/\partial\nu$ is the conormal derivative associated with the matrix (a^{ij}) :

$$\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{n} a^{ij} n_j \frac{\partial}{\partial x_i},$$

$\mathbf{n} = (n_1, \dots, n_n)$ being the unit exterior normal to Γ.

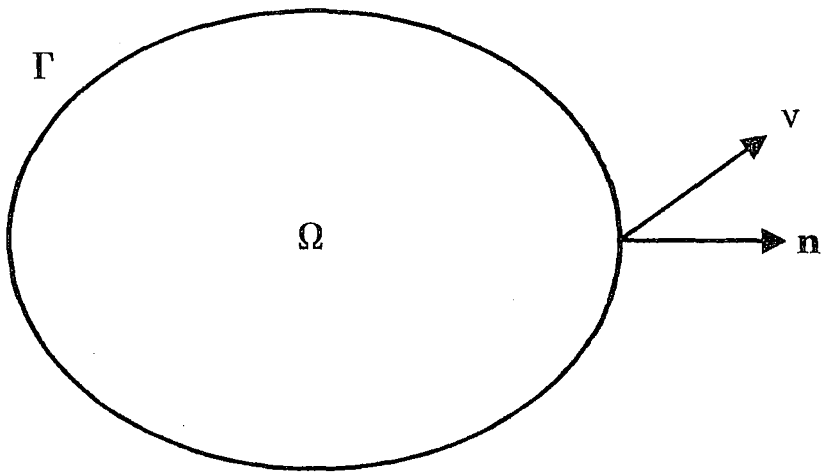

Figure 1

If $a \equiv 1$ and $b \equiv 0$ on Γ (resp. $a \equiv 0$ and $b \equiv 1$ on Γ), then the boundary condition B is the so-called Neumann (resp. Dirichlet) condition. We remark that problem (*) is *elliptic* (or *coercive*) if and only if the function a never vanishes on Γ.

In this note, under the condition $a \geqq 0$ on Γ, we shall consider the problem of existence and uniqueness of solutions of problem (*) in the framework of Sobolev spaces of L^p style when $|\lambda|$ tends to $+\infty$.

If $1 \leqq p < \infty$, we let

$L^p(\Omega)$ = the space of (equivalence classes of) Lebesgue measurable functions f on Ω such that $|f|^p$ is integrable on Ω.

The space $L^p(\Omega)$ is a Banach space with the norm

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}.$$

If m is a non-negative integer, we define the Sobolev space

$H^{m,p}(\Omega)$ = the space of (equivalence classes of) functions $u \in L^p(\Omega)$ whose derivatives $D^\alpha u$, $|\alpha| \leqq m$, in the sense of distributions are in $L^p(\Omega)$.

The space $H^{m,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{m,p} = \Big(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^p \, dx \Big)^{1/p} .$$

We remark that

$$H^{0,p}(\Omega) = L^p(\Omega) \; ; \; \|\cdot\|_{0,p} = \|\cdot\|_p .$$

Further we let

$B^{m-1/p,p}(\Gamma)$ = the space of the boundary values ϕ of functions $u \in H^{m,p}(\Omega)$.

In the space $B^{m-1/p,p}(\Gamma)$, we introduce a norm

$$|\phi|_{m-1/p,p} = \inf \|u\|_{m,p} ,$$

where the infimum is taken over all functions $u \in H^{m,p}(\Omega)$ which equal ϕ on the boundary Γ. The space $B^{m-1/p,p}(\Gamma)$ is a Banach space with respect to this norm ; more precisely, it is a *Besov space* (cf. Bergh-Löfström [4] ; Triebel [20]).

First we have the following :

THEOREM 1. *Assume that the following two conditions* (*A*) *and* (*B*) *are satisfied*:

(*A*) $a(x) \geqq 0$ *on* Γ.

(*B*) $b(x) > 0$ *on* $\Gamma_0 = \{x \in \Gamma \, ; \, a(x) = 0\}$.

Then, for any solution $u \in H^{2,p}(\Omega)$ $(1 < p < \infty)$ *of problem* (*) *with* $f \in L^p(\Omega)$ *and* $\phi \in B^{2-1/p,p}(\Gamma)$, *we have the* a priori *estimate*

(0.1) $\quad \|u\|_{2,p} \leqq C(\lambda)(\|f\|_p + |\phi|_{2-1/p,p} + \|u\|_p)$,

with a constant $C(\lambda) > 0$ *depending on* λ.

Here it is worth while pointing out that the *a priori* estimate (0.1) is the same one for the *Dirichlet* condition (cf. Agmon-Douglis-Nirenberg [3] ; Friedman [6] ; Lions-Magenes [12]).

We associate with problem (*) a unbounded linear operator A from $L^p(\Omega)$ into itself as follows :

(a) The domain of definition $D(\mathbf{A})$ is the space

$$D(\mathbf{A}) = \{ u \in H^{2,p}(\Omega) \,;\, Bu \equiv a\frac{\partial u}{\partial \nu} + bu|_\Gamma = 0 \text{ on } \Gamma \}.$$

(b) $\mathbf{A}u = Au\,,\ u \in D(\mathbf{A})$.

The next result is an L^p-version of Theorem 1 of Taira [17]:

THEOREM 2. *Assume that conditions (A) and (B) are satisfied. Then we have the following*:

(i) *For every* $\varepsilon > 0$, *there exists a constant* $r(\varepsilon) > 0$ *such that the resolvent set of* $\mathbf{A}$ *contains the set* $\Sigma_\varepsilon = \{ \lambda = r^2 e^{i\theta} \,;\, r \geqq r(\varepsilon)\,,\ -\pi+\varepsilon \leqq \theta \leqq \pi - \varepsilon \}$, *and that the resolvent* $(\mathbf{A} - \lambda I)^{-1}$ *satisfies the estimate*

$$\| (\mathbf{A} - \lambda I)^{-1} \| \leqq \frac{c(\varepsilon)}{|\lambda|}\,, \quad \lambda \in \Sigma_\varepsilon\,,$$

where $c(\varepsilon) > 0$ *is a constant depending on* ε.

(ii) *The operator* $\mathbf{A}$ *generates a semigroup* $U(z)$ *on* $L^p(\Omega)$ *which is* analytic *in the sector* $\Delta_\varepsilon = \{ z = t + is\,;\ z \neq 0\,, |\arg z| < \pi/2 - \varepsilon \}$ *for any* $0 < \varepsilon < \pi/2$.

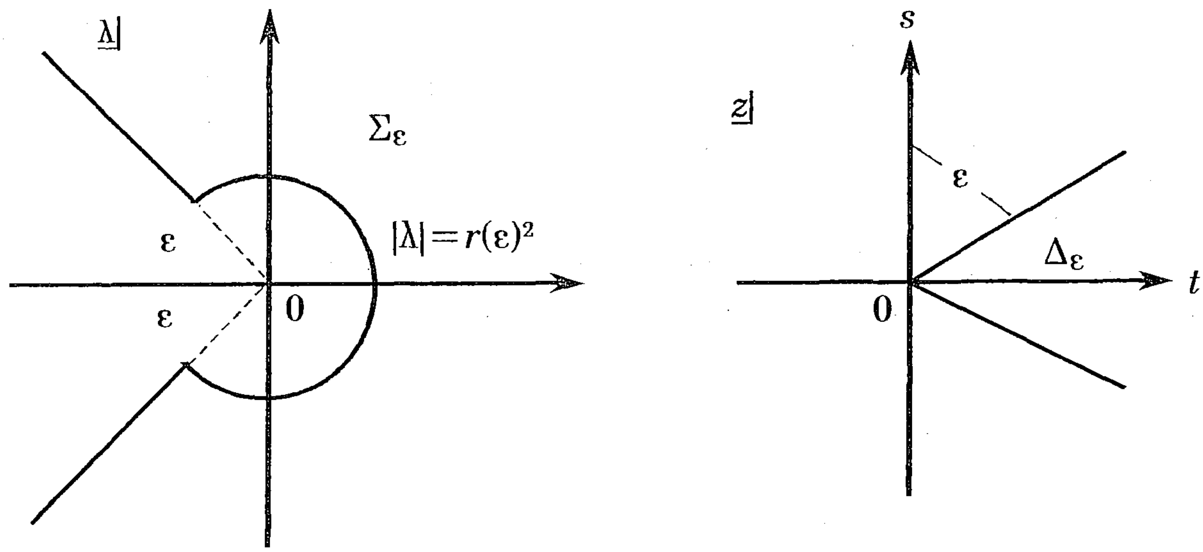

Figure 2

As an application of Theorem 2, we consider the following *semilinear* initial-boundary value problem : Given functions f and u_0 defined in $\Omega \times [0, T) \times \mathbf{R} \times \mathbf{R}^n$ and in Ω respectively, find a function u in $\Omega \times [0, T)$ such that

$$(**) \quad \begin{cases} (\frac{\partial}{\partial t} - \mathrm{A})u(x,t) = f(x,t,u,\mathrm{grad}\, u) \text{ in } \Omega \times (0, T), \\ Bu(x,t) = 0 \text{ on } \Gamma \times [0, T), \\ u(x,0) = u_0(x) \text{ in } \Omega . \end{cases}$$

By using the operator A , one can formulate problem (**) in terms of the *Cauchy problem* in the space $L^p(\Omega)$ as follows :

$$(**)' \quad \begin{cases} \frac{du}{dt} = \mathrm{A}u(t) + F(t,u(t)), \ 0 < t < T, \\ u|_{t=0} = u_0 . \end{cases}$$

Here $u(t) = u(\cdot,t)$ and $F(t,u(t)) = f(\cdot,t,u(t),\mathrm{grad}\, u(t))$ are functions defined on the interval $[0, T)$, taking values in the space $L^p(\Omega)$.

First we consider the case $p > n$:

THEOREM 3. *Assume that conditions (A) and (B) are satisfied. Let $p > n$ and let $f(x,t,u,\xi)$ be a locally* Lipschitz *continuous function of all its variables with the possible exception of the x variables. Then, for every function u_0 of $D(\mathrm{A})$, problem (**)' has a unique solution $u \in C([0,T'] ; L^p(\Omega)) \cap C^1((0,T') ; L^p(\Omega))$ where $T' = T'(p,u_0) > 0$.*

Here $C([0,T'] ; L^p(\Omega))$ denotes the space of continuous functions on $[0, T']$ taking values in $L^p(\Omega)$, and $C^1((0,T') ; L^p(\Omega))$ denotes the space of continuously differentiable functions on $(0, T')$ taking values in $L^p(\Omega)$, respectively.

In the case $p < n$, the domain $D(\mathrm{A})$ is very small compared with the case $p > n$. Hence we must impose some growth conditions on the function f :

THEOREM 4. *Assume that conditions (A) and (B) are satisfied. Let $n/2 < p < n$ and let $f(x,t,u,\xi)$ be a locally* Lipschitz *continuous function of all its variables with the possible exception of the x variables. Further assume that there exist a non-negative continuous function $\rho(t,r)$ on $\mathbf{R} \times \mathbf{R}$ and a constant $1 \leqq \gamma < n/(n-p)$ such that :*

(*a*) $|f(x,t,u,\xi)| \leqq \rho(t,|u|)(1 + |\xi|^r)$.

(*b*) $|f(x,t,u,\xi) - f(x,s,u,\xi)| \leqq \rho(t,|u|)(1 + |\xi|^r)|t - s|$.

(*c*) $|f(x,t,u,\xi) - f(x,t,u,\eta)| \leqq \rho(t,|u|)(1 + |\xi|^{r-1} + |\eta|^{r-1})|\xi - \eta|$.

(*d*) $|f(x,t,u,\xi) - f(x,t,v,\xi)| \leqq \rho(t,|u|+|v|)(1 + |\xi|^r)|u - v|$.

Then, for every function u_0 *of* $D(\mathrm{A})$, *problem* (**)' *has a unique local solution* $u \in C([0,T'] \,;\, L^p(\Omega)) \cap C^1((0,T') \,;\, L^p(\Omega))$ *where* $T' = T'(p,u_0) > 0$.

§2. Proof

The idea of proof is essentially the same as in Chapter 8 of Taira [18].

1) First we consider the following *Neumann* problem :

$$(N)\quad \begin{cases} (A - \lambda)v = f \ \text{ in } \Omega , \\ \dfrac{\partial v}{\partial \nu}\Big|_{\Gamma} = 0 \quad \text{on } \Gamma . \end{cases}$$

The existence and uniqueness theorem for problem (N) is well established in the framework of Sobolev spaces of L^p style (cf. Seeley [15], Taylor [19]). We let

$$v = G(\lambda)f.$$

The operator $G(\lambda)$ is the *Green* operator for the Neumann problem. Then it follows that a function u is a solution of problem (*) if and only if the function $w = u - v$ is a solution of the problem :

$$\begin{cases} (A - \lambda)w = 0 \ \text{ in } \Omega , \\ Bw = - Bv = - bv|_{\Gamma} \ \text{ on } \Gamma . \end{cases}$$

But we know that every solution w of the equation

$$(A - \lambda)w = 0 \ \text{ in } \Omega$$

can be expressed by means of a single layer potential as follows :

$$w = P(\lambda)\psi .$$

The operator $P(\lambda)$ is the *Poisson* operator for the Dirichlet problem. Thus, by using the Green and Poisson operators, one can reduce the study of problem (*) to that of the equation

$$T(\lambda)\psi \equiv BP(\lambda)\psi = - bv|_{\Gamma} , \quad v = G(\lambda)f.$$

This is a generalization of the classical *Fredholm* integral equation.

2) It is well known (cf. Seeley [15], Taylor [19]) that the operator $T(\lambda) = BP(\lambda)$ is a pseudo-differential operator of first order on the boundary Γ. The theory of pseudo-differential operators may be considered as a generalization of the classical potential theory. We study the boundary value problem (*) in the framework of Sobolev spaces of L^p style, by using the L^p theory of pseudo-differential operators. We can prove that the *a priori* estimate (0.1) of Theorem 1 is entirely equivalent to the corresponding *a priori* estimate for the pseudo-differential operator $T(\lambda)$ on the boundary.

3) We study the pseudo-differential operator $T(\lambda)$ in question, and prove that conditions (A) and (B) are sufficient for the validity of the *a priori* estimate (0.1). More precisely, we construct a *parametrix* $S(\lambda)$ for $T(\lambda)$ in the Hörmander class $L^{0}_{1,1/2}(\Gamma)$ (cf. Hörmander [9], Kannai [10]), and then apply *Besov-space* boundedness theorems of Bourdaud [5] to the parametrix $S(\lambda)$.

4) We study the operator $\mathbf{A}$, and prove fundamental *a priori* estimates for $\mathbf{A} - \lambda I$ which play an important role in the proof of Theorem 2. In the proof, we make use of a method essentially due to Agmon [2]. This is a technique of treating a spectral parameter as a second-order differential operator of an extra variable and relating the old problem to a new one with the additional variable. The method of Agmon plays an important role in the proof of the *surjectivity* of $\mathbf{A} - \lambda I$.

5) We study the imbedding properties of the domains of the *fractional powers* $(-\mathbf{A})^{\alpha}$ $(0 < \alpha < 1)$ into Sobolev spaces of L^p style. This allows us to solve by successive approximations the semilinear initial-boundary value problem (**)', proving Theorems 3 and 4. We remark that our semigroup approach to semilinear initial-boundary value problems can be traced back to the pioneering work of Fujita-Kato [7].

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Analyticity of solutions for semi-linear heat equations in one space dimension

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1. Introduction.

We consider the initial value problem for the semi-linear heat equation :

$$\partial_t u - \frac{1}{2}\partial_x^2 u = f(u,\bar{u}), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.1}$$

$$u(0,x) = \phi(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $\phi \in L^2(\mathbb{R})$ and $f(\lambda,\mu) = \sum_{1 \leqq j+k \leqq 2} a_{jk}\lambda^j\mu^k$ (a_{jk}, λ, $\mu \in \mathbb{C}$).

We show that the local solution in time of (1.1)-(1.2) is analytic in space variable and has an analytic continuation to a strip containing the real axis, provided the initial function ϕ belongs to $L^2(\mathbb{R})$.

We state notations and function spaces. We let $L^p(\mathbb{R}) = \{ f(x)$; $f(x)$ is measurable on $\mathbb{R}$, $|f|_p < \infty \}$, where $|f|_p = (\int |f(x)|^p dx)^{1/p}$ if $1 < p < \infty$ and $|f|_\infty = \sup\{ |f(x)|$; $x \in \mathbb{R} \}$. Let $H^{m,p}(\mathbb{R}) = \{$ $f(x) \in L^p(\mathbb{R})$; $|f|_{m,p} = \sum_{j=0}^{m} |\partial_x^j f(x)|_p < \infty \}$. For each $r > 0$ we denote by $S(r)$ the strip $\{-r < \operatorname{Im} z < r\}$ in the complex $z = x + iy$ plane. We let, for each $r > 0$, $AL^p(r) = \{ f(z)$; $f(z)$ is analytic on $S(r)$,

$|f|_{AL^p(r)} < \infty$ }, where $|f|^p_{AL^p(r)} = \frac{1}{2r}\int_{-r}^{r}\int_{\mathbb{R}} |f(z)|^p dxdy$ if $1 < p < \infty$ and $|f|_{AL^\infty(r)} = \sup\{ |f(z)| ; z \in S(r)\}$. Let $AH^{m,p}(r) = \{ f(z) \in AL^p(r) ; |f|_{AH^{m,p}(r)} = \sum_{j=0}^{m} |\partial_z^j f|_{AL^p(r)} < \infty \}$. If a complex valued function f(x) has an analytic continuation to S(r), then we denote the continuation by the same letter f(z); and if g(z) is an analytic function on S(r), then we denote the restriction of g(z) to the real axis by g(x). For an interval I of $\mathbb{R}$ and a Banach space B with norm $|\cdot|_B$, we let $C(I;B) = \{f(t) ; f(t)$ is continuous from I to B, $\sup\{ |f(t)|_B ; t \in I\} < \infty \}$ and $C^m(I;B) = \{ f(t) \in C(I;B) ; \sup\{ \sum_{j=0}^{m} |\partial_t^j f(t)|_B ; t \in I \} < \infty \}$. Positive constants will be denoted by the same letter C and will change from line to line. If necessary, by $C(*,\cdots,*)$ we denote constants depending only on the quantities appearing in parentheses. With these notations we state our

Theorem. We assume that $\phi \in L^2(\mathbb{R})$. Then there exists a positive constant $T = T(|\phi|_2)$ such that (1.1)-(1.2) has a unique solution $u(t,x) \in C([0,T];L^2(\mathbb{R}))$ which has an analytic continuation to $S(\sqrt{t})$. Moreover the extension u(t,z) satisfies for each δ, $0 < \delta < T$,

$$u(t,z) \in C([\delta,T];AL^2(\sqrt{\delta})),$$

and

$$u(t,z) \in C([\delta,T];AH^{m+2,2}(\alpha\sqrt{\delta})) \cap C^1([\delta,T];AH^{m,2}(\alpha\sqrt{\delta})),$$

for $0 < \alpha < 1$ and $m \in \mathbb{N} \cup \{0\}$.

We recall the isometrical identity obtained in [1], which is powerful for estimating the nonlinear term $f(u,\bar{u})$ in (1.2).

Lemma 1.1 ([1]). We let $m \in \mathbb{N} \cup \{0\}$. Suppose $f(z) \in AH^{m,2}(r)$. Then the formal power series

$$\sum_{k=0}^{m} \sum_{n=0}^{\infty} \frac{(2r)^{2n}}{(2n+1)!} |\partial_x^{n+k} f|_2^2 \tag{1.3}$$

converges and coincides with $|f|^2_{AH^{m,2}(r)}$. Conversely, suppose $f(x) \in H^{\infty,2}(\mathbb{R})$ and (1.3) is finite. Then $f(x)$ has an analytic extension $f(z) \in AH^{m,2}(r)$ and (1.3) coincides with $|f|^2_{AH^{m,2}(r)}$.

2. Preliminary estimates.

The next estimate corresponds to the Gagliardo-Nirenberg inequality for analytic functions.

Lemma 2.1. We assume that $h(z) \in AH^{1,2}(r)$. Then we have for $2 < q \leqq 6$

$$|h|_{AL^q(r)} \leqq C\cdot(|h|_2 + r|\partial_z h|_{AL^2(r)})^{\frac{q+2}{2q}} |\partial_z h|^{\frac{q-2}{2q}}_{AL^2(r)} .$$

Let us estimate the L^2-norm on the real axis of the difference of nonlinear terms.

Lemma 2.2. We assume that $v_j(x) \in H^{1,2}(\mathbb{R})$ for $j = 1,2$. Then we have

$$|f(v_1,\bar{v}_1) - f(v_2,\bar{v}_2)|_2 \leqq C(f)(1 + \sum_{j=1}^{2} |v_j|_2^{\frac{1}{2}} |\partial_x v_j|_2^{\frac{1}{2}})|v_1 - v_2|_2.$$

It is well-known that if v is analytic on $S(r)$, then so is $v^*(z)$

$= \overline{v(\bar{z})}$. Moreover, if $v \in AL^2(r)$, then so is v^* and $|v|_{AL^2(r)} = |v^*|_{AL^2(r)}$. The restriction of $v^*(z)$ on the real axis coincides with $\overline{v(x)}$, and hence $f(v(z), v^*(z))$ is an analytic continuation of $f(v(x), \overline{v(x)})$. We estimate the AL^2-norm on the strip $S(r)$ of difference of nonlinear terms.

Lemma 2.3. We assume that $v_j(z) \in AH^{1,2}(r)$ for $j = 1,2$. Then we have

$$|f(v_1, v_1^*) - f(v_2, v_2^*)|_{AL^2(r)}$$

$$\leqq C(f)\Big(|v_1 - v_2|_{AL^2(r)}$$

$$+ \Big(\sum_{j=1}^{2} \big(r^{\frac{3}{4}} |\partial_z v_j|_{AL^2(r)} + |\partial_z v_j|_{AL^2(r)}^{\frac{1}{4}} |v_j|_2^{\frac{3}{4}} \big) \Big) \times |\partial_z (v_1 - v_2)|_{AL^2(r)}^{\frac{1}{4}} |v_1 - v_2|_2^{\frac{3}{4}}$$

$$+ \Big(\sum_{j=1}^{2} \big(r^{\frac{3}{4}} |\partial_z v_j|_{AL^2(r)}^{\frac{1}{4}} |v_j|_2^{\frac{3}{4}} + r^{\frac{3}{2}} |\partial_z v_j|_{AL^2(r)} \big) \Big) \times |\partial_z (v_1 - v_2)|_{AL^2(r)} \Big).$$

3. Proof of Theorem.

We first consider the linear heat equation :

$$\partial_t u - \tfrac{1}{2}\partial_x^2 u = g(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{3.1}$$

$$u(0,x) = \phi(x), \quad x \in \mathbb{R}. \tag{3.2}$$

Lemma 3.1. We assume that $\phi \in L^2(\mathbb{R})$ and $\int_0^T |g(s)|_2 \, ds < \infty$. Then there exists a unique solution $u(t,x)$ of (3.1)-(3.2) which belongs to $C([0,T];L^2(\mathbb{R}))$. Moreover $u(t,x)$ satisfies

$$|u(t)|_2^2 + \int_0^t |\partial_x u(s)|_2^2 \, ds \leqq C\cdot(|\phi|_2^2 + (\int_0^t |g(s)|_2 \, ds)^2),$$

for $0 \leqq t \leqq T$.

Lemma 3.2. We assume that $\phi \in L^2(\mathbb{R})$ and $\int_0^T |g(s)|_{AL^2(\sqrt{s})} \, ds < \infty$ for $T > 0$. Then there exists a unique solution $u(t,x)$ of (3.1)-(3.2) such that $u(t,x)$ has an analytic continuation to $S(\sqrt{t})$ for $t \in (0,T]$. Moreover $u(t,z)$ belongs to $C([\delta,T];AL^2(\sqrt{\delta}))$ for $0 < \delta < T$ and satisfies

$$|u(t)|^2_{AL^2(\sqrt{t})} + \int_0^t |\partial_z u(s)|^2_{AL^2(\sqrt{s})} \, ds$$

$$\leqq C\cdot(|\phi|_2^2 + (\int_0^t |g(s)|_{AL^2(\sqrt{s})} \, ds)^2).$$

Proof of Theorem. We define the Banach space $B(T)$ by

$$B(T) = \{ v(t) \in C([0,T];L^2(\mathbb{R})) \ ; \ |v|_{B(T)} < \infty \},$$

where

$$|v|^2_{B(T)} = \sup_{0\leq t\leq T} |v(t)|^2_2 + \int_0^T |\partial_x v(t)|^2_2 \, dt$$

$$+ \sup_{0\leq t\leq T} |v(t)|^2_{AL^2(\sqrt{t})} + \int_0^T |\partial_z v(t)|^2_{AL^2(\sqrt{t})} \, dt.$$

We denote by $B^\rho(T)$ the closed ball in $B(T)$ with radius $\rho > 0$ and center at the origin. For $v \in B^\rho(T)$ we consider the equation :

$$\partial_t u - \frac{1}{2}\partial_x^2 u = f(v,\bar{v}), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R},$$

$$u(0,x) = \phi(x), \qquad x \in \mathbb{R}.$$

Define the mapping M by $u = Mv$. Then we can prove that there exists a positive constant T_1 such that M is a contraction mapping from $B^\rho(T_1)$ to itself by using Lemma 2.1-2.3 and Lemma 3.1-3.2. This implies the theorem.

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The Schrödinger operators with periodic magnetic fields

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1 Introduction

Let us consider a two-dimensional Schrödinger operator

$$L = -\sum_{j=1}^{2}(\partial_j - ib_j)^2 + V,$$

where $\partial_j = \partial/\partial x_j$, $i = \sqrt{-1}$, and b_j and V are the operators of multiplication by C^∞ real-valued functions on $\boldsymbol{R}^n$, $b_j(x)$ and $V(x)$, respectively. V and $\boldsymbol{b} = (b_1, b_2)$ are called a scalar potential and a (magnetic) vector potential, respectively, and the corresponding magnetic field is a *scalar* function $B \equiv \operatorname{curl} \boldsymbol{b}$

(1.1) $$B(x) = \partial_1 b_2(x) - \partial_2 b_1(x).$$

The operator L describes the motion of an electron confined in a 2 dimensional plane under the influence of the magnetic field perpendicular to the plane whose intensity is $B(x)$ at the point x.

In this lecture, we shall consider the periodic fields. More precisely, we shall assume

(H1) There exist constants $T_1, T_2 > 0$ such that

$$\begin{aligned} V(x_1+T_1, x_2) &= V(x_1, x_2+T_2) = V(x_1, x_2) \\ B(x_1+T_1, x_2) &= B(x_1, x_2+T_2) = B(x_1, x_2) \end{aligned}$$

for all $x = (x_1, x_2) \in \boldsymbol{R}^2$.

We are going to treat three topics concerning periodic fields:

1. Formulation of magnetic Bloch theory.
2. A sufficient condition for the absolute continuity of L.
3. Stability of gaps under variation of the magnetic field.

As is well known, in the case where the system is free of magnetic field B, the spectral properties of $L = -\Delta + V$ are studied by using Bloch wave function, which is a periodic function multiplied by a plane wave, and the operator $-\Delta + V$ is absolutely continuous for *any* periodic electric potential (see [O-K] and [T]). However, in the presence of magnetic fields, the situation becomes more complicated, and it seems that not much has been studied so far.

It is known (see, e.g., [Z1]) that an analysis using magnetic Bloch waves similar to the case of $-\Delta + V$ is possible if the magnetic flux penetrating a unit cell is an integal multiple of 2π:

(H2) $\iint_\Omega B(x)dx = 2\pi N$, where N is an integer and Ω is the domain $[0,\ T_1] \times [0,\ T_2]$ in $x = (x_1,\ x_2) \in \boldsymbol{R}^2$.

In Section 2, we shall give a formulation of magnetic Bloch theory appropriate for operator theoretical treatment.

In Section 3 and Section 4, we consider a rather special case, namely, we assume that the magnetic field $B(x)$ is uniform $\equiv B_0$ in $\boldsymbol{R}^2$. In Section 3, we show that L is absolutely continuous generically when a small perturbation V is turned on, which relates to a fact known as Landau level broadening in the physics literature (see, e.g., [Z2]). In Section 4, we present a result of [A-S], which shows that gaps of the spectrum of $L = L(B_0)$ are stable when B_0 varies.

2 Magnetic Bloch Theory

The choice of the magneic vector potential $\boldsymbol{b}$ is arbitrary as far as it satisfies (1.1) by gauge invariance (see. e.g., [L]). It is known that we can take

$$
(2.1)\qquad \begin{cases} b_1(x) = -\dfrac{B_0}{2}x_2 + a_1(x) \\ b_2(x) = \dfrac{B_0}{2}x_1 + a_2(x) \\ a_j(x_1 + T_1, x_2) = a_j(x_1, x_2 + T_2) = a_j(x_1, x_2) \quad (j = 1,\ 2) \end{cases}
$$

where B_0 is the density of the magnetic flux,

$$
B_0T_1T_2 = \iint_\Omega B(x)dx.
$$

Note that, unless B_0 equals 0, the vector potential cannot be taken to be periodic, even though the magnetic field is periodic.

Next, consider the operators S_1, S_2:

$$
\begin{aligned} S_1u(x_1,\ x_2) &\equiv e^{-\frac{i}{2}B_0T_1x_2}u(x_1 + T_1, x_2), \\ S_2u(x_1,\ x_2) &\equiv e^{\frac{i}{2}B_0T_2x_1}u(x_1, x_2 + T_2), \end{aligned}
$$

which are known as magnetic translations ([Z1]). These operators commute with L:

$$LS_j = S_jL \quad (j = 1, 2),$$

while simple translations do not commute with L because of the linear terms $(-B_0x_2/2,\ B_0x_1/2)$ in the vector potential $\boldsymbol{b}$. Moreover we have $S_1S_2 = e^{iB_0T_1T_2}S_2S_1$ by direct calculation. The commutativity of S_1 and S_2 is essential to the construction of magnetic Bloch theory and we shall assume (H2) (which is equivalent to $S_1S_2 = S_2S_1$).

Define the space of the magnetic Bloch functions by

$$\mathcal{E}(p) \equiv \{\ u \in C^\infty(\boldsymbol{R}^2)\ ;\ S_ju = e^{ip_jT_j}u\ (j = 1, 2)\ \}$$

with quasi-momenta $p = (p_1,\ p_2)$. $\mathcal{E}(p)$ depends periodically on the parameters $(p_1,\ p_2)$ for which a fundamental domain is $\Omega^* \equiv [0,\ 2\pi/T_1) \times [0,\ 2\pi/T_2)$. Define further the space $\mathcal{E}$ by

$$\begin{aligned}\mathcal{E} \ \equiv\ \Big\{ & u \in C^\infty(\boldsymbol{R}^2_x \times \boldsymbol{R}^2_p)\ ; u(x,p) \in \mathcal{E}(p) \text{ for all } p \in \boldsymbol{R}^2, \\ & u(x,p) = u(x, p_1 + \frac{2\pi}{T_1}, p_2) = u(x, p_1, p_2 + \frac{2\pi}{T_2}) \Big\}\end{aligned}$$

equipped with the norm

$$\|u\|^2_{\mathcal{E}} \equiv \int_\Omega dx \int_{\Omega^*} dp |u(x,p)|^2.$$

Then we have the following

Theorem 2.1 *Suppose that (H1) and (H2) hold. Then there exists a bijective correspondence* $U : \mathcal{S}(\boldsymbol{R}^2) \to \mathcal{E}$*, where* U *and* U^{-1} *is given by*

$$\begin{aligned} Uf(x,p) &= \frac{1}{\sqrt{T_1T_2}} \sum_{k,m\in\mathbf{Z}} e^{-i(kp_1T_1 + mp_2T_2)} S_1^k S_2^m f(x) \\ U^{-1}u(x) &= \frac{1}{\sqrt{T_1T_2}} \int_{\Omega^*} u(x,p)dp, \end{aligned}$$

for $f \in \mathcal{S}(\boldsymbol{R}^2)$ *(= the space of rapidly decreasing functions in* $\boldsymbol{R}^2$*) and* $u \in \mathcal{E}$*. Moreover* U *is unitary, i.e.,* $\|Uf\|_{\mathcal{E}} = \|f\|_{L^2(\mathbf{R}^2)}$*. If we put* $\tilde{L} \equiv ULU^{-1}$*, we have*

$$(\tilde{L}u)(x,p) = \left\{\tilde{L}(p)u(\cdot,p)\right\}(x)$$

where $\tilde{L}(p)$ *is an operator on* $\mathcal{E}(p)$ *given by* $(\tilde{L}(p)u)(x) = Lu(x)$ *for* $u \in \mathcal{E}(p)$*.*

Because $\tilde{L}(p)$ is an elliptic operator acting essentially in a compact domain Ω, $\tilde{L}(p)$ has purely discrete spectrum. Hence the study of the spectral property of L is reduced to that of an eigenvalue problem of a family of the operators $\tilde{L}(p)$ with parameters $p = (p_1,\ p_2) \in \Omega^*$. While $\tilde{L}(p)$ have different domains of definition $\mathcal{E}(p)$, we can show the following

Proposition 2.2 *$e^{i(p_1x_1+p_2x_2)}u(x) \in \mathcal{E}(p)$ for $u \in \mathcal{E}(0)$. Let $A(p)$ be an operator in $\mathcal{E}(0)$ defined by $A(p)u(x) \equiv e^{-i(p_1x_1+p_2x_2)}\tilde{L}(p)\{e^{i(p_1x_1+p_2x_2)}u(x)\}$. Then we have*

$$\begin{aligned} A(p) &= -(D_1+ip_1)^2-(D_2+ip_2)^2+V \\ &= A(0)-2i(p_1D_1+p_2D_2)+p_1^2+p_2^2 \end{aligned}$$

where $D_j = \partial_j - ib_j$ ($j = 1, 2$). Moreover D_j are infinitesimally small with respect to $A(0)$, i.e., for all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\|D_ju\| \le \varepsilon\|A(0)u\| + C_\varepsilon\|u\|$$

for all $u \in \mathcal{E}(0)$ where $\|u\| = \int_\Omega |u(x)|^2dx$.

Hence the operator $A(p)$, which is unitarily equivalent to $\tilde{L}(p)$, has a common domain of definition $\mathcal{E}(0)$ and depends analytically on p (it forms an entire analytic family of type A in the sense of Kato).

3 A Sufficient Condition for the Absolute Continuity of L

First, we give another unitary transform of $\tilde{L}(p)$ appropriate for our present purpose.

Lemma 3.1 *Define the operator $W(p)$ for $p = (p_1, p_2)$ by*

$$W(p)u(x) = e^{i(p_1x_2+p_2x_2)/2}u(x_1-\frac{p_2}{B_0}, x_2+\frac{p_1}{B_0})$$

for $u \in \mathcal{E}(0)$. Then $W(p)u \in \mathcal{E}(p)$ for $u \in \mathcal{E}(0)$ and

$$\begin{aligned} \hat{L}(p) &\equiv W(p)^*\tilde{L}(p)W(p) \\ &= -(D_1^0-ia_1^p)^2-(D_2^0-ia_2^p)^2+V^p, \end{aligned}$$

where $D_1^0 = \partial_1 + iB_0x_2/2$, $D_2^0 = \partial_2 - iB_0x_1/2$ and $a_1^p(x_1, x_2) = a_1(x_1 + p_2/B_0, x_2 - p_1/B_0)$ etc. (a_2^p and V^p are defined similarly.)

In the case where $a_1 = a_2 = V \equiv 0$, the operators $\hat{L}(p)$ are identical to an operator $\hat{L}_0 = -(D_1^0)^2 - (D_2^0)^2$ independent of p. It is known (see, e.g., [CV]) that the spectrum of operator $\hat{L}_0$ is given by

$\sigma(\hat{L}_0) = \{\ (2n-1)B_0 \mid n : \text{integer} \ge 1\ \}$ and all the eigenvalues have multiplicity N $(= \dfrac{B_0T_1T_2}{2\pi})$.

In this case, the corresponding operator L is the Schrödinger opeator with uniform magnetic field B_0

(3.1) $$L = L_0 = L_0(B_0) = -(\partial_1 + iB_0x_2/2)^2 - (\partial_2 - iB_0x_1/2)^2,$$

and $\sigma(L_0) = \sigma(\hat{L}_0)$ but each point of $\sigma(L_0)$ has an infinite number of degeneracy. Now we consider a perturbation of L_0 by a small electric potential V.

Theorem 3.2 *Suppose that (H1) holds and* $B_0T_1T_2 = 2\pi$ *(i.e., (H2) hold with* $N = 1$*). Let all the Fourier coefficient of* V *do not vanish and* $\|V\|_\infty \equiv \sup_x |V(x)| \le B_0/4$. *Then there exists a countable set* Σ_V *in the interval* $[-1,\ 1]$ *such that* $L = L_\kappa = L_0(B_0) + \kappa V$ *is absolutely continuous for* $\kappa \in [-1,\ 1] \setminus \Sigma_V$.

Sketch of proof. By Theorem 2.1 and Lemma 3.1 L_κ is unitarily equivalent to a decomposable operator by the direct integral decomposition:

$$\int_{\Omega^*}^{\oplus} \hat{L}_\kappa(p)dp,$$

where $\hat{L}_\kappa(p) = \hat{L}_0 + \kappa V^p$ (see, e.g., [R-S2, XIII.16] for direct integral decomposition). Since $\hat{L}_0$ has eigenvalues $(2n-1)B_0$ with multiplicity 1 for integers $n \ge 1$, $\hat{L}_\kappa(p)$ has a unique eigenvalue $\lambda_n(p,\kappa)$ contained in each circle $\{\ \lambda \in \boldsymbol{C} \mid |\lambda - (2n-1)B_0| = 3B_0/4\ \}$ if $\|\kappa V\|_\infty \le B_0/2$. Thus $\lambda_n(p,\kappa)$ is analytic in p and κ if $|\kappa| < 2$. Therefore L_κ is absolutely continuous if $\lambda_n(p,\kappa)$ is not a constant function of p for all n, because of the direct integral decomposition. By first order perturbation theory, we have

$$\begin{aligned} \lambda_n(p,\kappa) &= (2n-1)B_0 + \tilde{\lambda}_n(p)\kappa + \cdots, \\ \tilde{\lambda}_n(p_1,\ p_2) &= \int_\Omega V(x_1 + \frac{p_2}{B_0}, x_2 - \frac{p_1}{B_0})|\psi_n(x)|^2 dx, \end{aligned}$$

where ψ_n is an n-th eigenfunction of $\hat{L}_0$. By this formula, $\tilde{\lambda}_n(p)$ is, roughly speaking, given by something like the convolution of V and $|\psi_n|^2$, and the Fourier coefficient of $\tilde{\lambda}_n$ is given by a product of that of V and $|\psi_n|^2$. It is known that, as a property of the functions $\in \mathcal{E}(0)$, $|\psi_n(x)|^2$ is not a constant function of x. Therefore, if all the Fourier coefficient of V are nonzero, $\tilde{\lambda}_n(p)$ is a non-constant function of p for all n. Hence, for each n, $\lambda_n(p,\kappa)$ is not a constant function of p except for a finite number of $\kappa \in [-1,\ 1]$, which implies the assertion of the theorem. □

In a word, generically for small perturbation by a periodic electric potential V, broadening occurs at all the Landau levels, from which follows the absolute continuity of L.

4 Stability of Gaps Under Variation of a Magnetic Field

Consider the case where we assume (H1) but not (H2). Note that the Bloch wave analysis is applicable also to rational $N = B_0T_1T_2/2\pi$ by considering a unit cell of size T_1q by T_2 if $N = p/q$ (p, q are integers). In the case where N is irrational, it seems that not much has been studied so far.

[A-S] has given a result as to this problem in the case of uniform magnetic fields $B(x) \equiv B_0$ which can vary continuously, while V is fixed:

Theorem 4.1 ([A-S]) *Let $L(B_0) = L_0(B_0) + V$ where $L_0(B_0)$ is as in (3.1). Suppose that (H1) holds. Let $-\infty < a < b < \infty$. Then we have*

(a) *If $(a,\ b) \cap \sigma(L(B_0)) \neq \emptyset$, then there exists $\delta > 0$ such that $|B_0' - B_0| < \delta$ implies $(a,\ b) \cap \sigma(L(B_0')) \neq \emptyset$.*
(b) *If $[a,\ b] \cap \sigma(L(B_0)) = \emptyset$, then there exists $\delta > 0$ such that $|B_0' - B_0| < \delta$ implies $[a,\ b] \cap \sigma(L(B_0')) = \emptyset$.*

(a) of this theorem is an abstract functional analytic consequence of the fact that $L(B_0') \to L(B_0)$ as $B_0' \to B_0$ in the strong resolvent sense (see, e.g., [R-S1, VIII.7]). As for the proof of (b), [A-S] proves and exploits the equivalence of the relation $E \in \sigma(L(B_0))$ and the existence of a bounded (generalized) eigenfunction of $L(B_0)$ with eigenvalue E with the use of the Rellich theorem.

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On the initial-boundary value problems for the discrete Boltzmann equation

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1. Introduction

The discrete Boltzmann equation is the fundamental equation describing the time-evolution of a discrete velocity gas which consists of particles with a finite number of velocities ([5]). The aim of this note is to survey the author's recent works ([8,9,10]) concerning the global existence and asymptotic behavior of solutions to the initial-boundary value problems for the discrete Boltzmann equation on a bounded interval $0 < x < d$.

The general form of the discrete Boltzmann equation in one space dimension is written as

$$(1) \qquad c_i\left(\frac{\partial F_i}{\partial t} + v_i\frac{\partial F_i}{\partial x}\right) = Q_i(F), \qquad i \in \Lambda,$$

where Λ is a finite set $\{1,\cdots,m\}$, c_i are positive constants, and each $F_i = F_i(x,t)$ denotes the mass density of gas particles with the i-th velocity (whose x-component is denoted by v_i) at time t and position x. Collision terms $Q_i(F)$ on the right side of (1) are given by

$$Q_i(F) = \sum_{j,k,\ell}(A^{ij}_{k\ell}F_kF_\ell - A^{k\ell}_{ij}F_iF_j), \qquad i \in \Lambda,$$

where the summation is taken over all $j,k,\ell \in \Lambda$, and where the coefficients $A^{ij}_{k\ell}$ are nonnegative constants satisfying

(A1) $\quad A^{ji}_{k\ell} = A^{ij}_{k\ell} = A^{ij}_{\ell k}$, $\quad$ (A2) $\quad A^{ij}_{k\ell}(v_i - v_j - v_k - v_\ell) = 0$,

(A3) $\quad A^{ij}_{k\ell} = A^{k\ell}_{ij}$,

for any $i,j,k,\ell \in \Lambda$. (A2) means the conservation of momentum (in the x-direction) in the microscopic collision process and (A3) is called the micro-reversibility condition.

We prescribe the initial data :

(2) $\quad F_i(x,0) = F_{i0}(x)$, $\quad i \in \Lambda$.

Let $\Lambda_+ = \{ i \in \Lambda ; v_i > 0 \}$ and $\Lambda_- = \{ i \in \Lambda ; v_i < 0 \}$, and we impose the boundary conditions as follows : on the left boundary $x = 0$, either

(3) $\quad F_i(0,t) = B^0_i$, $\quad i \in \Lambda_+$, $\quad$ or

(3)' $\quad c_i F_i(0,t) = \sum^-_j B^0_{ij} F_j(0,t)$, $\quad i \in \Lambda_+$,

and on the right boundary $x = d$, either

(4) $\quad F_i(d,t) = B^1_i$, $\quad i \in \Lambda_-$, $\quad$ or

(4)' $\quad c_i F_i(d,t) = \sum^+_j B^1_{ij} F_j(d,t)$, $\quad i \in \Lambda_-$.

Here the boundary data B^0_i and B^1_i are positive constants, the coefficients B^0_{ij} and B^1_{ij} are nonnegative constants, and $\sum^\pm_j$ mean the summations taken over all $j \in \Lambda_\pm$, respectively. For (4)', we require as in [6],

(B1) $\quad \sum^-_i v_i B^1_{ij} + c_j v_j \geq 0$, $\quad j \in \Lambda_+$,

(B2) $\quad c_i M^1_i = \sum^+_j B^1_{ij} M^1_j$, $\quad i \in \Lambda_-$,

where $M^1 = (M^1_i)_{i \in \Lambda}$ is some constant Maxwellian, that is, M^1_i are positive constants satisfying $A^{ij}_{k\ell}(M^1_i M^1_j - M^1_k M^1_\ell) = 0$ for any $i,j,k,\ell \in \Lambda$. (Analogous conditions are required also for (3)'.) The requirement (B1) implies that the macroscopic flow velocity (in the x-direction) is not inward at $x = d$. In fact, we have $\sum_i c_i v_i F_i \geqq 0$ on $x = d$ under the natural situation $F_i > 0$ for $i \in \Lambda$.

2. Global solutions

We can set up essentially the following three initial-boundary value problems ; Problem (I) : {(1),(2),(3),(4)}, Problem (II) : {(1),(2),(3),(4)'} and Problem (III) : {(1),(2),(3)',(4)'}. Let $C^1(\Omega)$ be the space of continuously differentiable functions on a set Ω, and denote by $C^1_+(\Omega)$ the totality of positive functions in $C^1(\Omega)$. Our global existence result is then stated as follows.

Theorem 1. *Suppose that* $F_0 = (F_{i0})_{i \in \Lambda} \in C^1_+([0,d])$. *Then the problem* (I), (II) *or* (III) *has a unique global solution* $F = (F_i)_{i \in \Lambda}$ *in* $C^1_+([0,d] \times [0,\infty))$, *provided that* F_0 *satisfies the corresponding compatibility conditions up to order one.*

Remark. A similar global existence result is obtained also for the initial-boundary value problem (1),(2),(3) (or (3)') on $0 < x < \infty$, [9]. For the pure initial value problem (1),(2) on $-\infty < x < \infty$, see [1,2].

For the proof, we introduce

$$E_0 = \max_i \sup_{0 \leqq x \leqq d} F_{i0}(x) , \qquad E(t) = \max_i \sup_{\substack{0 \leqq x \leqq d \\ 0 \leqq \tau \leqq t}} F_i(x,\tau) , \tag{5}$$

$$\Phi(t,r) = \sup_{|I| \leqq r} \int_I \sum_i c_i F_i(x,t)\, dx ,$$

where the sup in the last expression is taken over all the intervals I in $[0,d]$, with the length $|I| \leqq r$. The standard method based on the contraction mapping principle shows that each problem has a unique local solution in $C^1_+([0,d]\times[0,T_0])$ for $T_0>0$ depending only on the sup-norm E_0 of the initial data. Therefore, the key of the proof of Theorem 1 is to derive a suitable a priori estimate for the sup-norm $E(t)$ of solutions in $C^1_+([0,d]\times[0,T])$ for any fixed $T>0$. The desired a priori estimate is obtained by the difference inequality (6) for $E(t)$, which involves $\Phi(t,r)$, combined with the estimate (7) for $\Phi(t,r)$:

Lemma 1. *Let* $F\in C^1_+([0,d]\times[0,T])$ *be a solution to the problem* (I), (II) *or* (III). *Then there is a positive constant* C *such that for any* $t \geqq 0$ *and* $h>0$ *satisfying* $2\tilde{v}h \leqq d$ *(where* $\tilde{v}=\max_i |v_i|$ *) and* $t+h\leqq T$,

$$E(t+h) \leqq CE(t) + C\{E_0 h + \Phi(t,2\tilde{v}h)\}E(t+h) \ . \tag{6}$$

Moreover, for any $0\leqq t\leqq T$ *and* $0<r\leqq d$,

$$\Phi(t,r) \leqq CE_0(1+T)\delta(r) \ , \tag{7}$$

where $\delta(r)$ *is a continuous function with the property that* $\delta(r)\to 0$ *as* $r\to 0$. *For the problem* (III), *the term* E_0h *in* (6) *is unneccessary.*

In deriving (6), we use the characteristic method and identities obtained by integrating the following equations (conservation of mass and momentum) over various regions in the rectangle $[0,d]\times[0,T]$.

$$\frac{\partial}{\partial t}\sum_i c_iF_i + \frac{\partial}{\partial x}\sum_i c_iv_iF_i = 0 \ , \tag{8}$$

$$\frac{\partial}{\partial t}\sum_i c_iv_iF_i + \frac{\partial}{\partial x}\sum_i c_iv_i^2F_i = 0 \ . \tag{9}$$

On the other hand, the estimate (7) is obtained by the argument employed in [13], which is based on the integral identities for (8) and for the following modified version of the H-theorem.

$$(10)\qquad \frac{\partial}{\partial t}\sum_i c_i F_i \log (F_i/M_i) + \frac{\partial}{\partial x}\sum_i c_i v_i F_i \log (F_i/M_i)$$
$$= -\frac{1}{4}\sum_{ijk\ell} A^{ij}_{k\ell}(F_iF_j - F_kF_\ell)\log (F_iF_j/F_kF_\ell) - \sum_i c_i v_i F_i \frac{\partial}{\partial x}\log M_i \ ,$$

where $M = (M_i)_{i\in\Lambda}$ is a Maxwellian depending smoothly in x and is chosen according to the boundary conditions.

3. Stationary solutions

We consider the corresponding stationary problems (pure boundary value problems) :

$$(11)\qquad c_i v_i \frac{dF_i}{dx} = Q_i(F) \ , \qquad i \in \Lambda \ ,$$

$$(12)\qquad F_i(0) = B_i^0 \ , \qquad i \in \Lambda_+ \ , \qquad \text{or}$$

$$(12)'\qquad c_i F_i(0) = \sum_j^- B^0_{ij} F_j(0) \ , \qquad i \in \Lambda_+ \ ,$$

$$(13)\qquad F_i(d) = B_i^1 \ , \qquad i \in \Lambda_- \ , \qquad \text{or}$$

$$(13)'\qquad c_i F_i(d) = \sum_j^+ B^1_{ij} F_j(d) \ , \qquad i \in \Lambda_- \ .$$

There are essentially three boundary value problems ; Problem (i) : {(11),(12),(13)}, Problem (ii) : {(11),(12),(13)'}, and Problem (iii) : {(11),(12)',(13)'}. The first two problems are solved in [4] under the restriction that $v_i \neq 0$ for $i \in \Lambda$. This restriction can be removed by assuming the following hypothesis. Let $\Lambda_0 = \{ i \in \Lambda \ ; \ v_i = 0 \}$.

Hypothesis : For any given $\{ F_i > 0 \ ; \ i \notin \Lambda_0 \}$, the system of algebraic equations $Q_i(F) = 0$, $i \in \Lambda_0$, admit a solution $\{ F_i > 0 \ ; \ i \in \Lambda_0 \}$. In addition, the resulting mapping $\{ F_i > 0 \ ; \ i \notin \Lambda_0 \} \to \{ F_i > 0 \ ; \ i \in \Lambda_0 \}$ is defined globally and is continuously differentiable.

In fact, we have :

Theorem 2. *Under the above hypothesis, the problem* (i) *or* (ii) *has a solution* $F = (F_i)_{i \in \Lambda}$ *in* $C^1_+([0,d])$.

Remark. A similar existence result is obtained also for the problem (11),(12) on $0 < x < \infty$, [3]. Uniqueness of these solutions are, however, unknown in the generality. The problem (iii) and the problem (11),(12)' (on $0 < x < \infty$) are unsolved.

As in [4], we can prove the above theorem by applying the following fixed point theorem of Leray-Schauder type (see [12]).

Fixed point theorem by Browder-Potter. *Let* S *be a closed convex subset of a normed space* X. *Let* $\Psi_\mu(F)$ *be a continuous mapping of* $(F,\mu) \in X \times [0,1]$ *into a compact subset of* X *such that*

(a) $\Psi_0(\partial S) \subset S$,

(b) *for* $0 \leqq \mu \leqq 1$, $\Psi_\mu(\cdot)$ *has no fixed point on* ∂S.

Then $\Psi_1(\cdot)$ *has a fixed point in* S.

In our problems, we take $X = C^0([0,d])$ and $S = \{ F \in C^0([0,d])\ ;\ 0 \leqq F_i \leqq R,\ i \in \Lambda \}$ for some large $R > 0$. The mapping $F = \Psi_\mu(G)$ is defined by solving the problem :

$$(14) \qquad \begin{cases} c_i v_i \dfrac{dF_i}{dx} = \mu\{q_i(G) - r_i(G)F_i\}, & i \notin \Lambda_0, \\ Q_i(F) = 0, & i \in \Lambda_0, \end{cases}$$

with the corresponding boundary conditions, where $q_i(F) = \sum_{jk\ell} A^{ij}_{k\ell} F_k F_\ell$ and $r_i(F) = \sum_{jk\ell} A^{k\ell}_{ij} F_j$. In application, a key point is to check condition (b) and this is done by deriving a priori estimate of solutions to the problem (14) with $G = F$, which is based on the identities obtained by integrating the following equations (conservation of mass and momentum) over $[0,d]$.

(15) $$\frac{d}{dx}\sum_i c_i v_i F_i = 0 , \qquad \frac{d}{dx}\sum_i c_i v_i^2 F_i = 0 .$$

4. Stationary solutions near Maxwellian

We can show that under the stability condition below, the solutions to the stationary problem (i) or (ii) in the preceding section are unique in a neighborhood of a Maxwellian.

Stability condition (stationary case) : Let $\psi \in \mathcal{M}$ and let $V\psi = 0$. Then $\psi = 0$.

Here $V = \mathrm{diag}(v_i)_{i\in\Lambda}$, and $\mathcal{M}$ is the space of collision invariants, that is, $\mathcal{M}$ consists of vectors $\psi = (\psi_i)_{i\in\Lambda}$ satisfying $A_{k\ell}^{ij}(\psi_i/c_i - \psi_j/c_j - \psi_k/c_k - \psi_\ell/c_\ell) = 0$ for any $i,j,k,\ell \in \Lambda$.

Theorem 3. *We consider the problem* (i) *under the above stability condition. Let* $M = (M_i)_{i\in\Lambda}$ *be any fixed constant Maxwellian and put* $\delta = \sum_i^+ |B_i^0 - M_i| + \sum_i^- |B_i^1 - M_i|$. *If* δ *is small enough, then there exists a unique solution* $F \in C^1([0,d])$ *satisfying* $\|F - M\|_1 \leqq C\delta$ *for some constant* C, *where* $\|\cdot\|_1$ *denotes the norm of the Sobolev space* $H^1(0,d)$.

Remark. A similar result holds true also for the problem (ii) : in this case we take $M = M^1$ and $\delta = \sum_i^+ |B_i^0 - M_i^1|$.

For the proof of Theorem 3, it is convenient to introduce a new unknown $f = (f_i)_{i\in\Lambda}$ by $F_i = M_i(1 + f_i)$, $i \in \Lambda$, or equivalently, by $F = M + D_M f$, where $D_M = \mathrm{diag}(M_i)_{i\in\Lambda}$, and transform the problem (i) into

(16) $$\tilde{D}_M V \frac{df}{dx} + L_M f = \Gamma_M(f,f) ,$$
$$f_i(0) = b_i^0 , \quad i \in \Lambda_+ , \qquad f_i(d) = b_i^1 , \quad i \in \Lambda_- .$$

Here $\tilde{D}_M = \mathrm{diag}(c_i M_i)_{i\in\Lambda}$, $b_i^0 = (B_i^0 - M_i)/M_i$ and $b_i^1 = (B_i^1 - M_i)/M_i$; L_M and Γ_M are defined by

$$L_M f = -2Q(M, D_M f) , \qquad \Gamma_M(f,g) = Q(D_M f, D_M g) ,$$

where $Q(F,G) = (Q_i(F,G))_{i \in \Lambda}$ and each $Q_i(F,G)$ is the extension of $Q_i(F)$ as a bilinear form. Recall that L_M (linearized collision operator) is real symmetric and nonnegative definite such that the null space $N(L_M)$ coincides with the space $\mathcal{M}$ of collision invariants ([5,7]).

We can solve the problem (16), equivalent to the original problem (i), by applying the contraction mapping principle, for we obtain the following result concerning the existence and regularity of solution to the linearized problem of (16).

Lemma 2. *Let* $g \in H^1(0,d)$. *Then the linearized problem* (16) *with* $\Gamma_M(f,f)$ *replaced by* g *has a unique solution* $f \in H^1(0,d)$. *Moreover, we have the estimate* $\|f\|_1 \leqq C(\tilde{\delta} + \|g\|_1)$ *for some constant* C, *where* $\tilde{\delta} = \sum_i^+ |b_i^0| + \sum_i^- |b_i^1|$.

The homogeneous boundary conditions $f_i(0) = 0$, $i \in \Lambda_+$, and $f_i(d) = 0$, $i \in \Lambda_-$, are maximal nonnegative for the boundary matrices $-\tilde{D}_M V$ and $\tilde{D}_M V$, respectively, so that the above lemma is trivial if V is non-singular and L_M is positive definite. But this is not the case. We need to use the stability condition (stationary case), dissipation property of the boundary conditions and the Poincare inequality to prove Lemma 2.

5. Large-time behavior of solutions

We shall show that the stationary solution in Theorem 3 is time-asymptotically stable if the following stability condition ([11]) is satisfied.

Stability condition (nonstationary case) : Let $\psi \in \mathcal{M}$ and let $V\psi = \lambda\psi$ for $\lambda \in \mathbb{R}$. Then $\psi = 0$.

Theorem 4. *We consider the problem* (I) *under the above stability con-*

dition. Let M *be any fixed constant Maxwellian and suppose that* $F_0 - M \in H^1(0,d)$ *and* F_0 *satisfies the compatibility conditions of order zero. Then, if* $\|F_0 - M\|_1$ *is small enough, there exists a unique global solution* F *satisfying* $F - M \in C^0([0,\infty);H^1(0,d)) \cap C^1([0,\infty);L^2(0,d))$. *Moreover, this solution* $F(x,t)$ *converges, uniformly in* x, *to the solution* $F^\infty(x)$ *of the corresponding stationary problem* (i), *which is obtained in Theorem* 3, *at an exponential rate* $e^{-\alpha t}$, $\alpha > 0$, *as* $t \to \infty$.

Remark. Asymptotic behavior of solutions is unknown for the problem (II) or (III).

Letting $F^\infty = M + D_M f^\infty$ be the solution to the stationary problem (i), we introduce a new unknown f by $F = F^\infty + D_M f = M + D_M(f^\infty + f)$. Then, as in the preceding section, we can transform the problem (I) into

$$\text{(17)} \quad \begin{aligned} &\tilde{D}_M \frac{\partial f}{\partial t} + \tilde{D}_M V \frac{\partial f}{\partial x} + L_M f = \Gamma_M(2f^\infty + f, f) , \qquad f(x,0) = f_0(x) , \\ &f_i(0,t) = 0 , \quad i \in \Lambda_+ , \qquad f_i(d,t) = 0 , \quad i \in \Lambda_- , \end{aligned}$$

where $f_0(x) = D_M^{-1}(F_0(x) - F^\infty(x)) = D_M^{-1}(F_0(x) - M) - f^\infty(x)$. We denote the space $C^0([0,T];H^1(0,d)) \cap C^1([0,T];L^2(0,d))$ by X_T^1 and put $|||f|||_1 = \|f\|_1 + \|\partial f/\partial t\|$ for $f \in X_T^1$, where $\|\cdot\|$ is the $L^2(0,d)$-norm. Theorem 4 is essentially based on the following result concerning the existence, regularity and exponential decay of solution to the linearized problem of (17).

Lemma 3. *Consider the linearized problem* (17) *with* g *in place of* $\Gamma_M(2f^\infty + f, f)$, *where* $g \in X_T^1$. *Suppose that* $f_0 \in H^1(0,d)$ *satisfies the compatibility conditions of order zero. Then there exists a unique solution* f *in* X_T^1. *Moreover, we have the estimate*

$$\text{(18)} \quad e^{\alpha t} |||f(t)|||_1^2 + \int_0^t e^{\alpha\tau} |||f(\tau)|||_1^2 \, d\tau \leqq C |||f(0)|||_1^2 + \int_0^t e^{\alpha\tau} |||g(\tau)|||_1^2 \, d\tau ,$$

for $t \in [0,T]$, *where* α *and* C *are positive constants.*

To prove this lemma, we use the argument similar to the one employed in the proof of Lemma 2, and also the technique developed in [7] for the pure initial value problem in a neighborhood of a Maxwellian.

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