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**Proceedings of the Meeting and the workshop  
“Algebraic Geometry and Hodge Theory”**

**Vol. I**

Meeting: August 23 – 28, 1989, Hokkaido University  
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**Edited by K. Konno, M.-H. Saito and S. Usui**

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# A Resolution of Normal Surfaces of Triple Section Type

TADASHI ASHIKAGA

**Introduction.** In this note, "a surface" means an algebraic surface defined over the complex number field. In [5, §2], Horikawa introduced a method for resolving singularities on normal surfaces of double section type in the completion of the total space of a line bundle over a surface. This method is sometimes useful not only for studying surfaces globally ([6], [13], [14] etc.) but also for studying isolated singularities locally ([17] etc.).

In fact, let  $(V, P)$  be a normal 2-dimensional double point. Since the embedded dimension of  $(V, P)$  is 3 by [1], an analytic equation of  $P$  is given by  $\xi^2 + f(x, y) = 0$  from a suitable change of the coordinate. Then one can obtain a resolution of  $P$  from a resolution of the plane curve singularity  $f(x, y) = 0$  by this method. The origin of this idea is classical ([4], [7], [10], etc.), but the essential point of [5, §2] is to obtain a formula for analytic invariants of  $P$  at the same time as resolving  $P$  ([5, p.50]).

Now in this note, we extend this method to normal surfaces of triple section type. (A special form of this is already seen in [2, §3].) The local version of our method is the following : Let  $(V, P)$  be a normal 2-dimensional hypersurface triple point. By a suitable change of the coordinate, the equation of  $P$  is given by  $\xi^3 + g(x, y)\xi + h(x, y) = 0$ . Then, from the pair of the resolution of plane curve singularities  $g(x, y) = 0$  and  $h(x, y) = 0$ , one can descend the multiplicity of any infinitely near point of  $P$  ; that is, one can reduce this triple point to some isolated double points and simple codimension 1 double points. (We call them compound nodes and compound cusps.)

In §1, we produce the above process by twisting the line bundles similar as in [5, §2], and obtain a certain formula. In §2, we give some examples. In §3, we show the following : Let  $(V, P)$  be an isolated singularity defined by  $\xi^3 + f(x, y) = 0$ . Let  $\mu$  be the Milnor number

of  $P$ , and let  $p_g$  be the geometric genus of  $P$ . Then we have  $\mu \geq 6p_g$ . Durfee's conjecture [3, p.97] is true in this case. (Compare [20] and [19].)

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## 1 Process of resolution.

**1.1** Let  $Y$  be a nonsingular surface, and let  $L$  be a line bundle on  $Y$ . Let  $\pi : X = P(\mathcal{O}_Y(L) \oplus \mathcal{O}_Y) \rightarrow Y$  be a  $\mathbb{P}^1$ -bundle, and set  $T = \mathcal{O}_X(1)$  the 0-section of  $\pi$  and set  $T_\infty$  the  $\infty$ -section of  $\pi$ .  $T_\infty$  is linearly equivalent to  $T - \pi^*L$ . Let  $S$  be a normal surface on  $X$  which is linearly equivalent to  $3T$ . We choose  $X_0 \in |T|$  (the complete linear system of  $T$ ) and  $X_1 \in |T_\infty|$  such that  $(X_0, X_1)$  gives a system of homogeneous fiber coordinate of  $\pi$ . Then the equation of  $S$  is given by  $\sum_{i=0}^3 \psi_{iL} X_0^{3-i} X_1^i = 0$  where  $\psi_{iL}$  is an element of  $H^0(Y, \mathcal{O}(iL))$ . By setting  $Y_0 = X_0 + (1/3)\psi_{2L}X_1 \in |T|$ ,  $Y_1 = X_1$ ,  $g = \psi_{2L} - (1/3)\psi_L^2 \in H^0(Y, \mathcal{O}(2L))$  and  $h = \psi_{3L} - (1/3)\psi_{2L}\psi_L + (2/27)\psi_L^3 \in H^0(Y, \mathcal{O}(3L))$ , the equation of  $S$  is given by  $Y_0^3 + gY_0Y_1^2 + hY_0^3 = 0$ . The surface  $S$  does not intersect with  $T_\infty$ . So for convenience, setting  $\xi = Y_0/Y_1$  the inhomogeneous fiber coordinate of  $\pi$ , we write the equation of  $S$  by

$$(1) \quad f := \xi^3 + g\xi + h = 0.$$

We call the divisors  $G = (g)$  and  $H = (h)$  on  $Y$  *the first and the second assistant curve* for  $S$  respectively.

Let  $\tilde{P}$  be a singular point on  $S$ . Set  $P = \pi(\tilde{P})$ . Let  $m_P$  be the maximal ideal at  $P$ , and let  $\Delta = 4g^3 + 27h^2 \sim 6L$  be the discriminant divisor for  $f$ . ( $\sim$  is the linear equivalence.) By [12, Lemma 5.1],  $\tilde{P}$  satisfies one of the following two conditions ;

- (a)  $g \in m_P$  and  $h \in m_P^2$  or,
- (b)  $g \notin m_P, h \notin m_P$  and  $\Delta \in m_P^2$ .

When (b) is the case, set  $\tilde{P} = (\xi, x, y) = (\xi_0, x_0, y_0)$  for  $\xi_0 \neq 0$  where  $(x, y)$  is a local coordinate at  $P$ . Then by putting  $\eta = \xi - \xi_0$ , we have

$$f = \eta^3 + 3\xi_0\eta^2 + (g + 3\xi_0^2)\eta + (\xi_0^3 + g\xi_0 + h),$$

where  $g(x_0, y_0) + 3\xi_0^2 = 0$ ,  $g_x(x_0, y_0)\xi_0 + h_x(x_0, y_0) = 0$  and  $g_y(x_0, y_0) + h_y(x_0, y_0) = 0$ . Hence  $\tilde{P}$  is a double point. We call it *an inner double point*. When (a) is the case,  $\tilde{P}$  is on the 0-section  $\xi = 0$ . We call it *a target singularity*. From now on, we consider to reduce the target singularities of multiplicity 3 on  $S$  to some isolated double points and simple codimension one double points.

**1.2** Let  $\tilde{P}$  be a target triple point on  $S$ . Assume  $\tilde{P} = \{(\xi, x, y) = (0, 0, 0)\}$  and the equation at  $\tilde{P}$  is of the form (1). Set  $m_1 = \text{mult}_P G \geq 2$  and  $n_1 = \text{mult}_P H \geq 3$  where  $\text{mult}_P G$  is the multiplicity of  $G$  at  $P$ . Let  $\tau_1 : Y_1 \rightarrow Y$  be the blow up at  $P$ , and set  $E_1 = \tau^{-1}(P)$ . Let  $\widetilde{G}_1, \widetilde{H}_1$  be the proper transform of  $G$  and  $H$  by  $\tau_1$  respectively. Then we have  $\tau_1^*G = \widetilde{G}_1 + m_1E_1$  and  $\tau_1^*H = \widetilde{H}_1 + n_1E_1$ . Now we set

$$l_1 = \min([m_1/2], [n_1/3]),$$

where  $[m_1/2]$  is the greatest integer not exceeding  $m_1/2$ . Put  $L_1 = \tau_1^*L - l_1E_1$ . Let  $\pi_1 : X_1 = \mathbf{P}(\mathcal{O}_{Y_1}(L_1) \oplus \mathcal{O}_{Y_1}) \rightarrow Y_1$  be the  $\mathbf{P}^1$ -bundle, and set  $T_1 = \mathcal{O}_{X_1}(1)$ . Put  $G_1 = \tau_1^*G - 2l_1E_1 = \widetilde{G}_1 + (m_1 - 2l_1)E_1 \sim 2L_1$  and  $H_1 = \tau_1^*H - 3l_1E_1 = \widetilde{H}_1 + (n_1 - 3l_1)E_1 \sim 3L_1$ . We have either  $0 \leq m_1 - 2l_1 \leq 1$  or  $0 \leq n_1 - 3l_1 \leq 2$ . Now we set

$$f_1 = \xi_1^3 + g_1\xi + h_1,$$

where  $\xi_1$  is the inhomogeneous fiber coordinate of  $\pi_1$ ,  $(g_1) = G_1$  and  $(h_1) = H_1$ . The surface  $S_1$  on  $X_1$  defined by  $(f_1)$  is linearly equivalent to  $3T_1$ . Let  $X_1 \rightarrow X$  be the morphism associated with the composition of sheaf homomorphisms

$$\mathcal{O}_Y(L) \xrightarrow{\tau_1^*} \mathcal{O}_{Y_1}(\tau_1^*L) \rightarrow \mathcal{O}_{Y_1}(L_1),$$

where the last map is obtained by tensoring  $\mathcal{O}_{Y_1}(-l_1E_1)$ . Let  $\tilde{\tau}_1 : S_1 \rightarrow S$  be the restriction of this morphism to  $S_1$ . We complete the first step of the following commutative diagram ;

$$(2) \quad \begin{array}{ccccccc} X & & X_1 & & & & X_r \\ \cup & & \cup & & & & \cup \\ S & \xleftarrow{\tilde{\tau}_1} & S_1 & \leftarrow \cdots \leftarrow & \tilde{\tau}_n & & S_r \\ \downarrow \pi & & \downarrow \pi_1 & & & & \downarrow \pi_r \\ Y & \xleftarrow{\tau_1} & Y_1 & \leftarrow \cdots \leftarrow & \tau_n & & Y_r. \end{array}$$

We continue this process. Then :

**Lemma 1.3** *After a finite step of this process (say  $r$ -times), the obtained surface  $S_r$  has no singularity of multiplicity 3.*

In the diagram (2), we assume that  $\tau_i : Y_i \longrightarrow Y_{i-1}$  is the blow up at  $P_{i-1} \in Y_{i-1}$ , and set  $m_i = \text{mult}_{P_{i-1}} G_{i-1}$ ,  $n_i = \text{mult}_{P_{i-1}} H_{i-1}$  and  $l_i = \min(\lfloor m_i/2 \rfloor, \lfloor n_i/3 \rfloor)$ . Let  $\omega_{S_i}$  be the dualizing sheaf on  $S_i$ .

**Lemma 1.4**

$$\begin{aligned} \chi(\mathcal{O}_{S_r}) - \chi(\mathcal{O}_S) &= -(1/2) \sum_{i=1}^r l_i (5l_i - 3), \\ \omega_{S_r}^2 - \omega_S^2 &= -3 \sum_{i=1}^r (2l_i - 1)^2. \end{aligned}$$

**1.5** We study the assistant curves of  $S_r$ . We may assume that the reduced part of  $G_r$  and  $H_r$  is normal crossing. Let  $\mathbb{E} = \sum_{i=1}^k E_i$  be the decomposition to the irreducible components of the total transform of the curve  $G \cup H$  by  $\tau = \tau_r \circ \cdots \circ \tau_1$ . Assume that  $\mathbb{E}' = \sum_{i=1}^r E_i$  is the exceptional curve for  $\tau$ . For each  $E_i \in \mathbb{E}$ , we give a double  $\mathbf{Z}$ -weighting  $(a_i : b_i)$  such that  $E_i$  is the component of  $G_r$  (resp.  $H_r$ ) of multiplicity  $a_i$  (resp.  $b_i$ ). By this way, the components of  $\mathbb{E}$  are classified into the following six types ;

$$\begin{array}{ll} (C) (\alpha, 2), \alpha \geq 2 & (N) (1, \beta), \beta \geq 2 \\ (I) (\gamma, 1), \gamma \geq 1 & (II) (\delta, 0), \delta \geq 1 \\ (III) (0, \epsilon), \epsilon \geq 1 & (\phi) (0, 0). \end{array}$$

Moreover, continue our process (2) if necessary. Then we can classify the type of points of intersection of the components of  $\mathbb{E}$  in the following ;

$$\begin{aligned}
& P(C \cap II), \quad P(N^2 \cap II), \quad P(N \cap III), \quad P(I \cap II), \\
& P(I \cap III^1), \quad P(I^1 \cap III), \quad P(II \cap II), \quad P(II \cap III^2), \\
& P(II \cap III^1), \quad P(II^1 \cap III), \quad P(III \cap III), \quad P(\phi \cap \text{any type}),
\end{aligned}$$

where  $P(C \cap II)$  is the point of intersection of the components of types  $(C)$  and  $(II)$ ,  $P(N^2 \cap II)$  is that of types  $(N)$  with  $\beta = 2$  and  $(II)$ , etc.

**1.6** We proceed our resolution process.

1) Let  $C_i$  ( $1 \leq i \leq n(C)$ ) be the component of  $\mathbf{E}'$  of type  $(C)$ . Set  $\widetilde{C}_i = \pi_r^{-1}(C_i)$ . The singularity of  $S_r$  is  $\xi^3 + x^\alpha \xi + x^2 = 0$  at a general point of  $\widetilde{C}_i$ , and is  $\xi^3 + x^\alpha y^\delta \xi + x^2 = 0$  at the point of type  $\pi_r^{-1}(P(C_i \cap II))$ . We call it a *compound cusp*.

Let  $\tau_{r+1} : X_{r+1} \longrightarrow X_r$  be the blow up with the center  $\sum_{i=1}^{n(C)} \widetilde{C}_i$ . Let  $S_{r+1}$  be the strict transform of  $S_r$  by  $\tau_{r+1}$ . Set  $\bar{\tau}_{r+1} = \tau_{r+1}|_{S_{r+1}}$  be the restriction of  $\tau_{r+1}$  to  $S_{r+1}$ . The surface  $S_{r+1}$  is nonsingular along  $(\bar{\tau}_{r+1})^{-1}(\widetilde{C}_i)$ ,  $1 \leq i \leq n(C)$ . Let  $C'_i$  be the reduced part of  $(\bar{\tau}_{r+1})^{-1}(\widetilde{C}_i)$ .  $C'_i$  is naturally isomorphic to  $\mathbf{P}^1$ . And we have

$$\begin{aligned}
\chi(\mathcal{O}_{S_{r+1}}) - \chi(\mathcal{O}_{S_r}) &= n(C) - \sum_{i=1}^{n(C)} (C'_i)^2, \\
\omega_{S_{r+1}}^2 - \omega_{S_r}^2 &= 8n(C).
\end{aligned}$$

2) Let  $N_i$  ( $1 \leq i \leq n(N)$ ) be the component of  $\mathbf{E}'$  of type  $(N)$ . Assume that  $N_i$  has  $\mathbf{Z}$ -weighting  $(1, 2)$  (resp.  $(1, \beta)$  with  $\beta \geq 3$ ) for  $1 \leq i \leq n(N^2)$  (resp.  $n(N^2) + 1 \leq i \leq n(N)$ ). Set  $\tau'_{r+1} = \bar{\tau}_{r+1} \circ \pi_r$  and  $\widetilde{N}_i = (\tau'_{r+1})^{-1}(N_i)$ . The singularity along  $\widetilde{N}_i$  is the following ;

$$\begin{aligned}
& \xi^3 + x\xi + x^\beta = 0 \text{ at a general point of } \widetilde{N}_i, \\
& \xi^3 + x\xi + x^\beta y^\epsilon = 0 \text{ at } (\tau'_{r+1})^{-1}(P(N_i \cap III)) \text{ for } 1 \leq i \leq n(N), \\
& \xi^3 + xy^\delta \xi + x^2 = 0 \text{ at } (\tau'_{r+1})^{-1}(P(N_i \cap II)) \text{ for } 1 \leq i \leq n(N^2).
\end{aligned}$$

We call it a *compound node*.

Let  $\tau_{r+2} : X_{r+2} \longrightarrow X_{r+1}$  be the blow up with the center  $\sum_{i=1}^{n(N)} \widetilde{N}_i$ . Let  $S_{r+2}$  be the strict transform of  $S_{r+1}$  by  $\tau_{r+2}$ . We set  $\bar{\tau}_{r+2} = \tau_{r+2}|_{S_{r+2}}$  and  $\tau'_{r+2} = \bar{\tau}_{r+2} \circ \tau'_{r+1}$ . The pull back  $(\bar{\tau}_{r+2})^*(\widetilde{N}_i)$  is a union with two

nonsingular rational curves  $N'_i$  and  $N''_i$ . The surface  $S_{r+2}$  is nonsingular along  $N'_i$  and  $N''_i$ . Moreover, for  $n(N^2) < i \leq n(N)$ ,  $N'_i$  and  $N''_i$  does not intersect each other. For  $1 \leq i \leq n(N^2)$ , we let  $II_1, \dots, II_{n(N_i II)}$  be the components of  $\mathbb{E}$  of type  $(II)$  which intersect  $N_i$ . Assume that the  $\mathbb{Z}$ -weighting of  $II_j$  is  $(\delta_j, 0)$  for  $1 \leq j \leq n(N_i II)$ . Then the curves  $N'_i$  and  $N''_i$  intersect each other at  $(\tau'_{r+2})^{-1}(P(N_i \cap II_j))$  with intersection multiplicity  $\delta_j$ . And we have

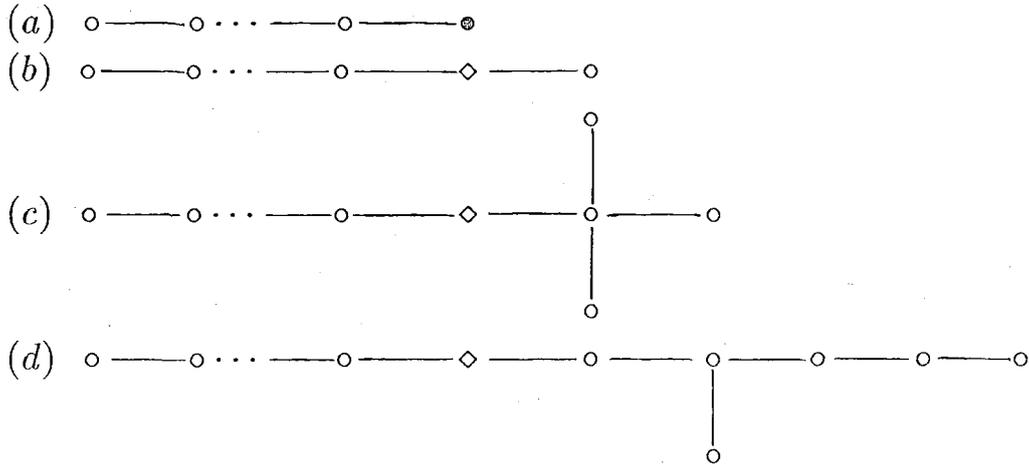
$$\begin{aligned} \chi(\mathcal{O}_{S_{r+2}}) - \chi(\mathcal{O}_{S_{r+1}}) &= n(N) - \sum_{i=1}^{n(N^2 II)} \sum_{j=1}^{n(N_i II)} \delta_j, \\ \omega_{S_{r+2}}^2 - \omega_{S_{r+1}}^2 &= 8n(N) + \sum_{i=1}^{n(N)} ((N'_i)^2 + (N''_i)^2) \\ &\quad - 2 \sum_{i=1}^{n(N^2 II)} \sum_{j=1}^{n(N_i II)} \delta_j. \end{aligned}$$

3) At the point of  $(\tau'_{r+2})^{-1}(P(I^r \cap III^\epsilon))$ , the surface  $S_{r+2}$  has a rational double point (a RDP for short) of type  $A_{3\epsilon-1}$ . Put  $\bar{\tau}_{r+3} : S_{r+3} \rightarrow S_{r+2}$  the minimal resolution of these singularities. We have

$$\chi(\mathcal{O}_{S_{r+3}}) = \chi(\mathcal{O}_{S_{r+2}}), \quad \omega_{S_{r+3}}^2 = \omega_{S_{r+2}}^2.$$

4) At the point of  $(\bar{\tau}_{r+3} \circ \tau'_{r+2})^{-1}(P(II^\delta \cap III^2))$ , the surface  $S_{r+3}$  has a singularity  $\xi^3 + x^\delta \xi + y^2 = 0$ . Let  $\bar{\tau}_{r+4} : S_{r+4} \rightarrow S_{r+3}$  be the minimal resolution. If  $\delta \leq 3$ , then this is a RDP. If  $\delta \geq 4$ , then the dual graph of the exceptional curve of this singularity is the following

:



In the above graph, the symbol  $\circ$  (resp.  $\diamond$ ) is a nonsingular rational curve with self-intersection number  $-2$  (resp.  $-3$ ). The symbol

• is an elliptic curve with self-intersection number  $-3$ . The graphs (a), (b), (c), (d) correspond to  $\delta \equiv 0, 1, 2, 3 \pmod{4}$ , respectively. The number of  $(-2)$ -curves of the head part  $\circ - - \circ \cdots \circ$  in (a)  $\sim$  (d) is  $\lfloor \delta/4 \rfloor - 1$ . Moreover we have

$$\chi(\mathcal{O}_{S_{r+4}}) - \chi(\mathcal{O}_{S_{r+3}}) = \omega_{S_{r+4}}^2 - \omega_{S_{r+3}}^2 = \sum_{i=1}^{n(III^2II)} [\delta_i/4],$$

where the summation moves all the points of type  $P(III^2 \cap II^{\delta_i})$ .

We set  $\bar{\tau} = \bar{\tau}_{r+4} \circ \cdots \circ \bar{\tau}_1 : S^* = S_{r+4} \longrightarrow S$ , and call it the *canonical resolution* of  $S$ .

Let  $\phi_i$  be a component of  $\mathbf{E}'$  of type  $(\phi)$ . Let  $E_1, \dots, E_{l(i)}$  be the components of  $\mathbf{E}$  which intersect to  $\phi_i$ . Assume that  $E_j$  has  $\mathbf{Z}$ -weighting  $(a_j, b_j)$  for  $1 \leq j \leq l(i)$ . Then we have  $3 \sum_{i=1}^{l(i)} a_j = 2 \sum_{i=1}^{l(i)} b_j$ . Let  $\Delta_r$  be the strict transform of the discriminant divisor  $\Delta$  by  $\tau$ . If  $\Delta_r$  intersect  $\phi_i$  at a point such that  $\Delta_r \in m_P^2$ , there is a possibility that  $S^*$  is singular at  $(\bar{\tau}_{r+4} \circ \cdots \circ \bar{\tau}_{r+1} \circ \pi_r)^{-1}(P)$ . We call it an *infinitely near inner double point*.

**Proposition 1.7** *The surface  $S^*$  has only inner double and infinitely near inner double points as its singularities (if they exists). Moreover we have*

$$\begin{aligned} \chi(\mathcal{O}_{S^*}) - \chi(\mathcal{O}_S) &= -(1/2) \sum_{i=1}^r l_i(5l_i - 3) + n(C) + n(N) \\ &\quad - \sum_{i=1}^{n(C)} (C_i')^2 - \sum_{i=1}^{n(N^2II)} \delta_i - \sum_{i=1}^{n(III^2II)} [\delta_i/4], \\ \omega_{S^*}^2 - \omega_S^2 &= -3 \sum_{i=1}^r (2l_i - 1)^2 + 8n(C) + 8n(N) \\ &\quad + \sum_{i=1}^{n(N)} ((N_i')^2 + (N_i'')^2) - 2 \sum_{i=1}^{n(N^2II)} \delta_i - \sum_{i=1}^{n(III^2II)} [\delta_i/4]. \end{aligned}$$

## 2 Examples.

Let  $(V, P)$  be a 2-dimensional hypersurface singularity of multiplicity 3. By the Weierstrass preparation theorem and the Tschirnhaus-transformation, the equation of  $(V, P)$  is given by  $\xi^3 + g(x, y)\xi + h(x, y) =$

0. Let  $\sigma : \widetilde{V} \rightarrow V$  be the minimal resolution of  $(V, P)$ . We fix a natural compactification  $\bar{\sigma} : \widehat{V} \rightarrow \bar{V}$  of  $\sigma$ . There exists an effective divisor  $Z_P$  supported on  $\bar{\sigma}^{-1}(P)$  such that  $\omega_{\widehat{V}} = \bar{\sigma}^* \omega_{\bar{V}} \otimes \mathcal{O}_{\widehat{V}}(-Z_P)$ . Then we have

$$\chi(\mathcal{O}_{\widehat{V}}) - \chi(\mathcal{O}_{\bar{V}}) = p_g(P), \quad \omega_{\widehat{V}}^2 - \omega_{\bar{V}}^2 = -Z_P^2,$$

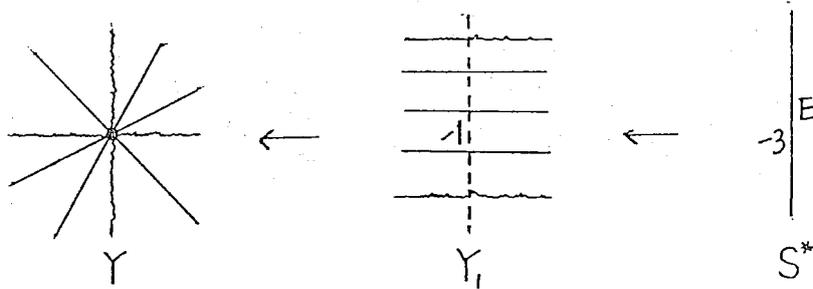
where  $p_g(P)$  is the geometric genus of  $(V, P)$ . We call  $(p_g(P) : -Z_P^2)$  the type of singularity  $(V, P)$ . For constructing a resolution process and for calculating the type of singularity  $(V, P)$ , the method in §1 is useful. (Compare [15] and [18, §2].)

In this section, we give three examples. In the figures below, a broken line (resp. a usual line) means a component of the first (resp. the second) assistant curve at each step from  $S$  to  $S_r$ . For instance, the curve  $E_3$  on  $Y_4$  in Example 2.2 has  $\mathbf{Z}$ -weighting  $(1, 3)$ . A dotted line is a curve of type  $(\phi)$ . A brack spot is the point for blow up in the next step. The number beside a line is the self-intersection number of the curve.

For all these examples, the type of the singularity is  $(1 : 3)$  by Proposition 1.6. (See also [8].)

*Example 2.1*  $\xi^3 + xy\xi + x^3 + y^3$ .

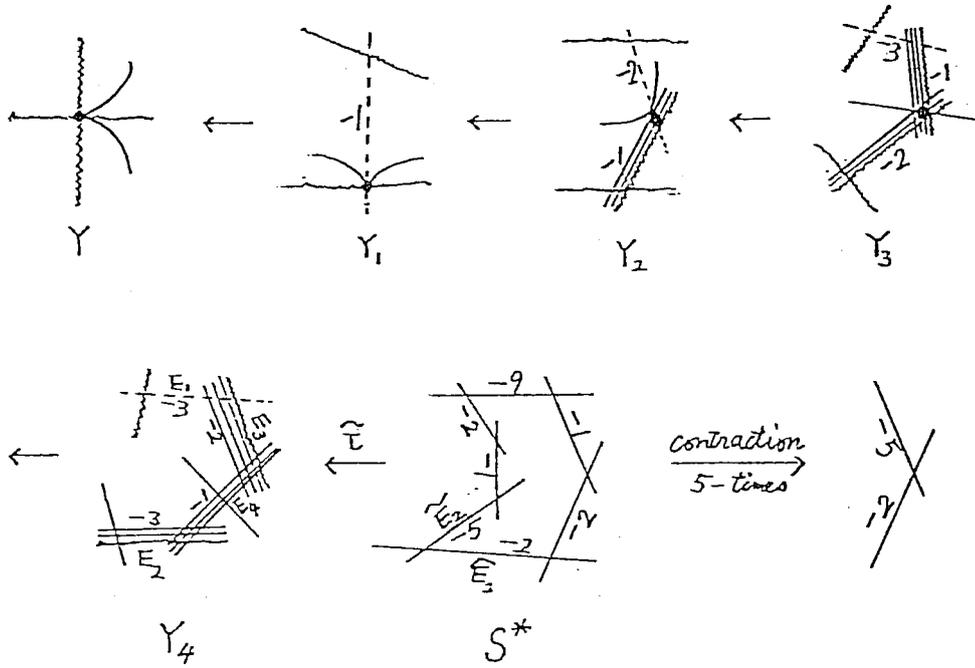
This is a simple elliptic singularity of type  $\widetilde{E}_6$ . ([16]) Since the degree of  $\Delta_1$  is 6,  $E$  is an elliptic curve.



*Example 2.2*  $\xi^3 + xy\xi + x^5 + y^3$ .

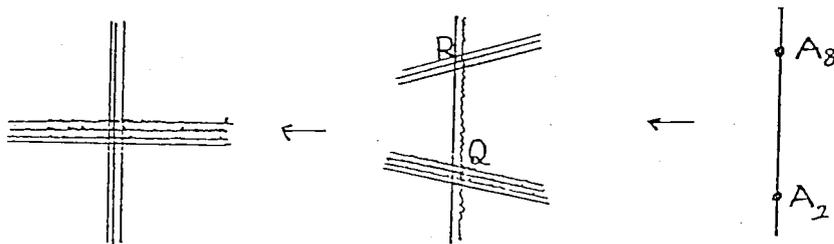
All the components of exceptional curves are rational. Over the curves  $E_2$  and  $E_4$  on  $Y_4$ ,  $S_4$  has compound nodes. The calculation for the self-intersection number is, for example, the following ; Since

$-3 = E_2^2 = (\tilde{\tau}_* \widetilde{E}_2) \widetilde{E}_2 = \widetilde{E}_2(\tilde{\tau}^* E_2) = \widetilde{E}_2(\widetilde{E}_2 + 2\widehat{E}_2) = \widetilde{E}_2^2 + 2$ , we have  $\widetilde{E}_2^2 = -5$ .



*Example 2.3*  $\xi^3 + x^3\xi + xy^3$ .

Since the equation over  $R$  is  $\xi^3 + z\xi + y^3z = z(\xi + y^3) + \xi^3 = z\zeta + (\zeta - y^3)^3 = \zeta(z + \dots) - y^9$ , this is a RDP of type  $A_8$ .



### 3 An inequality for a singularity $\xi^3 + h(x, y)$ .

Let  $(V, P)$  be an isolated triple point. Assume that the equation of  $(V, P)$  is given by  $\xi^3 + h(x, y) = 0$  after a suitable change of the coordinate. Let  $\mu(P)$  be the Milnor number of  $(V, P)$ . ([11]) The aim in this section is to calculate the number  $\mu(P) - 6p_g(P)$ . (For the motivation, see Durfee [3, p.97].)

We prepare the following definition : In general, let  $Q$  be a point on a nonsingular surface. Let  $F_1, \dots, F_s$  be irreducible reduced curves passing through  $Q$ . For  $s_1 \leq s$ , we consider a divisor

$$D = \sum_{i=1}^{s_1} F_i + 2 \sum_{j=s_1+1}^s F_j.$$

Let  $t$  be the number of tangent line at  $Q$  of the reduced part  $D_{red}$  of  $D$ . We have automatically  $t \leq s$ . We set  $m = \text{mult}_P(D)$  and  $l = \lfloor m/3 \rfloor$ .

**Definition 3.1** We define the Durfee's number  $d_Q(D)$  of  $D$  at  $Q$  by the following ;

- 1) If  $m \equiv 0 \pmod{3}$ , then put  $d_Q(D) = 3l(l+1) - 3s + 2s_1 - 2t + 3$ .
- 2) If  $m \equiv 1$  or  $2 \pmod{3}$ , then put  $d_Q(D) = 3l(l+1) - 3s + 2s_1 - 1$ .

Now setting  $G = \phi$  (empty) and  $H = (h(x, y))$ , we consider the diagram (2) in §1 of cyclic type at  $P$ . (See [2, §3].) The second assistant curve  $H_i$  coincides with the branch curve of  $\pi_i : S_i \rightarrow Y_i, 1 \leq i \leq r$ . Let  $b$  be the number of irreducible components of  $H$ . Set  $P_0 = P$  and  $H_0 = H$ .

**Proposition 3.2**  $\mu(P) - 6p_g(P) = \sum_{i=0}^{r-1} d_{P_i}(H_i) + b - 1$ .

For the proof of this proposition, we use Laufer's formula [9].

For the Durfee's number  $d_{P_i}(H_i)$ , we have the following :

**Lemma 3.3** 1) If  $m_i = \text{mult}_{P_i}(H_i) \geq 6$  or  $m_i = 3$ , then we have  $d_{P_i}(H_i) > 0$ .

2) If  $m_i = 4$  or  $3$ , then we have  $d_{P_i}(H_i) \geq -2$ . Moreover, if  $P_i$  is the point with  $d_{P_i}(H_i) = -1$  (resp.  $= -2$ ), then we have  $\sum_{j=i+1}^{r-1} d_{P_j}(H_j) \geq 1$  (resp.  $\geq 2$ ).

Combining this with Tomari's result [19], we obtain the following :

**Corollary 3.4** *Let  $(V, P)$  be an isolated singularity defined by  $\xi^3 + h(x, y) = 0$ . Then we have  $\mu(P) \geq 6p_g(P)$ .*

*Problems* 1) Show the analogous argument as in §3 for any 2-dimensional hypersurface triple point.

2) Construct the process of canonical resolution for surfaces of n-section type.

The detail will be published in our forthcoming paper.

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# A Remark on the Geography of Surfaces with Birational Canonical Morphisms

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**Introduction.** Let  $S$  be a minimal algebraic surface of general type defined over the field of complex numbers  $\mathbf{C}$ . Then by the inequalities of Noether and Bogomolov-Miyaoka-Yau, we have

$$2\chi(\mathcal{O}_S) - 6 \leq c_1^2(S) \leq 9\chi(\mathcal{O}_S), \chi(\mathcal{O}_S) > 0, c_1^2(S) > 0.$$

Thus a natural problem is to construct a surface  $S$  such that  $\chi(\mathcal{O}_S) = x$  and  $c_1^2(S) = y$  for any integers  $(x, y)$  in the above region. For this problem, Persson [15] proved that all the invariants  $(\chi, c_1^2)$  with  $2\chi - 6 \leq c_1^2 \leq 8\chi - 20$  occur for some surface carrying a pencil of curves of genus 2. Recently Xiao extended Persson's method and constructed many surfaces carrying pencils of hyperelliptic curves ([22],[24],[7]). We remark that the canonical maps of these surfaces are not birational because the relative canonical maps are already generically  $2 : 1$ .

In this paper, we are interested in surfaces whose canonical maps are birational onto their images, because we believe that the "general" surface of general type has this property in view of the theory of curves. (For this motivation, see also [6], [23]). Since the lower bound of such surfaces is  $c_1^2 = 3\chi - 10$  by Castelnuovo's second inequality ([3]), our problem is ;

"Construct surfaces with birational canonical map in the region  $3\chi - 10 \leq c_1^2 \leq 9\chi$  naturally from the geographical point of view."

Many works are related to this problem. For instance, see [10],[5],[14],[25],[19], [8] etc.(Consult [16] for other references.)

Now our approach is in some sense a nonhyperelliptic version of Persson's original one. His way was natural in the following sense: For any surface  $S$  on the Noether line  $c_1^2 = 2\chi - 6$  ( $\chi \geq 7$ ), Horikawa [12, I] proved that  $S$  has a genus 2 fibration such that the "degenerate" genus 2 fibers come from the resolution of rational double points in

some sense. So it seems that the simplest way to have many surfaces beyond the Noether line is to construct on  $S$  degenerate genus 2 fibers which come from the simple Gorenstein singularities not equal to RDP. Persson used the simple elliptic singularities of type  $\widetilde{E}_8$ .

Thus for our purpose, we first observe the surface on the Castelnuovo line  $c_1^2 = 3\chi - 10$ . By the classical idea of Castelnuovo [4], such a surface with birational canonical map ( $\chi \geq 9$ ) always has a fibration of nonhyperelliptic curves of genus 3 and the "degenerate" genus 3 fibers come from the resolution of RDP. Therefore it is natural for our purpose to construct many simple degenerate genus 3 fibers which do not come from RDP. We use simple elliptic singularities of type  $\widetilde{E}_7$ . Our result is the following :

**Theorem** *Let  $x, y$  be any integers with  $3x - 10 \leq y \leq 8x - 78$ . Then there exists a minimal surface  $S$  such that (1)  $\chi(\mathcal{O}_S) = x$ ,  $c_1^2(S) = y$ , (2) the canonical map of  $S$  is a birational holomorphic map onto its image and (3)  $S$  has a fibration of nonhyperelliptic curves of genus 3.*

One can also find such surfaces in a subregion of  $8\chi - 77 \leq c_1^2 \leq 8\chi - 35$  (see §3). The key step of the proof is to study the resolution process of singularities arising from certain cyclic quadruple coverings.

*Acknowledgement and Notes:* I express my special thanks to Dr. Kazuhiro Konno. Konno and I held seminars every week in 1988/89, and he gave me many essential advices. Now, after his recent beautiful work [13] was established (December 1989), the approach here is not "legitimate" in some sense (see [13, Remark 9.9]). For more natural way to this problem, please see his appendix to this paper.

I also thank Prof. Eiji Horikawa who communicated with me on the occasion of his visit in Sendai (March 1989). He had already studied surfaces carrying pencils of curves of genus 3 in detail about ten years ago. But his paper have not appeared yet unfortunately. He also gave us the question about the region of existence of such surfaces. Considering our result, one can ask;

"If  $S$  has a pencil of nonhyperelliptic curves of genus 3, then do we have  $3\chi(\mathcal{O}_S) - 10 \leq c_1^2(S) \leq 8\chi(\mathcal{O}_S)$ ?? "

I also thank Prof. Sampei Usui who explained me the relation between our method and a way to the Torelli problem and inspired me. I also thank Dr. Noboru Nakayama who gave me useful comment on irregularity.

## 1 Construction.

1.1 Let  $m, n$  be nonnegative integers satisfying

$$(1) \quad m + n \geq 2, \quad n \leq 3.$$

Let  $C$  be a nonsingular curve of genus  $g$ , and let  $D_m, D_n$  be divisors of degree  $m, n$  on  $C$  respectively. We let

$$\delta : W = \mathbf{P}(\mathcal{O}_C \oplus \mathcal{O}_C(D_m)) \longrightarrow C$$

be the geometrically ruled surface, and denote by  $C_0, C_\infty$  and  $f$  the 0-section ( $C_0^2 = m$ ), the  $\infty$ -section and a fiber on  $W$  respectively. We set

$$\pi : X = \mathbf{P}(\mathcal{O}_W \oplus \mathcal{O}_W(C_0)) \longrightarrow W$$

the  $\mathbf{P}^1$ -bundle, and set  $T = \mathcal{O}_X(1)$ .

Let  $S$  be an irreducible divisor on  $X$  which is linearly equivalent to  $4T + \pi^*\delta^*D_n$ . We denote by  $\omega_S$  the dualizing sheaf on  $S$ , and set  $p_g(S) = h^0(S, \omega_S)$  and  $q(S) = h^1(S, \mathcal{O}_S)$ .

### Lemma 1.2

$$p_g(S) = 4m + 3n + 3(g - 1), \quad q(S) = 0, \quad \omega_S^2 = 12m + 8n + 16(g - 1).$$

*Proof.* Denoting by  $K_X$  the canonical bundle on  $X$ , we have  $K_X + S \sim -2T + \pi^*(K_W + S) + 4T + \pi^*\delta^*D_n \sim 2T + \pi^*(-C_0 + \delta^*(K_C + D_m + D_n))$  where " $\sim$ " is the linear equivalence. Thus we have  $h^0(X, K_X + S) = \sum_{i=0}^2 h^0(W, (i-1)C_0 + \delta^*(K_C + D_m + D_n)) = 2h^0(C, K_C + D_m + D_n) + h^0(C, K_C + 2D_m + D_n) = 4m + 3n + 3(g - 1)$  by (1). Since

$R^p\pi_*\mathcal{O}(K_X + S) = 0$  for  $p \geq 1$ , we have similarly  $H^p(X, K_X + S) = 0$  for  $p = 0, 1$ . Hence by the long exact sequence associated with

$$0 \longrightarrow \mathcal{O}(K_X) \longrightarrow \mathcal{O}(K_X + S) \longrightarrow \mathcal{O}(\omega_S) \longrightarrow 0,$$

we have  $p_g(S) = h^0(X, K_X + S)$  and  $q(S) = h^2(X, K_X) = h^1(C, \mathcal{O}_C) = g$ .

Since  $\omega_S^2 = (K_X + S)^2 S$  and  $T^2 = c_1(\mathcal{O}_W \oplus \mathcal{O}_W(C_0))T$ , we also have the desired formula for  $\omega_S^2$  by an easy calculation. *q.e.d.*

**1.3** From now on, we fix a fiber coordinate  $(Y_0, Y_1)$  of  $\pi : X \longrightarrow W$ . Denote by  $T_\infty$  the section defined by  $Y_1 = 0$ .  $T_\infty$  is linearly equivalent to  $T - \pi^*C_0$ . If we choose  $D_n$  general, then the divisor  $S|_{T_\infty}$  (the restriction of  $S$  to  $T_\infty$ ) is  $n$  distinct fibers on  $T_\infty \simeq W \longrightarrow C$ . On the other hand, the self-intersection number of the divisor  $T_\infty|_S$  on  $S$  is  $-n$ . Thus  $S$  contains  $n$  exceptional curves of first kind. Let  $\nu : S \longrightarrow S'$  be the contraction of these curves. We remark that  $\nu$  is induced by the contraction of the ambient space

$$\Phi_T : X \longrightarrow X' := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C \oplus \mathcal{O}_C(D_m)),$$

where  $\Phi_T$  is the morphism defined by the linear system  $|T|$ . Then we have  $p_g(S') = p_g(S)$ ,  $q(S') = q(S)$  and  $\omega_{S'}^2 = \omega_S^2 + n = 3\chi(\mathcal{O}_{S'}) + 10(g - 1)$  by Lemma 1.2.

Now assume that (1)  $S$  has  $k$  simple elliptic singularities of type  $\widetilde{E}_7$  (see [18]), (2) other singularities of  $S$  are at most rational double points (RDP for short) and (3)  $S$  is smooth along  $T_\infty|_S$ . Let  $\tilde{S}$  be the minimal resolution of singularities of  $S'$ . Then by [17] or [1] we have

$$\chi(\mathcal{O}_{\tilde{S}}) = 4m + 3n + 2(g - 1) - k, \quad \omega_{\tilde{S}}^2 = 12m + 9n + 16(g - 1) - 2k.$$

In order to realize the above assumption, we set up the following situation.

Considering that any  $\phi \in H^0(X, 4T + \pi^*\delta^*D_n)$  can be written as  $\phi = \sum_{i=0}^4 \phi_{iC_0 + \delta^*D_n} Y_0^{4-i} Y_1^i$  where  $\phi_{iC_0 + \delta^*D_n} \in H^0(W, iC_0 + \delta^*D_n)$ , we set

$$|4T + \pi^*\delta^*D_n|_C = \{(\phi) \in |4T + \pi^*\delta^*D_n|; \phi_{\delta^*D_n} Y_0^4 + \phi_{4C_0 + \delta^*D_n} Y_1^4\},$$

where  $(\phi)$  is the divisor defined by  $\phi$ . We call it the *cyclic subsystem* of  $|4T + \pi^*\delta^*D_n|$ . For any member  $(\phi) \in |4T + \pi^*\delta^*D_n|_C$ , we call the divisor  $B_\phi = (\phi_{4C_0+\delta^*D_n})$  and  $B'_\phi = (\phi_{\delta^*D_n})$  the branch and the assistant branch locus of  $(\phi)$ , respectively. With this notation, we define the *good cyclic subsystem*  $|4T + \pi^*\delta^*D_n|_{GC}$  by the following two properties:

- 1)  $B_\phi$  is reduced and  $B'_\phi$  is nonsingular.
- 2)  $B'_\phi$  passes through no singular points of  $B_\phi$ , and they meet transversally.

**Lemma 1.4** *Let  $S = (\phi)$  be a member of  $|4T + \pi^*\delta^*D_n|_{GC}$  and set  $\pi_S := \pi|_S : S \rightarrow W$ . Then we have*

- (a)  $S$  is irreducible and normal,
- (b)  $Sing(S) = \pi_S^{-1}(Sing(B_\phi)) \cap Support(T)$ ,
- (c) the general fiber of  $\pi_S$  is a nonhyperelliptic curve of genus 3, where  $Sing(S)$  is the singular locus of  $S$ , etc.

*Proof* On  $\pi_S^{-1}(W \setminus B_\phi \cap B'_\phi)$ ,  $\pi_S$  is a finite morphism and the assertion (b) is clear. For any  $P \in B_\phi \cap B'_\phi$ , denote by  $(x, y)$  a local (analytic) parameter at  $P$  on  $W$ . Then  $\pi_S^{-1}$  is defined by the equation  $xY_0^4 + yY_1^4 = 0$  in  $X$ . Thus  $S$  is nonsingular along  $\pi_S^{-1}(B_\phi \cap B'_\phi)$ . ( $\pi_S^{-1}(P)$  is a nonsingular (-4) curve.) Hence (b) holds. Especially  $S$  is normal.

Let  $\tilde{\tau} : \tilde{S} \rightarrow S$  be the resolution of  $S$  and  $\tilde{S} = \cup_i S^{(i)}$  be the decomposition to its connected components. Since  $B_\phi \cap B'_\phi$  is a set of points,  $\pi_S \tilde{\tau}(S^{(i)})$  coincides with  $W$  for any  $i$ . On the other hand, for a general point  $Q$  of  $B_\phi$ ,  $\pi_S^{-1}(Q)$  consists of one point and nonsingular in  $S$ . Hence  $i = 1$ , i.e.  $S$  is irreducible.

The assertion (c) is clear. *q.e.d.*

**Definition 1.5** A curve singularity  $P$  is called an *infinitely close double point* iff it consists of two tangent branches no longer simultaneously tangent after one blow up. (See the point  $P_i$  in Fig. 1).

**1.6** Let  $S$  be a surface defined by  $(\phi) \in |4T + \pi^*\delta^*D_n|_{GC}$ . If  $P \in B_\phi \setminus B'_\phi$  is an infinitely close double point (resp. an ordinary double point) of  $B_\phi$ , then  $Q = \pi_S^{-1}(P)$  is an  $\widetilde{E}_7$ -singularity (resp. a RDP of type  $A_3$ ). Therefore we try to construct a member  $B := B_\phi \in |4C_0 + \delta^*D_n|$  such that  $B$  has  $k$  infinitely close double points and other singularities of  $B$  are at most ordinary double points. The following method is essentially due to Persson [15].

Let  $D_\beta$  be a divisor of degree  $\beta \geq 0$  on  $C$ . Assume that there is a divisor  $D_r \geq 0$  (effective or zero) on  $C$  such that

$$(2) \quad D_r \sim D_n - 2D_\beta.$$

Let  $P_1, \dots, P_k$  be points on  $C_0$  such that

$$(3) \quad h^0(C, 2D_m + D_\beta - \widehat{P}_1 - \dots - \widehat{P}_k) \geq 2,$$

where  $\widehat{P}_i = \delta(P_i)$ . Let  $B^{(1)}$  and  $B^{(2)}$  be general members of Persson's subsystem  $|2C_0 + \delta^*D_\beta|_P$  ([1]) passing through  $P_1, \dots, P_k$ . Assume that

$$(4) \quad \delta^*D_r \text{ does not pass through the singular points of the reducible curve } B^{(1)} + B^{(2)}.$$

We set

$$B := B^{(1)} + B^{(2)} + \delta^*D_r.$$

Then  $B$  is linearly equivalent to  $4C_0 + \delta^*D_n$ ,  $B$  has infinitely close double points at  $P_i$  ( $1 \leq i \leq k$ ) and other singularities of  $B$  are at most ordinary singular points (see the argument in [1]). From the above argument :

**Proposition 1.7** *Let  $m, n$  be as in (1). Assume that*

- (a)  $0 \leq k \leq 2m + [n/2] - 1$  when  $g = 0$ ,
- (b)  $0 \leq k \leq 2m + [n/2] - 2$  when  $g \geq 1$ ,

where  $[n/2]$  is the greatest integer not exceeding  $n/2$ . Then there exists a nonsingular surface  $\widetilde{S}$  such that

- 1)  $\chi(\mathcal{O}_{\widetilde{S}}) = 4m + 3n + 2(g - 1) - k$ ,  $c_1^2(\widetilde{S}) = 12m + 9n + 16(g - 1) - 2k$ ,
- 2)  $\widetilde{S}$  has a (base point free) pencil of nonhyperelliptic curves of genus 3 over a curve of genus  $g$ .

*Proof* When  $C = \mathbf{P}^1$  : By (a), we can choose  $D_\beta$  and  $D_\gamma$  such that the conditions (2), (3) and (4) are satisfied.

When  $g(C) \geq 1$  : Let  $C$  be a hyperelliptic curve of genus  $g \geq 1$  and let  $\iota : C \rightarrow \mathbf{P}^1$  be the hyperelliptic involution. We choose divisors  $D_m$  such that ;

(c) When  $m$  is even,  $D_m := \iota^* \mathcal{O}_{\mathbf{P}^1}(m/2)$  ,

(d) When  $m$  is odd,  $D_m := \iota^* \mathcal{O}_{\mathbf{P}^1}([m/2]) + Q$  for a ramification point  $Q \in C$ .

We choose divisors  $D_n, D_\beta, D_\gamma$  and  $\widehat{P}_1 + \cdots + \widehat{P}_k$  in a similar way carefully. Then the conditions (2), (3) and (4) are satisfied by (b).

*q.e.d.*

**Remark 1.8** In the proof of Proposition 1.7, if  $C$  is a general curve of genus  $g \geq 1$  and  $D_m, D_n$  are general divisors on  $C$ , then the invariants  $(\chi, c_1^2)$  of the resulting surfaces  $\tilde{S}$  do not cover wider area than the above ones in the surface geography.

## 2 Process of resolution.

In this section, first we resolve singularities of  $S$  explicitly. This method is analogous to that in [11, §2] and [2, §3]. Second we calculate the invariants of surfaces in each process of the resolution.

**2.1** We go back to the situation in 1.6. Our way of resolving singularities of  $S$  consists of the following six steps :

1) Let  $f_1, \cdots, f_k$  be the fibers of  $W \rightarrow C$  containing  $P_1, \cdots, P_k$  respectively. Let  $\tau_1 : W_1 \rightarrow W$  be the composition of blow ups at  $P_i (1 \leq i \leq k)$  and set  $g'_i = \tau_1^{-1}(P_i)$ . Denote by  $f'_i$  the strict transform of  $f_i$  and by  $Q'_i$  the point on intersection of  $f'_i$  and  $g'_i$ .

2) Let  $\tau_2 : W_2 \rightarrow W_1$  be the composition of blow ups at  $Q'_i, 1 \leq i \leq k$ . Set  $h''_i = \tau_1^{-1}(Q'_i)$  and denote by  $f''_i$  and  $g''_i$  the strict transform of  $f'_i$

and  $g'_i$  by  $\tau$  respectively. Put  $\tau = \tau_2 \circ \tau_1$ . We set

$$\begin{aligned} L_2 &= \tau^* \mathcal{O}_W(C_0) \otimes \mathcal{O}_{W_2} \left( - \sum_{n=1}^k h''_n \right), \\ B_2 &= \tau^* B_\phi - 4 \sum_{i=1}^k h''_i = \widetilde{B}_2 + 2 \sum_{i=1}^k g''_i, \quad B'_2 = \tau^* B'_\phi, \end{aligned}$$

where  $\widetilde{B}_2$  is the strict transform of  $B_\phi$  by  $\tau$ . Put

$$\pi_2 : X_2 = \mathbf{P}(\mathcal{O}_{W_2} \oplus \mathcal{O}_{W_2}(L_2)) \longrightarrow W_2$$

and set  $T_2 = \mathcal{O}_{X_2}(1)$ . We define a member  $S_2$  of the cyclic subsystem  $|4T_2 + \pi_2^* \tau^* \delta^* D_n|_C$  such that the branch and the assistant branch locus are  $B_2$  and  $B'_2$  respectively. Then by the same argument as in [2, §3], there is a natural morphism  $\mu : X_2 \longrightarrow X$  such that  $\widehat{\mu} = \mu|_{S_2} : S_2 \longrightarrow S$  is a birational morphism. The singular locus of  $S_2$  (except RDP) coincides with  $\cup_{i=1}^k (\pi_2|_{S_2})^{-1}(g''_i)$ . The local analytic equation of this is of the form  $\xi^4 + x^2 = 0$ . We set  $\widetilde{g}_i = (\pi_2|_{S_2})^{-1}(g''_i)_{red}$  the reduced curve of  $(\pi_2|_{S_2})^{-1}(g''_i)$ .

3) Let  $\mu_3 : X_3 \longrightarrow X_2$  be the composition of blow ups at  $\widetilde{g}_i, 1 \leq i \leq k$ . Denote by  $S_3$  the strict transform of  $S_2$  by  $\mu_3$ , and set  $E_i = \mu_3^{-1}(\widetilde{g}_i)$ ,  $\widehat{g}_i = (S_3 \cap E_i)_{red}$  and  $\widehat{\mu}_3 = \mu|_{S_3}$ .

4) Let  $\mu_4 : X_4 \longrightarrow X_3$  be the composition of blow ups at  $\widehat{g}_i, 1 \leq i \leq k$ . Denote by  $S_4$  and  $E'_i$  the strict transforms of  $S_3$  and  $E_i$  by  $\tau_4$  respectively. Set  $\widetilde{E}_i = \mu_4^{-1}(\widehat{g}_i)$  and  $\widehat{\mu}_4 = \mu_4|_{S_4}$ .  $S_4$  contains  $2k$  exceptional curves of first kind  $(\widehat{\mu}_4)^{-1}(\widehat{g}_i), 1 \leq i \leq k$ .

5) Let  $\mu_5 : S_4 \longrightarrow S_5$  be the composition of contraction maps of  $(\widehat{\mu}_4)^{-1}(\widehat{g}_i)$  for  $1 \leq i \leq k$  and of  $n$  exceptional curves  $T_\infty|_S$ . Let  $h'''_i$  be the image by  $\mu_5$  of the strict transform of  $h''_i$  by  $\widehat{\mu}_4 \circ \widehat{\mu}_3$ .  $h'''_i$  is a nonsingular elliptic curve of self-intersection number  $(-2)$ . (see Fig. 1)

6) Let  $\mu_6 : \widetilde{S} \longrightarrow S_5$  be the minimal resolution of all RDP on  $S_5$ .  $\widetilde{S}$  is nonsingular.

By the above process, we obtain the following diagram :

$$\begin{array}{ccccccccc}
X & \xleftarrow{\mu} & X_2 & \xleftarrow{\mu_3} & X_3 & \xleftarrow{\mu_4} & X_4 & & \\
\cup & & \cup & & \cup & & \cup & & \\
S & \xleftarrow{\widehat{\mu}} & S_2 & \xleftarrow{\widehat{\mu}_3} & S_3 & \xleftarrow{\widehat{\mu}_4} & S_4 & \xrightarrow{\mu_5} & S_5 & \xleftarrow{\mu_6} & \widehat{S} \\
\downarrow \pi_S & & \downarrow \pi_{S_2} & & & & & & & & \\
W & \xleftarrow{\tau} & W_2 & & & & & & & & 
\end{array}$$

(Fig. 1)

(Fig. 2)

Next for calculating  $p_g(S_j)$  and  $q(S_j)$  for  $2 \leq j \leq 4$ , we prepare the following two lemmas.

**Lemma 2.2** *We set*

$$\Theta := k - h^0(C, K_C + 2D_m + D_n) + h^0(C, K_C + 2D_m + D_n - \widehat{P}_1 \cdots - \widehat{P}_k).$$

*Then we have*

$$\begin{aligned}
(a) \quad & h^p(W_2, \tau^*(-C_0 + \delta^*(K_C + D_m + D_n)) + \sum_{i=1}^k (g_i'' + h_i'')) = 0, (\forall p), \\
(b) \quad & h^p(W_2, \tau^*\delta^*(K_C + D_m + D_n) + \sum_{i=1}^k g_i'') \\
& = \begin{cases} m + n + g - 1 & (p = 0) \\ k & (p = 1) \\ 0 & (p = 2) \end{cases} \\
(c) \quad & h^p(W_2, \tau^*(C_0 + \delta^*(K_C + D_m + D_n)) + \sum_{i=1}^k (g_i'' - h_i'')) \\
& = \begin{cases} 3m + 2n + 2(g - 1) - k + \Theta & (p = 0) \\ 2k + \Theta & (p = 1) \\ 0 & (p = 2) \end{cases}
\end{aligned}$$

*Proof.* We write  $g_i = g_i''$ ,  $h_i = h_i''$  and  $\Sigma = \sum_{i=1}^k$  for simplicity.

(a) : Put  $D_1 = \tau^*(-C_0 + \delta^*(K_C + D_m + D_n))$ , and consider the exact sequences

$$\begin{aligned}
0 & \longrightarrow \mathcal{O}_{W_2}(D_1) \longrightarrow \mathcal{O}_{W_2}(D_1 + \Sigma(g_i + h_i)) \\
& \longrightarrow \mathcal{O}_{\Sigma(g_i + h_i)}(D_1 + \Sigma(g_i + h_i)) \longrightarrow 0,
\end{aligned}$$

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\sum h_i}(D_1 + \sum h_i) \longrightarrow \mathcal{O}_{\sum(g_i+h_i)}(D_1 + \sum(g_i + h_i)) \\ &\longrightarrow \mathcal{O}_{\sum g_i}(D_1 + \sum(g_i + h_i)) \longrightarrow 0, \end{aligned}$$

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\sum_{i=1}^{j-1} h_i}(D_1 + \sum_{i=1}^{j-1} h_i) \longrightarrow \mathcal{O}_{\sum_{i=1}^j h_i}(D_1 + \sum_{i=1}^j h_i) \\ &\longrightarrow \mathcal{O}_{h_j}(D_1 + \sum_{i=1}^j h_i) \longrightarrow 0, \end{aligned}$$

for  $2 \leq j \leq k$ . By using

$$(D_1 + \sum h_i)h_j = (D_1 + \sum(g_i + h_i))g_j = -1$$

inductively, we have

$$\begin{aligned} H^p(W_2, D_1 + \sum(g_i + h_i)) &\simeq H^p(W_2, D_1) \\ &\simeq H^p(W, -C_0 + \delta^*(K_C + D_m + D_n)) = 0, \end{aligned}$$

for any  $p$ .

(b) : Since

$$(\tau^*\delta^*(K_C + D_m + D_n) + \sum g_i)g_j = -2,$$

we obtain (b) similarly.

(c) : Set  $D_2 = \tau^*(C_0 + \delta^*(K_C + D_m + D_n))$  and consider the exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{W_2}(D_2 - \sum h_i) \longrightarrow \mathcal{O}_{W_2}(D_2 + \sum(g_i - h_i)) \\ &\longrightarrow \mathcal{O}_{\sum g_i}(D_2 + \sum(g_i - h_i)) \longrightarrow 0, \end{aligned}$$

$$0 \longrightarrow \mathcal{O}_{W_2}(D_2 - \sum h_i) \longrightarrow \mathcal{O}_{W_2}(D_2) \longrightarrow \mathcal{O}_{\sum h_i} \longrightarrow 0.$$

Since  $(D_2 + \sum(g_i - h_i))g_j = -3$  and  $D_2|_{h_j} \simeq \mathcal{O}_{h_j}$ , we have

$$H^0(W_2, D_2 - \sum h_i) \simeq H^0(W_2, D_2 + \sum(g_i - h_i)),$$

$$(5) \quad \begin{aligned} 0 &\longrightarrow H^1(W_2, D_2 - \sum h_i) \longrightarrow H^1(W_2, D_2 + \sum(g_i - h_i)) \\ &\longrightarrow \mathbf{C}^{2k} \longrightarrow 0 \text{ (exact)}, \end{aligned}$$

$$\begin{aligned}
0 &\longrightarrow H^0(W_1, D_2 - \sum h_i) \longrightarrow H^0(W_1, D_2) \xrightarrow{\mathbf{r}} \mathbf{C}^k \\
&\longrightarrow H^1(W_1, D_2 - \sum h_i) \longrightarrow H^1(W_1, D_2) \longrightarrow 0 \text{ (exact)}.
\end{aligned}$$

We denote by  $\Theta'$  the dimension of the cokernel of the map  $\mathbf{r}$ . Then by (5), we have

$$\begin{aligned}
(6) \quad & h^0(W_2, D_2 + \sum(g_i - h_i)) \\
& = h^0(W, C_0 + \delta^*(K_C + D_m + D_n)) - k + \Theta', \\
& h^1(W_2, D_2 + \sum(g_i - h_i)) \\
& = h^1(W, C_0 + \delta^*(K_C + D_m + D_n)) + 2k + \Theta'.
\end{aligned}$$

Now we consider the map  $\mathbf{r}$ .

Any element  $\psi' \in H^0(W_2, D_2)$  is written as  $\psi' = \tau^*\psi$  for some  $\psi \in H^0(W, C_0 + \delta^*(K_C + D_m + D_n))$ . Then we have

$$\mathbf{r}(\psi') = (\psi'|_{h_1}, \dots, \psi'|_{h_k}) = (\psi(P_1), \dots, \psi(P_k)) \in \mathbf{C}^k$$

where  $\psi'|_{h_i} \in H^0(h_i, D_2|_{h_i}) \simeq \mathbf{C}$  and  $\psi(P_i)$  is the value of  $\psi$  at  $P_i$ . We set  $M = H^0(W, C_0 + \delta^*(K_C + D_m + D_n))$  and denote by  $M^i$  the vector subspace of  $M$  consisting of the elements which pass through  $P_1, \dots, P_i$ . We have a descending filtration

$$(7) \quad M = M^0 \supset M^1 \supset \dots \supset M^k.$$

We have  $\dim M^i = \dim M^{i-1}$  or  $\dim M^i = \dim M^{i-1} - 1$  according as  $P_i$  is a base point of  $M^{i-1}$  or not. Then the number  $\Theta'$  is equal to the cardinality of  $i$  such that  $\dim M^i = \dim M^{i-1}$ .

Let  $(Z_0 : Z_1)$  be a fiber coordinate of  $\pi : W \longrightarrow C$  such that  $C_0 = \{Z_0 = 0\}$ . Then any  $\psi \in H^0(W, C_0 + \delta^*(K_C + D_m + D_n))$  is written by

$$\psi = \psi_{K_C + D_m + D_n} Z_0 + \psi_{K_C + 2D_m + D_n} Z_1$$

where  $\psi_{K_C + jD_m + D_n} \in H^0(C, K_C + jD_m + D_n)$  for  $j = 1, 2$ . Since  $P_i (1 \leq i \leq k)$  is on  $C_0$ ,  $\psi(P_i) = 0$  is equivalent to  $\psi_{K_C + 2D_m + D_n}(\widehat{P}_i) = 0$ . We set  $N = H^0(C, K_C + 2D_m + D_n)$  and denote by  $N^i$  the subspace of  $N$  consisting of the elements which pass through  $\widehat{P}_1, \dots, \widehat{P}_i$ . We have a filtration

$$(8) \quad N = N^0 \supset N^1 \supset \dots \supset N^k.$$

By the above argument, the number  $\Theta'$  is equal to the cardinality of  $i$  such that  $\dim N^i = \dim N^{i-1}$ . Hence  $\Theta'$  coincides with  $\Theta$ . Thus by (6), we obtain (c). *q. e. d.*

**Lemma 2.3** *For any  $j$  ( $1 \leq j \leq k$ ), we have*

- (a)  $(K_{X_2} + S_2)\widehat{g}_j = -3$ ,
- (b)  $(K_{X_3} + S_3)\widehat{g}_j = -2$ .

*Proof.* Denote by  $T_2^\infty$  the  $\infty$ -section of  $\pi_2 : X_2 \rightarrow W_2$ . Then we have

$$(9) \quad T_2\widehat{g}_j = (T_2^\infty + \pi_2^*(\tau^*C_0 - \sum_{i=1}^k h_i''))\widehat{g}_j = -h_j''g_j'' = -1.$$

Since

$$(10) \quad \begin{aligned} K_{X_2} + S_2 &\sim -2T_2 + \pi_2^*(K_{W_2} + L_2) + 4T_2 + \pi_2^*\tau^*\delta^*D_n \\ &\sim 2T_2 + \pi_2^*(\tau^*(-C_0 + \delta^*(K_C + D_m + D_n) + \Sigma(g_i'' + h_i''))), \end{aligned}$$

we obtain (a).

For  $1 \leq i \leq k$ , let  $\widetilde{T}_2, \widetilde{G}_i, \widetilde{H}_i$  and  $\widetilde{F}_i$  be the proper transforms of  $T_2, G_i = \pi_2^{-1}(g_i''), H_i = (\pi_2'')^{-1}(h_i'')$  and  $F_i = \pi_2^{-1}(f_i'')$  by  $\mu_3$  respectively (see Fig. 2). Then by (10), we have

$$(11) \quad \begin{aligned} (K_{X_3} + S_3)\widehat{g}_j &= (\mu_3^*(K_{X_2} + S_2) - \sum_{i=1}^k E_i)\widehat{g}_j \\ &= (2(\widetilde{T}_2 + \Sigma E_i) + \Sigma(\widetilde{G}_i + E_i + \widetilde{H}_i) - \Sigma E_i)\widehat{g}_j \\ &= 2E_j\widehat{g}_j + \widetilde{G}_j\widehat{g}_j + 1. \end{aligned}$$

Now we consider the exact sequence

$$0 \rightarrow N_{\widetilde{g}_j/T_2} \rightarrow N_{\widetilde{g}_j/X_2} \rightarrow (N_{T_2/X_2})|_{\widetilde{g}_j} \rightarrow 0,$$

where  $N_{\widetilde{g}_j/T_2}$  is the normal bundle of  $\widetilde{g}_j$  in  $T_2$ , etc. We have  $N_{\widetilde{g}_j/X_2} \simeq \mathcal{O}_{\mathbf{P}^1}(-2)$  and  $(N_{T_2/X_2})|_{\widetilde{g}_j} \simeq \mathcal{O}_{\mathbf{P}^1}(-1)$  by (9). Since  $\text{Ext}^1(\mathcal{O}(-1), \mathcal{O}(-2))$  vanishes, we have  $N_{\widetilde{g}_j/X_2} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$ . Thus  $E_j$  is isomorphic to the Hirzebruch surface of degree 1, and  $\widehat{g}_j$  is the  $\infty$ -section of  $E_j$ . Hence we have  $\widetilde{G}_j\widehat{g}_j = (\widetilde{G}_j|_{E_j})^2 = -1$ .

On the other hand, the divisor  $E_j + \widetilde{G}_j + 2\widetilde{H}_j + \widetilde{F}_j$  is a singular fiber of the degeneration  $X_3 \rightarrow C$ . Thus we have

$$0 = (E_j + \widetilde{G}_j + 2\widetilde{H}_j + \widetilde{F}_j)\widehat{g}_j = E_j\widehat{g}_j + 1.$$

Hence by (11), we obtain (b).

*q.e.d.*

**Proposition 2.4** For  $2 \leq j \leq 4$ , we have ;

- 1)  $H^0(\omega_{S_j}) \simeq H^0(X_j, K_{X_j} + S_j)$ ,  
 $h^1(\omega_{S_j}) = h^1(X_j, K_{X_j} + S_j) - h^2(X_j, K_{X_j} + S_j) + g \ (\forall j)$ ,
- 2)  $h^0(\omega_{S_2}) = h^0(\omega_S) - k + \Theta$ ,  $h^1(\omega_{S_2}) = g + 3k + \Theta$ ,
- 3)  $H^0(\omega_{S_3}) \simeq H^0(\omega_{S_2})$ ,  $h^1(\omega_{S_3}) = h^1(\omega_{S_2}) - 2k$ ,
- 4)  $H^0(\omega_{S_4}) \simeq H^0(\omega_{S_3})$ ,  $h^1(\omega_{S_4}) = h^1(\omega_{S_3}) - k$ .

*Proof.* Since  $H^p(K_{X_j}) = 0 (p = 0, 1)$ ,  $h^2(K_{X_j}) = g$  and  $H^2(\omega_{S_j}) \simeq H^3(K_{X_j}) \simeq \mathbf{C}$ , the long exact sequence associated with

$$0 \longrightarrow \mathcal{O}(K_{X_j}) \longrightarrow \mathcal{O}(K_{X_j} + S_j) \longrightarrow \mathcal{O}(\omega_{S_j}) \longrightarrow 0,$$

implies 1).

By (10), we have  $R^p(\pi_2)_* \mathcal{O}(K_{X_2} + S_2) = 0$  for  $p \geq 1$ . Thus we have

$$\begin{aligned} H^p(X_2, K_{X_2} + S_2) &\simeq H^p(W_2, (\pi_2)_* \mathcal{O}(K_{X_2} + S_2)) \\ &\simeq H^p(W_2, \tau^*(-C_0 + \delta^*(K_C + D_m + D_n)) + \sum_i (g_i + h_i)) \\ &\quad \oplus H^p(W_2, \tau^* \delta^*(K_C + D_m + D_n) + \sum_i g_i) \\ &\quad \oplus H^p(W_2, \tau^*(C_0 + \delta^*(K_C + D_m + D_n)) + \sum_i (g_i - h_i)). \end{aligned}$$

Hence by Lemma 2.2 , we obtain 2).

Since  $K_{X_3} \sim \mu_3^* K_{X_2} + \sum_i E_i$  and  $S_3 \sim \mu_3^* S_2 - 2 \sum_i E_i$ , we have the exact sequence

$$(12) \quad \begin{aligned} 0 &\longrightarrow \mathcal{O}(K_{X_3} + S_3) \longrightarrow \mathcal{O}(\mu_2^*(K_{X_2} + S_2)) \\ &\longrightarrow \mathcal{O}_{\sum E_i}(\mu_3^*(K_{X_3} + S_3)|_{\sum E_i}) \longrightarrow 0. \end{aligned}$$

By Lemma 2.3 (a), we have

$$H^p(E_i, \mu_3^*(K_{X_2} + S_2)|_{E_i}) \simeq H^p(\mathbf{P}^1, \mathcal{O}(-3)) \simeq \begin{cases} 0 & (p \neq 1) \\ \mathbf{C}^2 & (p = 1) \end{cases}$$

Thus by the long exact sequence associated with (12) and by 1), we obtain 3).

By Lemma 2.3 (b), we obtain 4) similarly.

*q.e.d.*

**Corollary 2.5**  $p_g(\tilde{S}) = p_g(S) - k$ ,  $q(\tilde{S}) = g$ .

*Proof.* By Proposition 2.4, we have  $p_g(\tilde{S}) = p_g(S_4) = p_g(S) - k + \Theta$  and  $q(\tilde{S}) = q(S_4) = g + \Theta$ . On the other hand,  $\Theta$  equals to zero by Proposition 1.7. *q.e.d.*

### 3 Canonical mapping.

In this section, we study the canonical mapping of the surface  $\tilde{S}$  constructed in the previous section, and prove our theorem.

**Proposition 3.1** *Assume that*

- (a)  $m + n \geq 3$  and  $2m + n + 2g - 2 \geq k$ , or
- (b)  $C = \mathbb{P}^1$ ,  $m + n = 2$  and  $m - 1 \geq k$ .

*Then  $\tilde{S}$  is relatively minimal and the canonical map  $\Phi_{K_{\tilde{S}}}$  of  $\tilde{S}$  is a birational holomorphic map onto its image.*

*Proof.* Since  $H^0(K_{\tilde{S}})$  is naturally isomorphic to  $H^0(X_2, K_{X_2} + S_2)$  by Proposition 2.4, we consider this space. The divisor  $K_{X_2} + S_2$  is linearly equivalent to  $T_2^\infty + T_2 + \pi_2^*(\tau^*\delta^*(K_C + D_m + D_n) + \sum_{i=1}^k g_i)$ , and  $T_2^\infty$  is a part of the fixed component of this system by the proof of Lemma 2.2(a). We set

$$D := T_2 + \pi_2^*(\tau^*\delta^*(K_C + D_m + D_n) + \sum_i g_i).$$

If we fix a fiber coordinate  $(\widehat{Y}_0 : \widehat{Y}_1)$  of  $\pi_2 : X_2 \rightarrow W_2$ , then any  $\phi \in H^0(X_2, D)$  is written as

$$(13) \quad \phi = \phi^{(0)}\widehat{Y}_0 + \phi^{(1)}\widehat{Y}_1,$$

where  $\phi^{(0)} \in H^0(W_2, \tau^*\delta^*(K_C + D_m + D_n) + \sum g_i)$  and  $\phi^{(1)} \in H^0(W_2, \tau^*(C_0 + \delta^*(K_C + D_m + D_n)) + \sum(g_i - h_i))$ . By the proof of Lemma 2.2, we have  $H^0(W_2, \tau^*\delta^*(K_C + D_m + D_n) + \sum g_i) \simeq H^0(W_2, \tau^*\delta^*(K_C + D_m + D_n)) \xrightarrow{\tau^*} H^0(W, \delta^*(K_C + D_m + D_n))$ . Moreover  $\phi^{(1)}$  is written as

$$(14) \quad \phi^{(1)} = \tau^*\psi, \quad \psi = \psi^{(0)}Z_0 + \psi^{(1)}Z_1,$$

where  $\psi \in M^k \subset H^0(W, C_0 + \delta^*(K_C + D_m + D_n))$ ,  $\psi^{(0)} \in H^0(C, K_C + D_m + D_n)$  and  $\psi^{(1)} \in N^k$  (see (7) and (8)).

Hence by the expression (13) and (14), the curve  $A = \sum_{i=1}^k (g_i'' + h_i'')$  is a part of the base locus of  $D$ .

Set  $X_2^{(0)} = X_2 \setminus A$  the Zariski open set of  $X_2$ . We prove that the linear system  $|D|$  separates points on  $X_2^{(0)}$ ; i.e., For any points  $R_1, R_2 \in X_2^{(0)}$ , one can find  $\phi \in H^0(X_2, D)$  such that  $\phi(R_1) = 0$  and  $\phi(R_2) \neq 0$ . Put  $R'_i = \pi_2(R_i)$  and  $R''_i = \delta\tau(R'_i)$  for  $i = 1, 2$ .

1) When  $R''_1 \neq R''_2$ : If the condition (a) holds, then  $K_C + D_m + D_n$  is a very ample divisor on  $C$ . Thus  $|\tau^*\delta^*(K_C + D_m + D_n) + \sum g_i|$  separates  $R'_1$  and  $R'_2$ . Hence  $|D|$  separates  $R_1$  and  $R_2$ . If the condition (b) holds, then the system  $N^k$  separates  $R''_1$  and  $R''_2$ . Hence  $|D|$  also separates  $R_1$  and  $R_2$ .

2) When  $R'_1 \neq R'_2$  and  $R''_1 = R''_2$ : Since  $K_C + D_m + D_n \geq 0$  and  $K_C + 2D_m + D_n - \widehat{P}_1 - \cdots - \widehat{P}_k \geq 0$ , the system  $|\tau^*(C_0 + \delta^*(K_C + D_m + D_n) + \sum (g_i - h_i))|$  separates  $R'_1$  and  $R'_2$ . Hence  $|D|$  separates  $R_1$  and  $R_2$ .

3) When  $R''_1 = R''_2$ : We consider the restriction map

$$\lambda : H^0(X, D) \longrightarrow H^0(\pi_2^{-1}(R'_1), D|_{\pi_2^{-1}(R'_1)}) \simeq \mathbf{C}^2.$$

By (13), the map  $\lambda$  is given by

$$\phi \longmapsto (\phi^{(0)}(R'_1), \phi^{(1)}(R'_1)).$$

Thus by the same argument as above,  $\lambda$  is surjective. Hence  $|D|$  separates  $R_1$  and  $R_2$ .

By 1), 2) and 3),  $|D|$  separates points on  $X_2^{(0)}$ . Therefore  $|K_{X_2} + S_2|$  induce a birational map on  $X_2$ , and this map is holomorphic on  $X_2^{(0)}$ . Since the generic point of  $S_2$  is contained in  $X_2^{(0)}$ ,  $|\omega_{S_2}|$  also induces a birational map on  $S_2$ . Hence  $\Phi_{K_{S_2}}$  is a birational map.

Next we prove that the base locus  $Bs|\omega_{S_5}|$  is empty. By the above argument, we have  $Bs|\omega_{S_5}| \subset \cup_i h_i'''$ . Since  $\omega_{S_5} = (\mu_5)_*(\widehat{\mu}_4 \circ \widehat{\mu}_3 \circ \widehat{\mu})^*\omega_S \otimes \mathcal{O}_{S_5}(-\sum_i h_i''')$ , the divisor  $\sum_i h_i'''$  is not a fixed component of  $|\omega_{S_5}|$ . For  $1 \leq i \leq k$ , let  $P'_i, P''_i$  be the points on  $h_i'''$  which are the image of two exceptional curves  $(\widehat{\mu}_4)^{-1}(\widehat{g}_i)$  by  $\mu_5$ . For a general member

$\psi \in M^k$ , let  $C_\psi$  be the curve on  $W \simeq T \subset X$  defined by  $\psi$ . Then the strict transform of  $C_\psi$  by  $\mu_4 \circ \mu_3 \circ \mu$  does not intersect with  $(\widehat{\mu}_4)^{-1}(\widehat{g}_i)$ . Therefore the points  $P'_i$  and  $P''_i$  ( $1 \leq i \leq k$ ) are not the base points of  $|\omega_{S_5}|$ . Thus  $B_S|\omega_{S_5}|$  is empty. Hence  $B_S|K_{\widetilde{S}}|$  is empty.

Epecially  $\widetilde{S}$  is relatively minimal, and  $\Phi_{K_{\widetilde{S}}}$  is a holomorphic map.  
q.e.d.

From Propositions 1.7 and 3.1, we have the following by an easy computation :

**Theorem 3.2** *Let  $x, y$  be any integers with the following :*

(a) *If  $g = 0$ , then one of (a.1)  $\sim$  (a.4) is satisfied ;*

(a.1)  $3x - 10 \leq y \leq 4x - 16,$

(a.2)  $y = 4x - 15, x \geq 5$  and  $x \equiv 1 \pmod{2},$

(a.3)  $y = 4x - 14$  and  $x \geq 5,$

(a.4)  $y = 4x - i, i = 13$  or  $12, x \geq 6$  and  $x \equiv 0 \pmod{2}.$

(b) *If  $g \geq 1$ , then one of (b.1)  $\sim$  (b.5) is satisfied ;*

(b.1)  $3x + 10g - 10 \leq y \leq 4x + 8g - 18,$

(b.2)  $y = 4x + 8g - 17, x \geq 2g + 8$  and  $x \equiv 0 \pmod{2},$

(b.3)  $y = 4x + 8g - 16,$

(b.4)  $y = 4x + 8g - i, i = 15$  or  $14, x \geq 2g + 7$  and  $x \equiv 0 \pmod{2},$

(b.5)  $y = 4x + 8g - 12, x \geq 2g + 6$  and  $x \equiv 0 \pmod{2}.$

*Then there exists a minimal algebraic surface  $S$  such that*

1)  $\chi(\mathcal{O}_S) = x, c_1^2(S) = y$  and  $q(S) = g,$

2) *the canonical map  $\Phi_{K_S}$  is a birational holomorphic map,*

3)  *$S$  has a fibration of nonhyperelliptic curves of genus 3 over a curve of genus  $g$ .*

From Theorem 3.2, we have the following by an easy computation :

**Theorem 3.3** *Let  $x, y$  be any integers with ;*

(a)  $3x - 10 \leq y \leq 8x - 78,$  or

(b)  $y = 8x - i$  ( $35 \leq i \leq 77, i \neq 39, 41, 47$ ) and  $x$  satisfies the condition in the following table ;

$i$	35	36	37	38	40	42	43	44	45	46	48	49	50	51
$x$	5	E	6	6	E	D	D	A	7, 8	8	E	E	A	D

52	53	54	55	56	57	58	59	60	61	62	63
A	9, 10	D, 10	D	E	E	A	A	A	11, 12	D, 12	D

64	65	66	67	68	69	70	71	72	73	74	75	76	77
A	A	A	A	A	E, 13	A	D	A	A	A	A	A	E, 15

where "A" (resp. "E", resp. "D") means any (resp. any even, resp. any odd) integer with  $\geq (i/5) - 2$ . (For instance, if  $y = 8x - 54$ , then  $x$  is any odd integer with  $\geq 9$  or  $x = 10$ .)

Then there exists a minimal algebraic surface  $S$  such that

- 1)  $\chi(\mathcal{O}_S) = x$ ,  $c_1^2(S) = y$ ,
- 2)  $\Phi_{K_S}$  is a birational holomorphic map,
- 3)  $S$  has a fibration of nonhyperelliptic curves of genus 3.

**Remark 3.4** There exists a minimal algebraic surface  $S$  of general type such that

- 1)  $c_1^2(S) = 4p_g(S) - 6$ ,  $q(S) = 0$ ,  $p_g(S) \geq 4$  and  $p_g(S) \equiv 1 \pmod{2}$ ,
- 2)  $\Phi_{K_S}$  is generically 4 : 1 map,
- 3)  $S$  has a fibration  $\eta : S \rightarrow \mathbf{P}^1$  of nonhyperelliptic curves such that  $\Phi_{K_S}$  is induced by a relativation of the generic projection of the plane quartic curve which is the general fiber of  $h$ .

In fact, in Proposition 1.7, we set  $m \geq 3$ ,  $n = 0$  and  $k = 2m - 1$ . Then the assertion is verified by the similar argument as in the proof of Proposition 3.1.

**Remark 3.5** There exists a degeneration  $\rho : \mathcal{S} \rightarrow \Delta = \{|t| < 1\}$  of surfaces of general type such that the main component  $S^{(0)}$  of the degenerate fiber  $S_0 = \rho^{-1}(0)$  satisfies

$$p_g(S^{(0)}) = p_g(S_t) - 1, \quad c_1^2(S^{(0)}) = c_1^2(S_t) - 2, \quad q(S^{(0)}) = q(S_t),$$

where  $S_t = \rho^{-1}(t)$  ( $t \neq 0$ ) is a smooth fiber of  $\rho$ . This degeneration is observed in the same region as in Theorem 3.2. The construction is similar to [2, §4].

Other examples of "tame" degenerations can be seen in [9], [20], [21], [2], etc.

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Fig. 1

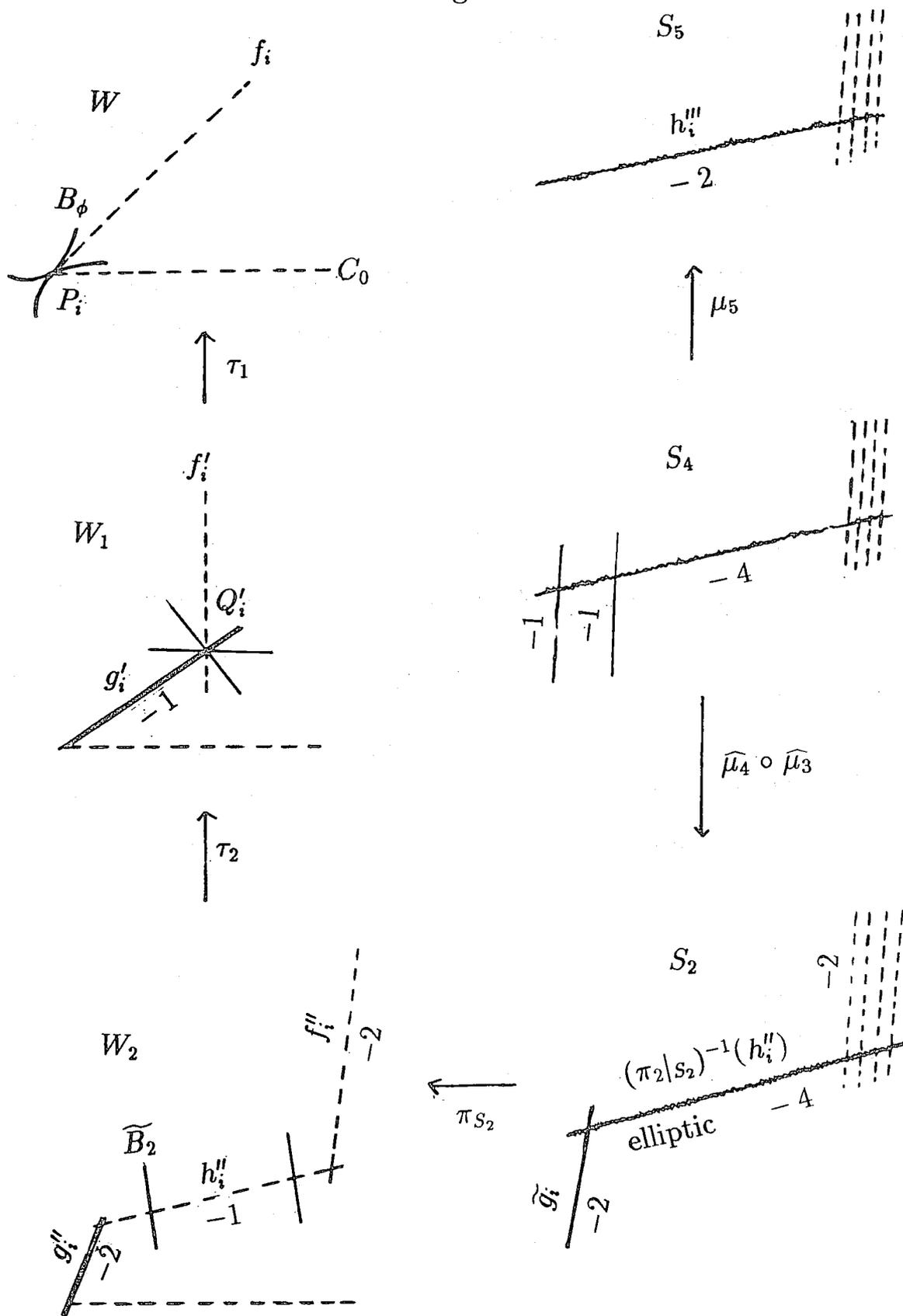
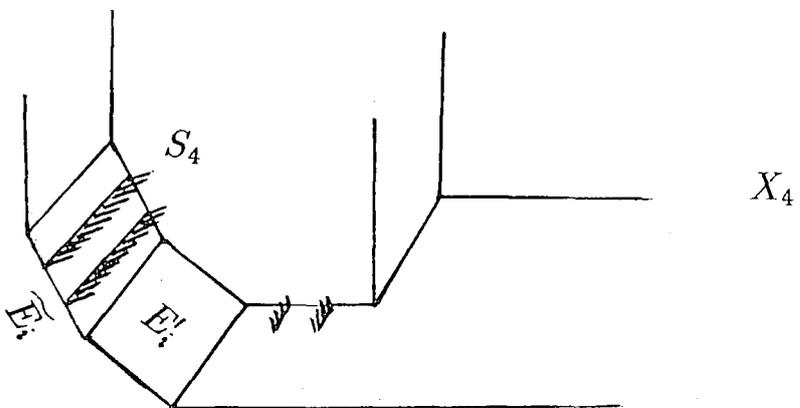
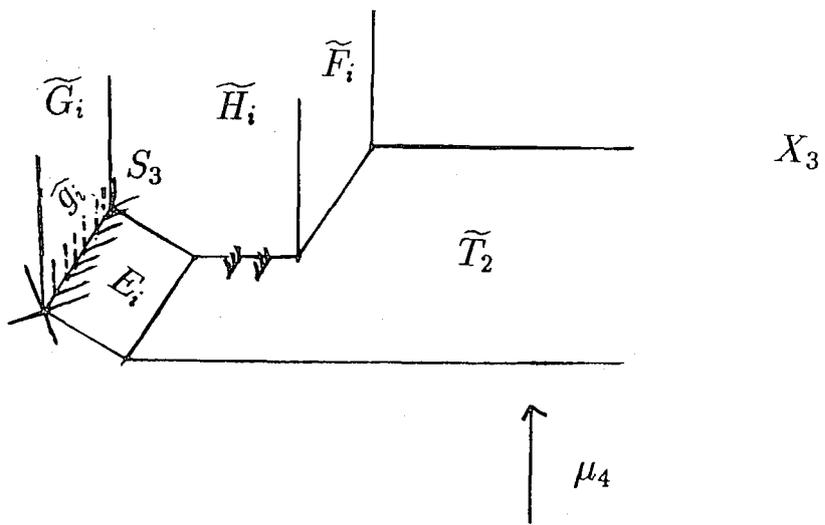
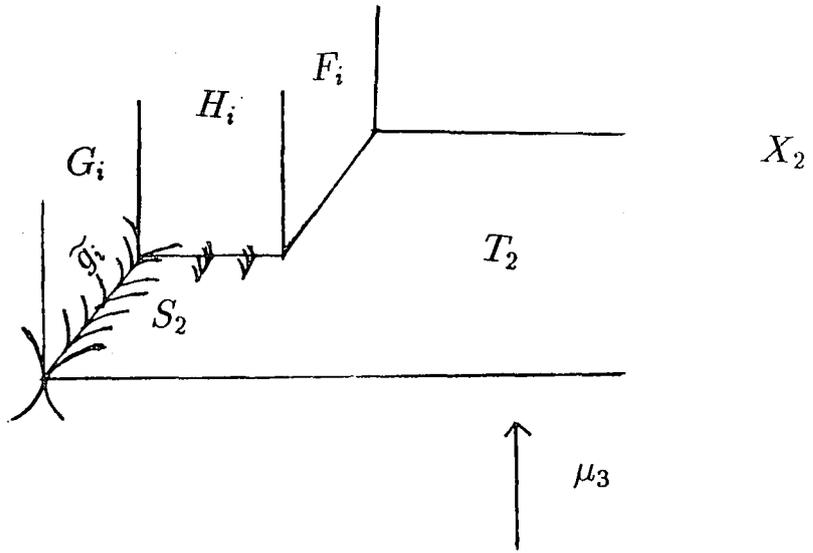


Fig. 2



## Appendix to T. Ashikaga's paper

KAZUHIRO KONNO

### Introduction.

The purpose of this appendix is to give another construction of surfaces of general type with pencils of nonhyperelliptic curves of genus 3, whose canonical map is a birational morphism onto its image. We restrict ourselves to regular surfaces here, whereas our method can be applied to irregular ones as well (with some more effort).

Let  $S$  be a smooth surface with a morphism  $f : S \rightarrow C$  onto a smooth curve  $C$ . Recall that, if a general fiber is a nonhyperelliptic curve of genus three, there is a canonical birational map of  $S$  into a  $\mathbf{P}^2$ -bundle  $W$  on  $C$  (see, [H, §1]). We let  $X$  be its image, and consider the fibration  $g : X \rightarrow C$  induced by the projection of  $W$ . Then the difference of the invariants  $(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_S), \omega_X^2 - \omega_S^2)$  can be considered as the contribution of the singular fibers of  $g$ . Though we do not have a complete list of possible singular fibers, we at least can expect that they are quite similar to those in [K]. However, the singular fiber arising from a simple elliptic singularity of type  $\tilde{E}_7$ , which Ashikaga has constructed, seems to be a “special” one ([K, §9]). What this means may be guaranteed by the fact that the canonical bundle of  $S$  cannot be ample in this case. Thus there should be a way to construct “general” ones. This is the motivation of the present note.

### Construction.

We let  $W$  denote the total space of the  $\mathbf{P}^2$ -bundle

$$\pi : \mathbf{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \rightarrow \mathbf{P}^1,$$

where  $a, b, c$  are integers satisfying  $0 \leq a \leq b \leq c$ . We let  $T$  and  $F$  denote the relatively ample tautological divisor and a fiber of  $\pi$ , respectively. Then the Picard group of  $W$  is a free abelian group generated

by them. Further, we have  $T^3 = (a + b + c)T^2F$  in the Chow ring of  $W$ . We put

$$p = a + b + c + 3 \quad (1.1)$$

and assume that  $p \geq 4$ . Let  $s$  be an integer, and let  $Q$  be a general member of the linear system  $|2T + sF|$ . We remark that  $Q$  is irreducible and has only rational double points of type  $A_1$  if

$$a + c + s \geq 0, \quad 2b + s \geq 0. \quad (1.2)$$

Choose general  $k$  fibers  $F_1, \dots, F_k$  of  $\pi$ . We assume that  $Q$  and  $F_i$  meet transversally, and that the intersection  $Q_i = Q \cap F_i$  is an irreducible conic (in  $F_i \simeq \mathbf{P}^2$ ) for each  $i$ .

We let  $\nu : \hat{W} \rightarrow W$  be the blowing-up along  $\cup Q_i$ , and put  $\mathcal{E}_i = \nu^{-1}(Q_i)$ . Since the normal sheaf of  $Q_i \simeq \mathbf{P}^1$  in  $W$  is isomorphic to  $\mathcal{O} \oplus \mathcal{O}(4)$ , each  $\mathcal{E}_i$  is isomorphic to  $\Sigma_4$ , the Hirzebruch surface of degree 4. On  $\hat{W}$ , we consider the linear system  $|L|$ , where

$$L = \nu^*(4T - (p - 5 - k)F) - 2 \sum_{i=1}^k \mathcal{E}_i. \quad (1.3)$$

If we denote by  $\hat{Q}$  and  $\hat{F}_i$  the proper transforms of  $Q$  and  $F_i$ , respectively, then we get

$$\begin{aligned} L &\sim \nu^*(4T - (p - 5 + k)F) + 2 \sum_{i=1}^k \hat{F}_i \\ &\sim 2\hat{Q} + (k + 5 - p - 2s)\nu^*F, \end{aligned}$$

where the symbol  $\sim$  means the linear equivalence of divisors. Since  $\hat{Q} \cap \hat{F}_i = \emptyset$ , we have

**Lemma 1:**  $Bs|L| = \emptyset$  if the following conditions are satisfied.

- 1)  $k \geq 2s + p - 5$ .
- 2)  $Q$  does not meet  $Bs|4T - (p - 5 + k)F|$ .

**Lemma 2:** The condition 2) of Lemma 1 is satisfied if one of the following conditions is satisfied:

- 1)  $k \leq 4a - p + 5 = 3a - b - c + 2$ .
- 2)  $s = -2a$ ,  $k \leq 4b - p + 5 = 3b - a - c + 2$ .

*Proof.* If 1) holds, then  $Bs|4T - (p - 5 + k)F| = \emptyset$ . We assume that 2) holds. We let  $X_0, X_1$  and  $X_2$  be sections on  $W$  of  $\mathcal{O}(T - aF)$ ,  $\mathcal{O}(T - bF)$  and  $\mathcal{O}(T - cF)$ , respectively, such that they form a system of homogeneous coordinates on each fiber of  $\pi : W \rightarrow \mathbf{P}^1$ . If  $4b \geq p - 5 + k$ , then  $|4T - (p - 5 + k)F|$  has no base point outside the rational curve  $B$  defined by  $X_1 = X_2 = 0$ . On the other hand, the equation of  $Q$  can be written as

$$q_{00}X_0^2 + q_{10}X_0X_1 + q_{01}X_0X_2 + q_{20}X_1^2 + q_{11}X_1X_2 + q_{02}X_2^2 = 0,$$

where  $q_{ij}$  is a homogeneous form on  $\mathbf{P}^1$  of degree  $(2 - i - j)a + ib + jc + s$ . Thus, if  $s = -2a$ , then we can assume that  $q_{00}$  is a nonzero constant. Then  $Q$  does not meet  $B$ . Thus 2) is also sufficient to imply 2) of Lemma 1. *q.e.d.*

**Lemma 3:** *Suppose that  $|L|$  contains an irreducible nonsingular member  $S$ . Then  $S$  is a minimal surface satisfying:*

- 1) *The canonical map of  $S$  is a birational morphism.*
- 2)  *$p_g(S) = p$ ,  $q(S) = 0$  and  $c_1^2(S) = 3p_g(S) - 7 + k$ , where  $p_g$ ,  $q$  and  $c_1^2$  denote the geometric genus, the irregularity and the Chern number of  $S$ , respectively.*

*Proof.* For any complex manifold  $M$ , we denote by  $K_M$  the canonical line bundle of  $M$ . Then, by the adjunction formula, we have

$$K_S \sim (K_{\hat{W}} + S)|_S \sim (\nu^*T + \sum_{i=1}^k \hat{F}_i)|_S.$$

Since  $\hat{F}_i$ ,  $1 \leq i \leq k$ , does not meet  $S$ , we have  $\mathcal{O}(K_S) = \mathcal{O}_S(\nu^*T)$ . We next consider the cohomology exact sequences derived from

$$0 \rightarrow \mathcal{O}(K_{\hat{W}}) \rightarrow \mathcal{O}(\nu^*T + \sum_{i=1}^k \hat{F}_i) \rightarrow \mathcal{O}(K_S) \rightarrow 0, \quad (1.4)$$

$$0 \rightarrow \mathcal{O}(\nu^*T) \rightarrow \mathcal{O}(\nu^*T + \sum_{i=1}^k \hat{F}_i) \rightarrow \bigoplus_{i=1}^k \mathcal{O}_{\hat{F}_i}(-1) \rightarrow 0. \quad (1.5)$$

Since  $\hat{W}$  is rational, we have  $H^q(\hat{W}, \mathcal{O}(K_{\hat{W}})) = 0$  for  $q < 3$ . Thus we get  $H^q(S, \mathcal{O}(K_S)) \simeq H^q(\hat{W}, \mathcal{O}(\nu^*T + \sum \hat{F}_i))$  for  $q < 2$  from (1.4). Then, by (1.5), we get  $H^q(K_S) \simeq H^q(W, \mathcal{O}(T))$  for  $q < 2$ . This shows the formulae for  $p_g(S)$  and  $q(S)$ . Note that we in particular have shown that  $|\nu^*T|$  is restricted to  $|K_S|$  isomorphically. Thus  $Bs|K_S| = Bs|\nu^*T| = \emptyset$ . This implies that  $S$  is minimal and we get 1). Finally, we calculate  $c_1^2$  to get:

$$c_1^2(S) = (\nu^*T)^2L = T^2(4T - (p - 5 - k)F) = 3p - 7 + k.$$

*q.e.d.*

Now, varying  $a, b, c, s$  and  $k$  under the conditions as above, we obtain a series of surfaces to show the following:

**Proposition:** *Let  $x$  and  $y$  be positive integers satisfying*

$$x \geq 4, \quad 3x - 7 \leq y \leq \begin{cases} 4x - 8 & \text{if } x \text{ is odd,} \\ 4x - 10 & \text{if } x \text{ is even.} \end{cases}$$

*Then there exists a minimal, regular surface of general type  $S$  with the following properties:*

- 1)  $p_g(S) = x, \quad c_1^2(S) = y.$
- 2) *The canonical linear system of  $S$  has neither fixed components nor base points, and the canonical map is a birational morphism onto its image.*
- 3)  *$S$  has a pencil of nonhyperelliptic curves of genus three.*
- 4) *The canonical image of  $S$  is contained in a threefold of minimal degree.*

*Sketch of Proof.* Put  $(a, b, c) = (a, a, a), (a, a, a + 1), (a, a + 1, a + 1)$  according to  $x \equiv 0, 1, 2$  modulo 3, respectively. Then, by 1)

of Lemma 2, we can cover the region

$$3x - 7 \leq y \leq \frac{10}{3}x - \begin{cases} 6, & \text{if } x \equiv 0, \\ \frac{22}{3}, & \text{if } x \equiv 1, \\ \frac{26}{3}, & \text{if } x \equiv 2, \end{cases} \pmod{3} \quad (1.6)$$

for a suitably chosen  $s$  satisfying (1.2).

We next put  $s = -2a$  and consider 2) of Lemma 2. Putting  $a = 0$ ,  $b = c$ , we can cover the region  $4x - 12 \leq y \leq 4x - 8$  with  $x$  odd. Similarly, putting  $a = 1$ ,  $b = c$ , we can cover the region  $4x - 16 \leq y \leq 4x - 10$  with  $x$  even. In this way, by increasing  $a$  and putting  $b = c$ , we can cover the region outside (1.6) as well. The other statements follow from Lemma 1.3 and the construction. *q.e.d.*

*Remark:* The inverse image of  $Q_i$  on  $S$  is a hyperelliptic curve of genus 3 (see, [K, §9]). Further, it can be checked that most surfaces we have constructed have ample canonical bundle.

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Smooth projective varieties dominated by homogeneous rational projective variety.

By Koji Cho and Eiichi Sato

In this paper we consider the following problem due to Remmert, Van de Ven [R,V] and Lazarsfeld [L].

**Problem** Let  $X$  be a smooth projective variety defined over the complex number field  $\mathbb{C}$ . Let  $f : G/P \longrightarrow X$  be a surjective morphism where  $G$  is a simple classical group and  $P$  a maximal parabolic subgroup of  $G$ . Then unless  $f$  is an isomorphism,  $X$  is isomorphic to a projective space.

Here we study the property of such a variety  $X$  and have

**Main Theorem** Let  $X$  be a smooth projective variety,  $Q$  a smooth quadric hypersurface of  $\dim \geq 3$  and  $f$  a surjective morphism defined over an algebraically closed field of any characteristic ( $\text{char } k \neq 2$ ). Assume that  $f$  is separable. Then unless  $f$  is an isomorphism,  $X$  is isomorphic to a projective space.

Mori showed in [M] that a smooth projective variety with ample tangent bundle is isomorphic to a projective space.

As an important application Lazarsfeld in [L] proved that if  $G/P$  is a projective space, the above problem is valid (even in any characteristic. Use Proposition 1.1 in his proof [L])

In our case a key for the proof of Main Theorem is to apply Mori's Theorem and to study the behavior of a extremal rational curve on  $X$  and the property about the tangent bundle  $T_Q$  of a

quadric hypersurface  $Q$ .

Precisely speaking, in §1, we observe a general nature of the variety  $X$  dominated by  $G/P$  and see that  $v^*T_X \simeq \mathcal{O}_{P^1}(2) \oplus \mathcal{O}_{P^1}(1)^{\oplus r} \oplus \mathcal{O}_{P^1}^{\oplus s}$  for the normalization  $P^1 \longrightarrow C$  (= a general extremal rational curve in  $X$ ) (Proposition 1.13)

In §2, we study the property of the tangent bundle  $T_Q$  (Theorem 2.1), by which we obtain a result (Theorem 2.7) about the vector bundle  $F$  on  $Q$  which is a non-trivial extension of an ample line bundle on a hyperplane section of  $Q$  by  $T_Q$ .

Namely theorem 2.7 says that there is a line  $L$  on  $Q$  such that  $F|_L$  is ample.

In §3, for an extremal rational curve  $C$  on  $X$ , we observe the possibility of a vector bundle  $v^*T_X$  with the normalization  $v: P^1 \longrightarrow C$ . Thus the case with the conditions below is left to us as an complicated one:

(1) the morphism  $f: Q \longrightarrow X$  has the ramification locus  $R_Q$  which is a hyperplane section of  $Q$  with  $f^{-1}f(R_Q) = R_Q$ ,

(2) for any line  $L$  on  $Q$  the restricted map  $f|_L: L \longrightarrow f(L)$  is contained in one component  $H$  ( $\ni [v]$ ) in  $\text{Hom}(P^1, X)$

(3) the restriction of  $f^*T_X$  to any line in  $Q$  is not ample and finally

(4)  $f^*T_X$  is obtained as an extension of the line bundle  $\mathcal{O}_{R_Q}(2)$  on  $R_Q$  by  $T_Q$ .

But the condition (3) (4) contradicts Theorem 2.7 and this case does not occur, which asserts Main Theorem.

After finishing the present paper we found that K.H.Paranjape and V.Srinivas got the same result (= Proposition 8) in the following paper:

Self maps of homogeneous aspcas.

Invent.math.98. 425-444(1989).

Our method of the proof is essentially same as the one by them except for the last part of the proof. The last part seems to be most complicated one. That is why we have the argument in any characteristic ( $\neq 2$ ). We must study the property of the homomorphism  $T_Q \longrightarrow f^* T_X$  on the branched locus  $R_Q$  in details.

Notation. Basically we use customary terminologies of algebraic geometry. A variety means a separated, reduced irreducible algebraic  $k$ -scheme where  $k$  is an algebraically closed field of any characteristic.  $\mathcal{O}_{\mathbb{P}^n}(1)$  denotes the line bundle corresponding to the hyperplanes in  $\mathbb{P}^n$  and in case of  $n=1$  it is abbreviated simply as  $\mathcal{O}(1)$  very often. When a vector bundle  $E$  on a variety is generated by its global sections, for the simplicity we say that  $E$  is GS. For a variety  $X$  and a closed subscheme  $Y$  which is locally complete intersection in  $X$ ,  $N_{Y/X}$  means the normal bundle of  $Y$  in  $X$ .

### §.1 General properties of a smooth target of $G / P$ .

In this section we study the property of the smooth projective variety which is a target of homogeneous complete variety  $G / P$ .

Let us begin with easy propositions about vector bundle on a curve. For the proof we use several facts in [H].

Proposition 1.1. ( Lemma 4.5 in [L]) Let  $E, F$  be vector bundles on a smooth projective curve  $C$  with the same rank and  $j: E \longrightarrow F$  an injective homomorphism. If  $E$  is an ample vector bundle, so is  $F$ .

Proof. Take an integer  $m$  such that  $S^m(E)$  is ample, GS (= is generated by its global sections) and its first cohomology group vanishes. (see Proposition 2.4 and Proposition 3.3 [H]) Moreover consider the homomorphism

$\bar{j}: S^m(E) \longrightarrow S^m(F)$  induced by

$j: E \longrightarrow F$ , which is also an injective homomorphism. Set  $T = \text{Cokernel of } \bar{j}$ . Since  $T$  is GS, so is  $S^m(F)$ . Now assume  $S^m(F)$  is not ample, namely there is a section  $C'$  (with respect to the projection  $P(S^m(F)) \longrightarrow C$ ) in  $P(S^m(F))$  which maps to a point via the morphism induced by the tautological line bundle of  $P(S^m(F))$ . Thus we infer that there is a surjective homomorphism  $S^m(F) \longrightarrow \mathcal{O}_{C'}$ , which gives rise to the non-trivial homomorphism  $S^m(E) \longrightarrow \mathcal{O}_{C'}$ . But since  $S^m(E)$  is ample, we have a contradiction. Hence  $S^m(F)$  is ample and therefore  $F$  is ample.

q.e.d.

Moreover we have

Proposition 1.2. Let  $E$  and  $F$  be vector bundles on  $P^1$  and  $j: E \rightarrow F$  a generically surjective homomorphism. Then we have

1) If  $E$  is GS, so is  $F$

2) If  $E$  is ample, so is  $F$ .

3) Particularly let  $E = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$  and  $\text{rank } F = n$ .

Suppose that  $F$  is the pull back <sup>of</sup> a vector bundle  $G$  on  $P^1$  via a finite morphism  $f: P^1 \rightarrow P^1$  and  $\text{deg } G \leq n + 1$ . Then  $G$  is one of the following:

$\alpha)$   $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1}$

$\beta)$   $\mathcal{O}(3) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$ .

$\gamma)$   $\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus n-3} \oplus \mathcal{O}$ .

$\delta)$   $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$ .

Proof. Noting that any vector bundle on  $P^1$  is a direct sum of line bundles, this proposition is trivial.

Now let us consider a separable, surjective morphism  $f: Y \rightarrow X$  between smooth projective varieties  $X, Y$  and let  $R_Y$  be the ramification divisor of  $Y$  via  $f$ . Then we have

Proposition 1.3. Under the above condition let  $Y$  be a Fano variety (i.e.  $-K_Y$  is ample). Then we have

1) If  $f$  is finite and  $R_Y$  is numerically effective i.e. the intersection number  $R_Y \cdot C$  is non-negative for any integral curve  $C$  on  $Y$ , then  $X$  is a Fano variety.

2) Assume that  $\text{Pic } Y \simeq Z$ . Then the assumptions in 1) hold good.

Proof. We have the equality:  $K_Y = f^*K_X + R_Y$ . Since the product of an ample line bundle and a numerically effective line bundle is ample,  $-K_X$  is ample by the finiteness of  $f$ . For 2) note that any effective divisor is ample. Unless  $f$  is finite there is a curve  $C$  in  $Y$  such that  $f(C)$  is a point. Take a divisor  $D$  in  $X$  which does not contain  $f(C)$ . Since  $f^{-1}(D)$  is ample, it intersects with the curve  $C$  which induces a contradiction.

q.e.d.

From now on let us consider

(1.4) a finite, separable and surjective morphism:  $f: G/P (= W) \longrightarrow X$  where  $R_W$  is the branched locus on  $W$  with respect to the morphism  $f$  and  $R_X = f(R_W)$ .

Then it is well-known that  $W$  is a Fano variety and any effective divisor on  $W$  is numerically effective. Hence by Proposition 1.3.1  $X$  is a Fano variety.

Hereafter the property of the variety  $X$  is studied.

First let us treat the case that  $R_W = \emptyset$ . For the purpose we state this case in more general form.

Proposition 1.5. Let  $Z$  and  $U$  be smooth projective varieties and  $f: U \longrightarrow Z$  an étale finite morphism. Assume that  $\chi(U, \mathcal{O}_U) = 1$ . Then,  $f$  is an isomorphism.

Proof. The assumption says that  $f^*T_Z = T_U$ . Thus, Hirzebruch Atiyah-Singer Riemann-Roch theorem implies that  $\deg f \chi(Z, \mathcal{O}_Z) = \chi(U, \mathcal{O}_U) = 1$ . Hence  $f$  is a isomorphism.

q.e.d.

Corollary 1.5.1. Any smooth projective Fano variety  $Z$  defined over the complex number field is algebraically simply connected.

Proof. Let  $f: U \longrightarrow Z$  be a finite etale morphism from an algebraic scheme. Then we see that  $U$  is a smooth projective variety. Since  $f^*K_Z = K_U$ ,  $U$  is a Fano variety. Hence by virtue of Kodaira's vanishing Theorem, we get  $H^i(Z, \mathcal{O}_Z) = 0$  for  $1 \leq i \leq \dim Z - 1$ . Thus, Proposition 1.5 asserts that  $f$  is an isomorphism.

q.e.d.

Now let us recall a characterization of a uniruled variety.

Proposition 1.6. Let  $M$  be a smooth projective variety. Then, the following two conditions are equivalent to each other:

- (1)  $M$  is separably uniruled namely, there is a separable surjective rational map from some projective space to  $M$ .
- (2) there is a rational curve  $C$  in  $M$  such that for the normalisation  $\varphi: \bar{C} \longrightarrow C$ ,  $\varphi^*T_X$  is generated by its global sections.

For the proof, for example, see Lemma 1.2 in [S].

Proposition 1.7. Under the notation and the condition in 1.4, let  $Z$  be an integral curve on  $X$ . Assume  $Z$  is not in  $R_X$ . Then for the normalization  $g: \bar{Z} \longrightarrow Z$ ,  $g^*T_X$  is generically generated by its global sections. Particularly assume that  $Z$  is a rational curve. Then  $g^*T_X$  is GS.

Proof. Let  $f^{-1}(Z) = Z_1 \cup \dots \cup Z_r$  be the irreducible

decomposition by the finiteness of  $f$ . Fix  $Z_1$  and take the normalization  $h: \bar{Z}_1 \rightarrow Z_1$ . Then there is a natural morphism  $\bar{f}: \bar{Z}_1 \rightarrow \bar{Z}$ . Now the morphism  $f$  induces the homomorphism:  $T_G / P \rightarrow f^* T_X$  which is generically surjective on  $Z_1$ . Since the tangent bundle of homogeneous space is generated by its global sections, so is  $h^* T_G / P$ . Thus we get the former. The latter is trivial.

q.e.d.

Combining Proposition 1.6 and Proposition 1.7, we get Corollary 1.7.1. The variety  $X$  in 1.4 is separably uniruled.

(1.8) Thus let  $S$  be a set:  $\{ \text{a rational curve } C \text{ in } X \mid C \not\subset R_X \}$  and  $c = \min \{ (C, -K_X) \mid C \in S \}$ . Now take a curve  $C$  in  $S$  with  $(C, -K_X) = c$  and the normalisation of  $C$ :  $\varphi: \bar{C} \rightarrow C$ . Then by Proposition 1.7,  $\varphi^* T_X$  is GS.

Let  $H$  be the irreducible component of the Hilbert scheme  $\text{Hom}(P^1, X)$  containing the point  $[\varphi]$ . (Note the  $H^1(P^1, \varphi^* T_X) = 0$ , therefore  $\text{Hom}(P^1, X)$  is smooth at  $[\varphi]$ )

Then we have an important

Lemma 1.8. Let  $H$  and  $R_X$  be as above. Then, for each element  $v$  in  $H$  with  $v(P^1) \not\subset R_X$ ,  $v: P^1 \rightarrow v(P^1)$  is birational.

Proof. Assume that there is an element  $v$  in  $H$  such that  $v: P^1 \rightarrow v(P^1)$  is not birational. Then the morphism  $v$  factors to a finite morphism  $h: P^1 \rightarrow P^1$  ( $\deg h \geq 2$ ) and a birational morphism  $v_1: P^1 \rightarrow X$ . Then

$$\deg v^* T_X = \deg h \cdot X \cdot \deg v_1^* T_X > \deg v_1^* T_X = -v_1(P^1) \cdot K_X$$

This contradicts the minimality of  $c$ .

q.e.d.

(1.9) Now let  $P$  be a smooth point of the image  $\varphi(P^1)$  ( $= C$ ) (1.8) and let  $\iota: o \longrightarrow P(\in X)$  a map with a point  $o$  in  $P^1$ . Then, it is known that  $\text{Hom}(P^1, X; \iota)$  is a closed subscheme in  $\text{Hom}(P^1, X)$  by Proposition 1 in [Mo].

Hence, letting  $H_P = \{v \in H \mid v(o) = P\}$ , we must remark that  $H_P = H \cap \text{Hom}(P^1, X; \iota)$  as a set.

Now we have

(1.9.1) Remark. 1) Let  $v$  be an element in  $H$  with  $v(P^1) \not\subset R_X$ . Then,  $v^*T_X$  is GS and hence  $H$  is smooth at  $[v]$  because of the fact  $H^1(P^1, v^*T_X \otimes \mathcal{O}_{P^1}(-1)) = 0$  by Proposition 1.7.

Hence if  $P$  is not in  $R_X$ , then  $H_P$  is smooth. In this case  $H_P$  may consist of finitely many components.

Hereafter till the end of this paper, we use the notations  $\varphi, H, P$  and  $H_P$  just above very often.

Now we shall state a sufficient condition for a rational curve in a projective variety to deform to a union of rational curves (which may not be distinct) which is implicitly shown in Theorem 4 in [Mo].

(1.10) Proposition (Theorem 4 in [Mo]) Let  $M$  be a non-singular projective variety,  $U$  a variety and  $g: P^1 \times U \longrightarrow M$  a morphism satisfying the following properties:

1) for each point  $u$  in  $U$ ,  $g(P^1, u)$  ( $= C_u$ ) is a curve in  $M$  and for

each pair  $(u_1, u_2)$  in  $U \times U$  ( $u_1 \neq u_2$ ),  $C_{u_1} \neq C_{u_2}$ .

2)  $\dim g(P^1 \times U) \leq \dim U$  ( $\geq 2$ ).

3) for each point  $u$  in  $U$ , there is a point  $m$  in  $M$  such that  $C_u$  passes through a point  $m$  and is smooth at  $m$ .

Then  $C_u$  deforms to a cycle which is a sum of rational curves (which may not be distinct). Precisely speaking, there is an open subset  $\bar{Z}$  in  $g(P^1 \times U)$  (this is a constructible subset in  $M$ ) satisfying the following:

1)  $\bar{Z}$  does not contain the point  $m$ .

2) for every point  $z$  in  $\bar{Z}$ , there exists a cycle  $\bar{C}$  as below to which some subfamily of  $\{C_u\}$  deforms :

letting  $\bar{C} = \bigcup_{i=1}^k a_i C_i$  where  $C_i$  is an irreducible component of  $C$

we have two cases:

1) there are two components  $C_i, C_j$  of  $\bar{C}$  with  $m \in C_i$  and  $z \in C_j$ .

2) there is a component  $C_i$  containing two points  $m, z$  with  $a_i \geq 2$ .

Proof. The properties 1) 2) say that for a general point  $n$  in  $g(P^1 \times U)$  there is a closed curve  $V \subset U$  so that every curve  $C_u$  ( $u \in V$ ) passes through the point  $n$ . Thus the proof of Théorem 4 in [Mo] implies the desired fact.

q.e.d.

(1.11) Let  $M$  be a smooth projective variety and  $C$  a rational curve in  $M$ . Let  $\phi: \bar{C} \rightarrow C$  be the normalisation of  $C$ . Assume that  $\phi^* T_M = \bigoplus_{P^1} \mathcal{O}_{P^1}(a_i)$  with  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ . Moreover let  $\iota: o \rightarrow m$  be a map from a point  $o$  in  $P^1$  to a point  $m$  ( $\in C$ ) in  $M$  where  $C$  is smooth at the point  $m$  and let us

consider an irreducible component  $N$  of  $\text{Hom}(P^1, M; \iota)$  containing the morphism  $\phi$ . For a point  $v$  in  $H$ , let  $v^* T_M = \bigoplus \mathcal{O}(a_i^v)$  with  $a_1^v \geq a_2^v \geq \dots \geq a_n^v$ .

Then Proposition 1.10 provides us with a

Proposition 1.12 Let the notation and condition be as in 1.11. Assume that there is an open subset  $N_0$  in  $N$  enjoying three conditions 1) 2) 3) or 1) 2) 3')

1)  $\phi$  is contained in  $N_0$ .

For every point  $v$  in  $N_0$ ,

2)  $a_n^v \geq 0$

3)  $a_1^v + a_2^v \geq 4$ .

3')  $a_1^v \geq 3$  and  $a_2^v = 0$ .

(note that for every point  $v$  in  $N$ ,  $a_1^v \geq 2$ .)

Finally suppose that the characteristic of the base field is zero.

Then  $C$  deforms to a cycle which is a sum of rational curves.

Proof. Set as  $r + 1 = \min \{i \mid a_i^v = 0 \text{ for any } v \in N_0\}$ .

By the semi-continuity of coherent sheaf, there is an open subset  $N'$  of  $N_0$  such that  $a_r^v > a_{r+1}^v = 0$  for any  $v \in N'$ .

Let  $\bar{W} (\subset M \times P^1 \times N')$  be the universal scheme of  $N'$  with the first projection  $p: \bar{W} \rightarrow M$  and the third projection  $q: \bar{W} \rightarrow N'$ .

Now take a general point  $P$  in  $M$  and let  $D$  be an component of  $qp^{-1}(P)$  in  $N'$ .

Since the characteristic of the base field is zero, the closure of  $pq^{-1}(D)$  is of  $r$  dimension by virtue of Proposition 1.11 in [S].

On the other hand for such a point  $v \in N'$ ,  $\dim H^0(P^1, v^*T_M \otimes \mathcal{O}(-1)) = \sum_{i=1}^r a_i^v \geq r + 2$  and  $\text{Aut}(P^1, o) (= \{\sigma \in \text{Aut } P^1 \mid \sigma(o) = o\})$  is of 2-dimension, Thus we can construct a subscheme  $U$  in  $D$  and a morphism  $g: P^1 \times U \longrightarrow X$  enjoying the three properties in Proposition 1.10. Hence we are done.

q.e.d.

Thus we get

Proposition 1.13. Let the notations  $R_X$ ,  $H$  and  $H_p$  be as in 1.4 1.8 and 1.9. Assume that the characteristic of the ground field is zero. Then there is a Zariski open subset  $H^0$  of  $H$  such that for each point  $u$  in  $X - R_X$  and every point  $v$  in  $H^0 \cap H_u$ ,  $v^*T_X$  is isomorphic to  $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r} \oplus \mathcal{O}^{\oplus s}$ .

Proof. The above corollary asserts that there is a point  $P$  outside the subset  $R_X$  enjoying the following condition: there is a birational morphism  $\varphi: P^1 \longrightarrow C \subset X$  so that  $P \in C$  and  $v^*T_X$  is isomorphic to  $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r} \oplus \mathcal{O}^{\oplus s}$ .

Let  $W (\subset X \times H \times P^1)$  be the universal scheme of  $H$  with the first projection  $p: W \longrightarrow X$ , the second projection  $q: W \longrightarrow H$  and  $r: W \longrightarrow P^1$  the third projection.

Then since  $q$  is proper and  $p^{-1}R_X$  is a closed subset in  $W$ ,  $qp^{-1}R_X$  is a properly closed set in  $H$ . Now set  $A = \{v \in H \mid q^{-1}(v) \subset p^{-1}R_X\}$ . Then we infer that  $A$  is closed in  $H$ . Letting  $H_0 = H - A$ , for any  $v$  in  $H_0$ ,  $v^*T_X$  is GS.

Remark that the projection  $q: W \longrightarrow H$  is  $P^1$ -bundle, therefore flat and  $p^*\Omega_X|_{q^{-1}(v)} \simeq v^*\Omega_X$ .

Now let  $s = \max\{h^0(P^1, v^* \Omega_X) \mid v \in H_0\}$  and  $H^0 = \{v \in H_0 \mid h^0(P^1, \Omega_X) = s\}$ . Then we infer that  $H^0$  is open in  $H_0$ . Next doing the same procedure for the vector bundle  $p^* \Omega_X \otimes r^* \mathcal{O}_{P^1}(1)$  on the  $P^1$ -bundle  $q^{-1}(H^0) \longrightarrow H^0$ , we complete the proof.

q.e.d.

Finally we finish this section to state a proposition which is implicitly shown in [Mo].

(1.14) Let  $\bar{H}_P$  be an irreducible component of the above  $H_P$ .

Hereafter we follow the notations written in §3 of [Mo].

Let  $G = \{g \in \text{Aut } P^1 \mid g(o) = o\}$ . Since the natural action of  $G$  on  $\text{Hom}_k(P^1, X; \iota)$  induces the action  $\sigma$  of  $G$  on the connected component  $\bar{H}_P$ :

$$\sigma : G \times \bar{H}_P \longrightarrow \bar{H}_P, \quad \sigma(g, v)x = v(g^{-1}x), \quad g \in G, v \in V, x \in P^1,$$

$G$  also acts on  $\bar{H}_P \times P^1$ :

$$\tau : G \times \bar{H}_P \times P^1 \longrightarrow \bar{H}_P \times P^1, \quad \tau(g, v, x) = (\sigma(g, v), gx).$$

Let  $\text{Chow}^d X$  be the Chow variety parameterising 1-dimensional effective cycles  $C$  of  $X$  with  $C \cdot K_X^{-1} = d$ . Then by Lemma 1.8, we have a morphism  $\alpha : \bar{H}_P \longrightarrow \text{Chow}^m X$  with  $v(P^1) \cdot K_X^{-1} = m$  ( $v \in \bar{H}_P$ ).

Then Mori proved in [Mo]

Proposition 1.14. (Lemma 9 [Mo])

1)  $\sigma$  is free action. 2)  $(Y, \Gamma)$  is the geometric quotient of  $\bar{H}_P$  by  $\sigma$  in the sense of [Mu] where  $Y$  is the normalization of the closure of  $\Gamma(Y)$  in  $\text{Chow}^m X$ .

§.2 The property of the tangent bundle of an  $n$ -dimensional smooth quadric hypersurface  $Q$  ( $n \geq 3$ ).

In this section we study the property of the tangent bundle  $T_Q$  of a non-singular quadric hypersurface in  $P^{n+1}$  ( $= P$ ).

What we want to state here is

Theorem 2.1. 1)  $T_Q$  is GS. For each line  $L$  in  $Q$ , the restriction  $T_Q|_L$  is isomorphic to  $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$ .  
 2) Let  $C$  be an integral curve in  $Q$ . If  $T_{Q|C}$  is not ample, then  $C$  is a line in  $Q$ .

To show the above theorem, we make several reviews about Grassmann variety and Flag variety (see [OSS])

For an  $(n+1)$ -dimensional projective space  $P^{n+1}$  ( $= P$ ), let  $G$  be a Grassmann variety parameterising lines in  $P$  and a flag variety  $F = \{(x, y) \in P \times G \mid x \in L_y\}$  where  $L_y$  is the line corresponding to a point  $y$  in  $G$ . Then we have canonical projections  $p: F \rightarrow P$ ,  $q: F \rightarrow G$  where  $p$  is a  $P^n$ -bundle and  $q$  a  $P^1$ -bundle.

Now the following is known.

Proposition 2.2.  $F \simeq P(\Omega_P(2))$ . The morphism  $q$  is a morphism induced naturally by the complete linear system of the tautological line bundle of  $\Omega_P(2)$ . (Of course,  $\Omega_P(2)$  is GS). Moreover the restriction of the morphism  $q$  on each fiber of  $p$  ( $\simeq P^1$ ) is an embedding.

Then we can show an important proposition which seems to be known. Here for a vector bundle  $E$  on  $Q$ ,  $E(m)$  denotes  $E \otimes (\mathcal{O}_P(m)|_Q)$ .

Proposition 2.3.  $T_Q \simeq \Omega_Q(2)$ .

Proof. First a smooth quadric hypersurface  $Q$  can be described as  $G_{n+2} / Pa$  in  $P^{n+1}$  where  $G_{n+2} = \{M \in SL(n+2, k) \mid {}^t M M = E\}$  is the orthogonal group and  $Pa$  is a maximal Parabolic subgroup of  $G_{n+2}$ .

Then  $V(T_Q(-1))$  is isomorphic to  $G_{n+2} \times k^n$  modulo the following relation:

let  $R$  be the radical part of  $Pa$  and  $\varphi : Pa \rightarrow GL(n, k)$  be the homomorphism where  $\varphi$  is trivial on  $R$  and the induced homomorphism  $\bar{\varphi}$  yields the irreducible representation  $(Pa / R (\simeq G_n), k^n)$  which is a canonical injective homomorphism  $G_n \rightarrow GL(n, k)$ . Then the relation  $(c, v) \sim (d, w)$  means that  $c d^{-1} \in Pa$  and  $v = \varphi(p)w$  with some point  $p$  in  $Pa$ .

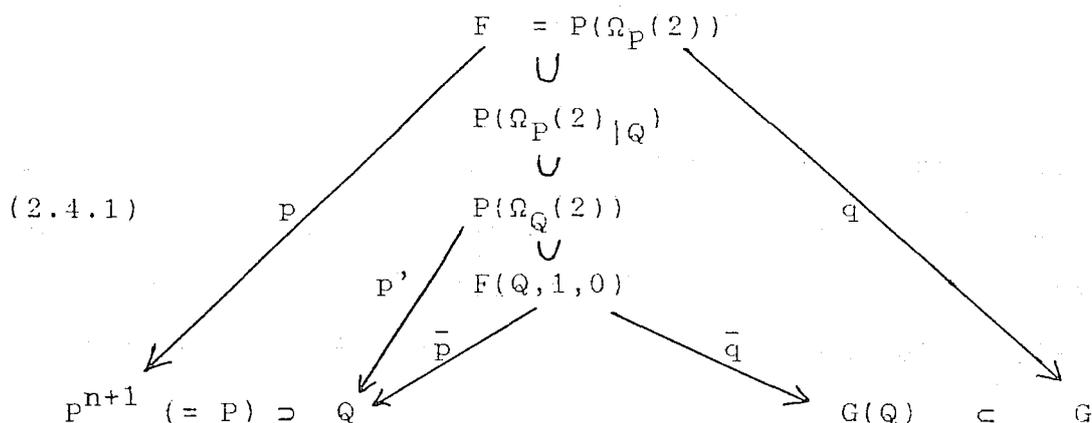
On the other hand  $V(\Omega_Q(1))$  is isomorphic to  $G_{n+2} \times k^n$  modulo the relation obtained by replacing the homomorphism  ${}^t \varphi^{-1}$  for the homomorphism  $\varphi$  defined as above. Remark that the property of the orthogonal group gives rise to the fact that  ${}^t \varphi^{-1} = \varphi$  (defined as above). Thus we complete our proof.

q.e.d.

Thus an embedding  $i: Q \rightarrow P^{n+1} (= P)$  naturally induces an important exact sequence:

$$(2.4) \quad 0 \longrightarrow \mathcal{O}_Q \longrightarrow \Omega_P(2)|_Q \longrightarrow \Omega_Q(2) (= T_Q) \longrightarrow 0$$

Hence we have a diagram:



where  $G(Q)$  denotes the set of lines in  $Q$  and  $F(Q,1,0)$  denotes the set  $\{(x,y) \in Q \times G(Q) \mid x \in L_y (= \text{the line corresponding to a point } x)\}$  ( $\subset F$ ). Then, a canonical projections  $\bar{p}$ :  $F(Q,1,0) \longrightarrow Q$  is a  $Q^{n-3}$  ( $= (n-3)$ -dimensional smooth quadric hypersurface) -bundle and the other canonical projection  $\bar{q}$ :  $F(Q,1,0) \longrightarrow G(Q)$  is a  $P^1$ -bundle.

Proof of Theorem 2.1.

The former of 1) is obvious since  $Q$  is a homogenous space. Next a line  $L$  of a smooth quadric surface  $Q_2$  induces an exact sequence:

$$0 \longrightarrow T_L (= \mathcal{O}_{P^1}(2)) \longrightarrow T_{Q_2|L} \longrightarrow N_L / Q_2 (=0) \longrightarrow 0.$$

Thus we see  $T_{Q_2|L} \simeq \mathcal{O} \oplus \mathcal{O}_{P^1}(2)$ . Letting  $\bar{Q}$  a smooth hyperplane section of  $Q$ , we get an exact sequence:

$$0 \longrightarrow T_{\bar{Q}} \longrightarrow T_{Q|\bar{Q}} \longrightarrow N_{\bar{Q}} / Q (= \mathcal{O}_{\bar{Q}}(1)) \longrightarrow 0.$$

Hence, the induction on the dimension of  $Q$  gives the latter of 1).

Now assume there is a curve  $C$  such that  $T_{Q|C}$  is not ample. Then by the exact sequence 2.4,

$\Omega_P(2)|_C$  is not ample. On the other hand, we have

$$\mathcal{O}_{P(\Omega_P(2))}^{(1)}|_{P^{-1}(C)} \cong \mathcal{O}_{P(\Omega_P(2)|_C)}^{(1)}.$$

Note that if a linear system of an ample line bundle has no base point, its induced morphism is finite. Thus the assumption  $T_{Q|C}$  is not ample and Proposition 2.2 give rise to a section  $D$  (with respect to the  $p$ ) in  $P(\Omega_P(2))|_C$  such that  $q(D)$  is a point in  $G$ . Thus the diagram show that  $D$  is some fiber of  $q$  and therefore  $p(D)$  is a line in  $P$  (in  $Q$ ).

q.e.d.

Let us consider an exact sequence of coherent sheaves on the smooth quadric hypersurface  $Q$  which is used in the final part in §3.

$$(2.5) \quad 0 \longrightarrow T_Q \xrightarrow{h} F \longrightarrow G \longrightarrow 0$$

where  $F$  is a torsion free sheaf on  $Q$  ( $\dim Q \geq 3$ ) and  $G$  is a line bundle on an irreducible hyperplane section of  $Q$  ( $= R$  possibly singular).

Remark 2.5.1. if  $\dim Q \geq 4$  or  $\dim Q = 3$  and  $R$  is singular, then  $\text{Pic } R \cong \mathbb{Z}$ .

Remark.2.6.1. Assume  $G$  is GS. Then since  $H^1(Q, T_Q) = 0$ ,  $F$  is GS.

Remark 2.6.2. Let  $b; Q \longrightarrow P^n$  be a separable double covering with the branched locus  $B$ . Then  $B$  is a smooth hyperplane section of  $Q$  and there is the following exact sequence:

$$0 \longrightarrow T_Q \longrightarrow b^* T_{P^n} \longrightarrow \mathcal{O}_Q(2)|_B \longrightarrow 0.$$

Remark 2.6.3. Under the exact sequence 2.5 and  $G = \mathcal{O}_Q(2)|_R$

$$\begin{aligned} \text{Ext}^1(\mathcal{O}_Q(a)|_R, T_Q) &\simeq H^0(Q, T_Q) & (a = 1) \\ &\simeq k & (a = 2) \\ &= 0 & (a \geq 3) \end{aligned}$$

Proof. There is an exact sequence on  $Q$ :

$$0 \longrightarrow \mathcal{O}_Q(a-1) \longrightarrow \mathcal{O}_Q(a) \longrightarrow \mathcal{O}_Q(a)|_R \longrightarrow 0,$$

which induces the long exact sequence of cohomologies

$$\begin{aligned} \longrightarrow H^0(Q, T_Q(-a+1)) \longrightarrow \text{Ext}^1(\mathcal{O}_Q(a)|_R, T_Q) \longrightarrow H^1(Q, T_Q(-a)) \longrightarrow \\ H^1(Q, T_Q(-a+1)) \end{aligned}$$

On the other hand there are two exact sequences:

$$(2.6.4) \quad 0 \longrightarrow T_Q(b) \longrightarrow T_P(b)|_Q \longrightarrow \mathcal{O}_Q(b+2) \longrightarrow 0,$$

$$(2.6.5) \quad 0 \longrightarrow T_P(b-2) \longrightarrow T_P(b) \longrightarrow T_P(b)|_Q \longrightarrow 0$$

where  $Q$  is a smooth quadric hypersurface in the projective space  $P$ .

If  $\dim P \geq 4$ ,  $H^i(P, T_P(b)) = 0$  for any integer  $b$  and  $i = 1, 2$  and therefore  $H^1(Q, T_P(b)|_Q) = 0$ . Thus, we get

$$H^0(Q, T_P(b)|_Q) \longrightarrow H^0(Q, \mathcal{O}_Q(b+2)) \longrightarrow H^1(Q, T_Q(b)) \longrightarrow 0.$$

Noting that  $H^i(Q, T_Q(-1)) = 0$  for  $i = 0, 1$ ,  $H^0(Q, T_Q(-2)) = 0$  and  $\dim H^1(Q, T_Q(-2)) = 1$ , we get the conclusion.

q.e.d.

Next we show a theorem which is important for the proof of Main Theorem.

Theorem 2.7 Under the condition 2.5, let us assume that

$F$  is locally free and the line bundle  $G$  is ample.

Then there is a line  $L$  on  $Q$  where  $F|_L$  is ample.

Proof. Assume that the conclusion is not true.

(2.7.1) for each line  $L$  on  $Q$ ,  $F|_L$  has a trivial line bundle as a

direct summand. (see Remark 2.6.1)

Note that for every line  $L$  on  $Q$ , there is a unique surjective homomorphism  $t_L: T_{Q|L} \longrightarrow \mathcal{O}_L \longrightarrow 0$  by Theorem 2.1.1 and  $\text{Ker } t_L$  is an ample vector bundle.

Now we have a

Claim 2.7.2. For every line  $L$  on  $Q$ ,  $F|_L$  has a unique trivial line bundle as a direct summand. Letting  $f_L: F|_L \longrightarrow \mathcal{O}_L \longrightarrow 0$  a unique surjective homomorphism,  $\text{Ker } f_L$  is an ample vector bundle and there is the following diagram with an isomorphism  $g_L$  of  $\mathcal{O}_L$ :

$$\begin{array}{ccc} T_{Q|L} & \xrightarrow{h|_L} & F|_L \\ \downarrow t_L & & \downarrow f_L \\ \mathcal{O}_L & \xrightarrow{g_L} & \mathcal{O}_L \end{array}$$

Proof. In case of  $L \not\subset R$ , the restriction of the exact sequence 2.5 to the line  $L$  yields a generically surjective homomorphism  $h|_L: T_{Q|L} \longrightarrow F|_L$ . Since  $T_{Q|L}$  has one trivial line bundle as a direct summand by Theorem 2.1.1,  $F|_L$  has one trivial line bundle as a direct summand by assumption 2.7.1. Hence since  $\text{Ker } f_L$  is ample,  $f_L h|_L(\text{Ker } t_L) = 0$ . Consequently  $h|_L$  yields the isomorphism  $g_L$  of  $\mathcal{O}_L$  as desired.

Next consider the case that  $L \subset R$ .

Restricting the above exact sequence 2.5 on  $R$ , we get two exact sequences:

$$(2.7.3) \quad 0 \longrightarrow H \longrightarrow T_{Q|R} \xrightarrow{j} E \longrightarrow 0$$

$$(2.7.4) \quad 0 \longrightarrow E \xrightarrow{k} F|_R \longrightarrow G \longrightarrow 0$$

where  $H$  is a linebundle on  $R$  and  $E$  a vector bundle of rank  $n-1$  on  $R$ .

Note that  $E|_L$  is GS and has at most one trivial line bundle as a direct summand by Theorem 2.1.1. Since the extension of ample vector bundle by an ample vector bundle is ample,  $E|_L$  has one trivial line bundle as a direct summand by 2.6.1 and its quotient bundle is ample, which yields the desired result.

q.e.d.

Hereafter we use the notations in the diagram 2.4.1.

Now there is a canonical exact sequences of vector bundle on  $P(T_Q)$ :

$$(2.7.5) \quad 0 \longrightarrow \text{Ker } u (= K_1) \longrightarrow p'^* T_Q \xrightarrow{u} I_Q \longrightarrow 0,$$

where  $I_Q$  is the tautological line bundle of  $T_Q$ .

Now we have a natural homomorphism:

$$(2.7.6) \quad \bar{q}^* \bar{q}_* \bar{p}^* F^V \xrightarrow{j} \bar{p}^* F^V. \text{ Then claim 2.7.2 and the base change theorem say that}$$

$\bar{q}_* \bar{p}^* F^V$  is a line bundle on  $G(Q)$  and  $j$  in (2.7.6)

is an injective homomorphism as a vector bundle.

Here dualizing the sequence (2.7.6), we get

$$(2.7.7) \quad 0 \longrightarrow \text{Ker } w (= K_2) \longrightarrow \bar{p}^* F \xrightarrow{w} \bar{q}^* (\bar{q}_* \bar{p}^* F^V)^V (= I_F) \longrightarrow 0$$

on  $F(Q,1,0)$ .

(2.7.8). Note that  $\bar{I}_Q$  and  $I_F$  are trivial on each fiber ( $\simeq P^1$ ) of  $\bar{q}: F(Q,1,0) \longrightarrow G(Q)$  in the diagram (2.4)

and  $K_1$  and  $K_2$  are ample vector bundles on it.

Moreover restrict the exact sequence 2.7.5 to  $F(Q,1,0)$  and consider the homomorphism  $\bar{p}^*(h): \bar{p}^* T_Q \longrightarrow \bar{p}^* F$  induced by the

homomorphism  $h: T_Q \longrightarrow F$ . Now let us study the following diagram on each fiber  $\ell$  of  $\bar{q}$ :

$$0 \longrightarrow \bar{K}_1|_{\ell} \xrightarrow{u'} \bar{p}^* T_Q|_{\ell} \xrightarrow{\bar{u}} \bar{I}_Q|_{\ell} \longrightarrow 0,$$

$$0 \longrightarrow K_2|_{\ell} \xrightarrow{w'} \bar{p}^* F|_{\ell} \xrightarrow{w} I_F|_{\ell} \longrightarrow 0$$

where  $\bar{K}_1$  and  $\bar{I}_Q$  are the restriction of two vector bundles  $K_1$  and  $I_Q$  on  $P(T_Q)$  to  $F(Q,1,0)$  respectively.

Then (2.7.8) implies that the composition  $u' \bar{p}^*(h) w$  of homomorphisms is zero map. Consequently the homomorphism  $\bar{p}^*(h)$  yields the homomorphism  $\bar{K}_1 \longrightarrow K_2$  and  $\bar{I}_Q \xrightarrow{k} I_F$  canonically. Thus we see that  $k$  is an isomorphism by claim 2.7.2.

Now note that

$$\mathcal{O}_{P(T_Q)}(F(Q,1,0)) \simeq I_Q^{\otimes 2} \otimes \mathcal{O}_{P^*M}$$

where  $M$  is a line bundle on  $Q$ .

The divisor  $F(Q,1,0)$  in  $P(T_Q)$  yields an exact sequence:

$$0 \longrightarrow \mathcal{O}_{P(T_Q)}(-F(Q,1,0)) \longrightarrow \mathcal{O}_{P(T_Q)} \longrightarrow \mathcal{O}_{F(Q,1,0)} \longrightarrow 0$$

Tensoring the tautological line bundle  $I_Q$  to the above exact sequence and taking the direct image  $R^i p_{*}$ , we see that

$$T_Q \simeq \bar{p}_* \bar{I}_Q.$$

Moreover remarking that  $\bar{p}_* \mathcal{O}_{F(Q,1,0)} \simeq \mathcal{O}_Q$  and taking the direct image  $\bar{p}_*$  of the isomorphism  $k: \bar{I}_Q \simeq I_F$ ,

we infer that  $T_Q \simeq F$ . This is absurd.

Hence we could prove Theorem 2.7.

q.e.d.

In case that  $R$  is singular and  $G = \mathcal{O}_Q(a)|_R$  with  $a \leq 2$ , the author do not know whether a non-trivial extension class  $F$  in the

exact sequence 2.5 is locally free or not.

But we can show

Corollary 2.8. Under the condition 2.5, assume that  $R$  is smooth and  $G = \mathcal{O}_Q(2)|_R$ . Then, there is a separable double covering  $h: Q \longrightarrow \mathbb{P}^n$  with the branched locus  $R$  and  $F \simeq h^*T_{\mathbb{P}^n}$ .

Proof. Remark 2.6.2 and 2.6.3 assert this corollary.

§.3. The proof of Main Theorem.

In this section we show Main Theorem. ( $\dim X \geq 3$ )

(3.1) Let  $f: Q \rightarrow X$  be a finite, separable surjective morphism from a smooth quadric hypersurface  $Q$  to a smooth projective variety  $X$  and  $R_Q, R_X$  as in 1.4. Moreover let  $C$  be a rational curve ( $\notin R_X$ ) in  $X$  which has the minimal degree with respect to  $-K_X$  as was stated in 1.8 (see Corollary 1.7.1). We maintain the notations  $\varphi, P, H, H_P$  in 1.8 and 1.9.1 under the case of  $W = Q$ .

Now we have an important

Proposition 3.1. Let  $R_X, H$  be as above. Then we have

1)  $\deg v^* T_X^*$  is independent of a choice of a point in  $H$  and the value is  $n$  or  $n+1$ .

2) Let  $v$  be an element in  $H$  such that  $v(P^1) \notin R_X$ .

If  $\deg v^* T_X^* = n + 1$ ,  $v^* T_X^*$  is one of the following:

$\alpha) \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1}$

$\beta) \mathcal{O}(3) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$ .

$\gamma) \mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus n-3} \oplus \mathcal{O}$ .

If  $\deg v^* T_X^* = n$ ,  $v^* T_X^*$  is isomorphic to

$\delta) \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$ .

Proof. It is obvious that  $\deg v^* T_X^*$  is constant for any  $v$ . Moreover Theorem 4 in [Mo] yields  $\deg v^* T_X^* \leq n + 1$ . Next let us consider 2). First assume that all the component's of  $f^{-1}(v(P^1))$  are lines. Then Theorem 2.1.1 and Proposition 1.2 apply. In case that  $C$  is a component of  $f^{-1}(v(P^1))$  which is not a line,

Proposition 1.2 (3) yields the desired fact.

q.e.d.

Hereafter, when  $v^*T_X$  is isomorphic to one ( $= \tau$ ) of  $\alpha, \beta, \gamma$  and  $\delta$  in Proposition 3.1,  $v$  is said to of  $\tau$ -type.

In the proof of Proposition 3.1, the following is shown

Corollary 3.2. Let  $v$  be an element of  $H$  with  $v(P^1) \not\subset R_X$ . Assume that  $v^*T_X$  is one of the three types:  $\beta, \gamma$  and  $\delta$ . Then  $f^{-1}(v(P^1))$  is a union of lines in  $Q$ .

(3.3) Let  $v$  be as in Corollary 3.2 and  $L_v$  a line in  $Q$  with  $f(L_v) = v(P^1)$ . We give a sufficient condition for  $f|_{L_v}: L_v \rightarrow v(P^1)$  to be birational.

Proposition 3.4. Under the notations 3.1 and 3.3, assume that there is a point  $P \in R_X$  and a subset  $H'$  in  $H_P$  consisting of an infinite  $G$ -orbit where each element  $v$  in  $H'$  is of  $\beta, \gamma$  or  $\delta$  type and  $G (= \{g \in \text{Aut } P^1 | g(o) = o\}$  in (1.11)) acts canonically on  $H_P$ .

Then for every line  $L$  on  $Q$  with  $f(L) \not\subset R_X$ , the restriction map  $f|_L: L \rightarrow f(L)$  is birational. Moreover the intersection number  $L \cdot f^*K_X$  is independent of a choice of a line  $L$  on  $Q$  and the value is  $-n$  or  $-n-1$ .

Proof. First we show

Claim. Assume that there is an element  $v$  in  $H'$  and an irreducible component  $L_0$  of  $f^{-1}(v(L_0))$  (by corollary 3.2) so that  $L_0$  is a line and  $f|_{L_0}: L_0 \rightarrow f(L_0)$  is birational. Then

for every line  $L$  with  $f(L) \not\subset R_X$   $f|_L : L \longrightarrow f(L)$  is birational.

Proof. For every line  $L$  in  $Q$ ,  $(-f^*K_X \cdot L) = (\deg f|_L)(f(L) \cdot K_X)$  and if  $f(L) \not\subset R_X$ ,  $f(L) \cdot K_X = n$  or  $n+1$  by virtue of Proposition 3.1. Hence

by the assumption we see that  $-f^*K_X \cdot L_0 = n$  or  $n+1$

and therefore for every line  $L$  in  $Q$  with  $f(L) \not\subset R_X$

$-f^*K_X \cdot L = n$  or  $n+1$  by means of Proposition 3.1 again.

This give us the desired result.

Now let us return the proof of our propostion.

Let  $f^{-1}(P) = \{q_1, \dots, q_r\}$  and  $S = \{L_S \mid \text{there is } v \text{ in } H'_P \text{ where } L_S \text{ is a component of } f^{-1}(v(P^1)) \text{ (corollary 3.2)}\}$ . Noting that  $S$  is an infinite set and a line in  $Q$  is determined by two points, we see that a general line  $\bar{L}$  in  $S$  contains only one point  $q_i$  in  $f^{-1}(P)$ . Since  $p \not\subset P_X$ ,  $f$  is unramified at  $q_i$ , which implies that the restricted morphism:  $f_{\bar{L}}$  of  $f$  to such a line  $\bar{L}$  is birational. The latter is proved in Claim.

q.e.d.

Remark 3.4.1. For a point  $q$  in  $Q$ , let  $C(q)$  be a set:  $\{ \text{a line } L \text{ in } Q \mid q \in L \}$ . Then if  $n = \dim Q \geq 3$ ,  $C(q)$  is an  $(n-2)$  dimensional smooth quadric hypersurface in  $P(T_{Q,q}^*) (\simeq P^{n-1})$ .

Corollary 3.4.2. Let the notations and the assumption be as in Proposition 3.4. Then there are an irreducible component  $H_{P,0}$  in  $H_P$  and its divisor  $C_P$  is canonically induced by

$f(C(q))$  with some point  $P (= f(q))$ .

Proof. Proposition 3.4 says that there is an element  $v$  in the irreducible component  $H_{P,0}$  in  $H_P$ , a line  $L$  on  $Q$  and a point  $q$  on  $L$  such that  $f(L) = v(P^1)$ ,  $f(q) = P$  and  $f|_L$  is birational. Since  $H_P$  is an  $(n+1)$ -dimensional smooth variety,  $C(q)$  induces the closed subscheme in  $H_P$  by Proposition 3.4. Hence our corollary follows from the fact that  $\dim C_P = n$  by Remark 3.4.1 and that  $f$  is a finite morphism.

q.e.d.

Hence we have

Theorem 3.5. Let  $X$  be a smooth projective variety and  $f:Q \rightarrow X$  a separable surjective morphism. Assume that there is a point  $v$  in  $H$  (3.1) such that  $\deg v^*T_X = n$ . Then  $f$  is an isomorphism.

Proof. By Proposition 3.1.2, the assumption in Proposition 3.4 is satisfied, namely  $f^*K_X \cdot L = -n$  for any line  $L$  on  $Q$ . Hence from the equality:  $K_Q = f^*K_X + R_Q$  it follows that  $R_Q \cdot L = 0$ , and therefore,  $R_Q$  is empty because  $\text{Pic } Q \simeq \mathbb{Z}$ . Thus  $f$  is unramified, and therefore etale. Proposition 1.5 asserts this theorem.

q.e.d.

In the sequel we assume that

$$(3.6) \quad \deg v^*T_X = n+1. \quad (\text{Proposition 3.1})$$

Now let  $H_0 = \{v \in H \mid v(P^1) \notin R_X\}$ . Then we see easily that  $H_0$  is an open subset in  $H$ . Moreover let  $H(=) = \{v \in H_0 \mid v \text{ is of}$

$\neq$ -type} for  $\neq = \alpha, \beta, \gamma$ .

Now letting  $\# (\subset H \times X \times P^1)$  the universal scheme corresponding to the Hilbert scheme  $H$  where  $p: \# \rightarrow H$  is the first projection  $q: \# \rightarrow X$  the second projection and  $r: \# \rightarrow P^1$  the third projection. Then we note that  $q^* T_X|_{p^{-1}(v)} = v^* T_X$  for each  $v$  in  $H$ .

Hence we have

Proposition 3.7. Under the above notations,  $H(\alpha)$  is an open subset in  $H_0$  and  $H(\gamma)$  is an open subset in  $H_0 - H(\alpha)$ .

Proof. Since  $p: \# \rightarrow H$  is flat, we get the former by the semi-continuity of the coherent sheaf. Similarly by considering the coherent sheaf:  $q^* T_X \otimes r^* \mathcal{O}(-3)$  we get the latter.

q.e.d.

(3.8) Hence, first we consider the following case:

there is a point  $P$  outside the branched locus  $R_X$  and an irreducible component  $H_{P,0}$  in  $H_P$  such that each point  $v$  in  $H_{P,0}$  is  $\alpha$ -type and the image  $v(P^1)$  of  $P^1$  via some element  $v$  in  $H_{P,0}$  is smooth at the point  $P$ .

In this case  $X$  is a Fano variety by Proposition 1.3. Thus the proof of Hartshorne's conjecture in §.3 [Mo] shows that  $X$  is a projective space.

Next Corollary 3.4.2 and Proposition 3.7 states that the case below does not occur

(3.9) There is a point  $u$  in  $X - R_X$  such that every element  $v$  in some component of the smooth projective scheme  $H_u((2))$  of 1.9.1) is of  $\beta$  or  $\gamma$ -type.

Therefore the following case is left to us:

(3.10) For every point  $u$  in  $X - R_X$ , every component of  $H_u$  contains an element of  $\alpha$ -type and an element of  $\beta$  or  $\gamma$ -type.

Hereafter until the end of this paper we treat with this case which does not occur.

For the purpose we make several preliminaries.

(3.11) Let a point  $P$  and an irreducible component  $H_{P,0}$  be as in 3.10 and let us define a morphism  $\Phi : H_{P,0} \rightarrow V(T_{X,P}^*)$  induced by the canonical morphism  $H_{P,0} \times P^1 \rightarrow X$

$$H_{P,0} \ni v \longrightarrow dv_{*,o} \left( \frac{d}{dt} \right) \in V(T_{X,P}^*)$$

where  $t$  is a local parameter of  $P^1$  at the fixed point  $o$ .

First each element  $v$  in  $H_{P,0} \cap H(\alpha)$  yields the unramified morphism  $v: P^1 \rightarrow X$ . Next, an element  $v$  in  $H_{P,0} - H(\alpha)$  is induced by the image of some line in  $Q$  and

$f: Q \rightarrow X$  is etale at the points at  $f^{-1}(P)$ .

Hence we see that  $\Phi$  is defined as a morphism and  $\Phi(H_{P,0}) \subset V(T_{X,P}^*) - \{0\}$ . Thus we get morphism

$H_{P,0} \rightarrow P(T_{X,P}^*) (\simeq P^{n-1})$  which is  $G$ -invariant with  $G = \{g \in \text{Aut } P^1 \mid g(o) = o\}$  (see 1.14), which induces a canonical

surjective morphism  $\theta: Y \rightarrow P^{n-1}$  where  $Y$  is the geometric

quotien

We study a property of the morphism  $\theta$ .

For an element  $v$  in  $H_{P,0}$  set the closed subset  $\Phi^{-1}(\Phi(v))$  as  $F(v)$ . Then we have

Proposition 3.12. For an element  $v$  in  $H_{P,0}$ ,

let  $w$  be an element in  $F(v)$ .

1) if  $w$  is of  $\alpha$ -type. Then  $F(v)$  is smooth at the point  $w$  and it is of one-dimension at  $w$ .

2) if  $w$  is of  $\beta$  or  $\gamma$  type, then  $F(v)$  is of one-dimensional at the point  $w$ .

Consequently the morphism  $\Phi$  is equi-dimensional and hence flat.

Proof. 1) is proved in [Mo].

Let us consider (2). Assume that  $\dim_w F(v) > 1$ . Then by 1) there is a component  $V (\ni w)$  of  $F(v)$  such that every element of  $V$  is of  $\beta$  or  $\gamma$  type, which gives the infinite set of rational curves  $\{v(P^1) | v \in V\}$  by the fact that  $\Gamma : H_{P,0} \rightarrow Y$  is equi-one-dimensional by Proposition 1.14.

Moreover setting  $f^{-1}(P) = \{q_1, \dots, q_r\}$

we can find a point  $q_i$  and an infinite set of lines in  $Q$  :

$\{\text{line } L \text{ in } Q | L \text{ is an irreducible component of } f^{-1}(v(P^1)) \text{ for some point } v \text{ in } V \text{ and passes through the point } q_i\}$  by Corollary 3.2.

Now we remark

(#. ) for each  $i$  ( $1 \leq i \leq r$ ), there is a natural

isomorphism:  $(\cong) \quad df_{*,q_i}: T_{Q, q_i} \cong T_{X,P}$  and  $C(q)$  in Remark 3.4.1 is canonically contained in  $P(T_{X,P}^*)$  as a smooth quadric hypersurface.

Therefore we get a contradiction, which yields (2).

q.e.d.

Now let us consider the property of the morphism

$$\theta: Y \longrightarrow P(T_{X,P}^*).$$

Corollary 3.13. Under the condition and the notations in 3.11, the morphism  $\theta: Y \longrightarrow P(T_{X,P}^*)$  is a finite surjective morphism. Moreover  $\Gamma(H_{P,0} \cap (H(\beta) \cup H(\gamma))) (= J)$  in  $Y$  is of dimension  $\leq n-2$  and  $\theta$  is etale at each point in  $Y - J$ .

Proof. The first part is obvious by the above proposition 3.11. Hence since  $f$  is finite and surjective, the second is obtained by  $\cong$  in the proof of Proposition 3.12. The last part is proved in [Mo].

q.e.d.

(3.14) Now we divide the case 3.10 into two cases:

- (I) There is a point  $P (\notin R_X)$  such that the closed subset  $(H(\beta) \cup H(\gamma)) \cap H_{P,0}$  is of codimension  $\geq 2$  in  $H_{P,0}$ . (We have an argument till 3.18)
- (II) For every point  $P (\notin R_X)$  and every component  $H_{P,j}$  of  $H_P$ ,  $(H(\beta) \cup H(\gamma)) \cap H_{P,j}$  is of codim 1 in  $H_{P,j}$ . (This case is argued from 3.19)

We get contradictions to each case in 3.18 and 3.30.

Moreover we have

Corollary 3.15. Assume that the closed subset

$(H(\beta) \cup H(\gamma)) \cap H_{P,0}$  is of codimension  $\geq 2$  in  $H_{P,0}$ . (see Proposition 3.7). Then the morphism  $\theta: Y \longrightarrow P(T_{X,P}^*)$  is an isomorphism.

Proof. By Corollary 3.13 the branched part  $R_\theta$  is empty by the purity of the branched locus. Since  $P^{n-1}$  is simply connected, we get the desired fact.

q.e.d.

(3.16) Now the following is studied before the claim 8.2. in [Mo].

Under the above notations, we have a  $G$ -invariant morphism:

$$F: H_{P,0} \times P^1 \longrightarrow Y \times X, \quad F(v,x) = (\Gamma(v), v(x)), \quad v \in H_{P,0}, \quad x \in P^1.$$

Let  $Z = \text{Spec}_Y X \times X[(F_* \mathcal{O}_{H_{P,0} \times P^1})^G]$ . Since  $Y \simeq P^{n-1}$  and

$\Gamma: H_{P,0} \longrightarrow Y$  is equidimensional,  $\Gamma$  is flat and therefore universally geometric quotient. Hence

$Z$  is the geometric

quotient  $H_{P,0} \times P^1 / G$  and is a  $P^1$ -bundle  $\varphi: Z \longrightarrow Y$  for

Zariski topology because of a section  $S$  of  $Z$  over  $Y$ , via the

morphism  $H_{P,0} \longrightarrow (v,P) \in H_{P,0} \times P^1$ . Thus we can introduce a

proper morphism via the  $G$ -invariant morphism  $\pi: H_{P,0} \times P^1 \longrightarrow v(x) \in X,$

We study the branch locus of the morphism  $\pi: Z \longrightarrow X$ .

First the following is shown in [Mo].

Remark 3.17.  $\pi: Z \longrightarrow X$  is etale at each point of

$Z - S - \pi^{-1}((H(\beta) \cup H(\gamma)) \cap H_{P,0})$ .

At last we have come to the final stage of the case (I) in 3.14  
 Now we assume

(3.18) the closed subset  $(H(\beta) \cup H(\gamma)) \cap H_{P,0}$  is of  
 codimension  $\geq 2$  in  $H_{P,0}$ .

Then by the above remark, we see that  $\pi$  is etale at each point of  
 $Z - S$ . Moreover  $\pi(S) = P$ . Thus, the argument after 8.2 in [Mo]  
 says that  $\pi^{-1}(Z - S)$  is finite and etale over  $X - \{P\}$  and hence  
 $\pi: Z - S \longrightarrow X - \{P\}$  is an isomorphism and consequently  $X \simeq$   
 $P^n$ . Therefore we infer that  $T_X$  is an ample vector bundle,  
 which induces a contradiction to the condition I.

Next, we study case II in 3.14.

(3.19) For every point  $P (\notin R_X)$  and every component  $H_{P,j}$  of  
 $H_P$ ,  $(H(\beta) \cup H(\gamma)) \cap H_{P,j}$  is of codim 1 in  $H_{P,j}$ .

By Proposition 3.4 and the equality:  $K_Q = f^*K_X + R_Q$ , we see  
 that  $R_Q \cdot L = 1$  for any line  $L$  on  $Q$ . Thus we have an important

Proposition 3.20. The branched locus  $R_Q$  is a hyperplane  
 section of  $Q$ , namely, it is a smooth one or a cone with an isolated  
 singularity.

Moreover we have

Proposition 3.21. Assume the condition 3.19. Then  
 for every line  $L$  on  $Q$ ,  $\deg f^*T_{X|L} = n + 1$  and  
 for every line  $L (\notin R_Q)$  on  $Q$ ,  
 the restriction of  $f^*T_X$  on the line  $L$  is of  $\beta$  or  $\gamma$ -type.  
 Moreover there is no line  $L$  on  $Q$  so that  $f^*T_{X|L}$  is ample.

Proof. Proposition 3.4 gives rise to the former.

Next, for every line  $L (\notin R_Q)$  on  $Q$ ,  $f^*T_{X|L}$  is GS by  
 Proposition 1.7. Hence, Proposition 1.2 and the former yield the  
 fact that

( $\neq$ ) for every line  $L (\notin R_Q)$  on  $Q$ ,  $f^*T_{X|L}$  is of  $\alpha, \beta$  or  $\gamma$ -type.

Now set  $T$  as

{a line  $L \subset Q$  |  $f^*T_{X|L}$  is of  $\beta$  or  $\gamma$  type}.

Then we claim that

$T$  is a dense subset in the set  $\{L \subset Q$  |  $f(L) \notin R_X\}$

Proof. For each point  $P$  in  $X - R_X$ , we can take a point  $q$  in  
 $Q$  with  $f(q) = P$  and a divisor  $C_P$  in  $H_{P,j}$  induced by the scheme  
 $C(q)$  (Remark 3.4.1) by the assumption 3.19 and Corollary 3.4.2.

Then noting that  $C(q)$  is contained in  $T$ , we get the claim.

In view of the above ( $\neq$ ), the type  $\alpha$  in the set  $T$  is an open  
 condition from Proposition 3.7. Thus the claim yield the desired  
 fact.

q.e.d.

In the next place, we state a

Remark 3.22. Let  $V$  be a hyperplane section in a smooth quadric hypersurface  $Q$  in  $P^{n+1}$  ( $n \geq 3$ ). Then for each point  $P$  in the regular part of  $V$  there is a line passing through the point  $P$  and not in  $V$ .

Hereafter till the end of this paper let us study the exact sequence of the coherent sheaf on  $Q$ :

$$(3.23) \quad 0 \longrightarrow T_Q \longrightarrow f^*T_X \longrightarrow M \text{ (the quotient sheaf)} \longrightarrow 0$$

induced by the morphism  $f: Q \longrightarrow X$ .

Note that the support of the coherent sheaf  $M$  is  $R_Q$ .

Moreover we have

Proposition 3.23.  $M \otimes k(x) \simeq k(x)$  for every smooth point  $x$  in  $R_Q$ . Consequently  $M$  is a line bundle on the smooth part of  $R_Q$ .

Proof. Restricting the exact sequence 3.23 to a line  $L \subset R_Q$ , we

have the sequence: 
$$T_{Q|L} \xrightarrow{j_L} f^*T_{X|L} \longrightarrow M|L \longrightarrow 0.$$

Here  $f^*T_{X|L}$  is of  $\beta$  or  $\gamma$ -type. Since the homomorphism

$j_L$  is a generical isomorphism, we have the desired fact by Remark 3.22.

q.e.d.

As the matter of fact, we can check that  $M$  is a line bundle on  $R_Q$ . For the purpose we show

Proposition 3.24. Let  $U$  be an open subscheme ( $= A^n - \{\text{one point} = P\}$ ) and  $j: U \dashrightarrow A^n$  a natural open immersion. Then, if  $n \geq 3$ ,  $R^1 j_* \theta_U = 0$ . Thus, for an open immersion

$k: \mathbb{Q}^n - \{\text{one point}\} (= V) \longrightarrow \mathbb{Q}^n (n \geq 3), \quad R^1 k_* \mathcal{O}_V = 0.$

Proof. Since  $R^1 j_* \mathcal{O}_U$  is a quasi-coherent sheaf induced by the module  $H^1(U, \mathcal{O}_U)$ , it suffices to show that  $H^1(U, \mathcal{O}_U) = 0.$

By the theory of the local cohomology, we get the following exact sequence:

$$\begin{aligned} &\longrightarrow H_P^1(\mathbb{A}^n, \mathcal{O}) \longrightarrow H^1(\mathbb{A}^n, \mathcal{O}) (= 0) \longrightarrow H^1(U, \mathcal{O}_U) \\ &\longrightarrow H_P^2(\mathbb{A}^n, \mathcal{O}) (\simeq H_P^2(\mathbb{P}^n, \mathcal{O}) \text{ by the existence where } \mathbb{A}^n \text{ is contained} \\ &\quad \text{naturally in } \mathbb{P}^n) \end{aligned}$$

Let  $I$  be the sheaf of ideal defining the closed point  $P$  and  $X_m = \mathcal{O}_{\mathbb{P}^n} / I^m$  the closed subscheme of  $\mathbb{P}^n$ . Then there is an exact sequence:

$$0 \longrightarrow I^m / I^{m+1} \longrightarrow \mathcal{O}_{X_{m+1}} \longrightarrow \mathcal{O}_{X_m} \longrightarrow 0.$$

Since  $I^m / I^{m+1} = S^m(I / I^2)$  and the support of  $I / I^2$  is  $P$ ,  $H^{n-2}(X_{m+1}, I^m / I^{m+1}) = H^{n-1}(X_{m+1}, I^m / I^{m+1}) = 0$ , which implies that  $H^{n-2}(X_{m+1}, \mathcal{O}_{X_{m+1}}) \simeq H^{n-2}(X_m, \mathcal{O}_{X_m})$  for each  $m$ . Thus we infer that  $H^{n-2}(\hat{\mathbb{P}}^n, \mathcal{O}_{\hat{\mathbb{P}}^n}) \simeq H^{n-2}(X_1, \mathcal{O}_{X_1}) = 0$  where  $\hat{\mathbb{P}}^n$  is the formal completion of  $\mathbb{P}^n$  along the point  $P$ , which yields the desired fact by the formal duality.

As for the latter since  $\mathbb{Q}^n$  is covered with finitely many affine spaces, the latter is obtained.

q.e.d.

The above result immediately gives rise to

Corollary 3.24.1. The coherent sheaf  $M$  in 3.23 is a line bundle on  $R_Q$ .

Proof. If  $R_Q$  is smooth, we have nothing to prove. . Next

assume that  $R_Q$  has the isolated singularity  $v$ . Letting a canonical open immersion  $k: R_Q - \{v\} (= R^0) \longrightarrow R_Q$ , consider the following exact sequence:

$$0 \longrightarrow k_*k^*T_Q \longrightarrow k_*k^*f^*T_X|_{R_Q} \longrightarrow k_*k^*M \longrightarrow R^1k_*k^*T_Q.$$

Then Proposition 3.24 asserts this corollary.

q.e.d.

Thus in view of the above corollary, restricting the exact sequence 3.23 to  $R_Q$ , we get an exact sequence:

$$(3.24.2) \quad T_Q|_{R_Q} \xrightarrow{j'} f^*T_X|_{R_Q} \longrightarrow M \longrightarrow 0.$$

Letting  $\text{Ker } j' = L_1$  and  $\text{Cokernel of } j' = E$ , we obtain

$$(3.24.2) \quad 0 \longrightarrow L_1 \longrightarrow T_Q|_{R_Q} \longrightarrow E \longrightarrow 0, \text{ and}$$

$$(3.24.3) \quad 0 \longrightarrow E \longrightarrow f^*T_X|_{R_Q} \longrightarrow M \longrightarrow 0.$$

Hence, by Remark 3.22 we have a

Proposition 3.25. Under the condition 3.19,

$L_1$  in the sequence 3.24.2 is an line bundle and

$E$  a vector bundle of rank  $n-1$  on  $R_Q$ . Moreover  $M = L_1 \otimes \mathcal{O}_{R_Q}(1)$ .

To show that  $M \simeq \mathcal{O}_{R_Q}(2)$ , we make preparations.

Proposition 3.26. Under the above notations,  $f^{-1}f(R_Q) = R_Q$ .

Proof. First we have a

Claim. For a general curve  $\bar{C}$  on  $R_Q$ ,  $M|_{\bar{C}}$  is an ample line bundle.

Proof. Assume that  $f^{-1}f(R_Q) = R_Q \cup D \cup \dots$  with divisors  $D, \dots$ . Take a general curve  $C$  in  $f(R_Q)$ . Then there are two components  $C_1$  and  $C_2$  of  $f^{-1}(C)$  where  $C_1 \subset R_Q$  and  $C_2 (\subset D)$  is not a line,  $C_2 \not\subset R_Q$ . Since  $T_{Q|C_2}$  is ample by Theorem 2.1.1, so is  $f^*T_{X|C_2}$  (by Proposition 1.1) and therefore so is  $T_{X|C}$ . Thus since  $f^*T_{X|C_1}$  is ample, its quotient line bundle  $M|_{C_1}$  is ample.

To complete the proof of our proposition, we have only to show that (\$) there is a line  $L$  on  $Q$  so that  $f^*T_{X|L}$  is ample, (which yields a contradiction to Proposition 3.21)

Thus we divide into two cases:

- I)  $\dim Q \geq 4$  or  $\dim Q = 3$  and  $R_Q$  is singular..
- II)  $\dim Q = 3$  and  $R_Q$  is a smooth quadric surface ( $\simeq P^1 \times P^1$ ).

First consider the case I. Taking account of Remark 2.5.1, we infer that  $M$  is ample. Hence, Theorem 2.7 asserts the fact (\$).

Next consider the case II. If  $M$  is ample, Theorem 2.7 yields S.

In the second place assume that

$M$  is not ample. Then we see that  $M$  is GS. In fact since  $T_{Q|C_2}$

is GS, so is  $f^*T_{X|C_2}$ , hence so is  $T_{X|C}$  and  $M|C$ . Thus by the

claim we infer that  $M \simeq \mathcal{O}(d,0)$  with a positive integer  $d$  where  $p_i: R_Q \rightarrow P^1$  is the  $i$ -th projection and  $\mathcal{O}(a,b) = p_1^* \mathcal{O}_{P^1}(a) \otimes p_2^* \mathcal{O}_{P^1}(b)$ .

Then we can easily check that  $T_{Q|R_Q} \simeq T_{R_Q} \oplus N_{R_Q}/Q \simeq \mathcal{O}(2,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)$

By virtue of Proposition 3.25, we have the following sequence:

$$\begin{array}{ccccccc}
 (\#) & 0 & \longrightarrow & L_1 & \longrightarrow & T_{Q|R_Q} & \xrightarrow{h} E \longrightarrow 0, \\
 & & & (= \mathcal{O}(d-1,-1)) & & (= \mathcal{O}(2,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)) & \\
 (\#\#) & 0 & \longrightarrow & E & \longrightarrow & f^*T_{X|R_Q} & \longrightarrow M|_R (= \mathcal{O}(d,0)) \longrightarrow 0
 \end{array}$$

Now assume that the restriction homomorphism  $\bar{h}$  :

$\mathcal{O}(1,1) \oplus \mathcal{O}(2,0) \rightarrow E$  of  $h$  to the subbundle  $\mathcal{O}(1,1) \oplus \mathcal{O}(2,0)$  is generally surjective. Then the restriction  $\mathcal{O}(1) \oplus \mathcal{O}(2) \rightarrow E|_F$  of  $\mathcal{O}(1,1) \oplus \mathcal{O}(2,0) \rightarrow E$  to a general fiber  $F$  of  $P_2$  implies that  $E|_F$  is ample. Thus by  $(\#\#)$ , we see that  $f^*T_{X|F}$  is ample which gives \$.

Next assume that  $\bar{h}$  is not generally surjective. This give rise to an exact sequence:

$$0 \longrightarrow \mathcal{O}(d-1,-1) \longrightarrow \mathcal{O}(1,1) \oplus \mathcal{O}(2,0) \longrightarrow \mathcal{O}(4-d,2) \longrightarrow 0.$$

Immediately we have  $d = 1$  or  $2$ . Moreover

the above exact sequence gives a contradiction because of the fact  $H^1(R_Q, \mathcal{O}(a,b)) = 0$  for  $a \leq 0$  and  $b \leq 0$ . Therefore the case does not happen that  $M$  is not ample.

Thus we complete our proof.

q.e.d.

Since  $K_Q = f^*K_X + R_Q$ , the above proposition yields

Corollary 3.26.1.  $f^*O_X(R_X) = O_Q(2)$  with  $R_X = f(R_Q)$ .

Now before proving that  $f|_{R^\circ} : R^\circ \longrightarrow R_X$  is etale where  $R^\circ$  is the smooth part of  $R_Q$ , we consider

Proposition 3.27.  $R_X$  is smooth at each point of  $f(R^\circ)$ .

Moreover the inverse image of the smooth part of  $R_X$  is  $R^\circ$ .

Proof. Let  $A$  be a point in  $R^\circ$  and  $B = f(A)$  which yields a canonical local homomorphism  $\bar{f} : O_{X,B} \longrightarrow O_{Q,A}$ . Let  $x_n$  be an element in  $O_{Q,A}$  defining the divisor  $R$  (around the point  $A$ ). Then, by Proposition 3.23, we can

take a regular parameter  $(y_1, \dots, y_n)$  at the point  $B$  such that  $n$ -elements  $\bar{f}y_1, \dots, \bar{f}y_{n-1}, x_n$  generate  $m_A / m_A^2$  ( $m_A$  the maximal ideal of  $O_{Q,A}$ ), and therefore they are regular parameter of  $O_{Q,A}$ . Now let  $z = z(y_1, \dots, y_n)$  be the local equation of  $f(R^\circ)$  in  $X$  around the point  $B$ . Then from the fact that  $K_Q = f^*K_X + R_Q$  it follows that  $\frac{\partial y_n}{\partial x_n} = a x_n$  with a unit  $a$  in  $O_{Q,A}$ .

Moreover we have  $(\#) \bar{f} z = b x_n^2$  in  $O_{Q,A}$  with a unit  $b$ . To study the regularity of  $fR^\circ$  at the point  $B$ , we differentiate  $\#$  partially by  $x_n$ . Then we get

the left-hand side of  $\frac{\partial z}{\partial x_n} = \sum_i \frac{\partial z}{\partial y_i} \frac{\partial y_i}{\partial x_n} = \frac{\partial z}{\partial y_n} \frac{\partial y_n}{\partial x_n} = a x_n \frac{\partial z}{\partial y_n}$  and the right-hand side is  $= \frac{\partial b}{\partial x_n} x_n^2 + 2b x_n$ .

Thus we have  $\frac{\partial z}{\partial y_n} = (\frac{\partial b}{\partial x_n} x_n + 2b) / a$ , which is a unit at the point  $B$ . This is the desired fact. The latter is trivial.

q.e.d.

At last we have come to the final stage.

Letting  $\bar{f}$  the restriction map of  $f$  to  $R_Q$ , we have

Proposition 3.28. Under the above notations, we have  $\bar{f}^* \omega_{R_X} = \omega_{R_Q}$ . Consequently  $\bar{f} : R^0 \longrightarrow R_X$  is etale.

Proof. The former is trivial by the equality:  $K_Q = f^* K_X + \mathcal{O}_{R_Q}(2)$  and Corollary 3.26.1, which gives rise to the latter.

q.e.d.

Now letting  $j: R_Q \longrightarrow Q$  the closed embedding, the morphism  $jf: R^0 \longrightarrow X$  is unramified. Thus since we have a canonical exact sequence:  $0 \longrightarrow T_{R^0} \longrightarrow T_{Q|R^0} \longrightarrow N_{R^0/Q} (= \mathcal{O}_{R^0}(1)) \longrightarrow 0$ , with  $\mathcal{O}_{R^0}(1) = \mathcal{O}_{P^{n+1}}(1)|_{R^0}$  ( $R^0 \subset R_Q \subset P^{n+1}$ ), the exact sequences 3.24.1, 3.24.2 give rise to

Proposition 3.29.  $E_{|R^0} = T_{R^0}$  and  $L_1 = \mathcal{O}_{R_Q}(1)$ . Consequently  $M \simeq \mathcal{O}_{R_Q}(2)$ .

Proof. The former is obvious. The latter is obtained by Proposition 3.25.

q.e.d.

Thus the exact sequence 3.28 gives an important exact sequence

$$(3.30) \quad 0 \longrightarrow T_Q \longrightarrow f^* T_X \longrightarrow \mathcal{O}_{R_Q}(2) \longrightarrow 0$$

Theorem 2.7 asserts that there is a line  $L$  such that  $f^* T_X|_L$

is ample, which contradicts the fact of Corollary 3.21.1.

Hence we can show Main Theorem.

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# Logarithmic Transformations on Elliptic Fiber Spaces. II.

- elliptic bundle case -

By Yoshio Fujimoto

## § 0. Introduction.

In [11], Kodaira introduced the notion of *logarithmic transformation* and showed that any elliptic surface possessing multiple singular fibers can be reduced to an elliptic surface free from multiple fibers by means of logarithmic transformations.

Moreover he showed that a non - Kähler elliptic surface and a Kähler one can be changed into each other via logarithmic transformations.

Therefore, it is natural to ask whether a similar surgery exists on an elliptic fiber space. Ueno [13] [14] and the author [5][6] considered generalized logarithmic transformations along a non - singular divisor and constructed strange degenerations of surfaces.

In this paper, we shall study "*generalized logarithmic transformations on an elliptic bundle*".

Here, by an elliptic bundle  $f : Y \longrightarrow X$ , we mean that  $Y$  is a principal fiber bundle over a complex manifold  $X$  ( not necessarily compact) whose typical fiber and structure groups are non - singular elliptic curves. For an elliptic bundle  $f : Y \longrightarrow X$  over  $X$ , we shall define generalized logarithmic transformations along an arbitrary Cartier divisor  $D$  on  $X$ , which is *not necessarily assumed*

to be reduced, irreducible or effective.

Such an attempt was already done by Calabi and Eckmann [2], when they constructed complex structures on the product of two spheres of any odd dimensions. It is an elliptic bundle over the products of two complex projective spaces and cannot admit as an unramified covering a holomorphic  $\mathbb{C}^*$  - bundle, while the corresponding result holds for elliptic surfaces. The main purpose of this paper is to prove the counterpart of Kodaira's theorem for elliptic bundles and give explanations for these facts.

We state our main theorem.

**Theorem (A.)** Let  $f : Y \longrightarrow X$  be an elliptic bundle over a projective manifold  $X$ , which satisfies either of the following conditions.

(1)  $Y$  is Kähler.

or (2)  $h^{2,0}(X) = 0$ .

Then  $Y$  can be obtained from the trivial elliptic bundle over  $X$  by a succession of generalized logarithmic transformations in the sense of definition(1.1.).

To prove the above theorem, we need to characterize elliptic bundles which are Kähler. We show that an elliptic bundle  $(X \times E)^\gamma \longrightarrow X$  over  $X$  (, where  $E := \mathbb{C}/G$  is a non - singular elliptic curve and  $\gamma \in H^1(X, \mathcal{O}(E))$ ) is Kähler if and only if the first Chern class  $c(\gamma) \in H^2(X, G)$  of  $\gamma$  is of finite order. ( See Proposition(2.1.) below. )

However, theorem(A) does not necessarily hold if we drop the

assumption that  $Y$  is Kähler, as was shown by Moriwaki. ( See §3.)  
Furthermore, we can give the characterization of elliptic bundles  
which are obtained from the trivial bundle by performing generalized  
logarithmic transformations. (See proposition(1.9).)

Finally, let us explain briefly the contents of our paper.  
In § 1, we shall define generalized logarithmic transformations  
on an elliptic bundle and we give a characterization of elliptic  
bundles which are obtained from the trivial bundle by performing  
generalized logarithmic transformations.  
Moreover, we shall give a necessary and sufficient condition  
for an elliptic bundle to admit as an unramified covering a  
holomorphic  $\mathbb{C}^*$ - bundle. In § 2, we shall give a criterion of  
Kählerity for an elliptic bundle and prove theorem(A). In §3, we  
shall give an example of an elliptic bundle which cannot be obtained  
from the trivial bundle by performing logarithmic transformations,  
which was constructed by Moriwaki.

The author wishes to express his thanks to A.Moriwaki for useful  
suggestions.

### Notation and Convention.

By an elliptic fiber space  $g : V \longrightarrow W$ , we mean that  $g$  is a  
proper surjective morphism of a complex manifold  $V$  to a complex  
manifold  $W$ , where each fiber is connected and the general fibers are  
non - singular elliptic curves. By an elliptic bundle  $f : Y \longrightarrow X$ ,

we mean that  $Y$  is a principal fiber bundle over  $X$  whose typical fiber and structure groups are non - singular elliptic curves.

For a compact complex manifold  $X$ , we use the following notation.

$b_i(X)$  : the  $i$  - th Betti - number of  $X$ .

$\kappa(X)$  : the Kodaira dimension of  $X$ .

$K_X$  : the canonical bundle of  $X$ .

$h^{p,q}(X) = \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$ .

$q(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$ .

$\text{Pic}(X) := H^1(X, \mathcal{O}_X^*)$  : the Picard group of  $X$ , which has the natural structure of a commutative complex Lie group.

Let  $c_1 : H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z})$  be the first Chern class map.

$\text{Pic}^{\circ}(X) := \text{Ker}(c_1)$  : the identity component of  $\text{Pic}(X)$ .

$X$  is in the class  $\mathcal{E}$  in the sense of Fujiki if  $X$  is a meromorphic image of a compact Kähler manifold.

If  $D$  is a divisor on  $X$ , we set

$[D]$  : the line bundle on  $X$  determined by  $D$ .

$c_1([D])$  : the first Chern class of the line bundle  $[D]$ .

$\text{supp}(D)$  : the support of  $D$ .

For a non - singular elliptic curve  $E$ , we represent  $E$  as a quotient group :  $E = \mathbb{C}/G$ , where  $G$  is a discontinuous subgroup of the additive group  $\mathbb{C}$  generated by  $\tau$  and  $1$ ,  $\text{Im}(\tau) > 0$ , and for any  $\xi \in \mathbb{C}$ , we denote by  $[\xi]$  the corresponding element of  $E = \mathbb{C}/G$ .

$\mathcal{O}_X(E)$  : the sheaf over  $X$  of germs of holomorphic functions with values in  $E$ .

## § 1. Generalized Logarithmic Transformations on an Elliptic Bundle.

Let  $f : Y \longrightarrow X$  be an elliptic bundle over a complex manifold  $X$  (not necessarily compact) whose typical fiber and structure group is the elliptic curve  $E = \mathbb{C} / G$ ,  $G := \mathbb{Z} \oplus \mathbb{Z}\tau$ ,  $\text{Im}(\tau) > 0$ .

Then  $Y$  can be expressed in the form :  $Y = (X \times E)^\eta$  for some  $\eta \in H^1(X, \mathcal{O}(E))$ . (Here we follow the notation of Kodaira[11].) And let  $D$  be an arbitrary Cartier divisor on  $X$  and take  $\alpha \in G$  arbitrarily. Choose a sufficiently fine open covering  $\{U_i\}_{i \in I}$  of  $X$  such that  $s_i = 0$  is the local defining equation of  $D$  on  $U_i$ . The transition function of  $[D]$  is expressed by a cocycle  $\{f_{ij}\} (:= s_i/s_j)$  and  $\eta \in H^1(X, \mathcal{O}(E))$  is expressed by a cocycle  $\{\eta_{ij}\}$  with respect to the covering  $\{U_i\}$  of  $X$ .

Now, identify  $(z, [\eta_i]) \in U_i \times E$  with  $(z, [\eta_j]) \in U_j \times E$  if and only if  $[\eta_i] = [\eta_j + \frac{\alpha}{2\pi\sqrt{-1}} \log(f_{ij}) + \eta_{ij}]$ .

By patching  $U_i \times E$ 's in this way, we obtain a new elliptic bundle  $Z$  over  $X$ . Outside the support of  $D$ , there exists an isomorphism

$$\Lambda : \begin{array}{ccc} Z|_{X \setminus \text{supp}(D)} & \xrightarrow{\sim} & Y|_{X \setminus \text{supp}(D)} \\ \Downarrow & & \Downarrow \\ (z, [\eta_i]) & \longrightarrow & (z, [\eta_i - \frac{\alpha}{2\pi\sqrt{-1}} \log(s_i)]) \end{array}$$

In fact, on  $U_i \cap U_j$  we have :

$$[\eta_i - \frac{\alpha}{2\pi\sqrt{-1}} \log(s_i)] = [\eta_j - \frac{\alpha}{2\pi\sqrt{-1}} \log(s_j) + \eta_{ij}]$$

And the elliptic bundle over each irreducible component of  $D$  is determined by the restriction of the line bundle  $[D]$ .

Hence  $Z$  can be obtained from  $Y$  by performing surgeries along  $D$ .

Definition (1.1.) We write  $Z = L_D(\alpha)(Y)$  and call  $L_D(\alpha)$  a *generalized logarithmic transformation along D*.

Remark(1.2.) In the above definition,  $D$  is not necessarily assumed to be reduced, irreducible or effective. However by changing  $\alpha \in G$  suitably and performing successive logarithmic transformations,  $Z$  can also be written in the form:

$Z = L_{D_1}(\alpha_1) L_{D_2}(\alpha_2) \cdots L_{D_k}(\alpha_k)(Y)$ , where  $\alpha_i \in G$  and each  $D_i$  is an irreducible, reduced and effective divisor on  $X$ .

Remark (1.3.) Let  $H$  be a non - singular analytic arc in  $X$  which intersects with  $D$  transversally. Then when restricted to the elliptic surface over  $H$ , the above surgery just coincides with the original logarithmic transformations defined by Kodaira [11].

Remark (1.4.) By using generalized logarithmic transformations, Ueno[14] constructed non - Kähler degenerations of complex tori. Other examples of strange degenerations of surfaces can be found in Ueno[13] and Fujimoto[5].

Remark(1.5.) Calabi and Eckmann [2] constructed a complex structure  $M$  on  $S^{2m+1} \times S^{2n+1}$  with  $m \geq 1$  and  $n \geq 1$ , which is the simplest example of simply connected, compact homogeneous non - Kähler manifolds. It is an elliptic bundle over  $\mathbb{P}^m \times \mathbb{P}^n$  and can be written in the form :  $M = L_{H_1}(-1)L_{H_2}(-\tau)(\mathbb{P}^m \times \mathbb{P}^n \times E)$ ,  
 $H_1 = p_1^* \mathcal{O}_{\mathbb{P}^m}(1)$ ,  $H_2 = p_2^* \mathcal{O}_{\mathbb{P}^n}(1)$ , where  $p_1 : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^m$  (resp.  $p_2 : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^n$ ) is the projection to the first (resp. second) factor.

In order to prove theorem(A), we need the following proposition.

Proposition(1.6.) Let  $f : Y = (X \times E)^\eta \longrightarrow X$  be an elliptic bundle over a projective manifold  $X$  with  $E \simeq \mathbb{C}/(1, \tau)$ ,  $\text{Im}(\tau) > 0$  and  $\eta \in H^1(X, \mathcal{O}(E))$ . Assume that the first Chern class  $c(\eta) \in H^2(Y, \mathbb{G})$  of  $\eta$  is of finite order. Then  $Y$  can be obtained from the trivial elliptic bundle over  $X$  by a succession of generalized logarithmic transformations.

Proof. From the exact sequence  $0 \longrightarrow G \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(E) \longrightarrow 0$ , we have the following exact sequence of cohomology:

$$\longrightarrow H^1(X, G) \longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{p} H^1(X, \mathcal{O}_X(E)) \xrightarrow{c} H^2(X, G) \longrightarrow \dots$$

$\eta \in H^1(X, \mathcal{O}_X(E))$  is expressed by a cocycle  $\{\eta_{ij}\}$  with respect to the open covering  $\{U_i\}_{i \in I}$  of  $X$ . And  $Y$  is obtained by identifying  $(z, [\xi_i]) \in U_i \times E$  and  $(z, [\xi_j]) \in U_j \times E$  if and only if

$$[\xi_i] = [\xi_j + \eta_{ij}(z)].$$

Step 1. We first consider the case where  $c(\eta) = 0$ .

Then there exists  $\tilde{\eta} = \{\tilde{\eta}_{ij}(z)\} \in H^1(X, \mathcal{O}_X)$  with  $p(\tilde{\eta}) = \eta$ .

Since the structure of  $Y$  is uniquely determined by the cohomology class of  $\eta \in H^1(X, \mathcal{O}(E))$ , we may assume that  $[\tilde{\eta}_{ij}(z)] = \eta_{ij}(z)$ .

By identifying  $(z, \xi_i) \in U_i \times \mathbb{C}$  with  $(z, \xi_j) \in U_j \times \mathbb{C}$  if and only if  $\xi_i = \xi_j + \tilde{\eta}_{ij}(z)$ , we obtain an affine bundle  $W$  over  $X$ .  $\mathbb{Z}^2$  acts on  $W$  by

$$(k, \ell) : \begin{array}{ccc} W & \longrightarrow & W \\ \cup & & \cup \\ (z, \xi_i) & \longrightarrow & (z, \xi_i + k + \ell\tau) \end{array}$$

and  $Y$  is isomorphic to the quotient space  $Y/\mathbb{Z}^2$ .

Now, we have the following exact sequence:

$$\longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{e} H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}) .$$

If we put  $L := e(\tilde{\eta})$ , we have  $L \in \text{Pic}^\circ(X)$  and the transition function of  $L$  can be expressed by a cocycle  $\{ \exp(2\pi\sqrt{-1} \tilde{\eta}_{ij}(z)) \}$  with respect to the covering  $\{U_i\}$  of  $X$ .

Let  $\mathbb{L}$  be the total space of the line bundle  $L$  and define an automorphism  $g$  of  $\mathbb{L}^* := \mathbb{L} \setminus \{ \text{zero - section} \}$  by

$$g : \begin{array}{ccc} \mathbb{L}^* & \longrightarrow & \mathbb{L}^* \\ \cup & & \cup \\ (z, \eta_i) & \longrightarrow & (z, \rho\eta_i) \end{array} ,$$

where we put  $\rho = \exp(2\pi\sqrt{-1} \tau)$  ( $|\rho| < 1$ ) and  $\mathbb{L}|_{U_i}$  has local coordinates  $(z, \eta_i)$ . The group  $\langle g \rangle$  acts on  $\mathbb{L}^*$  freely and properly discontinuously, so the quotient space  $Z := \mathbb{L}^*/\langle g \rangle$  is non-singular.

$$\text{There is a natural morphism } \Phi : \begin{array}{ccc} Z & \longrightarrow & X \\ \cup & & \cup \\ \overline{(z, \eta_i)} & \longrightarrow & z \end{array} ,$$

where by  $\overline{(z, \eta_i)}$ , we denote the point of  $Z$  corresponding to a point  $(z, \eta_i) \in \mathbb{L}^*$ . By this morphism,  $Z$  is an elliptic bundle over  $X$  and

$$\text{we have an isomorphism } \Lambda : \begin{array}{ccc} Y & \xrightarrow{\sim} & Z \\ \cup & & \cup \\ (z, [\xi_i]) & \longrightarrow & \overline{(z, \exp(2\pi\sqrt{-1}\xi_i))} \end{array} .$$

Since  $X$  is projective, there exists a Cartier divisor  $D$  on  $X$  with  $L \simeq \mathcal{O}_X(D)$ . Hence by the above isomorphism  $\Lambda$  and definition(1.1.), we have  $Y \simeq L_D(1)(X \times E)$ , that is,  $Y$  can be obtained from  $X \times E$  by means of generalized logarithmic transformations along  $D$ .

Step 2. Next, we consider the general case.

There is a natural isomorphism  $G \simeq \mathbb{Z} \oplus \mathbb{Z}$  and  $H^2(X, G) \simeq H^2(X, \mathbb{Z}) \oplus$

$H^2(X, \mathbb{Z})$ . The first Chern class  $c(\eta) \in H^2(X, G)$  of  $\eta$  can be expressed by a cocycle  $\{c_{ijk}\}$ ,  $c_{ijk} = p_{ijk} + \tau \cdot q_{ijk}$ ,  $p_{ijk}, q_{ijk} \in \mathbb{Z}$ , with respect to the covering  $\{U_i\}$  of  $X$ .

Then  $c' := \{p_{ijk}\} \in H^2(X, \mathbb{Z})$  and  $c'' := \{q_{ijk}\} \in H^2(X, \mathbb{Z})$  are of finite order, since  $c(\eta) \in H^2(X, G)$  is of finite order.

Hence there exists Cartier divisors  $D_1, D_2$  on  $X$  with  $c_1([D_1]) = c'$ ,  $c_1([D_2]) = c''$ . By performing generalized logarithmic transformations along  $D_1$  and  $D_2$ , we obtain a new elliptic bundle

$Y' := L_{D_1}(-1)L_{D_2}(-\tau)(Y)$  over  $X$ . If we put  $Y' = (X \times E)^{\eta'}$ ,  $\eta' \in H^1(X, \mathcal{O}(E))$ , we have  $c(\eta') = 0$ . Hence from step 1,  $Y'$  can also be obtained from a trivial elliptic bundle by successive logarithmic transformations. Since  $Y = L_{D_1}(1)L_{D_2}(\tau)(Y')$ , the same is clearly true for the original  $Y$ . q.e.d.

Remark(1.7.) The assumption that  $c(\eta)$  is of finite order means that  $Y$  is Kähler. This will be shown in § 2.

Remark(1.8.) In case  $c(\eta) = 0$ , the elliptic bundle  $f : Y \longrightarrow X$  can be defined by locally constant transition functions.

By the same argument as in the proof of proposition(1.6.), we get the following proposition.

Proposition(1.9.) Let  $f: Y = (X \times E)^{\eta} \longrightarrow X$  be an elliptic bundle over a projective manifold  $X$  with  $E \simeq \mathbb{C}/(1, \tau)$ ,  $\text{Im}(\tau) > 0$  and  $\eta \in H^1(X, \mathcal{O}(E))$ . Then  $Y$  can be obtained from the trivial bundle over  $X$  by generalized logarithmic transformations if and only if the following condition is satisfied.

Condition(A). Via the natural isomorphism

$$\Lambda : H^2(X, G) \simeq H^2(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}), \quad c', c'' \in H^2(X, \mathbb{Z}) \text{ are both}$$

$$\begin{array}{ccc} \Psi & & \Psi \\ c(\eta) & \longrightarrow & (c', c'') \end{array}$$

algebraic cycles.

Corollary(1.10.) Let  $f: Y \longrightarrow X$  be an elliptic bundle over a projective manifold  $X$  with  $h^{2,0}(X) = 0$ . Then  $Y$  can be obtained from the trivial bundle over  $X$  by means of logarithmic transformations.

Proof. Since  $h^{2,0}(X) = 0$ ,  $H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z})$  is surjective and the claim follows immediately from proposition(1.9.).

Example(1.11.) The following example is due to M.Kato [8], p.55,

example 2. Let  $\ell = \{(z_0: z_1: z_2: z_3) \in \mathbb{P}^3 \mid z_0 = z_1 = 0\}$ ,  
 $\ell_\infty = \{(z_0: z_1: z_2: z_3) \in \mathbb{P}^3 \mid z_2 = z_3 = 0\}$  and put  $W := \mathbb{P}^3 \setminus \ell \setminus \ell_\infty$ .

Fix a constant  $\alpha \in \mathbb{C}$  with  $0 < |\alpha| < 1$  and define an automorphism  $g$  of  $W$  by :  $g : (z_0: z_1: z_2: z_3) \longrightarrow (z_0: z_1: \alpha z_2: \alpha z_3)$ .

Then the quotient space  $M := W / \langle g \rangle$  is a compact complex manifold of class  $L$ . By a natural holomorphic map  $p : M \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  sending  $\overline{(z_0: z_1: z_2: z_3)}$  to  $((z_0: z_1), (z_2: z_3))$ ,  $M$  is an elliptic bundle over  $\mathbb{P}^1 \times \mathbb{P}^1$ , where by  $\overline{(z_0: z_1: z_2: z_3)}$ , we denote the point of  $M$  corresponding to a point  $(z_0: z_1: z_2: z_3) \in W$ .

Since  $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(E)) \simeq H^2(\mathbb{P}^1 \times \mathbb{P}^1, G) \simeq G \oplus G$ ,  $M$  can be written in the form :  $M = L_{H_1}(-1) L_{H_2}(1) (\mathbb{P}^1 \times \mathbb{P}^1 \times E)$ ,  $H_i \simeq p_i^* \mathcal{O}_{\mathbb{P}^1}(1)$ , where  $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$  is the projection to the  $i$ -th factor. It is easy to see that an elliptic surface over the curve of type (1,1) on  $\mathbb{P}^1 \times \mathbb{P}^1$  is trivial and contains many rational curves.

Proposition (1.12.) Let  $Y = L_{D_1}(1) L_{D_2}(\tau)(X \times E)$  be an elliptic bundle over a projective manifold  $X$ , where  $E \simeq \mathbb{C}/(1, \tau)$ ,  $\text{Im}(\tau) > 0$  and  $D_1, D_2$  are Cartier divisors on  $X$ . Then  $Y$  is isomorphic to the  $\mathbb{Z}$ -quotient of some  $\mathbb{C}^*$ -bundle over  $X$  if and only if there exists a Cartier divisor  $H$  on  $X$  with  $c_1([D_1]) = \alpha \cdot c_1([H])$ ,  $c_1([D_2]) = \beta \cdot c_1([H])$  for some  $\alpha, \beta \in \mathbb{Z}$ .

Proof. (*necessity*) Assume that  $Y$  is isomorphic to the  $\mathbb{Z}$ -quotient  $Y'$  of some  $\mathbb{C}^*$ -bundle on  $X$ . Then  $Y'$  can be written in the form:  $Y' = L_D(1)(X \times E')$ , where  $E' \simeq \mathbb{C}/(1, \tau')$ ,  $\text{Im}(\tau') > 0$  and  $D$  is a Cartier divisor on  $X$ . Take an open covering  $\{U_i\}$  of  $X$  such that  $Y|_{U_i} \simeq U_i \times E$ ,  $Y'|_{U_i} \simeq U_i \times E'$  and  $Y|_{U_i}$  (resp.  $Y'|_{U_i}$ ) has coordinates  $(z, [\xi_i])$  (resp.  $(z, [\eta_i])$ ). On  $U_i \cap U_j$ , we have

$$(1) \quad [\xi_i] = \left[ \xi_j + \frac{1}{2\pi\sqrt{-1}} \log(f_{ij}) + \frac{\tau}{2\pi\sqrt{-1}} \log(g_{ij}) \right]$$

$$(2) \quad [\eta_i] = \left[ \eta_j + \frac{1}{2\pi\sqrt{-1}} \log(h_{ij}) \right],$$

where  $\{f_{ij}\}$  (resp.  $\{g_{ij}\}, \{h_{ij}\}$ ) is the transition function of the line bundle  $[D_1]$  (resp.  $[D_2], [D]$ ). Since the bundle structures of  $Y$  and  $Y'$  are uniquely determined by its cohomology class, we may assume that the above isomorphism over  $X$  can be written as follows.

$$(3) \quad [\xi_i] = \left[ \frac{\eta_i}{c\tau' + d} \right], \quad \tau = \frac{a\tau' + b}{c\tau' + d} \quad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).$$

From (2) and (3), we have  $[\xi_i] = \left[ \xi_j + \frac{-c\tau + a}{2\pi\sqrt{-1}} \log(h_{ij}) \right]$ ,

since  $\frac{1}{c\tau' + d} = -c\tau + a$ . In view of (1) and the natural

isomorphism  $H^2(X, G) \simeq H^2(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$ , we have

$$c_1([D_1]) = a \cdot c_1([H]), \quad c_1([D_2]) = -c \cdot c_1([H]), \quad a, c \in \mathbb{Z}.$$

( *sufficiency* ) We choose integers  $\alpha', \beta', c, d$  and  $k$  such that  $c\alpha' - \beta'd = 1, \alpha = k\alpha', \beta = k\beta'$  and introduce a new fiber coordinate  $\xi' = \frac{\xi}{\alpha' + \beta'\tau}$  on  $U_i \times \mathbb{C}$ . Correspondingly we replace  $\tau$  by  $\tau' = \frac{c\tau + d}{\beta'\tau + \alpha'}$  and put  $G' \simeq \mathbb{Z} \oplus \mathbb{Z}\tau', E' \simeq \mathbb{C}/G'$ . Then the formula (1) takes the form

$$(2) \quad [\xi'_i] = [\xi'_j + \eta'_{ij}],$$

where  $[\eta'_{ij}] = [ \frac{1}{\beta'\tau + \alpha'} \log(f_{ij}) + \frac{\tau}{2\pi\sqrt{-1}} \log(g_{ij}) ]$ ,  
 $\eta' = \{ \eta'_{ij} \} \in H^1(X, \mathcal{O}(E'))$ .

Hence  $Y$  is isomorphic to the elliptic bundle  $(X \times E')^{\eta'}$  and the first Chern class  $c(\eta') \in H^2(X, G')$  of  $\eta'$  is as follows.

$$\begin{aligned} c(\eta') &= \frac{1}{\alpha' + \beta'\tau} c_1([D_1]) + \frac{\tau}{\alpha' + \beta'\tau} c_1([D_2]) \\ &= \frac{\alpha + \beta\tau}{\alpha' + \beta'\tau} c_1([H]) \\ &= k \cdot c_1([H]), \quad k \in \mathbb{Z}. \end{aligned}$$

Therefore, by the proof of Proposition (1.6.), step 1, there exists a Cartier divisor  $D$  on  $X$  such that  $Y \simeq L_H(k)L_D(1)(X \times E'), k \in \mathbb{Z}$ . Hence if we put  $L = [kH + D]$ ,  $Y$  is isomorphic to the  $\mathbb{Z}$ -quotient of the  $\mathbb{C}^*$ -bundle associated to  $L$ . q.e.d.

Corollary (1.13.) Every elliptic bundle over a curve admits as its unramified covering a holomorphic  $\mathbb{C}^*$ -bundle, while the corresponding result does not hold for Calabi - Eckmann manifolds. (c.f. Remark(1.5.))

Proposition(1.14.) Let  $X$  be a complex manifold ( which are not

necessarily compact) and  $X^\circ \subset X$  be a Zariski open subset with  $\text{codim}(X \setminus X^\circ) \geq 3$ . Suppose that there exists an elliptic bundle  $f^\circ: Y^\circ \longrightarrow X^\circ$  over  $X^\circ$ . Then there exists a relative compactification  $f: Y \longrightarrow X$ .

Proof. From the exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(E) \longrightarrow 0,$$

we have the following commutative diagram.

$$\begin{array}{ccccccc} \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{O}_X(E)) & \longrightarrow & H^2(X, G) & \longrightarrow & H^2(X, \mathcal{O}_X) \\ & \downarrow & & \downarrow & & \parallel & & \downarrow \\ \longrightarrow & H^1(X^\circ, \mathcal{O}_{X^\circ}) & \longrightarrow & H^1(X^\circ, \mathcal{O}_{X^\circ}(E)) & \longrightarrow & H^2(X^\circ, G) & \longrightarrow & H^2(X^\circ, \mathcal{O}_{X^\circ}) \end{array}$$

Since  $\text{codim}(X \setminus X^\circ) \geq 3$ , by the theorem of Siu - Trautmann, we have an isomorphism  $H^1(X, \mathcal{O}_X) \cong H^1(X^\circ, \mathcal{O}_{X^\circ})$  and the restriction map  $H^2(X, \mathcal{O}_X) \longrightarrow H^2(X^\circ, \mathcal{O}_{X^\circ})$  is injective. Hence from the above diagram, the restriction map  $r: H^1(X, \mathcal{O}_X(E)) \longrightarrow H^1(X^\circ, \mathcal{O}_{X^\circ}(E))$  is surjective. If we put  $Y^\circ = (X^\circ \times E)^{\gamma^\circ}$ ,  $\gamma^\circ \in H^1(X^\circ, \mathcal{O}_{X^\circ}(E))$ , there exists  $\gamma \in H^1(Y, \mathcal{O}_Y(E))$  with  $r(\gamma) = \gamma^\circ$ . Then  $X := (X \times E)^\gamma \longrightarrow X$  is a relative compactification of  $f^\circ: Y^\circ \longrightarrow X^\circ$ . q.e.d.

The above proposition is not necessarily true if  $\text{codim}(Y \setminus Y^\circ) = 2$ . The following example is due to K.Ueno.

Example (1.15.) Let  $X = \mathbb{C}^2 \setminus 0$  and  $U_0 = \{(x, y) \in \mathbb{C}^2 \mid x \neq 0\}$ ,  $U_1 = \{(x, y) \in \mathbb{C}^2 \mid y \neq 0\}$  be open coverings of  $X$ .

By patching  $(x, [\xi]) \in U_0 \times E$  and  $(y, [\xi']) \in U_1 \times E$  if and only if  $[\xi'] = [\xi + \frac{1}{xy}]$ , we obtain an elliptic bundle  $Y$  over  $X$ .

Since  $\left\{\frac{1}{xy}\right\} \in H^1(X, \mathcal{O}) \simeq H^1(X, \mathcal{O}(E))$  and  $H^1(\mathbb{C}^2, \mathcal{O}(E)) = 0$ , the restriction map  $H^1(\mathbb{C}^2, \mathcal{O}(E)) \longrightarrow H^1(X, \mathcal{O}(E))$  is not surjective. It is not known whether it has a relative compactification or a Kähler metric.

## § 2. A Criterion of Kählerity for an elliptic bundle and a proof of Main Theorem(A).

In this section, we shall give a necessary and sufficient condition for an elliptic bundle to be Kähler and prove main theorem(A). The following theorem is fundamental.

Theorem (2.1.) (Fujiki [3], Proposition(4.3.), Lemma(4.4.))

Let  $f : X \longrightarrow Y$  be a fiber space of compact complex manifolds in the class  $\mathcal{E}$ . Let  $U \subset Y$  be a Zariski open subset over which  $f$  is smooth and  $f_U : X_U \longrightarrow U$  be the restriction of  $f$  to  $X_U$ .

If  $R^1 f_{U*} \mathbb{C}$  is a constant system, then  $q(X) = q(Y) + q(f)$ , where  $q(f)$  denotes the irregularity of the general fiber of  $f$ .

Proposition(2.2.) Let  $f : Y := (X \times E)^\eta \longrightarrow X$  be an elliptic bundle over a projective manifold  $X$  with  $E \simeq \mathbb{C}/G$ ,  $G \simeq \mathbb{Z} \oplus Z\tau$ ,  $\text{Im}(\tau) > 0$  and  $\eta \in H^1(X, \mathcal{O}(E))$ .

Then the following conditions are equivalent.

- (1)  $Y$  is Kähler.
- (2)  $Y$  is in the class  $\mathcal{E}$ .
- (3)  $b_1(Y) = b_1(X) + 2$ .

(4) The first Chern class  $c(\eta) \in H^2(X, G)$  of  $\eta$  is of finite order  $m$  in  $H^2(X, G)$ .

Moreover, if  $q(X) = 0$ , (1) is also equivalent to the following condition.

(5)  $Y$  is projective.

Proof. (1)  $\implies$  (2). Obvious.

(2)  $\implies$  (3). This follows immediately from theorem(2.1.).

(3)  $\implies$  (4). Assume the contrary.

Then we have  $\alpha(c(\eta)) \neq 0$ , where  $\alpha : H^2(X, G) \longrightarrow H^2(X, \mathbb{C})$  is the natural homomorphism. Hence by the same method as in the proof of [11], III, theorem(11.9.), we have  $b_1(Y) \leq b_1(X) + 1$ .

This is a contradiction.

(4)  $\implies$  (1). We follow the idea of Miyaoka [12].

Since the Kählerity is preserved under a finite, étale morphism, it suffices to prove that  $(X \times E)^{m\eta}$  is Kähler. Replacing  $\eta$  by  $m\eta$ , we may assume that  $c(\eta) = 0$ . Hence from the exact sequence

$$(*) \quad \begin{array}{ccccccc} H^1(X, G) & \xrightarrow{i^*} & H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{O}_X(E)) & \xrightarrow{c} & \\ H^2(X, G) & \longrightarrow & H^2(X, \mathcal{O}_X) & , & & & \end{array}$$

we have  $\eta \in H^1(X, \mathcal{O}_X) / i^*H^1(X, G)$ .

Since  $X$  is projective, we have

$$j^*H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \simeq H^1(X, \mathcal{O}_X), \text{ where } j^* : H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X)$$

is the natural homomorphism. Moreover, from the natural isomorphism  $H^1(X, G) \simeq H^1(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z})$ , it follows that

$$i^*H^1(X, G) \otimes_{\mathbb{Z}} \mathbb{R} \simeq H^1(X, \mathcal{O}_X). \text{ Then from the sequence } \{k\eta\}_{k=1,2,\dots}$$

we can take a subsequence  $\{\eta_k\}_{k=1,2,\dots}$ , which converges to  $\eta_0$

with a finite order. Hence for a suitable integer  $m$ ,  $B^{m\eta}$  is a small deformation of  $B^{\eta_0}$ , which is projective.

This implies that  $B^{m\eta}$  and hence  $B^\eta$  are Kähler.

(1)  $\implies$  (5). Moreover, if  $q(X) = 0$ , then the homomorphism  $c : H^1(X, \mathcal{O}(E)) \longrightarrow H^2(X, G)$  is injective and (4) implies that  $\eta \in H^1(X, \mathcal{O}(E))$  is of finite order. Hence  $Y$  is projective.

q.e.d.

Remark(2.3.) In general,  $H^2(X, G)$  has torsions, so the homomorphism  $\alpha : H^2(X, G) \longrightarrow H^2(X, \mathbb{C})$  is not necessarily injective.

The following corollary follows immediately from Proposition(2.2.).

Corollary(2.4.) Let  $f : Y = L_{D_1}(\alpha_1) \cdots L_{D_k}(\alpha_k)(X \times E) \longrightarrow X$

be an elliptic bundle over a projective manifold  $X$ , where

$\alpha_i = m_i + n_i \tau \in G$ , and  $D_1, \dots, D_k$  are Cartier divisors on  $X$ , (which are not necessarily reduced, irreducible or effective.) Then  $Y$  is

Kähler if and only if (\*)  $c_1([\sum_{i=1}^k m_i D_i])$ ,  $c_1([\sum_{i=1}^k n_i D_i]) \in H^2(X, \mathbb{Z})$

are of finite order.

Remark(2.5.) If  $q(X) > 0$ , (\*) does not necessarily imply that  $Y$  is projective. There exists a two dimensional complex torus of algebraic dimension one. An algebraic reduction gives it the structure of an elliptic bundle over an elliptic curve, which can be defined by locally constant transition functions.

Corollary(2.6.) (c.f. Katsura and Ueno [9], Appendix 1)

The elliptic surface  $f : S = L_{P_1}(m_1, \alpha_1) \cdots L_{P_\lambda}(m_\lambda, \alpha_\lambda)(C \times E) \longrightarrow C$

over a compact curve  $C$  is Kähler if and only if  $\sum_{i=1}^{\lambda} \alpha_i = 0$ .

Moreover if  $C \simeq \mathbb{P}^1$ , this implies that  $S$  is projective.

Proof. Let  $m$  be the least common multiple of  $m_i$ 's ( $1 \leq i \leq \lambda$ ). The multiplication map  $m : E \rightarrow E$  induces a finite surjective morphism  $\Phi : S \rightarrow Y$ , where  $Y = L_{p_1}(1, m\alpha_1) \cdots L_{p_\lambda}(1, m\alpha_\lambda)(C \times E)$  is an elliptic bundle. Hence  $S$  is Kähler if and only if  $Y$  is Kähler. Corollary(2.4.) implies that  $Y$  is Kähler if and only if  $\sum_{i=1}^{\lambda} \alpha_i = 0$ . Hence the claim follows. q.e.d.

Proposition(2.7.) Let  $f : Y \rightarrow X$  be an elliptic bundle over a projective manifold  $X$  with  $\dim(X) \geq 3$ . Assume that for any  $p$ -dimensional submanifold  $Z$  on  $X$  ( $p \geq 2$  is a fixed positive integer), the elliptic bundle  $Y|_Z \rightarrow Z$  over  $Z$  is Kähler. Then  $Y$  is also Kähler.

Proof. There exists  $\eta \in H^1(X, \mathcal{O}(E))$  with  $Y \simeq (X \times E)^\eta$ ,  $E \simeq \mathbb{C}/G$ ,  $G \simeq \mathbb{Z} \oplus \mathbb{Z}\tau$ ,  $\text{Im}(\tau) > 0$ . Via the natural isomorphism  $H^2(X, G) \simeq H^2(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$ , the first Chern class  $c(\eta) \in H^2(X, G)$  of  $\eta$  can be expressed as  $c(\eta) = (c', c'')$ , where  $c', c'' \in H^2(X, \mathbb{Z})$ . Let  $V$  be a smooth hyperplane section of  $X$ . Then by Lefschetz's hyperplane section theorem, the map  $H^2(X, \mathbb{Q}) \rightarrow H^2(V, \mathbb{Q})$  induced by the inclusion  $i : V \hookrightarrow X$  is an isomorphism if  $\dim(X) \geq 4$  and injective if  $\dim(X) = 3$ . Continuing this process successively, we find a non-singular surface  $S \subset X$  such that the natural homomorphism  $H^2(X, \mathbb{Q}) \rightarrow H^2(S, \mathbb{Q})$  is injective. Since  $Y|_S$  is Kähler, proposition(2.2.) implies that  $c'|_S, c''|_S \in H^2(S, \mathbb{Z})$  are of finite order. Hence  $c', c'' \in H^2(X, \mathbb{Z})$  are of finite order. Therefore, by proposition(2.2.),  $Y$  is Kähler. q.e.d.

Remark(2.8.) If we drop the assumption that  $p \geq 2$ , the above

proposition does not necessarily hold. (See Remark(3.3.))

Proof of theorem(A). This follows immediately from proposition (1.6.), corollary (1.10.) and proposition (2.2.).

Corollary (B.) Under the same assumptions as in theorem(A), there exists a Zariski open subset  $U$  of  $X$  such that the restriction  $Y|_U \rightarrow U$  over  $U$  is a trivial elliptic bundle.

Example(2.9.) Let  $f : Y = (X \times E)^\eta \rightarrow X$  be an elliptic bundle over an  $n$ -dimensional projective manifold  $X$ . The first Chern class  $c(\eta) \in H^2(X, G)$  can be expressed as  $c(\eta) = (c', c'') \in H^2(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$ . Put  $N_{\mathbb{Q}}(X) = (\{\text{one-cycles on } X\} / \equiv) \otimes_{\mathbb{Z}} \mathbb{Q}$ , where  $\equiv$  denotes the numerically equivalence class. Via the intersection pairing  $(\cdot, \cdot)$  of one-cycles and Cartier divisors,  $N_{\mathbb{Q}}(X)$  is dual to  $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ , where  $NS(X)$  denotes the Neron-Severi group of  $X$ . Then via the cup-product  $\langle \cdot, \cdot \rangle : H^2(X, \mathbb{Z}) \otimes H^{2n-2}(X, \mathbb{Z}) \rightarrow \mathbb{Z}$ ,  $-c'$  (resp.  $-c''$ ) is a linear functional on  $N_{\mathbb{Q}}(X)$  and can be identified with an element  $D'$  (resp.  $D''$ ) of  $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Namely,  
 (\*\*)  $\langle x, -c' \rangle = (x, D')$  (resp.  $\langle x, -c'' \rangle = (x, D'')$ ) for  
 for all  $x \in N_{\mathbb{Q}}(X)$ .

Choose a positive integer  $r$  such that  $D_1 = rD'$  and  $D_2 = rD''$  are integral divisors and put  $W := L_{D_1}(1)L_{D_2}(\tau)(X \times E)^{r\eta}$ . Then by proposition (2.2.), the elliptic bundle  $W|_C \rightarrow C$  is kähler for all non-singular curve  $C \subset X$ .

### § 3. Counterexamples.

In this section, we shall construct an elliptic bundle  $f : Y \longrightarrow X$  over a projective manifold  $X$  which cannot be obtained from the trivial bundle over  $X$  by generalized logarithmic transformations.

We first need the following lemma.

Let  $E \simeq \mathbb{C}/(1, \tau)$ ,  $\text{Im}(\tau) \neq 0$  be a non-singular elliptic curve and put  $G := \mathbb{Z} \oplus \mathbb{Z}\tau$ . There is the following exact sequence :

$$\longrightarrow H^1(X, \mathcal{O}(E)) \xrightarrow{c} H^2(X, G) \xrightarrow{q} H^2(X, \mathcal{O}_X) \longrightarrow$$

Lemma(3.1.) For  $\alpha, \beta \in H^2(X, \mathbb{Z})$ , we have  $\alpha + \beta\tau \in \text{Im}(c)$  if and only if  $(*) \alpha_{(2,0)} + \beta_{(2,0)}\bar{\tau} = 0$ , where  $\alpha_{(2,0)}$  (resp.  $\beta_{(2,0)}$ ) means the  $(2, 0)$ -component of  $\alpha$  (resp.  $\beta$ ) under the Hodge decomposition of  $H^2(X, \mathbb{C})$ .

Proof. We have the following commutative diagram.

$$\begin{array}{ccccc} \longrightarrow H^1(X, \mathcal{O}(E)) & \xrightarrow{c} & H^2(X, G) & \xrightarrow{p} & H^2(X, \mathcal{O}_X) \simeq H^{0,2} \\ & & \downarrow j & \nearrow & \uparrow \varphi \\ & & H^2(X, \mathbb{C}) \simeq H^{2,0} \oplus H^{1,1} \oplus H^{0,2} & & \end{array}$$

where  $\varphi : H^2(X, \mathbb{C}) \longrightarrow H^2(X, \mathcal{O}_X)$  is the projection to the  $(0, 2)$ -component. Hence we have  $\alpha + \beta\tau \in \text{Im}(c) = \ker(p)$  if and only if  $p(\alpha + \beta\tau) = \alpha_{(0,2)} + \beta_{(0,2)}\tau = 0$ . By taking complex conjugates, it is also equivalent to  $\alpha_{(2,0)} + \beta_{(2,0)}\bar{\tau} = 0$ . q.e.d.

The following example is due to A.Moriwaki.

Example(3.2.) (Moriwaki) Let  $X$  be a projective manifold which

enjoys the following conditions.

(1)  $h^{2,0}(X) = 1$ .

(2) Let  $\omega$  be a holomorphic 2 - form on  $X$ . Then there exists an automorphism  $g$  of  $X$  with  $g^*\omega = t\omega$ ,  $t \in \mathbb{C}$ ,  $t \notin \mathbb{R}$ .

( For example, put  $X = E_\rho \times E_\rho$ ,  $E_\rho \simeq \mathbb{C}/(1, \rho)$ ,  $\rho = \exp(2\pi i/3)$  and  $g : (x, y) \longrightarrow (\rho x, \rho y)$ . )

Next, choose a transcendental cycle  $\beta \in H^2(X, \mathbb{Z})$  arbitrarily and put  $\alpha := g^*\beta \in H^2(X, \mathbb{Z})$ . Since the action of  $\langle g \rangle$  on  $H^2(X, \mathbb{Z})$  is compatible with the Hodge decomposition, we have

$\alpha_{(2,0)} = t\beta_{(2,0)}$ . Put  $\tau := -\bar{t} \notin \mathbb{R}$  and  $E \simeq \mathbb{C}/(1, \tau)$ . Then by lemma (3.1.), there exists  $\eta \in H^1(X, \mathcal{O}(E))$  with  $c(\eta) = \alpha + \beta\tau \in H^2(X, G)$ .

Put  $Y := (X \times E)^\eta$ . From our construction, the first Chern class  $c(\eta) \in H^2(X, G)$  does not satisfy condition(A) in proposition(1.9.). Hence  $Y$  cannot be obtained from the trivial bundle over  $X$  by generalized logarithmic transformations.

Remark(3.3.) By theorem(A), example (2.9.) and example (3.2.), there exists a *non - Kähler* elliptic bundle  $f : Y \longrightarrow X$  over a projective manifold  $X$  such that  $Y|_C \longrightarrow C$  is Kähler for all non - singular curve  $C \subset X$ .

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A note on an example of a compactification of  $\mathbb{C}^3$

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§0. Introduction. Let  $(X, Y)$  be a (projective) compactification of  $\mathbb{C}^3$  with the second Betti number  $b_2(X) = 1$ . Then  $Y$  is an irreducible ample divisor on  $X$  and the canonical divisor is written as  $K_X = -rY$  ( $r > 0$ ,  $r \in \mathbb{Z}$ ). In the case of  $r = 2$ , the structure of  $(X, Y)$  is completely determined (cf. [1], [2], [3], [9]). In the case of  $r = 1$ ,  $Y$  must be non-normal (cf. [3]). Moreover, in this case, by Peternell-Schneider [9],  $X$  is a Fano 3-fold of first kind of index one, genus 12 and  $Y$  is a non-normal hyperplane section of  $X$ . In the paper [3], the author proved that such a compactification  $(X, Y)$  really exists. In fact,  $X = V'_{22}$  is a Fano 3-fold constructed by Mukai-Umemura [7], and  $Y = H'_{22}$  is a non-normal hyperplane section of  $V'_{22}$  whose singular locus is a line in  $V'_{22}$ .

Now, in this paper, we will study the double projection of  $V'_{22}$  from the singular locus of  $H'_{22}$ , which is a line, and give a geometric structure of the compactification  $(V'_{22}, H'_{22})$  of  $\mathbb{C}^3$  (Theorem in §2).

§1. Mukai - Umemura's construction.

Let  $\mathbb{C}[x,y]$  be the polynomial ring of two complex variables  $x$  and  $y$ .  $SL(2,\mathbb{C})$  operates on  $\mathbb{C}[x,y]$  as follows:

$$\begin{cases} x^\sigma = ax + by \\ y^\sigma = cx + dy \end{cases}$$

for  $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ .

Let us denote by  $R_n$  the vector space of the homogeneous polynomials of degree  $n$ . Let  $f(x,y) = \sum_{i=0}^n a_i \binom{n}{i} x^{n-i} y^i \in R_n$  be a non-zero homogeneous polynomial of degree  $n$ . We take  $(a_0 : \dots : a_n)$  as homogeneous coordinates on the projective space  $\mathbb{P}(R_n) \cong \mathbb{P}^n$  on which  $SL(2,\mathbb{C})$  operates. Let us denote by  $X(f)$  the closure  $\overline{SL(2,\mathbb{C}) \cdot f}$  of the  $SL(2,\mathbb{C})$ -orbit  $SL(2,\mathbb{C}) \cdot f$  of  $f$  in  $\mathbb{P}(R_n)$ . Then  $SL(2,\mathbb{C})$  operates on  $X(f)$ . Now, let us consider the following two polynomials:

$$\begin{cases} f_6(x,y) = xy(x^4 - y^4) \\ h_{12}(x,y) = xy(x^{10} + 11x^5 y^5 + y^{10}). \end{cases}$$

We put

$$\begin{cases} V_5 := X(f_6) \hookrightarrow \mathbb{P}(R_6) \cong \mathbb{P}^6 \\ V'_{22} := X(h_{12}) \hookrightarrow \mathbb{P}(R_{12}) \cong \mathbb{P}^{12} \end{cases}$$

Then,

Lemma 1 (Lemma 3.3 in [7]). (1)  $V_5$  is a Fano threefold of index 2, genus 21 and the hyperplane section of  $V_5 \hookrightarrow \mathbb{P}^6$  is the positive generator of  $\text{Pic } V_5 \cong \mathbb{Z}$ .

(2)  $V'_{22}$  is a Fano threefold of index 1, genus 12 and the hyperplane section of  $V'_{22} \hookrightarrow \mathbb{P}^{12}$  is the positive generator of  $\text{Pic } V'_{22} \cong \mathbb{Z}$ .

The defining equations for  $V_5$ ,  $V'_{22}$  are the following (p.505-p.506 in [7]) :

$$(V_5) \quad \begin{cases} a_0 a_4 - 4a_1 a_3 + 3a_2^2 = 0 \\ a_0 a_5 - 3a_1 a_4 + 2a_2 a_3 = 0 \\ a_0 a_6 - 9a_2 a_4 + 8a_3^2 = 0 \\ a_1 a_6 - 3a_2 a_5 + 2a_3 a_4 = 0 \\ a_2 a_6 - 4a_3 a_5 + 3a_4^2 = 0 \end{cases}$$

$$(V'_{22}) \quad \sum_{\lambda=0}^{\rho} \binom{8}{\lambda} \binom{8}{\rho-\lambda} (a_{\lambda} a_{\rho+4-\lambda} - 4a_{\lambda+1} a_{\rho+3-\lambda} + 3a_{\lambda+2} a_{\rho+2-\lambda}) = 0$$

$$(0 \leq \rho \leq 16)$$

Now, we put

$$H_5^{\infty} := V_5 \cap \{a_6=0\} \hookrightarrow \mathbb{P}^5$$

$$H'_{22} := V'_{22} \cap \{a_0=0\} \hookrightarrow \mathbb{P}^{11}$$

Let us denote by  $\text{Sing } H_5^{\infty}$  (resp.  $\text{Sing } H'_{22}$ ) the singular locus of  $H_5^{\infty}$  (resp.  $H'_{22}$ ). Then we have

Lemma 2. (1) ([2]).  $V_5 - H_5^\infty \cong \mathbb{C}^3$  and  $\text{Sing } H_5^\infty =: s$  is a line in  $V_5$  with the normal bundle  $N_{s|V_5} \cong 0(-1) \oplus 0(1)$ .

(2) ([3]).  $V'_{22} - H'_{22} \cong \mathbb{C}^3$  and  $\text{Sing } H'_{22} =: \ell$  is a line in  $V'_{22}$  with the normal bundle  $N_{\ell|V'_{22}} \cong 0(-2) \oplus 0(1)$ . In particular, there is no other line in  $V'_{22}$  which intersects the line  $\ell$ .

By a direct computation, we have

Lemma 3. (1)  $\text{Sing } H'_{22} =: \ell = \{a_0 = a_1 = \dots = a_{10} = 0\} \hookrightarrow \mathbb{P}^{12}$ .

(2)  $H'_{22} \cap \{a_1 \neq 0\} \cong \mathbb{C}^2$ .

(3)  $H'_{22} \cap \{a_{12} \neq 0\} \cong V(f) \hookrightarrow \mathbb{C}^3(x, y, z)$ , where

$$(*) \quad f = b_0 x^4 + (b_1 y z + b_2 z^3) x^3 + (b_3 y^3 + b_4 y^2 z^2 + b_5 y z^4) x^2 \\ + (b_6 y^4 z + b_7 y^3 z^3) x + b_8 y^6 + b_9 y^5 z^2,$$

$$\begin{cases} b_0 = -2^8 5^2 \\ b_1 = 2^9 3^3 5 \\ b_2 = -2^6 3^4 5 \\ b_3 = -2^8 3^3 7 \\ b_4 = -2^4 3^4 127 \end{cases} \quad \begin{cases} b_5 = 2^9 3^5 \\ b_6 = 2^2 3^6 89 \\ b_7 = -2^8 3^6 \\ b_8 = -3^6 5^3 \\ b_9 = 2^5 3^7 \end{cases}$$

(4)  $\text{Sing } V(f) = \{x = y = 0\} \cong \ell \cap \{a_{12} \neq 0\}$ . Furthermore, the multiplicity  $\text{mult}_\ell Y = 3$ , namely,  $Y$  is a unique element of the linear system  $|0_{V'_{22}}(1) \otimes I_\ell^3|$ , where  $I_\ell$  is the ideal sheaf of  $\ell$  in  $0_{V'_{22}}$ .

§2. Double projection.

We will study the double projection of  $V'_{22}$  from the singular locus  $\ell := \text{Sing } H'_{22}$  which is a line in  $V'_{22}$ . For simplicity, we put  $X := V'_{22}$  and  $Y := H'_{22}$ .

(I) Let  $\sigma_1 : X_1 \rightarrow X$  be the blowing up of  $X$  along the line  $\ell$  and put  $L_1 := \sigma_1^{-1}(\ell)$ . By Lemma 2-(2), we have  $L_1 \cong \mathbb{F}_3$  (Hirzebruch surface). Let  $Y_1$  be the proper transform of  $Y$  in  $X_1$ . By Lemma 3-(4), we have a linear equivalence  $Y_1 \sim \sigma_1^* H - 3L_1$ .

Let us consider the linear systems  $|H| := |O_X(1) \otimes I_\ell^2|$  and  $|H_1| := |\sigma_1^* H - 2L_1|$ . Then we have

Lemma 4 (Lemma 5.4 in [4]). (1)  $\dim_{\mathbb{C}} |H| = \dim_{\mathbb{C}} |H_1| = 6$  and  $\dim_{\mathbb{C}} |\sigma_1^* H - 3L_1| = \dim_{\mathbb{C}} |Y_1| = 0$  (namely,  $Y_1$  is a unique member of the linear system  $|\sigma_1^* H - 3L_1|$ ).

$$(2) (H_1^3) = 2$$

(3)  $Y_1 \cap L_1 \sim 3\ell_1 + 7f_1$  in  $L_1$ , where  $\ell_1, f_1$  is the negative section and a fiber of  $L_1$ .

Since  $K_{X_1} \sim -\sigma_1^* H + L_1$  and  $(L_1 \cdot \ell_1) = 1$ , we have  $(K_{X_1} \cdot \ell_1) = 0$ . Thus, by the following exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & N_{\ell_1|L_1} & \rightarrow & N_{\ell_1|X_1} & \rightarrow & N_{L_1|X_1}|_{\ell_1} \rightarrow 0 \\ & & \cong & & \cong & & \\ & & 0(-3) & & 0(a) \oplus 0(b) & & 0(1) \end{array},$$

where  $a + b = c_1(N_{\ell_1|X_1}) = -2$ , we have

Lemma 5.

$$N_{\ell_1|X_1} \cong \begin{cases} (a) & 0(-1) \oplus 0(-1) \\ (b) & 0(-2) \oplus 0 \\ (c) & 0(-3) \oplus 0(1) \end{cases}$$

Lemma 6.  $Bs H_1 = \ell_1$ .

Proof. Since  $(H_1 \cdot \ell_1) = -1 < 0$ ,  $\ell_1 \subseteq Bs H_1$ . By Lemma 2-(2), there is no other line in  $X$  which intersects  $\ell$ . Thus, by the same argument as in the proof of Lemma 5.4-(ii) in [4], we have the claim.

Let us denote by  $\pi_{2\ell}$  a rational map defined by the linear system  $H$ , which is called the double projection from  $\ell$ . By Lemma 4-(1), we have a diagram :

$$\begin{array}{ccc} X_1 & & \\ \sigma_1 \downarrow & \dots & \downarrow \Phi_1 \\ X & \dots \rightarrow & \mathbb{P}^6 \\ & & \pi_{2\ell} \end{array}$$

where  $\Phi_1 := \Phi_{H_1}$  is a rational map defined by the linear system  $H_1$ .

We will resolve the indeterminacy of the rational map  $\Phi_1 : X_1 \dots \rightarrow \mathbb{P}^6$ . For this, we need the following

Lemma 7. (1)  $Sing Y_1 = 2\ell_1$ , namely,  $Sing Y_1 = \ell_1$  as a set and  $\text{mult}_{\ell_1} Y_1 = 2$ .

(2)  $Y_1 \cap L_1 = A_1 + A_2 + A_3$ , where  $A_1 \sim 2\ell_1, A_2 \sim \ell_1 + 4f_1, A_3 \sim 3f_1$  in  $L_1$ .

Proof. One can obtain easily the (local) defining equation of  $Y_1$  in  $X_1$  from (\*) in Lemma 3-(3). Looking at this equation, we have the assertion.

(II) Let  $\sigma_2 : X_2 \rightarrow X_1$  be the blowing up of  $X_1$  along the negative section  $\ell_1$  of  $L_1$  and put  $L_2 := \sigma_2^{-1}(\ell_1)$ . By Lemma 5, we have the following three cases:

$$L_2 \cong \begin{cases} (a) & \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & \mathbb{F}_2 \\ (c) & \mathbb{F}_4 \end{cases}$$

Let  $Y_2$  be the proper transform of  $Y_1$  in  $X_2$ . By Lemma 7-(1) we have  $Y_2 \sim \sigma_2^* Y_1 - 2L_2$ . We put  $H_2 := \sigma_2^* H_1 - L_2$ . Let  $\ell_2, f_2$  be the non-positive section and a fiber of  $L_2$ . Since

$$(L_2^2) \sim \begin{cases} -\ell_2 - f_2 & \text{if } L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ -\ell_2 - 2f_2 & \text{if } L_2 \cong \mathbb{F}_2 \\ -\ell_2 - 3f_2 & \text{if } L_2 \cong \mathbb{F}_4 \end{cases},$$

we have easily

Lemma 8.

$$Y_2 \cap L_2 \sim \begin{cases} (a) & 2\ell_2 & \text{if } L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & 2\ell_2 + 2f_2 & \text{if } L_2 \cong \mathbb{F}_2 \\ (c) & 2\ell_2 + 4f_2 & \text{if } L_2 \cong \mathbb{F}_4 \end{cases}$$

On the other hand, by a direct computation, we have

Lemma 9. (1)  $Y_2 \cap L_2 = B_1 + B_2$ , where  $B_1 \sim 2\ell_2$ ,  $B_2 \sim 2f_2$  in  $L_2$ .

(2)  $\text{Sing } Y_2 = 2\ell_2$ .

Corollary 10.  $L_2 \cong \mathbb{F}_2$ , namely,  $N_{\ell_1|X_1} \cong 0(-2) \oplus 0$ .

Lemma 11.  $Bs|H_2| = \ell_2$ .

Proof. Since  $(H_2 \cdot \ell_2) = -1$ ,  $\ell_2 \subseteq Bs H_2$ . On the other hand, since  $|H_2| \cap L_2 \subseteq |\ell_2 + f_2|$ , we have the claim.

Since  $(K_{X_2} \cdot \ell_2) = 0 = (L_2 \cdot \ell_2)$ , by an exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & N_{\ell_2|L_2} & \rightarrow & N_{\ell_2|X_2} & \rightarrow & N_{L_2|X_2} \otimes \ell_2 & \rightarrow & 0 \\ & & \cong & & \cong & & \cong & & \\ & & 0(-2) & \rightarrow & 0(a) \oplus 0(b) & \rightarrow & 0 & & \end{array}$$

where  $a + b = -2$ , we have

Lemma 12.

$$N_{\ell_2|X_2} \cong \begin{cases} (a) & 0(-1) \oplus 0(-1) \\ (b) & 0(-2) \oplus 0 \end{cases}$$

(III) Let  $\sigma_3 : X_3 \rightarrow X_2$  be the blowing up of  $X_2$  along the section  $\ell_2$  of  $L_2$  and put  $L_3 = \sigma_3^{-1}(\ell_2)$ . By Lemma 11, we have the two cases:

$$L_3 \cong \begin{cases} (a) & \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & \mathbb{F}_2 \end{cases}$$

Let  $Y_3$  be the proper transform of  $Y_2$  in  $X_3$ . By Lemma 9-(2), we have  $Y_3 \sim \sigma_3^*(Y_2) - 2L_3$ . We put  $H_3 := \sigma_3^*H_2 - L_3$ . Let  $\ell_3, f_3$  be the non-positive section and a fiber of  $L_3$ . Since

$$(L_3^2) \sim \begin{cases} (a) & -\ell_3 - f_3 & \text{if } L_3 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & -\ell_3 - 2f_3 & \text{if } L_3 \cong \mathbb{F}_2 \end{cases}$$

Lemma 12.

$$Y_3 \cap L_3 \cong \begin{cases} (a) & 2\ell_3 & \text{if } L_3 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & 2\ell_3 + 2f_3 & \text{if } L_3 \cong \mathbb{F}_2 . \end{cases}$$

By a direct computation, one can easily have the following

Lemma 13.(1)  $Y_3 \cap L_3 = C_1 + C_2$  , where  $C_1 \sim 2\ell_3$  ,  $C_2 \sim 2f_3$  .

(2)  $\text{Sing } Y_3 = 2\ell_3 + 2f_3$  .

Corollary 14.  $L_3 \cong \mathbb{F}_2$  , namely,  $N_{\ell_3|X_3} \cong 0(-2) \oplus 0$  .

Moreover, we have

Lemma 15.  $Bs|H_3| = \ell_3$  .

Lemma 16.

$$N_{\ell_3|X_3} \cong \begin{cases} (a) & 0(-1) \oplus 0(-1) \\ (b) & 0(-2) \oplus 0 \end{cases}$$

The proof is similar to that of Lemma 11, Lemma 12.

(IV) Let  $\sigma_4 : X_4 \rightarrow X_3$  be the blowing up of  $X_3$  along the section  $\ell_3$  of  $L_3$  and put  $L_4 := \sigma_4^{-1}(\ell_3)$ . Let  $Y_4$  be the proper transform of  $Y_3$  in  $X_4$  and put  $H_4 := \sigma_4^*H_3 - L_4$  and let  $\ell_4$  ,  $f_4$  be the zero section , a fiber of  $L_4$ . By Lemma 16, we have the two cases:

$$L_4 \cong \begin{cases} (a) & \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & \mathbb{F}_2 \end{cases} .$$

By Lemma 13-(2), we have  $Y_4 \sim \sigma_4^*Y_3 - 2L_4$  . Since

$$(L_4^2) \sim \begin{cases} (a) & -\ell_4 - f_4 & \text{if } L_4 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & -\ell_4 - 2f_4 & \text{if } L_4 \cong \mathbb{F}_2 , \end{cases}$$

we have

Lemma 17.

$$Y_4 \cap L_4 \sim \begin{cases} (a) & 2\ell_4 & \text{if } L_4 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & 2\ell_4 + 2f_4 & \text{if } L_4 \cong \mathbb{F}_2. \end{cases}$$

On the other hand, by a direct computation, we have

Lemma 18.

$$Y_4 \cap L_4 = D \sim 2\ell_4 \text{ in } L_4.$$

Corollary 19.

$$L_4 \cong \mathbb{P}^1 \times \mathbb{P}^1, \text{ namely, } N_{\ell_3|X_3} \cong 0(-1) \oplus 0(-1).$$

Moreover, we have

Lemma 20. (1)  $Bs|H_4| = \phi$ .

(2)  $(H_4^3) = 5$ .

(3)  $(H_4^2 \cdot Y_4) = (H_4^2 \cdot L_4) = (H_4^2 \cdot L_j^{(4)}) = 0$  ( $j = 2, 3$ ), and

$(H_4^2 \cdot L_1^{(4)}) = 5$ , where  $L_j^{(4)}$  is the proper transform of  $L_j$  in  $X_4$ .

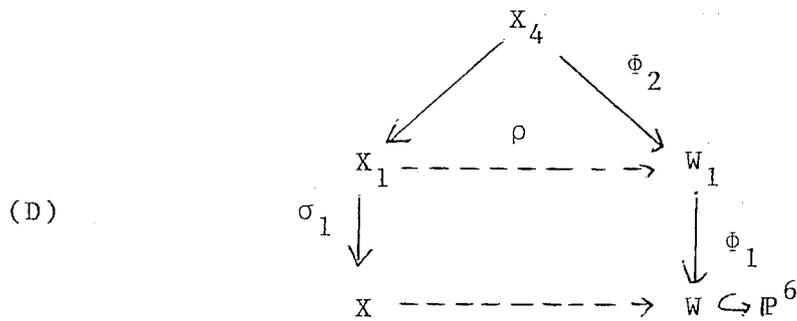
(4)  $(H_4 \cdot A_2^{(4)}) = 5$ ,  $(H_4 \cdot A_3^{(4)}) = 0$ ,  $(H_4 \cdot f_1^{(4)}) = 1$ , where  $A_j^{(4)}$ ,

$f_1^{(4)}$  are the proper transforms of  $A_j$  and a general fiber  $f_1$  in  $X_4$ .

By Lemma 20-(1), we have the morphism  $\Phi: X_4 \rightarrow \mathbb{P}^6$ , defined by the linear system  $|H_4|$ . We put  $W := \Phi(X_4)$ . By Lemma 20-(2), we have  $\deg W = 5$ . By construction,  $W$  is a compactification of  $\mathbb{C}^3$  with  $b_2(W) = 1$ .

Since  $N_{\ell_3|X_3} \cong 0(-1) + 0(-1)$ , by Reid [8],  $L_4$  can be blown down along  $\ell_4$ , and then blowing downs can be done step-by-step. Finally, we have a smooth projective threefold  $W_1$  with  $b_2(W_1) = 2$ , morphism  $\Phi_2: X_4 \rightarrow W_1$ ,  $\Phi_1: W_1 \rightarrow W$ , and a birational map  $\rho: X_1 \rightarrow W_1$  which is called a flop such that

- (i)  $\Phi = \Phi_1 \circ \Phi_2 \circ \rho$
- (ii)  $X_1 - \ell_1 \cong W_1 - g_1$ , where  $g_1 := \Phi_2(L_4 \cap L_1^{(4)})$ .



- (iii)  $W_1 - E_1 \cong W - \Gamma$ , where  $E_1 = \rho(Y_1), \Gamma = \Phi(Y_4)$   
 $= \Phi_1(Y_1)$ .

Since  $-K_{X_1} = Y_1 + 2L_1$  and  $\rho: X_1 - \ell_1 \cong W_1 - g_1$ , we have  $-K_{W_1} = E_1 + 2F_1$ , where  $F_1 = \rho(L_1)$ . We remark that  $\Gamma = \Phi_1(E_1)$ .

We put  $Z = \Phi_1(F_1)$ . By Lemma 20-(4),  $Z$  is swept out by lines in  $W \hookrightarrow \mathbb{P}^6$  and  $\Gamma \subset Z$ ,  $\Gamma \cap g = \{\text{one point}\}$ , where  $\deg \Gamma = 5$  and  $\Gamma$  is a smooth rational curve.

Let  $\gamma$  be a conic in  $X$  which intersects the line  $\ell$ . Then, we have  $\gamma \subset Y$ . Let  $\gamma_1$  be the proper transform of  $\gamma$  in  $X_1$ . We put  $\widehat{\gamma}_1 := \rho(\gamma_1)$ . Since  $(K_{W_1}, \widehat{\gamma}_1) = -(E_1, \widehat{\gamma}_1) - 2(F_1, \widehat{\gamma}_1) = 1 - 2 = -1$ , by K.M.M. [5],  $\Phi_1: W_1 \rightarrow W$  is the contraction of an extremal ray. Since the exceptional divisor  $E_1$  is contracted to a smooth

curve, by Mori [6],  $W$  is smooth. Thus  $-K_W = 2Z$ . Since  $W - Z \cong \mathbb{C}^3$  by construction,  $Z$  is ample, namely,  $W$  is a Fano threefold of first kind of index 2, genus 21. Since  $Z$  is swept out by lines on  $W$ ,  $Z$  is non-normal. Therefore, we have  $(W, Z) \cong (V_5, H_5)$  (see §1) (cf. [2]). Thus we have

Theorem. There is a birational map  $\pi_{2\ell}: V'_{22} \dashrightarrow V_5$ , called the double projection from the line  $\ell$ , such that the restriction  $\pi_{2\ell}: V'_{22} - H'_{22} \xrightarrow{\sim} V_5 - H_5 (\cong \mathbb{C}^3)$  is isomorphic. The inverse map  $\rho_2^{-1}: V_5 \dashrightarrow V'_{22}$  is given by the linear system  $|0_{V_5}(3) \otimes I_{\Gamma}^2|$ .

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# Algebraic Surfaces of General Type with $c_1^2 = 3p_g - 6$

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## Introduction.

Let  $S$  be a minimal algebraic surface of general type defined over the complex number field  $\mathbf{C}$ . Castelnuovo's second inequality says that the canonical map  $\Phi_K$  of  $S$  can be birational onto its image only when  $c_1(S)^2 \geq 3p_g(S) - 7$  (see, e.g. [11, II, Lemma(1.1)]). Surfaces with  $c_1^2 = 3p_g - 7$  have been studied by several authors (e.g. [8], [2]). The purpose of the present article is to determine, to some extent, the structure of minimal surfaces with  $c_1^2 = 3p_g - 6$ . We remark that Horikawa ([11, III and IV]) studied, among others, those with  $(p_g, c_1^2) = (3, 3), (4, 6)$  in detail. So we restrict ourselves to the case  $p_g \geq 5$ .

In §1, we classify them into three types according to the nature of  $\Phi_K$  which is generically finite onto its image. Namely, a minimal surface of general type with  $c_1^2 = 3p_g - 6$  is said to be of type I, type II or of type III, if  $\deg \Phi_K = 1, 2$  or  $3$ , respectively. We also state a result on regular surfaces of type II similar to one in [2]. In §2, we study type III surfaces  $S$  and show  $p_g(S) \leq 5, q(S) = 0$ . Those with  $p_g = 3, 4$  are known to exist (see, [11, III and IV]). We show that  $S$  with  $p_g = 5$  also exists. It has a pencil  $|D|$  of nonhyperelliptic curves of genus 3 with one base point, and the canonical map induces on  $D$  the projection from it to a line, if we identify  $D$  with a plane quartic.

The rest of the article, §§3–9, is devoted to surfaces of type I. Please recall that the canonical image of a type I surface with  $c_1^2 = 3p_g - 7$  is contained in a threefold of  $\Delta$ -genus 0 [2]. An analogous phenomena can be observed also in the present case. We show that the canonical image of a type I surface is contained in a threefold  $W$  of  $\Delta$ -genus  $\leq 1$ , which is, if  $p_g \geq 6$ , cut out by all quadrics through the canonical image (Theorem 3.1). As in the case of  $c_1^2 = 3p_g - 7$ , the proof is based on Castelnuovo's idea: We cut the canonical image twice by hyperplanes and count the number of quadrics through the resulting set of points. The key is a more recent result of Harris-Eisenbud [9] on the Hilbert functions of a special set of points in a projective space, which itself is a generalization of classical Castelnuovo's Lemma. We remark that varieties of  $\Delta$ -genus  $\leq 1$  are successfully classified by Fujita ([4], [5], [7]). This enables us to study type I surfaces as divisors on "known" threefolds, and to clarify their structure. We show, among others, every surface of type I has a pencil of nonhyperelliptic curves of genus 3 if  $p_g \geq 12$  (Theorem 6.2).

As mentioned above, surfaces of type I are further classified into two types by the  $\Delta$ -genus of the threefold  $W$ . We say that  $S$  is of type I-0 or type I-1, according to whether  $W$  is of  $\Delta$ -genus 0 or 1. In §4, we study surfaces of type I-1. We show Theorem 4.2, which says in particular that they satisfy  $p_g \leq 11$ . In §§5–9, we study

surfaces of type I-0. In §5, we discuss whether the canonical map can be lifted to a holomorphic map into a nonsingular model of  $W$ , and show Proposition 5.6. In §6, we determine the linear equivalence class on  $W$  of the canonical image, and prepare some Lemmas for the later use. It will be clarified that every surface of type I-0 has a pencil of nonhyperelliptic curves of genus 3 or 4. In the last case, we have  $p_g \leq 7$ . The existence of those with  $p_g = 7$  and  $p_g = 6$  is shown in §7 and §8, respectively. In §9, we study the case of genus 3. We sketch a member of the pencil arising from the double curve of the canonical image. In general, it is a hyperelliptic curve of genus 3.

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## Contents

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- §3. Quadrics through canonical surfaces
- §4. Surfaces of type I-1
- §5. Lifting of the canonical maps
- §6. Divisor classes
- §7. Surfaces of type I-0: The case (s.1)
- §8. Surfaces of type I-0: The case (s.2)
- §9. Surfaces of type I-0: The case (s.3)

## 1 Canonical maps.

1.1 Let  $S$  be a minimal algebraic surface of general type defined over the complex number field  $\mathbf{C}$ . We denote by  $p_g(S)$ ,  $q(S)$  and  $c_1(S)^2$  the geometric genus, the irregularity and the Chern number of  $S$ , respectively. The canonical divisor of  $S$  will be denoted by  $K$ . We let  $\sigma : \tilde{S} \rightarrow S$  be the composition of quadric transformations which is the shortest among those with the property that the variable part  $|L|$  of  $|\sigma^*K|$  is free from base points. We denote by  $\tilde{K}$  and  $E$  the canonical divisor of  $\tilde{S}$  and the

exceptional divisor of  $\sigma$ , respectively. In particular, we have  $\widetilde{K} \sim \sigma^*K + E$ , where the symbol  $\sim$  means the linear equivalence of divisors.

In what follows, we often use the standard fact that, if a surface admits a rational map of degree less than three onto a ruled surface, then its canonical map cannot be birational onto its image.

We recall a well-known result due to Castelnuovo (see, [9]).

**Lemma 1.2** (Castelnuovo's bound)

Let  $C \subset \mathbf{P}^r$  be an irreducible nondegenerate curve of degree  $d$  and geometric genus  $g(C)$ . Then

$$g(C) \leq \pi_0(d, r) = \binom{m_0}{2} (r-1) + m_0 \epsilon_0$$

where  $m_0$  and  $\epsilon_0$  are nonnegative integers satisfying

$$d = m_0(r-1) + \epsilon_0 + 1, \quad 0 \leq \epsilon_0 \leq r-2.$$

**Lemma 1.3** Let  $S$  be a minimal algebraic surface of general type with  $c_1(S)^2 = 3p_g(S) - 6$ . Let  $\Phi_K : S \rightarrow S_0 \subset \mathbf{P}^{p_g-1}$  be the canonical map of  $S$ . Then one of the following occurs:

- 1)  $\Phi_K$  induces a birational holomorphic map onto  $S_0$ .
- 2)  $\Phi_K$  induces a generically finite rational map of degree 2 onto  $S_0$  which is birationally equivalent to a ruled surface.
- 3)  $\Phi_K$  induces a holomorphic map of degree 3 onto  $S_0$  which is a surface of minimal degree  $p_g - 2$  in  $\mathbf{P}^{p_g-1}$ .

*Proof.* We remark that  $|K|$  is not composite with a pencil by [11, III, Theorem 1.1] and [3, p. 136]. Thus  $\Phi_K$  induces a generically finite rational map onto its image. Let  $\sigma : \widetilde{S} \rightarrow S$  be as in 1.1. Since  $S_0$  is an irreducible nondegenerate surface in  $\mathbf{P}^{p_g-1}$ , we see from [10, Lemma 1] that

$$3p_g - 6 = c_1^2 \geq L^2 = (\deg \Phi_K)(\deg S_0) \geq (\deg \Phi_K)(p_g - 2).$$

So we have  $\deg \Phi_K \leq 3$ . Furthermore, if  $\deg \Phi_K = 3$ , the equality holds everywhere. This implies that  $S_0$  is a surface of minimal degree and that  $|K|$  is free from base points [10, Lemma 1]. Thus we have 3). If  $\deg \Phi_K = 2$ , then we are in the case 2) and the statement for  $S_0$  can be found in [3]. We show that  $|K|$  is free from base points if  $\Phi_K$  is birational. For this purpose, we recall the inequality  $L^2 \geq 3p_g - 7$  ([11, II, Lemma(1.1)]). Thus we have either

- i)  $c_1^2 = L^2$ , or

ii)  $c_1^2 = L^2 + 1$ .

We show that ii) leads us to a contradiction. Assume that ii) is the case. Then, by the argument in [10, §1], we see that  $|K|$  has the unique base point  $P$  and  $\sigma$  is the blowing-up of  $S$  at  $P$ . Thus we have  $|\sigma^*K| = |L| + E$ , where  $E$  is the exceptional  $(-1)$ -curve and  $LE = 1$ . Let  $C$  be a general member of  $|L|$ , which we can assume irreducible and nonsingular. Since  $C_0 := \Phi_K(C)$  is an irreducible nondegenerate curve of degree  $3p_g - 7$  in  $\mathbf{P}^{p_g-2}$ , Castelnuovo's bound (Lemma 1.2) shows that its geometric genus  $g(C)$  is bounded from above by  $\pi_0(3p_g - 7, p_g - 2) = 3p_g - 6$ . On the other hand, by the virtual genus formula, we have

$$g(C) = \frac{1}{2}C(\widetilde{K} + C) + 1 = L^2 + LE + 1 = 3p_g - 5.$$

Thus  $\pi_0(3p_g - 7, p_g - 2) < g(C)$ , a contradiction. If i) is the case, the absence of base points of  $|K|$  follows from [10, Lemma 1]. *q.e.d.*

**Definition 1.4** Let  $S$  be as in Lemma 1.3. We say that it is of type I, type II or type III if  $\Phi_K$  satisfies 1), 2) or 3) in Lemma 1.3, respectively.

Throughout the paper, we assume that  $p_g \geq 5$ , since the cases  $p_g = 3, 4$  can be found in [11, III and IV].

We now consider type II surfaces. Since we shall show in Lemma 2.2 and Theorem 3.1 that surfaces of types I and III are regular, we restrict ourselves to regular surfaces of type II here. Then  $\Phi_K$  and the ruling of  $S_0$  induce on  $S$  a linear pencil of hyperelliptic curves. Though the structure is simple, it seems hard to classify them completely. As for regular surfaces of type II, we have the following:

**Proposition 1.5** *Let  $S$  be a minimal algebraic surface of general type with  $c_1^2 = 3p_g - 6$  and  $q = 0$  which is of type II in the sense of 1.4. If  $p_g > 50$ , then it has a unique pencil of hyperelliptic curves of genus less than 5. Conversely, take  $g \in \{2, 3, 4\}$  and fix it. Then, for any pair of integers  $(x, y)$  satisfying  $y = 3x - 6$  and  $x \geq 5$ , there exists a minimal regular surface  $S$  of type II with a pencil of hyperelliptic curves of genus  $g$  such that  $p_g(S) = x$ ,  $c_1(S)^2 = y$ .*

*Proof.* The first half follows from a result of [13]. The last half can be shown by the method given in [2]. *q.e.d.*

## 2 Surfaces of type III.

In this section, we study surfaces of type III. Those with  $p_g = 3, 4$  exist (see [11, II and III]). Among others, we shall use the following notation. For any nonnegative integer  $d$ , we denote by  $\Sigma_d$  the Hirzebruch surface of degree  $d$ . We let  $\Gamma$  denote a fiber of  $\Sigma_d$  and  $\Delta_0$  a section with  $\Delta_0^2 = -d$ .

**2.1** As we have seen in Lemma 1.3, the canonical image of a type III surface is a surface of minimal degree. Here we recall a result of [12] (see also [4]). Let  $V$  be an irreducible nondegenerate surface of minimal degree  $n - 1$  in  $\mathbf{P}^n$ . Then it is either

- a)  $\mathbf{P}^2$  ( $n = 2$ ),
- b)  $\mathbf{P}^2$  embedded into  $\mathbf{P}^5$  by the complete linear system of quadrics ( $n = 5$ ),
- c)  $\Sigma_d$  embedded into  $\mathbf{P}^n$  by  $|\Delta_0 + \frac{n-1+d}{2}\Gamma|$ , where  $n - d - 3$  is a nonnegative even integer ( $n \geq 3$ ), or
- d) a cone over a rational curve of degree  $n - 1$  in  $\mathbf{P}^{n-1}$ , that is, the image of  $\Sigma_{n-1}$  by  $|\Delta_0 + (n - 1)\Gamma|$  ( $n \geq 3$ ).

**Lemma 2.2** *Every type III surface  $S$  satisfies  $p_g(S) \leq 5$  and  $q(S) = 0$ . If  $p_g(S) = 5$ , then its canonical image is a cone over a rational curve of degree 3.*

*Proof.* We put  $n = p_g - 1$ . We assume  $n \geq 5$  and show that this leads us to a contradiction. We denote by  $f : S \rightarrow S_0$  the holomorphic map induced by  $\Phi_K$ . Then  $f$  is of degree 3 and  $S_0$  is one of the surfaces in 2.1. Since  $n \geq 5$ , we need not consider the case a). If b) is the case, then we have  $K \sim 2f^*l$ , where  $l$  is a line in  $\mathbf{P}^2$ . Thus  $(f^*l)(K + f^*l) = 3(f^*l)^2 = 9$ . This contradicts that  $(f^*l)(K + f^*l)$  is even. Similarly, if c) is the case, we have  $K \sim f^*(\Delta_0 + ((n - 1 + d)/2)\Gamma)$  and get a contradiction because  $(f^*\Gamma)(K + f^*\Gamma) = 3$ .

We consider the case d). Then, by the same argument as in the proof of [11, I, Lemma 1], we have  $K \sim (n - 1)D + G$ , where  $|D|$  is a pencil and  $G$  is an effective (possibly zero) divisor with  $KG = 0$ . In particular, we have  $G^2 \leq 0$  by Hodge's index theorem. Since  $3(n - 1) = K^2 = K((n - 1)D + G) = (n - 1)KD$ , we get  $KD = 3$ . Thus we have

- i)  $0 = KG = (n - 1)GD + G^2$ , and
- ii)  $3 = KD = (n - 1)D^2 + GD$ .

Further,  $D^2$  is a positive odd integer, because  $D(K + D) = 3 + D^2$  is even. Since  $G^2 \leq 0$ , we have  $GD \geq 0$  by i). From this and ii), we get  $3 \geq (n - 1)D^2 \geq n - 1$ , which contradicts the assumption  $n \geq 5$ .

Thus we have  $p_g(S) \leq 5$  for a type III surface  $S$ . Then the vanishing of  $q(S)$  follows from [11, V, Theorem 4.1]. *q.e.d.*

**Theorem 2.3** *Let  $S$  be a type III surface with  $p_g(S) = 5$ . Then  $S$  is the minimal nonsingular model of a surface  $S'$  defined by*

$$w^3 + \alpha_1 \zeta w^2 + \alpha_2 \zeta^2 w + \alpha_3 \zeta^3 = 0 \quad (1)$$

*in the total space of the line bundle  $[2\Delta_0 + 4\Gamma]$  on  $\Sigma_3$ , where  $w$  is the fiber coordinate and*

$$\begin{aligned} \alpha_j &\in H^0(\Sigma_3, \mathcal{O}((j + [j/3])\Delta_0 + 4j\Gamma)), \quad 1 \leq j \leq 3, \\ \zeta &\in H^0(\Sigma_3, \mathcal{O}(\Delta_0)). \end{aligned}$$

*Proof.* This is a verbatim translation of [11, III, §4]. By the proof of Lemma 2.2, we have a pencil  $|D|$  with  $K \sim 3D$ ,  $KD = 3$  and  $D^2 = 1$ . Thus it has the unique base point  $P$ . We let  $\sigma : \hat{S} \rightarrow S$  be the blowing-up with center  $P$  and put  $E = \sigma^{-1}(P)$ . Then the variable part  $|\hat{D}|$  of  $|\sigma^*D|$  defines a holomorphic map  $g : \hat{S} \rightarrow \mathbf{P}^1$ . Since  $|K|$  has no base point, there exists  $\eta \in H^0(\hat{S}, \mathcal{O}(\sigma^*K))$  which does not vanish on  $E$ . Further, we can take nonzero  $\omega \in H^0(\hat{S}, \mathcal{O}(3E))$ . Then the pair  $(\omega, \eta)$  defines a holomorphic map  $h : \hat{S} \rightarrow \Sigma_3$  with  $h^*\Delta_0 = 3E$  and  $\hat{K} \sim h^*(\Delta_0 + 3\Gamma) + E$ , where  $\hat{K}$  is the canonical divisor of  $\hat{S}$ . By the Riemann-Roch theorem,

$$\chi(\hat{K} + \hat{D}) = \chi(K + D) = D(K + D)/2 + \chi(\mathcal{O}_S) = 8.$$

Since  $H^p(-D) = 0$  for  $p < 2$  by Ramanujam's vanishing theorem, we get  $h^0(\hat{K} + \hat{D}) = h^0(K + D) = 8$ . A nonzero  $\xi \in H^0(2E)$  defines an injection  $H^0(\hat{K} + \hat{D}) \rightarrow H^0(\hat{K} + \hat{D} + 2E)$ . We have  $\hat{K} + \hat{D} + 2E \sim h^*(2\Delta_0 + 4\Gamma)$  and  $h^0(\Sigma_3, \mathcal{O}(2\Delta_0 + 4\Gamma)) = 7$ . Thus there exists  $\varphi \in H^0(\hat{K} + \hat{D})$  such that  $\varphi\xi$  is not induced by  $H^0(\Sigma_3, \mathcal{O}(2\Delta_0 + 4\Gamma))$ . We let  $W$  denote the total space of the line bundle  $[2\Delta_0 + 4\Gamma]$  on  $\Sigma_3$  and  $w$  its fiber coordinate. Then putting  $w = \varphi\xi$ , we get a holomorphic map  $\hat{h} : \hat{S} \rightarrow W$  over  $h$ .

Put  $S' = \hat{h}(\hat{S})$ . Since  $h$  is of degree 3, we see that  $S'$  is birational to  $\hat{S}$  by the choice of  $\varphi$ . We consider the subspace of  $H^0(\hat{S}, \mathcal{O}(h^*(6\Delta_0 + 12\Gamma)))$  consisting of those sections vanishing on  $6E$ . It contains

$$\begin{aligned} &\varphi^3 \xi^3, \\ &\varphi^2 \xi^2 \alpha \zeta \quad \text{with } \alpha \in H^0(\Sigma_3, \mathcal{O}(\Delta_0 + 4\Gamma)), \\ &\varphi \xi \beta \zeta^2 \quad \text{with } \beta \in H^0(\Sigma_3, \mathcal{O}(2\Delta_0 + 8\Gamma)), \\ &\gamma \zeta^3 \quad \text{with } \gamma \in H^0(\Sigma_3, \mathcal{O}(4\Delta_0 + 12\Gamma)). \end{aligned}$$

These represent 61 sections. On the other hand, since  $h^*(6\Delta_0 + 12\Gamma) - 6E \sim 4\sigma^*K$ , we have  $H^0(\hat{S}, \mathcal{O}(h^*(6\Delta_0 + 12\Gamma) - 6E)) = 60$ . Thus  $S'$  is defined by the equation of the form (1). Then we can show, as in [11, III, §4], that  $S'$  has a double curve along  $w = \zeta = 0$  but other singular points are at most rational double points, and that  $\hat{S}$  is the minimal resolution of  $S'$ . *q. e. d.*

### 3 Quadrics through canonical surfaces.

Recall that, for a projective variety  $X \subset \mathbf{P}^n$  with  $h^0(\mathcal{O}_X(1)) = n + 1$ , the  $\Delta$ -genus is defined by  $\Delta = \dim X + \deg X - n - 1$ . We refer the reader to Fujita [4], [5], [7] for the theory of  $\Delta$ -genus.

The purpose of this section is to prove the following:

**Theorem 3.1** *Let  $S$  be a surface of type I with  $p_g(S) \geq 5$ . Then the irregularity  $q(S)$  vanishes. The canonical image  $S_0$  of  $S$  is contained in an irreducible nondegenerate variety  $W_0$  of dimension 3 and  $\Delta$ -genus  $\leq 1$ .  $W_0$  is cut out by all quadrics through  $S_0$  unless it is a hypercubic in  $\mathbf{P}^4$ . Further, the following hold.*

- 1) *If  $W_0$  is of  $\Delta$ -genus 0, then it is a rational normal scroll.*
- 2) *If  $W_0$  is of  $\Delta$ -genus 1, then  $S_0$  is projectively normal and has only rational double points (RDP for short) as its singularity.*

**Remark 3.2** Prof. E. Horikawa has a result similar to Theorem 3.1 (unpublished). His proof is based on a detailed study of quadrics through  $S_0$  which is quite similar to "Petri's analysis".

**3.3** For any projective variety  $X \subset \mathbf{P}^r$ , we denote by  $\mathcal{I}_X$  the ideal sheaf of  $X$  in  $\mathbf{P}^r$ . We consider the cohomology exact sequence derived from

$$0 \rightarrow \mathcal{I}_X(n) \rightarrow \mathcal{O}_{\mathbf{P}^r}(n) \rightarrow \mathcal{O}_X(n) \rightarrow 0,$$

for any nonnegative integer  $n$ . The Hilbert function  $h_X$  of  $X$  is given by

$$h_X(n) = \dim_{\mathbf{C}} \operatorname{Im} \left\{ H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(n)) \rightarrow H^0(X, \mathcal{O}_X(n)) \right\}.$$

If  $Y$  is a general hyperplane section of  $X$ , then it follows from [9, Lemma(3.1)] that

$$\delta h_X(n) := h_X(n) - h_X(n-1) \geq h_Y(n), \quad n \geq 1. \quad (2)$$

We remark that  $X$  is projectively normal if  $\delta h_X(n) = h_Y(n)$  holds for any  $n \in \mathbf{N}$ .

We return to the situation we are interested in. Let  $S$  be a type I surface with  $p_g \geq 5$  and put  $r = p_g - 2$ . We choose a general member  $C \in |K|$ . Since  $|K|$  has no base points, we can assume that  $C$  is irreducible and nonsingular. Then the geometric genus  $g(C)$  of  $C$  is given by  $g(C) = \frac{1}{2}C(K+C) + 1 = K^2 + 1 = 3r + 1$ . Since  $2K|_C$  is the canonical divisor of  $C$ , we have

$$h^0(C, \mathcal{O}(nK|_C)) = \begin{cases} 3r + 1 & \text{if } n = 2, \\ 3r(n-1) & \text{if } n \geq 3. \end{cases} \quad (3)$$

If we put  $C_0 = \Phi_K(C)$ , then  $C_0$  is an irreducible nondegenerate curve of degree  $K^2 = 3r$  in  $\mathbf{P}^r$ . We remark that it is linearly normal because  $\pi_0(3r, r+1) < g(C)$  (see, Lemma 1.2). We take a general hyperplane section  $Z_0$  of  $C_0$ . It is a set of  $3r$  distinct points in  $\mathbf{P}^{r-1}$  and satisfies  $h_{Z_0}(1) = r$ . Further, since it enjoys the *uniform position property*, it follows from [9, Corollary(3.5)] that

$$h_{Z_0}(n+1) \geq \min \{3r, h_{Z_0}(n) + r - 1\}, \quad n \in \mathbf{N}. \quad (4)$$

Thus we have (for  $r \geq 4$ )

$$\begin{aligned} h_{Z_0}(1) &= r, \\ h_{Z_0}(n) &\geq n(r-1) + 1, \quad \text{if } 2 \leq n \leq 3, \\ h_{Z_0}(n) &= 3r, \quad \text{if } n \geq 4. \end{aligned}$$

Since  $C_0$  is nondegenerate, we have  $h_{C_0}(1) = r + 1$ . Thus (2) and (3) yield

$$3r + 1 = h^0(C, \mathcal{O}(2K|_C)) \geq h_{C_0}(2) \geq r + 1 + h_{Z_0}(2).$$

From this and (4), we get  $h_{Z_0}(2) = 2r$  or  $2r - 1$ .

We now recall two surprizing results due to Castelnuovo and Harris-Eisenbud, respectively. For the proof, see [9, p. 106].

**Lemma 3.4** (Castelnuovo)

*If  $Z \subset \mathbf{P}^{r-1}$  is a set of  $d \geq 2r + 1$  points in general position, then  $h_Z(2) = 2r - 1$  holds if and only if  $Z$  lies on a rational normal curve  $R$  of degree  $r - 1$ , cut out by all quadrics containing  $Z$ .*

**Lemma 3.5** (Harris-Eisenbud)

*Let  $Z \subset \mathbf{P}^{r-1}$ ,  $r \geq 3$ , be any finite set of  $d \geq 2r + 3$  points in uniform position with  $h_Z(2) = 2r$ . Then  $Z$  lies on an elliptic normal curve  $R$  of degree  $r$  in  $\mathbf{P}^{r-1}$ . Further,  $R$  is cut out by all quadrics containing  $Z$  if  $r \geq 4$ , and it is a plane cubic if  $r = 3$ .*

**3.6** We consider the case  $h_{Z_0}(2) = 2r$ . Since  $Z_0$  lies on an elliptic normal curve by Lemma 3.5, we get  $h_{Z_0}(3) = 3r - 1$ . Then, it follows from (4) that

$$6r = h^0(C, \mathcal{O}(3K|_C)) \geq h_{C_0}(3) \geq h_{C_0}(2) + h_{Z_0}(3) = 6r.$$

Thus  $h_{C_0}(3) = 6r$ . Similarly, we can show  $\delta h_{C_0}(n) = h_{Z_0}(n)$  for any  $n \in \mathbf{N}$ . Thus  $C_0$  is projectively normal.

We turn our attention to  $S_0$ . By (2) and the pluri-genus formula, we have

$$4r + 3 - q(S) = h^0(S, \mathcal{O}(2K)) \geq h_{S_0}(2) \geq h_{S_0}(1) + h_{C_0}(2) = 4r + 3.$$

Thus  $q(S) = 0$ ,  $h^0(2K) = h_{S_0}(2)$  and  $\delta h_{S_0}(2) = h_{C_0}(2)$ . Quite similarly, we get  $h^0(nK) = h_{S_0}(n)$  and  $\delta h_{S_0}(n) = h_{C_0}(n)$  for any  $n > 0$ . Thus  $S_0$  is also projectively normal, and we see that the multiplication map  $Sym^n H^0(S, \mathcal{O}(K)) \rightarrow H^0(S, \mathcal{O}(nK))$  is surjective for any  $n \geq 0$ . Thus the canonical ring of  $S$  is generated in degree 1. This implies that  $S_0$  is isomorphic to the canonical model. In particular, it has only RDP as its singularity.

We next show that  $S_0$  is contained in an irreducible threefold  $W_0$  of  $\Delta$ -genus 1. We first assume that  $r \geq 4$ . By Lemma 3.5, we have  $h^0(\mathcal{I}_{Z_0}(2)) = h^0(\mathcal{I}_R(2)) = r(r-3)/2$ . Since  $S_0$  is projectively normal, the linear system  $|\mathcal{I}_{S_0}(2)|$  is restricted to  $|\mathcal{I}_{Z_0}(2)|$  isomorphically. Thus  $W_0 = Bs|\mathcal{I}_{S_0}(2)|$  is an irreducible threefold of  $\Delta$ -genus 1, because  $R = Bs|\mathcal{I}_{Z_0}(2)|$  is an elliptic normal curve of degree  $r$ . If  $r = 3$ , then  $Z_0$  is contained in a plane cubic. Since  $S_0$  is projectively normal, it is contained in a hypercubic  $W_0$ . Thus we get 2) of Theorem 3.1.

**3.7** We next consider the case  $h_{Z_0}(2) = 2r - 1$ . We recall that  $C_0$  and  $S_0$  are linearly normal. Thus the linear system  $|\mathcal{I}_{S_0}(2)|$  is restricted to  $|\mathcal{I}_{Z_0}(2)|$  isomorphically. If we put  $W_0 = Bs|\mathcal{I}_{S_0}(2)|$ , then it is a threefold of  $\Delta$ -genus 0, because  $R = Bs|\mathcal{I}_{Z_0}(2)|$  is a rational normal curve of degree  $r - 1$  by Lemma 3.4.

We show  $q(S) = 0$ . Since  $h^0(\mathcal{I}_{Z_0}(2)) = h^0(\mathcal{I}_R(2)) = (r-1)(r-2)/2$ , we get  $h_{S_0}(2) = 4r + 2$ . Since  $h_{S_0}(2) \leq h^0(S, \mathcal{O}(2K)) = 4r + 3 - q(S)$ , we have either  $q(S) = 0$  or  $q(S) = 1$ . If  $q(S) = 1$ , then the Albanese map gives  $S$  a pencil of hyperelliptic curves of genus  $\leq 3$  by a result of Horikawa [11, V]. This is impossible, because the canonical map of  $S$  is birational. Thus  $q(S) = 0$ .

According to the classification of varieties of  $\Delta$ -genus zero ([4], [8]),  $W_0$  is either

- A)  $\mathbf{P}^3$ , ( $p_g = 4$ ),
- B) a hyperquadric  $\subset \mathbf{P}^4$ , ( $p_g = 5$ ),
- C) a cone over  $\mathbf{P}^2$  embedded into  $\mathbf{P}^5$  by  $|\mathcal{O}(2)|$ , ( $p_g = 7$ ), or

D) a rational normal scroll  $\subset \mathbf{P}^{p_g-1}$ , that is, the image of the total space of the  $\mathbf{P}^2$ -bundle

$$\pi : \mathbf{P}_{a,b,c} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(c)) \rightarrow \mathbf{P}^1$$

under the holomorphic map induced by  $|T|$ , where  $T$  is the tautological divisor and  $a, b, c$  are integers satisfying

$$0 \leq a \leq b \leq c, \quad a + b + c = p_g - 3.$$

Thus there are the following three subcases:

D.1)  $a > 0$ :  $W_0 \simeq \mathbf{P}_{a,b,c}$ .

D.2)  $a = 0, b > 0$ :  $W_0$  is a cone over the Hirzebruch surface  $\Sigma_{c-b}$  embedded into  $\mathbf{P}^{p_g-2}$  by  $|\Delta_0 + c\Gamma|$ .

D.3)  $a = b = 0$ :  $W_0$  is a generalized cone over the rational normal curve in  $\mathbf{P}^{p_g-3}$ . The ridge of  $W_0$  is a line (see [7, §1] for the terminology).

We need not consider A) by the assumption  $p_g \geq 5$ . We claim that  $W_0$  is singular in the case B). In fact, if  $W_0$  is nonsingular hyperquadric, then  $S_0$  is obtained as a hypersurface section of  $W_0$ . Then  $\deg S_0$  must be even. This contradicts  $\deg S_0 = 9$ . Thus  $W_0$  is singular. Then, since  $\text{rank}(W_0) \leq 4$ , it can be represented as a rational normal scroll. We next exclude the case C). If C) is the case, then  $C_0$  lies on the Veronese surface. Thus  $\deg C_0$  must be even, contradicting  $\deg C_0 = 15$ . In summary, we get 1) of Theorem 3.1.

**Definition 3.8** Let  $S$  and  $W_0$  be as in Theorem 3.1. We say that  $S$  is of type I-0 or of type I-1 according to whether  $W_0$  is of  $\Delta$ -genus 0 or 1.

## 4 Surfaces of type I-1.

In this section, we let  $S$  be of type I-1 and  $W_0$  the threefold of  $\Delta$ -genus 1 on which the canonical image  $S_0$  lies. We sometimes use the terminology in [6] and [7]. In particular, see [6, (5.6)] for the definition of a *Del Pezzo variety*.

4.1 By [7], if  $W_0 \subset \mathbf{P}^{p_g-1}$  is an irreducible nondegenerate threefold of degree  $p_g - 2$ , then it is either

- 1) a hypercubic ( $p_g = 5$ ),
- 2) a complete intersection of two hyperquadrics ( $p_g = 6$ ),
- 3) a cone over a surface  $V \subset \mathbf{P}^{p_g-2}$  of  $\Delta$ -genus 1, where  $V$  is either
  - 3a) the Veronese embedding into  $\mathbf{P}^8$  of a quadric in  $\mathbf{P}^3$  ( $p_g = 10$ ),
  - 3b) the image of  $\mathbf{P}^2$  by the rational map associated with the linear system  $|3l - \sum_{i=1}^k x_i|$ , where  $l$  is a line on  $\mathbf{P}^2$  and the  $x_i$  are points on  $\mathbf{P}^2$  which are possibly infinitely near ( $p_g = 11 - k, 0 \leq k \leq 6$ ),
  - 3c) a cone over a nonsingular elliptic curve, or
  - 3d) a projection of a surface of  $\Delta$ -genus 0 in  $\mathbf{P}^{p_g-1}$  from a point,

- 4) a non-conic normal Del Pezzo threefold ( $7 \leq p_g \leq 10$ ), or  
 5) a projection of a threefold of  $\Delta$ -genus 0 in  $\mathbf{P}^{p_g}$  from a point.

Not all varieties listed above can be  $W_0$  in Theorem 3.1. The cases 3d) and 5) are excluded, because the canonical image  $S_0$  and its general hyperplane section  $C_0$  are both projectively normal. The case 3c) is also excluded because  $S$  is regular: In fact, if it is the case, we can easily show that  $S$  has an irrational pencil parametrized by an elliptic curve (cf. 4.3 below).

Thus we have  $p_g(S) \leq 11$  for any surface  $S$  of type I-1, and our  $W_0$  is a normal Del Pezzo variety if  $p_g \geq 7$ . In particular, it is projectively normal (cf. [6, §5]).

**Theorem 4.2** *Every surface  $S$  of type I-1 satisfies  $5 \leq p_g(S) \leq 11$ . The canonical image  $S_0$  is a complete intersection of two hypercubics if  $p_g = 5$ , and it is a complete intersection of two hyperquadrics and a hypercubic if  $p_g = 6$ . If  $p_g \geq 7$ , then  $S_0$  is a hypercubic section of a normal Del Pezzo threefold  $W_0$ .*

*Proof.* We use the notation of 3.6. Since  $W_0$  is projectively normal, we can assume that  $R$  is nonsingular. Thus  $h^0(\mathcal{I}_R(3)) = r(r+1)(r+2)/6 - 3r$ . On the other hand, we have  $h^0(\mathcal{I}_{Z_0}(3)) = r(r+1)(r+2)/6 - 3r + 1$ . Thus there exists a hypercubic not containing  $R$  but  $Z_0$ . Since  $\deg R = r$  and  $\deg Z_0 = 3r$ , we see that  $Z_0$  is a hypercubic section of  $R$ . This gives Theorem 4.2, because  $W_0$  and  $S_0$  are both projectively normal. *q.e.d.*

In the following, we examine the existence of surfaces of type I-1 in the cases 3) and 4) of 4.1.

**4.3** Here we consider the case 3) of 4.1. Let  $v$  be the vertex of  $W_0$ . We denote by  $\Lambda_0$  the pull-back to  $S$  by  $\Phi_K$  of the linear system of hyperplanes through  $v$ . We let  $G$  be the fixed part of  $\Lambda_0$  and put  $\Lambda = \Lambda_0 - G$ . Then  $\Lambda$  defines a rational map  $\mu : S \rightarrow V$  and we have  $K \sim H + G$ ,  $H \in \Lambda$ . We note that  $KG = 0$ , since  $|K|$  has no base point. Then  $3p_g - 6 = K^2 = KH = H^2 + HG \geq H^2 \geq (\deg \mu)(\deg V) = (\deg \mu)(p_g - 2)$ . Thus  $\deg \mu \leq 3$ . Since  $S$  is of type I,  $\deg \mu \leq 2$  leads us to a contradiction. Thus  $\deg \mu = 3$ ,  $HG = 0$  and  $H^2 = 3p_g - 6$ . Since  $G^2 = 0$ , we get  $G = 0$  by Hodge's index theorem. Further, we see that  $\mu$  is holomorphic. In fact, if  $\Lambda$  has a base point  $P$ , blowing  $S$  up at  $P$  and considering the strict transform  $\tilde{\Lambda}$  of  $\Lambda$ , we would have  $\tilde{H}^2 < H^2$  for  $\tilde{H} \in \tilde{\Lambda}$  and conclude that the map induced by  $\tilde{\Lambda}$  is of degree less than three.

The case 3a): As we saw above, a subsystem of  $|K|$  induces a holomorphic map of degree 3 onto  $\Sigma_0$  or a quadratic cone in  $\mathbf{P}^3$ . In the latter case, we can lift it to a holomorphic map  $S \rightarrow \Sigma_2$  as in [10, p. 46]. Thus, in either case, we obtain a

holomorphic map  $h : S \rightarrow \Sigma_d$ , with  $d = 0$  or  $2$ , satisfying  $K \sim h^*(2\Delta_0 + (d+2)\Gamma)$ . Let  $W \rightarrow \Sigma_d$  be the line bundle associated with  $H_0 = 2\Delta_0 + (d+2)\Gamma$  and, let  $w$  be the fiber coordinate of  $W$ . We show that  $S$  is birationally equivalent to a surface  $S' \subset W$  defined by the equation

$$w^3 + \alpha_1 w^2 + \alpha_2 w + \alpha_3 = 0, \quad (5)$$

where  $\alpha_i \in H^0(\Sigma_d, \mathcal{O}(iH_0))$ . Since  $p_g(S) = 10$  and  $h^0(\Sigma_d, \mathcal{O}(H_0)) = 9$ , there exists  $\psi \in H^0(K)$  which is not induced by a section of  $\mathcal{O}(H_0)$ . Thus we get a holomorphic map  $f : S \rightarrow W$  over  $h$  by putting  $w = \psi$ . Since  $h$  is of degree 3,  $f$  is birational onto  $S' = f(S)$  by the choice of  $\psi$ . In  $H^0(3K)$ , we have the following elements:

$$\begin{aligned} & \psi^3, \\ & \psi^2 \alpha \quad \text{with } \alpha \in H^0(\Sigma_d, \mathcal{O}(H_0)), \\ & \psi \beta \quad \text{with } \beta \in H^0(\Sigma_d, \mathcal{O}(2H_0)), \\ & \gamma \quad \text{with } \gamma \in H^0(\Sigma_d, \mathcal{O}(3H_0)). \end{aligned} \quad (6)$$

These represents 84 sections. On the other hand, we have  $h^0(3K) = 83$ . Thus we can find a non-trivial relation of the form

$$\delta \psi^3 + \alpha \psi^2 + \beta \psi + \gamma = 0,$$

where  $\delta$  is a constant and  $\alpha, \beta, \gamma$  are as above. Since  $h$  is of degree 3, we see that  $\delta$  is non-zero and there is no further relation among the sections in (6). Thus  $S'$  is defined by the equation of the form (5). Conversely, if we choose the  $\alpha_i$  in (5) generic, then the obtained surface is nonsingular, minimal and satisfies  $p_g = 10$ ,  $q = 0$  and  $c_1^2 = 24$ .

The case 3b): Let  $\tilde{V}$  be the projective plane blown up at  $k = 11 - p_g$  points  $x_1, \dots, x_k$  and let  $\lambda : \tilde{V} \rightarrow \mathbf{P}^2$  be the natural map. Let  $H_0 = 3\lambda^*l - \sum \lambda^{-1}(x_i)$  be the pull-back of the hyperplane of  $\mathbf{P}^{p_g-2}$ . We denote by  $W$  the line bundle associated with  $H_0$ , and let  $w$  be its fiber coordinate. We show that  $S$  is birationally equivalent to a hypersurface  $S'$  of  $W$  defined by

$$w^3 + \alpha_1 w^2 + \alpha_2 w + \alpha_3 = 0, \quad (7)$$

where  $\alpha_i \in H^0(\tilde{V}, \mathcal{O}(iH_0))$ ,  $1 \leq i \leq 3$ .

Since  $\mathcal{O}(-H_0)$  is the canonical sheaf of  $\tilde{V}$ , we have

$$h^0(H_0) = p_g - 1, \quad h^0(2H_0) = 3p_g - 5, \quad h^0(3H_0) = 6p_g - 11,$$

by the Riemann-Roch theorem and the Kodaira vanishing theorem. Let  $\phi_0, \dots, \phi_{p_g-2}, \psi$  be a basis of  $H^0(S, \mathcal{O}(K))$  such that  $\phi_0, \dots, \phi_{p_g-2}$  span the module of  $\Lambda$ . Since  $h^0(3K) = 10p_g - 27$ , the following  $10p_g - 26$  products

$$\psi^3, \quad \phi_i \psi^2, \quad \phi_i \phi_j \psi, \quad \phi_i \phi_j \phi_k$$

in  $H^0(3K)$  are linearly dependent. Thus  $S$  is birationally equivalent to a triple covering of  $V$  defined by

$$u^3 + \alpha(z)u^2 + \beta(z)u + \gamma(z) = 0$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are homogeneous forms of respective degree 1, 2 and 3 in the homogeneous coordinates  $(z_0 : \cdots : z_{p_g-2})$  of  $\mathbf{P}^{p_g-2}$  and  $\psi = \Phi_K^* u$ . This triple covering induces via  $\tilde{V} \rightarrow V$  a triple covering  $S'$  of  $\tilde{V}$ . Then the equation of  $S'$  is of the form (7). Conversely, if we choose the  $\alpha_i$  in (7) generic, then the obtained surface is nonsingular and satisfies  $c_1^2 = 3p_g - 6$ .

**4.4** Here we consider the case 4) of 4.1 assuming that  $W_0$  is nonsingular. In this case, we can determine the divisor class of  $S_0$  by Theorem 4.2 once we know the structure of  $W_0$ . For the classification of polarized manifolds of  $\Delta$ -genus one, see Fujita [5].

i)  $p_g = 7$ : Let  $Gr(2, 5)$  be the Grassmannian of two planes in  $\mathbf{C}^5$  embedded into  $\mathbf{P}^9$  by the Plücker embedding. Then  $W_0$  is obtained by cutting  $Gr(2, 5)$  three times by hyperplanes.  $S_0$  is a hypercubic section of  $W_0$ .

ii)  $p_g = 8$ :

ii-1)  $W_0 = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$  embedded by  $|H_1 + H_2 + H_3|$ , where  $H_i$  is the pull-back of a point of the  $i$ -th factor.  $S_0 \sim 3(H_1 + H_2 + H_3)$ .

ii-2)  $W_0 = \mathbf{P}(\Theta_{\mathbf{P}^2})$ , where  $\Theta_{\mathbf{P}^2}$  is the tangent sheaf of  $\mathbf{P}^2$ . If  $H$  denotes the tautological divisor, then  $W_0$  is embedded by  $|H|$ .  $S_0 \sim 3H$ .

iii)  $p_g = 9$ :  $W_0$  is the blowing-up of  $\mathbf{P}^3$  at one point. If we denote by  $H$  and  $E$  the pull-back of a plane of  $\mathbf{P}^3$  and the exceptional divisor, respectively, then  $W_0$  is embedded by  $|2H - E|$ .  $S_0 \sim 6H - 3E$ .

iv)  $p_g = 10$ :  $W_0 = \mathbf{P}^3$  embedded by  $|\mathcal{O}(2)|$  and  $S_0$  is a sextic surface in  $\mathbf{P}^3$ .

It is clear that  $S_0$  with only RDP exists in each case.

**4.5** We consider the case 4) of 4.1 assuming that  $W_0$  is singular. From [7], we know that this case occurs only when  $p_g = 7, 8$ , and that  $W_0$  is represented as the image of the following  $W$ :

i)  $p_g = 7$ :

i-1) Consider the  $\mathbf{P}^2$ -bundle  $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(\Delta_0 + 2\Gamma)) \rightarrow \Sigma_1$ . If we denote by  $\tilde{T}$  the tautological divisor, then  $W \sim \tilde{T} + \tilde{\pi}^*(\Delta_0 + \Gamma)$ .

i-2) Consider the  $\mathbf{P}^1$ -bundle  $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(T)) \rightarrow \mathbf{P}_{1,1,1}$  (see 3.7 for the notation). If  $\tilde{T}$  denotes the tautological divisor, then  $W \sim \tilde{T} + \tilde{\pi}^*(T - F)$ , where  $F$  is a fiber of  $\mathbf{P}_{1,1,1} \rightarrow \mathbf{P}^1$ .

ii)  $p_g = 8$ :

ii-1) Consider the  $\mathbf{P}^2$ -bundle  $\mathbf{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$  over  $\mathbf{P}^2$ . If we denote by  $\tilde{T}$  and  $\tilde{F}$  the tautological divisor and the pull-back of a line in  $\mathbf{P}^2$ , respectively, then  $W \sim \tilde{T} + \tilde{F}$ .

ii-2) Consider the  $\mathbf{P}^2$ -bundle  $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(\Delta_0 + 3\Gamma)) \rightarrow \Sigma_2$ . If  $\tilde{T}$  denotes the tautological divisor, then  $W$  is a nonsingular member of  $|\tilde{T} + \tilde{\pi}^*(\Delta_0 + \Gamma)|$ .

ii-3) Consider the  $\mathbf{P}^1$ -bundle  $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(T)) \rightarrow \mathbf{P}_{1,1,2}$ . If  $\tilde{T}$  denotes the tautological divisor, then  $W \sim \tilde{T} + \tilde{\pi}^*(T - 2F)$ .

ii-4) Consider the  $\mathbf{P}^2$ -bundle  $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(\Delta_0 + 2\Gamma)) \rightarrow \Sigma_0$ . If  $\tilde{T}$  denotes the tautological divisor, then  $W$  is a nonsingular member of  $|\tilde{T} + \tilde{\pi}^*\Delta_0|$ .

In each case, the natural map  $W \rightarrow W_0$  is induced by  $|\tilde{T}|$ . We have  $H^0(W, \mathcal{O}(k\tilde{T})) \simeq H^0(W_0, \mathcal{O}(k))$  for any  $k > 0$ , and  $W$  has some mild singularities in general (see [7]). One can easily check the existence of a surface  $S' \in |3\tilde{T}|_W|$  with only RDP satisfying  $\omega_{S'}^2 = 3h^0(\omega_{S'}) - 6$ .

## 5 Lifting of the canonical maps.

From now on, we study surfaces of type I-0. Let  $S$  be of type I-0 and  $W_0$  the threefold of  $\Delta$ -genus zero on which the canonical image  $S_0$  lies. Then  $W_0$  is a rational normal scroll as we saw in §3. In this section, we assume that it is singular, and discuss whether we can lift the canonical map to a holomorphic map into a nonsingular model of  $W_0$ .

5.1 We consider the case D.2) of 3.7. That is,  $W_0$  is a cone over the Hirzebruch surface  $\Sigma_{c-b}$  embedded into  $\mathbf{P}^{p_g-2}$  by  $|\Delta_0 + c\Gamma|$ , where  $0 < b \leq c$  and  $p_g = b + c + 3$ . We denote by  $\hat{W}$  the total space of the  $\mathbf{P}^1$ -bundle  $\varpi : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\Delta_0 + c\Gamma)) \rightarrow \Sigma_{c-b}$ . Then it is a nonsingular model of  $W_0$ . We let  $H$  be the tautological divisor of  $\hat{W}$  and denote by  $H_\infty$  the unique divisor with  $H_\infty \sim H - \varpi^*(\Delta_0 + c\Gamma)$ . Then the natural map  $\hat{W} \rightarrow W_0$  is induced by  $|H|$ , and  $H_\infty$  is contracted to the vertex of  $W_0$ .

Let  $\Lambda_0$  be the pull-back to  $S$  by  $\Phi_K$  of the linear system of hyperplanes through the vertex of  $W_0$ . We let  $G$  be the fixed part of  $\Lambda_0$  and put  $\Lambda = \Lambda_0 - G$ . We have  $KG = 0$  since  $|K|$  is free from base points. The linear system  $\Lambda$  induces a surjective rational map  $\mu : S \rightarrow \Sigma_{c-b}$ . We let  $\sigma : \hat{S} \rightarrow S$  be the composition of quadric transformations which is the shortest among those with the property that the variable part of  $\sigma^*\Lambda$  is free from base points. We denote by  $\hat{K}$  the canonical divisor of  $\hat{S}$ . Then  $\hat{K} \sim \sigma^*K + E$ , where  $E$  is the exceptional divisor of  $\sigma$ . We have a holomorphic map  $\hat{\mu} : \hat{S} \rightarrow \Sigma_{c-b}$  and  $\sigma^*K \sim M + \hat{E} + \sigma^*G$ , where  $M = \hat{\mu}^*(\Delta_0 + c\Gamma)$  and  $\hat{E}$  is a sum of exceptional curves

of  $\sigma$  satisfying  $\text{Supp}(\hat{E}) = \text{Supp}(E)$  and  $\hat{E} \geq E$ . We remark that  $\deg \hat{\mu}$  is greater than two. Since

$$3(b + c + 1) = (\sigma^*K)^2 = (\sigma^*K)M \geq M^2 = (\deg \hat{\mu})(b + c),$$

we have either

- 1)  $\deg \hat{\mu} = 3$ , or
- 2)  $\deg \hat{\mu} = 4$  and  $b + c = 2, 3$ .

We also remark that  $(\sigma^*K)\hat{E} = M\hat{E} + \hat{E}^2 = 0$  and  $(\sigma^*K)(\sigma^*G) = M(\sigma^*G) + G^2 = 0$ .

**5.2** We consider the case 1) of 5.1. We have  $M(\hat{E} + \sigma^*G) = 3$ . Since  $KG = 0$ ,  $G^2$  is even and so is  $M(\sigma^*G)$ . Thus we have either

- i)  $M\hat{E} = 1$ ,  $M(\sigma^*G) = 2$ , or
- ii)  $M\hat{E} = 3$ ,  $M(\sigma^*G) = 0$ .

We first consider the case i). Then  $\hat{E}^2 = -1$ ,  $G^2 = -2$ . Let  $E_0$  be a  $(-1)$ -curve on  $\hat{S}$ . Then it is a common irreducible component of  $\hat{E}$  and  $E$ . Further, we have  $ME_0 > 0$ , because, if  $ME_0 = 0$ , we can contract  $E_0$  contradicting that  $\sigma$  is the shortest. Thus  $M\hat{E} = ME = ME_0 = 1$ . Then, similarly as in [10, §1], we can show that  $\hat{E} = E$  is a  $(-1)$ -curve  $E_0$ , which we represent by  $E$  for the sake of simplicity. Thus,  $\sigma$  is the blowing-up at  $\sigma(E)$ . Since  $ME = 1$ , we see that  $\hat{\mu}$  maps  $E$  biholomorphically onto a nonsingular rational curve. Put  $\hat{\mu}(E) \sim \alpha\Delta_0 + \beta\Gamma$ , where  $\alpha$  and  $\beta$  are nonnegative integers. Since  $(\Delta_0 + c\Gamma)\hat{\mu}(E) = 1$ , we get  $(\alpha, \beta) = (1, 0)$  or  $(0, 1)$ . We remark that  $(\alpha, \beta) = (1, 0)$  occurs only when  $b = 1$ , and that, if  $b = c = 1$ , we can assume  $(\alpha, \beta) = (0, 1)$  by considering another ruling of  $\Sigma_0$ .

We have  $\hat{K} \sim \hat{\mu}^*(\Delta_0 + c\Gamma) + 2E + \sigma^*G$ . Since  $(\hat{\mu}^*\Gamma)^2 = 0$ ,  $\hat{K}(\hat{\mu}^*\Gamma) = 3 + 2E(\hat{\mu}^*\Gamma) + (\sigma^*G)(\hat{\mu}^*\Gamma)$  is an even integer. Thus  $(\sigma^*G)(\hat{\mu}^*\Gamma)$  is a positive odd integer. Since  $KG = 0$ ,  $G$  consists of  $(-2)$ -curves. Thus any irreducible component of  $\sigma^*G$  is a nonsingular rational curve as well. Since  $M(\sigma^*G) = 2$ , we have at most two irreducible components of  $\sigma^*G$  having positive intersection number with  $M$ . If we have two such components  $G_0$  and  $G_1$  with  $MG_0 = MG_1 = 1$ , then  $\hat{\mu}(\sigma^*G - G_0 - G_1)$  cannot be a divisor, because  $M$  is the pull-back of an ample divisor and  $M(\sigma^*G - G_0 - G_1) = 0$ . Thus  $(\sigma^*G)\hat{\mu}^*\Gamma = (G_0 + G_1)\hat{\mu}^*\Gamma$  is odd. From this and  $MG_i = 1$ , we see that one of them, say  $G_0$ , is mapped to  $\Delta_0$  and  $G_1$  is mapped to a fiber  $\Gamma$ . In particular, we get  $b = 1$ . If we have a component  $G_0$  with  $MG_0 = 2$ , then  $\hat{\mu}(\sigma^*G - G_0)$  is not a divisor and we have either  $\hat{\mu}(G_0) = \Delta_0$ ,  $b = 2$  or  $\hat{\mu}(G_0) \sim \Delta_0 + \Gamma$ ,  $b = 1$  by a simple calculation. In the last case, we have  $c \leq 2$  because the irreducibility of  $G_0$  implies  $(\Delta_0 + \Gamma)\Delta_0 = 2 - c \geq 0$ . In either case, we have  $(\sigma^*G)(\hat{\mu}^*\Gamma) = 1$ .

As a consequence,  $|\hat{\mu}^*\Gamma|$  is a pencil of curves of genus 3 (if  $\hat{\mu}(E) = \Gamma$ ) or 4 (if  $\hat{\mu}(E) = \Delta_0$ ). Put  $D = \sigma_*\hat{\mu}^*\Gamma$ . In the former case,  $|D|$  is a pencil of curves of genus 3

without base points. As we shall see in Lemma 5.3, there is a lifting  $f : S \rightarrow \mathbf{P}_{0,b,c}$  of the canonical map. In the latter case,  $|D|$  is a pencil of curves of genus 4 with one base point. We show that there is a natural map  $f : \hat{S} \rightarrow \mathbf{P}_{0,b,c}$  over the holomorphic map induced by  $|\sigma^*K|$ . Let  $\xi \in H^0(G)$  and  $e \in H^0(\hat{S}, \mathcal{O}(E))$  be sections satisfying  $(\xi) = G$  and  $(e) = E$ . Then we can find a section  $x_0 \in H^0(K)$  such that  $\text{Supp}((\sigma^*x_0))$  does not intersect with  $\sigma^*G + E$ . Then the pair  $(\sigma^*x_0, e\sigma^*\xi)$  defines a holomorphic map  $\hat{f} : \hat{S} \rightarrow \hat{W}$  such that  $\hat{f}^*H_\infty = \sigma^*G + E$ . In particular, we have  $\sigma^*K \sim M + \hat{f}^*H_\infty \sim \hat{f}^*H$ . We remark that there is a holomorphic map  $\nu : \hat{W} \rightarrow \mathbf{P}_{0,b,c}$  which contracts the divisor  $H_\infty$  to a nonsingular rational curve and satisfies  $\nu^*T \sim H$ . Thus we get the desired map by putting  $f = \nu \circ \hat{f}$ .

We next consider the case ii). By Hodge's index theorem, we have  $G = 0$ . Since  $\hat{K}M + M^2 = 6(b+c) + 3 + ME$  is even,  $ME$  is odd. Then  $ME = 1$  or  $3$ , since  $M\hat{E} = 3$  and  $\hat{E} \geq E$ . If  $ME = 3$ , then we have  $M(\hat{E} - E) = 0$ . Thus  $\hat{\mu}(\hat{E} - E)$  is at most 0-dimensional. But then,  $\hat{K}(\hat{\mu}^*\Gamma) = 3 + 2E(\hat{\mu}^*\Gamma)$  is odd, a contradiction. If  $ME = 1$ , then we can find a  $(-1)$ -curve  $E_0$  with  $ME_0 = 1$ . No irreducible component of  $E_1 := E - E_0$  is a  $(-1)$ -curve, because  $ME_1 = 0$ . Thus  $\hat{E} = 3E_0 + \hat{E}_1$  with  $M\hat{E}_1 = 0$ . We have  $\hat{K} \sim M + 4E_0 + E_1 + \hat{E}_1$ . Since  $\hat{K}E_0 = -1$ , we have  $(E_1 + \hat{E}_1)E_0 = 2$ . On the other hand, we have  $-3 = \hat{E}^2 = (3E_0 + \hat{E}_1)^2 = -9 + 6E_0\hat{E}_1 + (\hat{E}_1)^2$ . Thus  $(\hat{E}_1)^2 = 6 - 6E_0\hat{E}_1$ . Since  $M\hat{E}_1 = 0$ , we get  $(\hat{E}_1)^2 \leq 0$  by Hodge's index theorem. Thus  $E_0\hat{E}_1 \geq 1$ . If  $E_0\hat{E}_1 = 2$ , then  $E_0E_1 = 0$ . This means  $E_1 = 0$ , because, otherwise,  $E_1$  must contain a  $(-1)$ -curve. Then  $\hat{E}_1$  is also zero, a contradiction. If  $E_0\hat{E}_1 = 1$ , then  $(\hat{E}_1)^2 = 0$ . Since  $M\hat{E}_1 = 0$ , we have  $\hat{E}_1 = 0$  by Hodge's index theorem, a contradiction. In summary, the case ii) is excluded.

**Lemma 5.3** *Let the situation be in the case i) of 5.2 and suppose that  $S$  has a pencil  $|D|$  of curves of genus 3 described there. Then the canonical map can be lifted to a holomorphic map  $f : S \rightarrow \mathbf{P}_{0,b,c}$ .*

*Proof.* We choose a general  $D \in |D|$  and put  $\hat{D} = \sigma^*D$ . Then  $\hat{D} \in |\hat{\mu}^*\Gamma|$ . Since  $E\hat{D} = 0$ ,  $\sigma^*K|_{\hat{D}}$  is the canonical divisor  $K_{\hat{D}}$  of  $\hat{D}$ . We consider the commutative diagram

$$\begin{array}{ccccc} 0 \rightarrow H^0(\sigma^*K - (i+1)\hat{D}) \rightarrow & H^0(\sigma^*K - i\hat{D}) \rightarrow & H^0(\hat{D}, K_{\hat{D}}) \\ & \uparrow & \uparrow \\ 0 \rightarrow H^0(\Delta_0 + (c-i-1)\Gamma) \rightarrow & H^0(\Delta_0 + (c-i)\Gamma) \rightarrow & H^0(\Gamma, \mathcal{O}(1)) \end{array}$$

for  $i \geq 0$ . We remark that the vertical maps are injective, since  $\sigma^*K \sim \mu^*(\Delta_0 + c\Gamma) + E + \sigma^*G$ .

We measure  $m(i) := \dim_{\mathbf{C}} \text{Im}\{H^0(\sigma^*K - i\hat{D}) \rightarrow H^0(K_{\hat{D}})\}$ . Since  $\Phi_K$  is birational, the natural map  $H^0(\sigma^*K) \rightarrow H^0(K_{\hat{D}})$  is surjective. Thus  $h^0(\sigma^*K - \hat{D}) = p_g - 3 = h^0(\Delta_0 + (c-1)\Gamma)$ . Then, by a diagram chasing, we see that  $H^0(\Delta_0 + (c-i)\Gamma) \rightarrow$

$H^0(\sigma^*K - i\hat{D})$  is bijective for  $i > 0$ . Thus 0 is the greatest integer such that  $m(i) = 3$ . Similarly,  $b$  (resp.  $c$ ) is the greatest integer such that  $m(i) = 2$  (resp.  $m(i) = 1$ ) if  $b < c$ , and  $b$  is the greatest integer such that  $m(i) > 0$  if  $b = c$ .

Since  $H^0(K - iD) \simeq H^0(\sigma^*K - i\hat{D})$  for any  $i$ , we can choose three sections  $x_0 \in H^0(K)$ ,  $x_1 \in H^0(K - bD)$  and  $x_2 \in H^0(K - cD)$  such that they span  $H^0(K_D)$ . Then the triple  $(x_0, x_1, x_2)$  defines a rational map  $f : S \rightarrow \mathbf{P}_{0,b,c}$ . We can assume it holomorphic, since  $|K|$  has no base point. *q.e.d.*

**5.4** Suppose we are in the case 2) of 5.1. We have  $M^2 = 4(b + c)$  and  $M(\hat{E} + \sigma^*G) = 3 - (b + c)$ , where  $(b, c) = (1, 1)$  or  $(1, 2)$ .

We first consider the case  $(b, c) = (1, 1)$ . If  $M\hat{E} = 0$  and  $M(\sigma^*G) = 1$ , then we get  $G^2 = -1$ . This contradicts that  $KG + G^2$  is even. Thus we have  $M\hat{E} = 1$ ,  $M(\sigma^*G) = 0$ . Then  $G^2 = 0$ . Thus  $G = 0$  by Hodge's index theorem. Further, we can show that  $\hat{E} = E$  is a  $(-1)$ -curve and  $\hat{\mu}(E) = \Delta_0$  or a fiber  $\Gamma$  as in 5.2. We can assume that  $\hat{\mu}(E)$  is a fiber by considering another ruling of  $\Sigma_0$  if  $\hat{\mu}(E) = \Delta_0$ . Then  $|\hat{\mu}^*\Gamma|$  is a pencil of curves of genus 3. Since  $E\hat{\mu}^*\Gamma = 0$ , we see that  $S$  also has a pencil  $|D|$ ,  $D = \sigma_*\hat{\mu}^*\Gamma$ , of curves of genus 3 without base points.

We next consider the case  $(b, c) = (1, 2)$ . Then we have  $\hat{E} = G = 0$  by Hodge's index theorem. Thus  $\sigma$  is the identity map and  $K \sim \mu^*(\Delta_0 + 2\Gamma)$ . This shows that  $S$  has a pencil  $|D|$ ,  $D = \mu^*\Gamma$ , of curves of genus 3.

In either case, we can show that the canonical map can be lifted to a map  $f : S \rightarrow \mathbf{P}_{0,b,c}$  as in Lemma 5.3.

**5.5** We consider the case D.3) of 3.7. Thus  $W_0$  is a generalized cone over a rational normal curve. By the argument similar to [11, I, Lemma 1], we have  $K \sim cD + G$ , where  $|D|$  is a pencil and  $G$  is a (possibly zero) effective divisor corresponding to the ridge of  $W_0$ . Since  $K^2 = 3c + 3 \geq cKD$ , we have  $KD \leq 4$ . By Hodge's index theorem and the fact that  $KD + D^2$  is even, we are left the following possibilities:

- i)  $KD = 3, D^2 = 1, DG = 1 \quad (c = 2)$ ,
- ii)  $KD = 4, D^2 = 0, DG = 4 \quad (c = 2, 3)$ .

We show that i) is impossible. We remark that a general  $D \in |D|$  is nonsingular, since  $D^2 = 1$ . Since  $S$  is of type I,  $\Phi_K$  maps  $D$  birationally onto its image. However, since  $KD = 3$ , we have  $h^0(D, \mathcal{O}(K|_D)) \leq 2$  by Clifford's theorem. Thus  $\Phi_K(D)$  is (at most) a rational curve, a contradiction.

In the case ii), we let  $\xi$  be the section of  $\mathcal{O}(G)$  with  $(\xi) = G$  and let  $x_0, x_1$  be two independent element in  $H^0(K)$  which correspond to hyperplanes not containing the ridge of  $W_0$ . We can assume that  $\text{Supp}((x_0)) \cap \text{Supp}((x_1)) \cap G = \emptyset$  since  $|K|$  has no base points. Then the triple  $(x_0, x_1, \xi)$  defines the holomorphic map  $f : S \rightarrow \mathbf{P}_{0,0,c}$ .

We summarize the above results in the following:

**Proposition 5.6** *Let  $S$  be a surface of type I-0 whose canonical image is contained in a singular rational normal scroll. Then it has a pencil  $|D|$  of nonhyperelliptic curves of genus 3 or 4.*

1) *If  $g(D) = 3$ , then  $|D|$  has no base point and there is a lifting  $f : S \rightarrow \mathbf{P}_{0,b,c}$  of the canonical map. In this case,  $0 \leq b \leq 2$  and, if  $b = 0$ , then  $c = 2, 3$ .*

2) *If  $g(D) = 4$ , then  $|D|$  has a base point  $P$ . Let  $\sigma : \hat{S} \rightarrow S$  be the blowing-up with center  $P$ . Then there is a natural map  $f : \hat{S} \rightarrow \mathbf{P}_{0,b,c}$  over the holomorphic map induced by  $|\sigma^*K|$ . In this case,  $b = 1$ .*

*Further, it can be assumed that  $g(D) = 3$  if  $p_g = 5$ .*

## 6 Divisor classes.

We assume that  $p_g \geq 5$  as usual.

**6.1** Let  $S$  be of type I-0. If  $a > 0$ , we denote by  $f : S \rightarrow \mathbf{P}_{a,b,c}$  the natural map induced by  $\Phi_K$ . We let  $f$  have the same meaning as in Proposition 5.6, if  $a = 0$ . Put  $S^* = f(S)$  (or  $f(\hat{S})$  in case 2), Proposition 5.6). We choose a general member  $C \in |K|$  and put  $C^* = f(C)$  (or  $f(\sigma^*C)$  in case 2), Proposition 5.6).

The Picard group of  $W := \mathbf{P}_{a,b,c}$  is generated by the tautological divisor  $T$  and a fiber  $F$ . In particular, we have  $K_W \sim -3T + (p_g - 5)F$ , since  $a + b + c = p_g - 3$ . We have  $T^3 = (p_g - 3)T^2F$  in the Chow ring of  $W$ .

We determine the linear equivalence class of  $S^*$ . Put  $S^* \sim \alpha T + \beta F$  and denote by  $g : S^* \rightarrow \mathbf{P}^1$  the holomorphic map induced by the projection map of  $W$ . Then every fiber of  $g$  is a plane curve of degree  $\alpha$ . Thus we get  $\alpha \geq 4$ , since, otherwise,  $S^*$  has a pencil of curves of genus less than 2 contradicting that it is birational to the surface  $S$  of general type. Since  $\deg S_0 = 3p_g - 6$ , we have

$$3p_g - 6 = T^2(\alpha T + \beta F) = \alpha(p_g - 3) + \beta$$

Thus  $\beta = (3 - \alpha)(p_g - 3) + 3$ . Since  $C$  is of genus  $3p_g - 5$ , the arithmetic genus of  $C^*$  can be written as  $p_a(C^*) = 3p_g - 5 + \delta$  with some nonnegative integer  $\delta$ . Since we have

$$\begin{aligned} p_a(C^*) &= \frac{1}{2}T((\alpha - 2)T + (\beta + p_g - 5)F)(\alpha T + \beta F) + 1 \\ &= \frac{1}{2}[3(\alpha - 2)(p_g - 2) + \alpha(\beta + p_g - 5)] + 1, \end{aligned}$$

we get  $(\alpha - 4)(\beta + 1) = 2\delta - 4$ . Recall that, if  $p_g = 5$ , we can assume  $\alpha = 4$  by Proposition 5.6. So we get the following list:

$$(s.1) \quad S^* \sim 5T - 5F \quad \text{if } p_g = 7, p_a(C^*) = 16.$$

$$(s.2) \quad S^* \sim 5T - 3F \quad \text{if } p_g = 6, p_a(C^*) = 14,$$

$$(s.3) \quad S^* \sim 4T - (p_g - 6)F \quad \text{if } p_a(C^*) = 3p_g - 3,$$

This and Theorem 4.2 in particular show the following:

**Theorem 6.2** *Let  $S$  be a surface of type I. If  $p_g(S) \geq 12$ , then it has a pencil of nonhyperelliptic curves of genus 3. If  $S$  is of type I-0, then the same holds for  $p_g(S) \geq 8$ .*

**Lemma 6.3** *The cases (s.1) and (s.2) cannot occur when  $a > 0$ .*

*Proof.* We assume  $a > 0$  and show that this leads us to a contradiction.

We first consider the case (s.1). Since  $p_g = 7$ , we have  $(a, b, c) = (1, 1, 2)$ . We can identify  $S^*$  with the canonical image  $S_0$ . It has only isolated singular points, because  $p_a(C^*) = g(C)$  implies that its general hyperplane section is nonsingular. Thus it is normal by Serre's criterion. By a direct calculation, we have  $\chi(\mathcal{O}_{S^*}) = \chi(\mathcal{O}_W) - \chi(\mathcal{O}_W(-S^*)) = 4$ . This is impossible, because  $S$  is the minimal resolution of  $S^*$  and  $\chi(\mathcal{O}_S) = 8 > \chi(\mathcal{O}_{S^*})$ .

We next consider the case (s.2). Since  $p_g = 6$ , we have  $(a, b, c) = (1, 1, 1)$ . Thus  $W \simeq \mathbf{P}^1 \times \mathbf{P}^2$ . Under this identification,  $T$  and  $F$  correspond to  $H_1 + H_2$  and  $H_1$ , respectively, where  $H_i$  is the pull-back of a hyperplane in  $\mathbf{P}^1$ . Thus  $S^* \sim 2H_1 + 5H_2$  on  $\mathbf{P}^1 \times \mathbf{P}^2$ . Then it is a double covering of  $\mathbf{P}^2$  via the projection map  $W \rightarrow \mathbf{P}^2$ . This contradicts that it is birational to a type I surface. *q.e.d.*

Let the situation be as in 6.1. In the cases (s.2) and (s.3), the curve  $C^*$  is singular since  $p_a(C^*) > g(C)$ . In the rest of this section, we study its singularity.

**6.4** We denote by  $\phi : C \rightarrow C_0 \subset \mathbf{P}^r$ ,  $r = p_g - 2$ , the restriction of the canonical map of  $S$  to  $C$ . Then  $C_0$  is contained in a surface scroll  $V$ . Recall that  $V$  is one of the surfaces in c) and d) of 2.1. If  $V$  is in d), then it is singular. In this case, however, there exists the lifting  $C \rightarrow \Sigma_{r-1}$  of  $\phi$  (see 5.5). Thus, in either case, we have a holomorphic map  $f_C : C \rightarrow \Sigma_d$  over  $\phi$ , where  $r - d - 1$  is a nonnegative even integer and the natural map  $\Sigma_d \rightarrow V \subset \mathbf{P}^r$  is induced by  $|H_0|$ ,  $H_0 = \Delta_0 + ((r - 1 + d)/2)\Gamma$ . Then  $C^*$  is the image of  $f_C$  and  $\Sigma_d$  can be identified with a member of  $|T|$ . Further,  $T|_{\Sigma_d} \sim H_0$  and  $F|_{\Sigma_d} \sim \Gamma$ . Since  $2K|_C$  is the canonical divisor  $K_C$  of  $C$ , we get  $K_C \sim 2f_C^*H_0$ .

Since  $C^*$  is irreducible, we have  $C^*\Delta_0 \geq 0$ . Then we get the following list corresponding to (s.1), (s.2) and (s.3):

- (c.1)  $C^* \sim 5\Delta_0 + 10\Gamma$  on  $\Sigma_2$ , ( $r = 5$ ,  $p_a(C^*) = 16$ ).
- (c.1)'  $C^* \sim 5\Delta_0 + 5\Gamma$  on  $\Sigma_0$ , ( $r = 5$ ,  $p_a(C^*) = 16$ ),
- (c.2)  $C^* \sim 5\Delta_0 + 7\Gamma$  on  $\Sigma_1$ , ( $r = 4$ ,  $p_a(C^*) = 14$ ),
- (c.3)  $C^* \sim 4\Delta_0 + (r + 2 + 2d)\Gamma$  on  $\Sigma_d$ , ( $p_a(C^*) = 3r + 3$ ),

**Lemma 6.5** *The case (c.1)' of 6.4 can be excluded.*

*Proof.* Since  $p_a(C^*) = g(C)$ , we can identify  $C^*$  with  $C$ . By the adjunction formula and  $K_C \sim 2H_0|_C$ , we have

$$K_C \sim (3\Delta_0 + 3\Gamma)|_C \sim (2\Delta_0 + 4\Gamma)|_C.$$

Thus we get  $\Delta_0|_C \sim \Gamma|_C$ . This is impossible, because  $C$  meets a general member of  $|\Delta_0|$  at distinct five points which are not on the same fiber  $\Gamma$ . *q.e.d.*

**Lemma 6.6** *Assume that  $C^*$  is of type (c.2) in 6.4, then it has a singular point  $P$  of multiplicity 2 on  $\Delta_0$ .*

*Proof.* Since  $p_a(C^*) - g(C) = 1$ ,  $P$  is the unique singular point of multiplicity 2. Let  $\lambda : V^* \rightarrow \Sigma_1$  be the blowing-up with center  $P$  and put  $E = \lambda^{-1}(P)$ . Then the proper transform of  $C^*$  is isomorphic to  $C$  and it is linearly equivalent to  $\lambda^*(5\Delta_0 + 7\Gamma) - 2E$ . Thus we have  $K_C \sim (\lambda^*(3\Delta_0 + 4\Gamma) - E)|_C$  by the adjunction formula. On the other hand, we have  $K_C \sim 2\lambda^*H_0|_C \sim \lambda^*(2\Delta_0 + 4\Gamma)|_C$ . From these, it follows  $(\lambda^*\Delta_0 - E)|_C = 0$ . This means that  $P$  is on  $\Delta_0$ . *q.e.d.*

**Lemma 6.7** *Assume that  $C^*$  is of type (c.3) in 6.4. Then it has two singular points  $P_1, P_2$  of multiplicity 2, which are possibly infinitely near. Further, they are on the same fiber of  $\Sigma_d$ .*

*Proof.* Since  $p_a(C^*) - g(C) = 2$ , the first assertion is clear. Assume that  $P_1$  and  $P_2$  are on the distinct fibers  $\Gamma_1$  and  $\Gamma_2$ , respectively. We let  $\lambda : V^* \rightarrow \Sigma_d$  be the blowing-up with center  $P_1 \cup P_2$ , and put  $E_i = \lambda^{-1}(P_i)$ ,  $1 \leq i \leq 2$ . Since the proper transform of  $C^*$  is isomorphic to  $C$ , we have  $C \sim \lambda^*(4\Delta_0 + (r + 2 + 2d)\Gamma) - 2E_1 - 2E_2$  and  $K_C \sim (\lambda^*(2\Delta_0 + (r + d)\Gamma) - E_1 - E_2)|_C$  by the adjunction formula. On the other hand, since  $K_C \sim 2\lambda^*H_0|_C$ , we have  $K_C \sim \lambda^*(2\Delta_0 + (r - 1 + d)\Gamma)|_C$ . Thus, if we denote the proper transform of  $\Gamma_i$  by  $\tilde{\Gamma}_i$ , we get  $\lambda^*\Gamma|_C \sim \tilde{\Gamma}_1|_C + \tilde{\Gamma}_2|_C$ . This implies that  $\tilde{\Gamma}_1|_C$

and  $\tilde{\Gamma}_2|_C$  are the pull-back of a point via the natural map  $C \rightarrow \mathbf{P}^1$  defined by  $|\lambda^*\Gamma|_C|$ . This contradicts our initial assumption  $\Gamma_1 \neq \Gamma_2$ .

We next suppose that  $C^*$  has an infinitely near double point  $P_1$ . We denote by  $\Gamma_1$  the fiber passing through  $P_1$ . Let  $\lambda_1 : V_1 \rightarrow \Sigma_d$  be the blowing up at  $P_1$  and denote by  $\hat{E}_1, \hat{\Gamma}_1$  and  $\hat{C}$  the exceptional curve, the proper transforms of  $\Gamma_1$  and  $C^*$ , respectively. Then  $\hat{C}$  still has a singular point  $P_2$  of multiplicity 2 on  $\hat{E}_1$ . We must show that  $\hat{\Gamma}_1$  contains  $P_2$ . Assume that this is not the case. Let  $\lambda_2 : V^* \rightarrow V_1$  be the blowing-up of  $V_1$  at  $P_2$  and put  $E_2 = \lambda_2^{-1}(P_2)$ . We denote by  $E_1$  and  $\tilde{\Gamma}_1$  the proper transforms by  $\lambda_2$  of  $\hat{E}_1$  and  $\hat{\Gamma}_1$ , respectively. The proper transform of  $\hat{C}$  by  $\lambda_2$  can be identified with  $C$ . We put  $\lambda = \lambda_2 \circ \lambda_1 : V^* \rightarrow \Sigma_d$ . Since  $C \sim \lambda^*C^* - 2E_1 - 4E_2$ , it does not intersect with  $E_1$ . Then, using  $K_C \sim 2\lambda^*H_0|_C$  and the adjunction formula, we get

$$\begin{aligned} K_C &\sim \lambda^*(2\Delta_0 + (r-1+d)\Gamma)|_C \\ &\sim \lambda^*(2\Delta_0 + (r+d)\Gamma - E_1 - 2E_2)|_C. \end{aligned}$$

Since  $\tilde{\Gamma}_1 \sim \lambda^*\Gamma - E_1 - E_2$  and  $CE_1 = 0$ , we have  $2\tilde{\Gamma}_1|_C \sim \lambda^*\Gamma|_C$ . This is impossible by the same reasoning as in the previous case. Thus  $P_2$  is contained in  $\hat{\Gamma}_1$ . *q.e.d.*

## 7 Surfaces of type I-0: The case (s.1).

By Lemma 6.3 and the definition of the map  $f$ , we see that the cases (s.1) and (s.2) of 6.1 correspond to 2) of Proposition 5.6. In this section, we prove the existence of surfaces of type (s.1).

**7.1** Since  $p_g = 7$ , we have  $(a, b, c) = (0, 1, 3)$  by Proposition 5.6 and Lemma 6.3. We choose sections  $X_0, X_1$  and  $X_3$  of  $\mathcal{O}(T), \mathcal{O}(T-F)$  and  $\mathcal{O}(T-3F)$ , respectively, such that they form a system of homogeneous fiber coordinates on each fiber of  $W \rightarrow \mathbf{P}^1$ . Then the equation of  $S^*$  can be written as

$$\sum_{i,j \geq 0, i+j \leq 5} \phi_{ij} X_0^{5-i-j} X_1^i X_2^j = 0,$$

where  $\phi_{ij}$  is a homogeneous form of degree  $i + 3j - 5$  on  $\mathbf{P}^1$ . Thus  $S^*$  is singular along the rational curve  $B$  defined by  $X_1 = X_2 = 0$ . Note that  $B$  is contracted to the vertex of  $W_0$  under the holomorphic map induced by  $|T|$ . Since  $p_a(C^*) = g(C)$ , we see that  $S^*$  has only isolated singularity except for  $B$ .

We let  $\nu : \hat{W} \rightarrow W$  be the blowing-up with center  $B$ . Since  $\hat{W}$  is nothing but the manifold appeared in §5, we use the notation there. In particular, we have  $\nu^*T \sim H$ ,  $\nu^*F \sim \varpi^*\Gamma$  and  $\nu^{-1}(G_0) = H_\infty$ . Let  $S'$  be the proper transform of  $S^*$  by  $\nu$ . Then we have  $S' = \hat{f}(\hat{S})$ , where  $\hat{f} : \hat{S} \rightarrow \hat{W}$  is the holomorphic map defined in 5.2. If the multiplicity of  $S^*$  at a generic point of  $B$  is  $k$ , then  $S'$  is linearly equivalent to

$5H - 5\varpi^*\Gamma - kH_\infty$ . From this, we infer readily  $k = 2$ , since  $\hat{\mu} : \hat{S} \rightarrow \Sigma_2$  is of degree 3. Thus  $S' \sim 3H + \varpi^*(2\Delta_0 + \Gamma)$  on  $\hat{W}$ .

We choose sections  $Y_0$  and  $Y_1$  of  $\mathcal{O}(H)$  and  $\mathcal{O}(H_\infty)$ , respectively, such that they form a system of homogeneous coordinates on each fiber of  $\varpi : \hat{W} \rightarrow \Sigma_2$ . Then the equation of  $S'$  can be written as

$$\psi_0\zeta^2Y_0^3 + \psi_1\zeta Y_0^2Y_1 + \psi_2\zeta Y_0Y_1^2 + \psi_3Y_1^3 = 0, \quad (8)$$

where  $\zeta, \psi_0, \psi_1, \psi_2$  and  $\psi_3$  are sections on  $\Sigma_2$  of  $\mathcal{O}(\Delta_0), \mathcal{O}(\Gamma), \mathcal{O}(2\Delta_0+4\Gamma), \mathcal{O}(3\Delta_0+7\Gamma)$  and  $\mathcal{O}(5\Delta_0+10\Gamma)$ , respectively. Thus  $S'$  is singular along the rational curve  $\Lambda$  defined by  $\zeta = Y_1 = 0$ . Since  $(5\Delta_0 + 10\Gamma)\Delta_0 = 0$  on  $\Sigma_2$ ,  $\psi_3$  is constant on  $\Delta_0$ . Since  $S'$  is irreducible,  $\psi_3$  does not vanish on  $\Delta_0$ . We also remark that  $\psi_0$  is not identically zero. Thus the multiplicity of  $S'$  at a generic point of  $\Lambda$  is 2. Moreover,  $S'$  has only isolated singular points except for the double curve  $\Lambda$ .

Let  $\tau : \tilde{W} \rightarrow \hat{W}$  be the blowing-up with center  $\Lambda$  and put  $\mathcal{E} = \tau^{-1}(\Lambda)$ . If  $\tilde{S}$  is the proper transform of  $S'$  by  $\tau$ , then it is linearly equivalent to  $\tau^*S' - 2\mathcal{E}$ . Since the normal sheaf of  $\Lambda \simeq \mathbf{P}^1$  in  $\hat{W}$  is  $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$ ,  $\mathcal{E}$  is isomorphic to  $\Sigma_1$ .

**7.2** We show that  $\tilde{S}$  is nonsingular in a neighbourhood of  $\tilde{S} \cap \mathcal{E}$ . For any point  $P \in \Lambda$ , we can assume that  $(\zeta, y, t)$  forms a local coordinate system in a neighbourhood  $U$  of  $P$ , where  $y = Y_1/Y_0$  and  $t$  is a local parameter of  $\Lambda$  at  $P$ . We regard the  $\psi_i$  in (8) as functions of  $(t, \zeta)$  in  $U$ . Then, on  $U$ ,  $S'$  is defined by

$$\psi_3y^3 + \psi_2\zeta y^2 + \psi_1\zeta y + \psi_0\zeta^2 = 0.$$

Since  $\psi_3$  does not vanish in a neighbourhood of  $\Lambda$ , we can assume  $\psi_2 = 0$  by replacing  $y$  by  $y + \psi_2\zeta/(3\psi_3)$ .

We cover  $\tau^{-1}(U)$  by open sets  $U_1, U_2$  with coordinates  $(u_1, v_1, t_1), (u_2, v_2, t_2)$ , respectively, where

$$\zeta = u_1v_1 = u_2, \quad y = u_1 = u_2v_2, \quad t = t_1 = t_2.$$

Then  $\tilde{S}$  is defined by

$$\begin{aligned} \psi_3u_1 + \psi_1v_1 + \psi_0v_1^2 &= 0 & \text{on } U_1, \\ \psi_3u_2v_2^3 + \psi_1v_2 + \psi_0 &= 0 & \text{on } U_2. \end{aligned} \quad (9)$$

Since  $\psi_3$  is a nonzero constant on  $v_1 = 0$ ,  $\tilde{S}$  is nonsingular in  $U_1$ . Since  $\psi_0$  is at most of multiplicity 1 at  $P$ ,  $\tilde{S}$  is nonsingular in  $U_2$ .

Thus  $\tilde{S}$  has only isolated singularities. Hence it is normal by Serre's criterion. We remark that  $\tilde{S}$  is not necessarily the normalization of  $S'$ : If  $\psi_1$  does not vanish identically on  $\Delta_0$ , then  $\mathcal{E} \cap \tilde{S}$  consists of two disjoint curves  $E$  and  $\tilde{G}$ , where  $E$  is defined by  $v_1 = 0$ , and  $\tilde{G}$  is defined by  $\psi_1 + \psi_0v_1 = 0$  on  $U_1$  and  $\psi_1v_2 + \psi_0 = 0$  on

$U_2$ . Thus  $\tilde{S}$  is the normalization of  $S'$  in this case. On the other hand, if  $\psi_1$  vanishes identically on  $\Delta_0$ , then we have  $\mathcal{E}|_{\tilde{S}} = 2E + \hat{G}$ , where  $\hat{G}$  is defined by  $\psi_0 = 0$ . Since  $\psi_0$  has a simple zero on  $\Delta_0$ ,  $\hat{G}$  is a fiber of  $\mathcal{E} \rightarrow \Lambda$ . Thus  $\tilde{S}$  is obtained by blowing up a point on the normalization of  $S'$ .

We show that  $E$  is a  $(-1)$ -curve on  $\tilde{S}$ . We let  $Z$  be the divisor on  $\hat{W}$  defined by  $\varpi^*\zeta$  and  $\tilde{Z}$  its proper transform by  $\tau$ . Then  $\tilde{Z} \sim \tau^*\varpi^*\Delta_0 - \mathcal{E}$  and it is given by  $v_1 = 0$ . Thus (9) shows that  $E = \tilde{Z}|_{\tilde{S}}$ . We identify  $\tilde{Z}$  with  $\Sigma_1$ . Then we have

$$\tau^*H|_{\tilde{Z}} \sim \Delta_0 + \Gamma, \tau^*\varpi^*\Gamma|_{\tilde{Z}} \sim \Gamma, \mathcal{E}|_{\tilde{Z}} \sim \Delta_0.$$

Further we have

$$\begin{aligned} \tilde{Z}|_{\tilde{Z}} &\sim (\tau^*\varpi^*(\Delta_0 + 2\Gamma) - 2\tau^*\varpi^*\Gamma - \mathcal{E})|_{\tilde{Z}} \\ &\sim -\Delta_0 - 2\Gamma, \end{aligned}$$

since we have  $\Delta_0(\Delta_0 + 2\Gamma) = 0$  on  $\Sigma_2$ . Thus we get

$$\begin{aligned} E^2 &= \tilde{Z}^2\tilde{S} \\ &= \tilde{Z}^2[\tau^*(3H + \varpi^*\Gamma) + 2\tilde{Z}] \\ &= -(\Delta_0 + 2\Gamma)[3(\Delta_0 + \Gamma) + \Gamma - 2(\Delta_0 + 2\Gamma)] \quad (\text{on } \tilde{Z} \simeq \Sigma_1) \\ &= -1. \end{aligned}$$

**Lemma 7.3** *The map  $\hat{f} : \hat{S} \rightarrow S'$  factors through  $\tilde{S}$ .*

*Proof.* Let  $\mathcal{I}$  be the ideal sheaf of  $\Lambda$  in  $S'$ . We show that  $\hat{f}^{-1}\mathcal{I} \cdot \mathcal{O}_{\hat{S}}$  is invertible. For this purpose, we freely use the notation of 5.2. Let  $\mathcal{G}$  be the greatest common divisor of  $\hat{f}^*H_\infty$  and  $\hat{f}^*\varpi^*\Delta_0 = \hat{\mu}^*\Delta_0$ . Recall that  $\hat{f}^*H_\infty = \sigma^*G + E$  and that  $\sigma^*G$  can be written as  $G_0 + G_1 + G'$  with  $\hat{\mu}(G_0) = \Delta_0$ ,  $\hat{\mu}(G_1) \sim \Gamma$ . Since  $\hat{\mu}(E) = \Delta_0$ , we have  $\mathcal{G} \geq E$  and  $\mathcal{G} \geq G_0$ . Thus it suffices for us to show that  $\hat{f}^*H_\infty - \mathcal{G}$  and  $\hat{\mu}^*\Delta_0 - \mathcal{G}$  do not meet.

Let  $G_2$  be an irreducible component of  $G'$ . Since  $\hat{\mu}(G_2)$  is a point, we have  $G_2(\hat{\mu}^*\Delta_0) = 0$ . Thus, if  $G_2$  intersects with a component of  $\hat{\mu}^*\Delta_0$ , it must be a component of  $\mathcal{G}$ . We next consider the curve  $G_1$ . Clearly, it is not a component of  $\hat{\mu}^*\Delta_0$ . Since  $G_1(\hat{\mu}^*\Delta_0) = 1$ , it suffices to show  $G_1\mathcal{G} > 0$ .

The case  $G_0 \neq E$ : We have  $\mathcal{G} \geq E + G_0$ . Since  $E$  is a  $(-1)$ -curve, we have

$$-1 = \hat{K}E = ME + 2E^2 + E(\sigma^*G). \quad (10)$$

Thus  $E(\sigma^*G) = 0$ . Then each irreducible component of  $\sigma^*G$  is a  $(-2)$ -curve. In particular,  $\hat{K}G_1 = 0$  and we have  $G_0G_1 + G'G_1 = 1$ . If  $G_0G_1 = 1$ , then we get  $G_1\mathcal{G} > 0$  since  $\mathcal{G} \geq E + G_0$ . We assume  $G_0G_1 = 0$  and  $G'G_1 = 1$ . Let  $G_2$  be the component of  $G'$  with  $G_1G_2 = 1$ . If it is a component of  $\mathcal{G}$ , then we are done. So we assume that  $G_2$

is not a component of  $\mathcal{G}$  and show that this leads us to a contradiction. Since  $G_2$  is a  $(-2)$ -curve, we have

$$0 = \hat{K}G_2 = MG_2 + 2EG_2 + G_0G_2 + G_1G_2 + G_2^2 + G_2(G' - G_2).$$

Since  $MG_2 = EG_2 = G_0G_2 = 0$ , we have  $G_2(G' - G_2) = 1$ . Thus we can find a component  $G_3$  of  $G'$  such that  $G_2G_3 = 1$ . Since  $G_2$  is not a component of  $\mathcal{G}$ , so is  $G_3$ . Then we have, as above,  $G_3(G' - G_2 - G_3) = 1$  since  $G_3$  is a  $(-2)$ -curve. Thus we can find a component  $G_4$  of  $G' - G_2 - G_3$  with  $G_3G_4 = 1$ , which is not a component of  $\mathcal{G}$ . Continuing this procedure, we would get infinite number of components  $G_i$ ,  $i \geq 2$ , of  $G'$ . This is impossible.

The case  $G_0 = E$ : We have  $E(G_1 + G') = 1$  by (10). If  $EG_1 = 1$ , then we are done. So we assume that  $EG_1 = 0$  and  $EG' = 1$ . Then  $G_1$  is a  $(-2)$ -curve. Thus, as in the previous case, we can find a component  $G_2$  of  $G'$  with  $G_1G_2 = 1$ . If  $G_2$  is a component of  $\mathcal{G}$ , we are done. So we assume that this is not the case. We in particular have  $G_2E = 0$ . Thus it is a  $(-2)$ -curve and we can find a component  $G_3$  of  $G' - G_2$  with  $G_2G_3 = 1$ . Then, as in the previous case, we get a contradiction by continuing this procedure. Thus we have a chain  $\mathcal{G}' = G_1 + G_2 + \cdots + G_n \leq \mathcal{G}$  of irreducible components of  $\sigma^*G$  such that  $G_iG_{i+1} = 1$ ,  $G_i$  is a  $(-2)$ -curve for  $i < n$ , and  $G_n$  is a  $(-3)$ -curve. *q.e.d.*

**7.4** We calculate the invariants of  $\tilde{S}$ . Since  $\tau^*(K_{\hat{W}} + S' - \mathcal{E}) \sim \tau^*H + \tilde{Z}$ , the dualizing sheaf of  $\tilde{S}$  is given by  $\omega_{\tilde{S}} = \mathcal{O}(\tau^*H|_{\tilde{S}} + E)$ . Since  $E(\tau^*H) = 0$  and  $E^2 = -1$ , we get  $\omega_{\tilde{S}}^2 = 14$  by

$$\begin{aligned} (\tau^*H|_{\tilde{S}})^2 &= (\tau^*H)^2(\tau^*(3H + \varpi^*(2\Delta_0 + \Gamma)) - 2\mathcal{E}) \\ &= H^2(3H + \varpi^*(2\Delta_0 + \Gamma)) \quad (\text{on } \hat{W}) \\ &= 15, \end{aligned}$$

where we have used the equality  $H^2 = H\varpi^*(\Delta_0 + 3\Gamma)$  in the Chow ring of  $\hat{W}$ .

In order to calculate  $\chi(\mathcal{O}_{\tilde{S}})$ , we use the exact sequences

$$0 \rightarrow \mathcal{O}(K_{\hat{W}}) \rightarrow \mathcal{O}(\tau^*H + \tilde{Z}) \rightarrow \omega_{\tilde{S}} \rightarrow 0,$$

and

$$0 \rightarrow \mathcal{O}(\tau^*H) \rightarrow \mathcal{O}(\tau^*H + \tilde{Z}) \rightarrow \mathcal{O}_{\tilde{Z}}(-\Gamma) \rightarrow 0.$$

From these, we get  $H^q(\omega_{\tilde{S}}) \simeq H^q(\mathcal{O}(\tau^*H + \tilde{Z})) \simeq H^q(\mathcal{O}(\tau^*H)) \simeq H^q(\hat{W}, \mathcal{O}(H))$  for  $q < 2$ . Thus  $h^0(\omega_{\tilde{S}}) = 7$ ,  $h^1(\omega_{\tilde{S}}) = 0$ . Since  $\tilde{S}$  is a normal surface on the nonsingular threefold  $\hat{W}$ , we see that  $\tilde{S}$  has only RDP as its singularity from the equality  $\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\hat{S}})$ .

7.5 Conversely, we start from  $S'$  defined by (8). We can assume that it has only RDP except for the double curve  $\Lambda$  if we choose the  $\psi_i$  in (8) general. Let  $\tilde{S}$  be its proper transform by  $\tau: \tilde{W} \rightarrow \hat{W}$  described above. Then it has only RDP. Let  $\hat{S}$  be the minimal resolution of  $\tilde{S}$ . Then  $\hat{S}$  has a  $(-1)$ -curve  $E$ . Let  $\sigma: \hat{S} \rightarrow S$  be the contraction of  $E$ . Then  $S$  is a minimal surface satisfying  $p_g = 7$ ,  $q = 0$  and  $c_1^2 = 15$ . Since  $H^0(K)$  is in bijection with  $H^0(\hat{W}, \mathcal{O}(H))$ , it is of type I.

## 8 Surfaces of type I-0: The case (s.2).

In this section, we study the case (s.2) of 6.1.

8.1 Since  $p_g = 6$ , we have  $(a, b, c) = (0, 1, 2)$  by Lemma 6.3. As in 7.1, we can show that  $S' = \hat{f}(\hat{S})$  is linearly equivalent to  $3H + \varpi^*(2\Delta_0 + \Gamma)$  on  $\hat{W} = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\Delta_0 + 2\Gamma))$  and that its equation can be written as

$$\psi_0 \zeta Y_0^3 + \psi_1 Y_0^2 Y_1 + \psi_2 Y_0 Y_1^2 + \psi_3 Y_1^3 = 0, \quad (11)$$

where  $(Y_0, Y_1)$  is a system of homogeneous coordinates on fibers of  $\varpi: \hat{W} \rightarrow \Sigma_1$ , and  $\zeta, \psi_0$  and  $\psi_i, 1 \leq i \leq 3$ , are sections of  $\mathcal{O}(\Delta_0), \mathcal{O}(\Delta_0 + \Gamma)$  and  $\mathcal{O}((i+2)\Delta_0 + (2i+1)\Gamma)$ , respectively. We let  $Z$  be the divisor on  $\hat{W}$  defined by  $\zeta$ .

We study the singular locus of  $S'$ . For this purpose, we let  $H \in |H|$  be a general member. Then it is isomorphic to  $\Sigma_1$  and  $H \cap S'$  can be identified with  $C^*$ . Since  $p_a(C^*) = g(C) + 1$ ,  $C^*$  has a singular point  $P$  of multiplicity 2. By Lemma 6.6,  $P \in H \cap Z$ . If we vary  $H$  in  $|H|$ , such  $P$  traces a curve  $\Lambda$  on  $Z \simeq \Sigma_1$ , which is a multiple curve of  $S'$ . Putting  $\zeta = 0$  in (11), we get  $Y_1(\psi_1 Y_0^2 + \psi_2 Y_0 Y_1 + \psi_3 Y_1^2) = 0$ . Thus, by identifying  $Z$  with  $\Sigma_1$ , we get  $S'|_Z = \Delta_0 + G'$  with  $G' \sim 2\Delta_0 + 2\Gamma$ . Since  $G' \geq 2\Lambda$ , we have either

- i)  $S'|_Z = \Delta_0 + 2\Lambda$ ,  $\Lambda \sim \Delta_0 + \Gamma$  is irreducible, or
- ii)  $S'|_Z = 3\Delta_0 + 2\Lambda$ ,  $\Lambda \sim \Gamma$ .

We let  $\hat{D}$  be a general member of  $|\hat{\mu}^*\Gamma|$  on  $\hat{S}$ , and put  $D' = \hat{f}(\hat{D})$ . Then  $D' \sim \varpi^*\Gamma|_{S'}$ . Since  $\omega_{D'} = \mathcal{O}_{D'}(H + \varpi^*(\Delta_0 + \Gamma))$ , we have

$$\deg \omega_{D'} = (H + \varpi^*(\Delta_0 + \Gamma))(3H + \varpi^*(2\Delta_0 + \Gamma))\varpi^*\Gamma = 8.$$

Thus  $D'$  is of arithmetic genus 5. Since  $\hat{D}$  is of genus 4,  $D'$  has a singular point of multiplicity 2. Thus  $S'$  has a multiple curve which intersects with  $D'$ . In the case i),  $\Lambda$  has such a property. In the case ii), there is a multiple curve other than  $\Lambda$ .

**8.2** Here we consider the case i) of 8.1. Since a general  $H \in |H|$  induces on  $Z \simeq \Sigma_1$  an irreducible divisor linearly equivalent to  $\Delta_0 + \Gamma$  and since the restriction map  $H^0(\hat{W}, \mathcal{O}(H)) \rightarrow H^0(Z, \mathcal{O}(H|_Z))$  is surjective, we can assume that  $\Lambda = Z \cap (Y_0)$  by a suitable change of  $(Y_0) \in |H|$ . Thus  $\psi_2$  and  $\psi_3$  in (11) can be divided by  $\zeta$ . Since  $\Lambda$  is a double curve,  $S'$  is defined by

$$\psi_0 \zeta Y_0^3 + \psi_1 Y_0^2 Y_1 + \psi_2' \zeta Y_0 Y_1^2 + \psi_3' \zeta^2 Y_1^3 = 0, \quad (12)$$

where  $\psi_i'$ ,  $i = 2, 3$ , is a section of  $\mathcal{O}(3\Delta_0 + (2i+1)\Gamma)$  on  $\Sigma_1$ . We remark that  $\psi_1$  must be a nonzero constant on  $\Delta_0$ , since  $S'$  is irreducible. In particular, the multiplicity of  $S'$  along  $\Lambda$  is 2. Further, we can assume that  $\psi_2'$  and  $\psi_3'$  do not vanish identically on  $\Delta_0$ . This can be seen as follows: Let  $\omega$  be a section of  $\mathcal{O}(2\Gamma)$  and put  $\phi = \zeta\omega$ . If we replace  $Y_0$  by  $Y_0 + \phi Y_1$ , then  $\psi_3'$ , for example, is replaced by  $\psi_0 \zeta^2 \omega^3 + \psi_1 \omega^2 + \psi_2' \zeta \omega + \psi_3'$ . Thus, by such a change of coordinates, we can assume that  $\psi_3'$  does not vanish identically on  $\Delta_0$ , since  $\psi_1 \neq 0$  on  $\Delta_0$ .

Let  $\tau : \tilde{W} \rightarrow \hat{W}$  be the blowing-up with center  $\Lambda$  and put  $\mathcal{E} = \tau^{-1}(\Lambda)$ . Let  $\tilde{S}$  be the proper transform of  $S'$  by  $\tau$ . Then  $\tilde{S} \sim \tau^* S' - 2\mathcal{E}$ . For any point  $Q \in \Lambda$ , we can assume that  $(\zeta, y, t)$  forms a local coordinate system in a neighbourhood  $U$  of  $Q$ , where  $y = Y_0/Y_1$  and  $t$  is a local parameter of  $\Lambda$  at  $Q$ . We cover  $\tau^{-1}(U)$  by open sets  $U_1, U_2$  with coordinates  $(u_1, v_1, t_1), (u_2, v_2, t_2)$ , respectively, where

$$\zeta = u_1 v_1 = u_2, \quad y = u_1 = u_2 v_2, \quad t = t_1 = t_2.$$

Then  $\tilde{S}$  is defined by

$$\begin{aligned} \psi_0 u_1^2 v_1 + \psi_1 + \psi_2' v_1 + \psi_3' v_1^2 &= 0 & \text{on } U_1, \\ \psi_0 u_2^2 v_2^3 + \psi_1 v_2^2 + \psi_2' v_2 + \psi_3' &= 0 & \text{on } U_2. \end{aligned}$$

Since  $\psi_1$  is a nonzero constant on  $\Delta_0$ ,  $\tilde{S}$  does not meet the divisor  $(v_1)$  on  $U_1$ . Since  $\psi_2'$  and  $\psi_3'$  do not vanish identically on  $\Delta_0$ ,  $\tilde{S}$  has at most isolated singularities on  $U_2$ . The curve  $\mathcal{E} \cap \tilde{S}$  is defined by  $u_1 = \psi_1 + \psi_2' v_1 + \psi_3' v_1^2 = 0$  on  $U_1$  and  $u_2 = \psi_1 v_2^2 + \psi_2' v_2 + \psi_3' = 0$  on  $U_2$ . Thus  $\tilde{S}$  is the normalization of  $S'$ . In particular, the map  $\tilde{f} : \tilde{S} \rightarrow S'$  factors through  $\tilde{S}$ .

Let  $\tilde{Z}$  be the proper transform of  $Z$ . Then we have  $\omega_{\tilde{S}} = \mathcal{O}_{\tilde{S}}(\tau^* H + \tilde{Z})$ . We recall that  $S'$  contains a rational curve  $E$  defined by  $\zeta = Y_1 = 0$  and that it is nonsingular in a neighbourhood of  $E$ . Thus the above local calculation shows in particular that  $\tilde{Z}$  induces  $E$  on  $\tilde{S}$ . We have the following by identifying  $\tilde{Z}$  with  $\Sigma_1$ .

$$\tau^* H|_{\tilde{Z}} \sim \mathcal{E}|_{\tilde{Z}} \sim \Delta_0 + \Gamma, \quad \tau^* \varpi^* \Gamma|_{\tilde{Z}} \sim \Gamma, \quad \tilde{Z}|_{\tilde{Z}} \sim -\Delta_0 - 2\Gamma.$$

Then

$$E^2 = \tilde{Z}^2 \tilde{S} = \tilde{Z}^2 [\tau^*(3H + \varpi^* \Gamma) + 2\tilde{Z}] = -1.$$

Thus  $E$  is a  $(-1)$ -curve. The invariants of  $\tilde{S}$  can be calculated as in 7.4. We get  $\omega_{\tilde{S}}^2 = 11$ ,  $h^0(\omega_{\tilde{S}}) = 6$  and  $h^1(\omega_{\tilde{S}}) = 0$ . We remark that  $H^0(\omega_{\tilde{S}})$  is in bijection with  $H^0(\hat{W}, \mathcal{O}(H))$ . Since  $\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\hat{S}})$ , we see that  $\tilde{S}$  has only RDP. The existence of surfaces of this type is now clear.

**8.3** We consider the case ii) of 8.1. In this case,  $\psi_i$ ,  $1 \leq i \leq 2$ , can be divided by  $\zeta$ . We let  $P$  be the point on  $\Sigma_1$  with  $\varpi^{-1}(P) = \Lambda$ . Then  $P$  lies on the zero section  $\Delta_0$ . Since  $\Lambda$  is a double curve, the equation of  $S'$  can be rewritten as

$$\phi_0 \zeta^2 Y_0^3 + \phi_1 \zeta Y_0^2 Y_1 + \phi_2 \zeta Y_0 Y_1^2 + \phi_3 Y_1^3 = 0, \quad (13)$$

where  $\phi_0, \phi_1, \phi_2$  and  $\phi_3$  are sections of  $\mathcal{O}(\Gamma)$ ,  $\mathcal{O}(2\Delta_0+3\Gamma)$ ,  $\mathcal{O}(3\Delta_0+5\Gamma)$  and  $\mathcal{O}(5\Delta_0+7\Gamma)$  on  $\Sigma_1$ , respectively, which satisfy the conditions:

- 0)  $\phi_0$  is not identically zero.
- 1)  $\phi_1$  and  $\phi_2$  vanish at  $P$ .
- 2)  $P$  is a singular point of multiplicity 2 of the curve  $(\phi_3)$ .

Thus, if  $\zeta_0$  is a section of  $\mathcal{O}(\Delta_0+\Gamma)$  such that  $(\zeta_0, \zeta)$  forms a system of homogeneous coordinates on fibers of  $\Sigma_1$ , then  $\phi_i$  can be written as

$$\begin{aligned} \phi_1 &= \alpha_0 \omega \zeta_0^2 + \alpha_1 \zeta_0 \zeta + \alpha_2 \zeta^2, \\ \phi_2 &= \beta_0 \omega \zeta_0^3 + \beta_1 \zeta_0^2 \zeta + \beta_2 \zeta_0 \zeta^2 + \beta_3 \zeta^3, \\ \phi_3 &= \omega^2 \zeta_0^5 + \gamma_1 \omega \zeta_0^4 \zeta + \gamma_2 \zeta_0^3 \zeta^2 + \gamma_3 \zeta_0^2 \zeta^3 + \gamma_4 \zeta_0 \zeta^4 + \gamma_5 \zeta^5, \end{aligned}$$

where  $\omega$  is a linear form on  $\mathbf{P}^1$  which gives  $P$  on  $\Delta_0$ , and  $\alpha_i, \beta_j, \gamma_k$  are homogeneous forms on  $\mathbf{P}^1$  whose degree can be determined uniquely by the linear equivalence classes of  $\phi$ 's. In particular,  $\alpha_0$  is a constant. Further, we can assume that  $(\phi_3)$  is irreducible and nonsingular except for  $P$ .

We let  $\Lambda_0$  be the curve on  $\hat{W}$  defined by  $\zeta = Y_1 = 0$ . Then  $S'$  has only isolated singular points except for the curve  $\Lambda + \Lambda_0$ . We let  $\tau_1 : W_1 \rightarrow \hat{W}$  be the blowing-up with center  $\Lambda_0$  and denote by  $\mathcal{E}'$  the exceptional divisor. Since the normal sheaf of  $\Lambda_0 \simeq \mathbf{P}^1$  in  $\hat{W}$  is  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ ,  $\mathcal{E}'$  is isomorphic to  $\Sigma_0$ . If we denote by  $S''$  the proper transform of  $S'$  by  $\tau_1$ , then we can show that  $\hat{f} : \hat{S} \rightarrow S'$  factors through it as in Lemma 7.3.

For any point  $Q \in \Lambda_0$ , we can assume that  $(\zeta, y, t)$  forms a local coordinate system in a neighbourhood  $U$  of  $Q$ , where  $y = Y_1/Y_0$  and  $t$  is a local parameter of  $\Lambda_0$  at  $Q$ . We cover  $\tau^{-1}(U)$  by open sets  $U_1, U_2$  with coordinates  $(u_1, v_1, t_1), (u_2, v_2, t_2)$ , respectively, where

$$\zeta = u_1 v_1 = u_2, \quad y = u_1 = u_2 v_2, \quad t = t_1 = t_2.$$

Then  $S''$  is defined by

$$\begin{aligned} \phi_0 v_1^2 + \phi_1 v_1 + \phi_2 u_1 v_1 + \phi_3 u_1 &= 0 \quad \text{on } U_1, \\ \phi_0 + \phi_1 v_2 + \phi_2 u_2 v_2^2 + \phi_3 u_2 v_2^3 &= 0 \quad \text{on } U_2. \end{aligned}$$

Since  $\phi_0$  is the function in  $t$  of degree 1,  $S''$  is nonsingular on  $U_2$ . On  $U_1$ , it is singular along the curve  $v_1 = \omega = 0$ , which is the proper transform  $\Lambda'$  of  $\Lambda$  by  $\tau_1$ .  $S'' \cap \mathcal{E}'$  is defined by  $u_1 = v_1(\phi_0 v_1 + \phi_1) = 0$  on  $U_1$  and  $u_2 = \phi_0 + \phi_1 v_2 = 0$  on  $U_2$ .

Let  $\tau_2 : \tilde{W} \rightarrow W_1$  be the blowing-up with center  $\Lambda'$ . Let  $\mathcal{E}_1$  be the proper transform of  $\mathcal{E}'$  by  $\tau_2$ , and let  $\mathcal{E}_2$  be the exceptional divisor of  $\tau_2$ . Then  $\mathcal{E}_1$  is  $\Sigma_0$  blown up at one point, and  $\mathcal{E}_2$  is isomorphic to  $\Sigma_1$ . The proper transform  $\tilde{S}$  of  $S'$  by  $\tau = \tau_1 \circ \tau_2$  is linearly equivalent to  $\tau^* S' - 2(\mathcal{E}_1 + \mathcal{E}_2)$ .

We let  $V$  be a sufficiently small open neighbourhood in  $W_1$  of a point  $Q \in \Lambda'$ . On  $V$ , we can regard  $\phi_0, \alpha_i, \beta_j, \gamma_k$  as functions of  $\omega$ . In particular, we can write  $\phi_0 = r_0\omega + r_1$  with some constants  $r_0, r_1$ . If  $Q \in \mathcal{E}'$ , then we can assume that  $(u_1, v_1, \omega)$  is a local coordinate system on  $V$ . We cover  $\tau_2^{-1}(V)$  by open sets  $V_1, V_2$  with coordinates  $(u_1, w_1, x_1)$  and  $(u_1, w_2, x_2)$ , respectively, where

$$\omega = w_1 x_1 = w_2, \quad v_1 = w_1 = w_2 x_2.$$

Then  $\tilde{S}$  is defined by

$$\begin{aligned} \phi_0 + \phi'_1 + \phi'_2 u_1 + \phi'_3 u_1 &= 0 && \text{on } V_1, \\ \phi_0 x_2^2 + \phi''_1 x_2 + \phi''_2 u_1 x_2 + \phi''_3 u_1 &= 0 && \text{on } V_2. \end{aligned}$$

where

$$\begin{aligned} \phi'_1 &= \alpha_0 x_1 + \alpha_1 u_1 + \alpha_2 u_1^2 w_1, \\ \phi'_2 &= \beta_0 x_1 + \beta_1 u_1 + \beta_2 u_1^2 w_1 + \beta_3 u_1^3 w_1^2, \\ \phi'_3 &= x_1^2 + \gamma_1 u_1 x_1 + \sum_{i=2}^5 \gamma_i u_1^i w_1^{i-2}, \\ \phi''_1 &= \alpha_0 + \alpha_1 u_1 x_2 + \alpha_2 u_1^2 w_2 x_2^2, \\ \phi''_2 &= \beta_0 + \beta_1 u_1 x_2 + \beta_2 (u_1 x_2)^2 w_2 + \beta_3 (u_1 x_2)^3 w_2^2, \\ \phi''_3 &= 1 + \gamma_1 u_1 x_2 + \sum_{i=2}^5 \gamma_i (u_1 x_2)^i w_2^{i-2}. \end{aligned}$$

Thus  $\tilde{S}$  is nonsingular on  $V_2$  and it has at most isolated singular points on  $V_1$ . Varying  $Q$  on  $\Lambda'$ , we find similarly that  $\tilde{S}$  has only isolated singular points.  $\tilde{S} \cap \mathcal{E}_1$  is defined by  $u_2 = r_0\omega + r_1 + \alpha_0\omega v_2 = 0$  on  $U_2$ ,  $u_1 = r_0 w_1 x_1 + r_1 + \alpha_0 x_1 = 0$  on  $V_1$  and  $u_1 = x_2((r_0 w_2 + r_1)x_2 + \alpha_0) = 0$  on  $V_2$ .  $\tilde{S} \cap \mathcal{E}_2$  is defined by

$$\begin{aligned} w_1 &= r_1 + \alpha_0 x_1 + u_1(\alpha_1 + \beta_0 x_1 + \beta_1 u_1 + x_1^2 + \gamma_1 u_1 x_1 + \gamma_2 u_1^2) = 0 && \text{on } V_1, \\ w_2 &= r_1 x_2^2 + \alpha_0 x_2 + u_1(\alpha_1 x_2^2 + \beta_0 x_2 + \beta_1 u_1 x_2^2 + 1 + \gamma_1 u_1 x_2 + \gamma_2 u_1^2 x_2^2) = 0 && \text{on } V_2. \end{aligned}$$

Thus  $\tilde{S}$  is the normalization of  $S''$  unless  $r_1 = \alpha_0 = 0$ . If  $r_1 = \alpha_0 = 0$ , then it is obtained by blowing up a point of the normalization.

**Lemma 8.4** *The map  $\hat{f} : \hat{S} \rightarrow S'$  factors through  $\tilde{S}$ .*

*Proof.* We can assume  $r_1 = \alpha_0 = 0$ . Let  $\Gamma_\omega$  be the fiber of  $\Sigma_1$  defined by  $\omega = 0$ . Let us use the notation of 5.2 and the proof of Lemma 7.3. We have the natural map  $\hat{\mu} : \hat{S} \rightarrow \Sigma_1$ . We let  $\mathcal{F}$  be the greatest common divisor of  $\hat{\mu}^* \Gamma_\omega$ ,  $\hat{\mu}^* \Delta_0$  and  $\hat{f}^* H_\infty$ . As we have seen above,  $\tilde{S}$  is obtained as the blowing up of a point of the normalization of  $S''$ . Thus it suffices to show that  $\mathcal{F}$  is not zero and that  $\hat{\mu}^* \Gamma_\omega - \mathcal{F}$ ,  $\hat{\mu}^* \Delta_0 - \mathcal{F}$  and  $\hat{f}^* H_\infty - \mathcal{F}$  do not meet simultaneously.

We first extract information from  $\hat{S} \rightarrow S''$ . Since  $U_2 \cap S'' \cap (v_2)$  is defined by  $\phi_0 = 0$ ,  $G_1$  is a common component of  $\hat{f}^* H_\infty$  and  $\hat{\mu}^* \Gamma_\omega$ . Similarly, considering  $S'' \cap \mathcal{E}'$ , we can find a common component  $G_2$  of them, which results from the curve  $\mathcal{E}' \cap \tau_1^* \varpi^* \Gamma_\omega$  and satisfies  $G_1 G_2 = 1$ .  $G_1$  meets no other component of  $\hat{f}^* H_\infty$ . In particular,  $G_1$  does not meet  $E$ . Thus it is a  $(-2)$ -curve. Further, it can be checked that  $G_0 = E$ . By the proof of Lemma 7.3, we have a chain of rational curves  $\mathcal{C} = G_2 + \cdots + G_n$  consisting of components of  $\sigma^* G$  satisfying  $G_i G_{i+1} = 1$ ,  $i < n$ ,  $G_i G_j = 0$  for  $|i - j| > 1$ ,  $G_i$  is a  $(-2)$ -curve for  $i < n$ , and  $G_n$  is a  $(-3)$ -curve with  $G_n E = 1$ . Note that  $G_i$  is the unique irreducible component of  $\sigma^* G$  with  $G_i G_{i-1} > 0$  and that the multiplicity of  $G_i$  in  $\sigma^* G$  is one. We have  $\mathcal{C} \leq \mathcal{G}$ . Since the support of  $\tau_1^* H_\infty \cap S''$  is contained in the union of  $\tau_1^* \varpi^* \Gamma_\omega$  and the proper transform  $Z'$  of  $Z$  by  $\tau_1$ ,  $\mathcal{C} - G_2$  results from the point  $\tau_1^* H_\infty \cap Z' \cap \tau_1^* \varpi^* \Gamma_\omega$ . Thus it follows that  $G_1 + \mathcal{C}$  coincides with the greatest common divisor of  $\hat{f}^* H_\infty$  and  $\hat{\mu}^* \Gamma_\omega$ . Therefore  $\mathcal{F} = \mathcal{C}$ .

Thus it suffices for our purpose to show that any irreducible component  $G^*$  of  $G'' := \sigma^* G - E - G_1 - \mathcal{C}$  does not meet  $\mathcal{C}$ . The curve  $G_i$ ,  $i < n$ , does not meet  $G^*$ . We show  $G_n$  also does not meet it. Since  $G_n$  is a  $(-3)$ -curve, we have  $1 = \hat{K} G_n = (M + 3E + G_1 + \mathcal{C} + G'') G_n = 3 + 1 - 3 + G'' G_n$ . Thus  $G'' G_n = 0$ . This shows the assertion. *q.e.d.*

**8.5** We let  $\tilde{Z}$  be the proper transform of  $Z$  by  $\tau$ . Then  $\tilde{Z} \sim \tau^* \varpi^* \Delta_0 - \mathcal{E}_1 - \mathcal{E}_2$ . Thus  $\omega_{\tilde{S}} = \mathcal{O}_{\tilde{S}}(\tau^* H + \tilde{Z})$ . We show that  $E := \tilde{Z}|_{\tilde{S}}$  is a  $(-1)$ -curve. For this purpose, we identify  $\tilde{Z}$  with  $\Sigma_1$ . Then we have

$$\tau^* H|_{\tilde{Z}} \sim \Delta_0 + \Gamma, \mathcal{E}_1|_{\tilde{Z}} \sim \Delta_0, \mathcal{E}_2|_{\tilde{Z}} \sim \Gamma.$$

Since  $\tilde{Z}|_{\tilde{Z}} \sim (\tau^* \varpi^*(\Delta_0 + \Gamma) - \tau^* \varpi^* \Gamma - \mathcal{E}_1 - \mathcal{E}_2)|_{\tilde{Z}} \sim -\Delta_0 - 2\Gamma$ , we have  $\tilde{S}|_{\tilde{Z}} \sim 3(\Delta_0 + \Gamma) + \Gamma + 2\tilde{Z}|_{\tilde{Z}} \sim \Delta_0$ . Thus  $E$  is a nonsingular rational curve. As we saw above,  $\tilde{S}$  is nonsingular in a neighbourhood of  $E$ . Since

$$E^2 = \tilde{Z}^2 \tilde{S} = \tilde{Z}^2 (\tau^*(3H + \varpi^* \Gamma) + 2\tilde{Z}) = -1,$$

we conclude that  $E$  is a  $(-1)$ -curve. Thus  $\omega_{\tilde{S}} \sim \tau^* H|_{\tilde{S}} + E$ . The invariants of  $\tilde{S}$  can be calculated as in 7.4. We have  $\omega_{\tilde{S}}^2 = 11$ ,  $h^0(\omega_{\tilde{S}}) = 6$  and  $h^1(\omega_{\tilde{S}}) = 0$ . Further,  $H^0(\omega_{\tilde{S}})$  is in bijection with  $H^0(\hat{W}, \mathcal{O}(H))$ . In particular, since we have  $\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\hat{S}})$ ,  $\tilde{S}$  has only RDP. The existence of surfaces of this type can be checked, if we choose  $\alpha_i, \beta_j, \gamma_k$  general.

## 9 Surfaces of type I-0: The case (s.3).

In this section, we study the case (s.3) of 6.1.

**Lemma 9.1** *Suppose that  $S^* \sim 4T - (p_g - 6)F$  on  $W = \mathbf{P}_{a,b,c}$ . Then*  
 1) *the dualizing sheaf  $\omega_{S^*}$  of  $S^*$  is given by  $\omega_{S^*} = \mathcal{O}_{S^*}(T + F)$  and*

$$\begin{aligned} h^0(S^*, \omega_{S^*}) &= p_g + 3, & h^1(S^*, \omega_{S^*}) &= 0, \\ \omega_{S^*}^2 &= 3h^0(S^*, \omega_{S^*}) - 7 = 3p_g + 2, \end{aligned}$$

2) *the singular locus  $\text{Sing}(S^*)$  of  $S^*$  has no horizontal component, i.e., the image via  $g : S^* \rightarrow \mathbf{P}^1$  of any connected component of  $\text{Sing}(S^*)$  is one point.*

*Proof.* 1): Since  $K_W = -3T + (p_g - 5)F$ , we get  $\omega_{S^*} = \mathcal{O}_{S^*}(T + F)$  by the adjunction formula. By using the exact sequence

$$0 \rightarrow \mathcal{O}(K_W) \rightarrow \mathcal{O}(T + F) \rightarrow \omega_{S^*} \rightarrow 0$$

and the fact that

$$\begin{aligned} H^q(W, \mathcal{O}(K_W)) &= 0 \quad \text{for } q < 3, \\ H^q(W, \mathcal{O}(T + F)) &\simeq H^q(\mathbf{P}^1, \mathcal{O}(a + 1) \oplus \mathcal{O}(b + 1) \oplus \mathcal{O}(c + 1)) \end{aligned}$$

for  $\forall q$ , we get the desired formulae for  $h^q(S^*, \omega_{S^*})$ . Further, an easy calculation shows  $\omega_{S^*}^2 = (T + F)^2(4T - (p_g - 6)F) = 3p_g + 2$ . Thus we get 1).

2): Suppose that  $\text{Sing}(S^*)$  has an horizontal component. Then any fiber of  $g$  is a singular plane quartic curve. Thus the normalization of  $S^*$  has a pencil of curves of genus  $\leq 2$ . This contradicts that it is birational to the type I surface  $S$ . Thus  $\text{Sing}(S^*)$  is vertical. *q.e.d.*

**Lemma 9.2** *Let  $S^*$  be as in Lemma 9.1. Then the double curve of  $S^*$  is the unique fiber  $C_s$  of  $g : S^* \rightarrow \mathbf{P}^1$ . Its support is a (possibly singular) conic curve or a line viewed in the fiber  $\mathbf{P}^2$  of  $W$ .*

*Proof.* It follows from Lemma 6.7 that  $C^* \sim 4\Delta_0 + (p_g + 2d)\Gamma$  on  $\Sigma_d$  and that  $C^*$  has two singular points  $P_1$  and  $P_2$  of multiplicity 2 on the same fiber  $\Gamma_s$ . Thus  $C^*$  meets  $\Gamma_s$  at no other points. We regard  $\Sigma_d$  as a member of  $|T|$ , and let  $F_s$  be the fiber of  $\pi : W \rightarrow \mathbf{P}^1$  which induces  $\Gamma_s$  on  $\Sigma_d$ . The rational curve  $\Gamma_s$  is a line in  $F_s \simeq \mathbf{P}^2$ . If we vary  $\Sigma_d$  in  $|T|$ , then, by 2) of Lemma 9.1, singular points of  $C^*$  traces a curve on  $F_s$ , which is a double curve of  $S^*$ . Since  $C^*$  is nonsingular except for  $P_1$  and  $P_2$ , there is no other multiple curve. Thus the double curve is a line or a conic (in  $F_s$ ) according to whether the singular points of  $C^*$  are infinitely near or not. *q.e.d.*

**Lemma 9.3** *Let  $S^*$  be as in Lemma 9.1. Then the integers  $a, b, c$  satisfy the conditions*

- 1)  $a + c \leq 3b + 3$ , and
- 2)  $b \leq 2a + 2$ .

*Proof.* We take sections  $X_0, X_1$  and  $X_2$  of  $\mathcal{O}(T - aF)$ ,  $\mathcal{O}(T - bF)$  and  $\mathcal{O}(T - cF)$ , respectively, such that they form a system of homogeneous coordinates on each fiber of  $W = \mathbf{P}_{a,b,c} \rightarrow \mathbf{P}^1$ . Then the equation of  $S^*$  can be written as

$$\sum_{i,j \geq 0, i+j \leq 4} \phi_{ij} X_0^{4-i-j} X_1^i X_2^j = 0, \quad (14)$$

where  $\phi_{ij}$  is a homogeneous form of degree  $(4 - i - j)a + ib + jc - (p_g - 6)$  on  $\mathbf{P}^1$ . If  $4b < p_g - 6$ , then the left hand side of (14) can be divided by  $X_2$ , which is impossible since  $S^*$  is irreducible. Thus we get 1). If  $3a + c < p_g - 6$ , then the curve  $B$  defined by  $X_1 = X_2 = 0$  is a multiple curve of  $S^*$ . This is impossible by Lemma 9.2. Thus we get  $3a + c \geq p_g - 6$ , i.e.,  $b \leq 2a + 3$ . If  $b = 2a + 3$ , then  $\phi_{10}$  and  $\phi_{01}$  are constants. As we have seen above,  $\phi_{01}$  cannot be zero. But then  $S^*$  is nonsingular along  $B$ . Since the double curve  $C_s$  meets  $B$ , this is impossible. Thus we get 2). *q.e.d.*

**9.4** We call  $C_s$  in Lemma 9.2 the *singular fiber* of  $S^*$ . We have the following list of singular fibers (as plane quartics):

- 1)  $C_s = 2L$ , where  $L$  is a nonsingular conic.
- 2)  $C_s = 2L$ ,  $L = L_1 + L_2$ , where  $L_1, L_2$  are distinct lines.
- 3)  $C_s = 4L$ , where  $L$  is a line.

We let  $F_s$  be the fiber of  $\pi : W = \mathbf{P}_{a,b,c} \rightarrow \mathbf{P}^1$  containing  $C_s$  as in the proof of Lemma 9.2. Take an affine coordinate  $t$  on  $\mathbf{P}^1$  such that  $t = 0$  defines  $F_s$  on  $W$ . Then the equation (14) can be rewritten as

$$\sum_{i=0}^{4c-p_g+6} \psi_i(X_0, X_1, X_2)t^i = 0, \quad (15)$$

where the  $\psi_i$  are homogeneous forms of degree 4 in  $X_0, X_1, X_2$ . Then  $\psi_0$  defines  $C_s$  on  $F_s \simeq \mathbf{P}^2$ .

We denote by  $\nu : \hat{W} \rightarrow W$  the blowing-up with center  $L$ , and let  $S'$  be the proper transform of  $S^*$  by  $\nu$ . Let  $\mathcal{I}$  be the ideal sheaf of  $L$  in  $S^*$ . Since  $f^{-1}\mathcal{I} \cdot \mathcal{O}_S$  is clearly invertible,  $f : S \rightarrow S^*$  factors through  $S'$ .

In the following, we shall sketch the curve  $D_s$  on  $S$  coming from  $C_s$ , which we call the *canonical degenerate fiber* of  $h = g \circ f : S \rightarrow \mathbf{P}^1$ .

9.5 We assume here that  $S^*$  has the singular fiber of type 1). For any  $P \in L$ , we can take a small open neighbourhood  $U$  of  $P$  in  $W$  with coordinates  $(t, x, y)$ , where  $t$  is as in 9.4 and  $(x, y)$  is a local coordinate system on  $F_s \simeq \mathbf{P}^2$  such that  $L$  is defined by  $x = 0$ . Then, on  $U$ , the equation (15) can be rewritten as

$$x^2 + f_1xt + \sum_{i=2}^{4c-(p_g-6)} f_i t^i = 0,$$

where the  $f_i = f_i(x, y)$  are holomorphic functions in  $x, y$ .

Let  $\nu : \tilde{W} \rightarrow W$  and  $S'$  be as in 9.4. We put  $\mathcal{E} = \nu^{-1}(L)$ . Since the normal sheaf  $N_L$  of  $L \simeq \mathbf{P}^1$  is given by  $\mathcal{O}(4) \oplus \mathcal{O}$ ,  $\mathcal{E}$  is isomorphic to  $\Sigma_4$ . We show  $\mathcal{E}|_{\mathcal{E}} \sim -\Delta_0$ : Since  $\det N_L = \mathcal{O}(4)$ , we have  $\mathcal{O}_L(K_W) = \mathcal{O}(-6)$ . Then we get

$$K_{\mathcal{E}} \sim (\nu^* K_W + 2\mathcal{E})|_{\mathcal{E}} \sim -6\Gamma + 2\mathcal{E}|_{\mathcal{E}}.$$

On the other hand, we have  $K_{\mathcal{E}} \sim -2\Delta_0 - 6\Gamma$ . Thus we get  $\mathcal{E}|_{\mathcal{E}} \sim -\Delta_0$ .

We have  $S' \sim \nu^* S^* - 2\mathcal{E} \sim \nu^*(4T - (p_g - 6)F) - 2\mathcal{E}$ . Thus  $S'|_{\mathcal{E}} \sim 2\Delta_0 + 8\Gamma$  on  $\mathcal{E} \simeq \Sigma_4$ . Let  $F'$  be the proper transform of  $F_s$ . Then  $F' \sim \nu^* F - \mathcal{E}$ , and it does not meet  $S'$ . Further,  $S'$  is the normalization of  $S^*$ . These can be seen as follows. We cover  $\nu^{-1}(U)$  by open sets  $U_1, U_2$  with coordinates  $(u_1, v_1, y_1)$  and  $(u_2, v_2, y_2)$ , respectively, where

$$t = u_1 v_1 = u_2, \quad x = u_1 = u_2 v_2, \quad y = y_1 = y_2.$$

Then  $S'$  is defined by

$$\begin{aligned} 1 + f_1 v_1 + \sum_{i \geq 2} f_i u_1^{i-2} v_1^i &= 0 \quad \text{on } U_1, \\ v_2^2 + f_1 v_2 + \sum_{i \geq 2} f_i u_2^{i-2} &= 0 \quad \text{on } U_2. \end{aligned}$$

Since  $F'$  is defined by  $v_1 = 0$ , it does not meet  $S'$ . We remark that  $F'$  induces on  $\mathcal{E}$  the 0-section  $\Delta_0$ . Thus  $\nu|_{S'} : S' \rightarrow S^*$  is finite.  $\mathcal{E} \cap S'$  is defined by  $u_1 = 1 + f_1(0, y_1)v_1 + f_2(0, y_1)v_1^2$  on  $U_1$  and  $u_2 = v_2^2 + f_1(0, y_2)v_2 + f_2(0, y_2) = 0$  on  $U_2$ . Thus there are the following possibilities:

i)  $S'|_{\mathcal{E}}$  is irreducible.

ii)  $S'|_{\mathcal{E}}$  is reduced and consists of two irreducible components each of which is linearly equivalent to  $\Delta_0 + 4\Gamma$ .

iii)  $S'|_{\mathcal{E}} = 2L'$ , where  $L' \sim \Delta_0 + 4\Gamma$  is irreducible.

If  $S'$  has a multiple curve, then it must be contained in  $\nu^* F_s$  as we saw in Lemma 9.2. Thus  $S'$  may not be normal when iii). Since the singularities of  $C^*$  are not infinitely near, and since  $\nu^* T$  meets  $L'$ , we see that  $L'$  cannot be a multiple curve. It follows that  $S'$  is normal. The cases i) and ii) occur when the singular points of  $C^*$  are ordinary double points. On the other hand, the case iii) occurs when they are simple cusps.

Since  $(\nu^* T + F - \mathcal{E})|_{S'} \sim \nu^* T|_{S'}$ , we have  $\omega_{S'} = \mathcal{O}_{S'}(\nu^* T)$ . Then

$$\omega_{S'}^2 = (\nu^* T)^2 (\nu^* S^* - 2\mathcal{E}) = T^2 (4T - (p_g - 6)F) = 3p_g - 6.$$

In order to calculate  $\chi(\mathcal{O}_{S'})$ , we use the exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{O}(K_{\hat{W}}) \rightarrow \mathcal{O}(\nu^*T + F') \rightarrow \omega_{S'} \rightarrow 0, \\ 0 \rightarrow \mathcal{O}(\nu^*T) \rightarrow \mathcal{O}(\nu^*T + F') \rightarrow \mathcal{O}_{F'}(-1) \rightarrow 0, \end{aligned}$$

where  $F'$  is identified with  $\mathbf{P}^2$ . From the cohomology exact sequence derived from the first sequence, we get  $H^q(\omega_{S'}) \simeq H^q(\hat{W}, \mathcal{O}(\nu^*T + F'))$  for  $q < 2$ . From the second one, we get  $H^p(\mathcal{O}(\nu^*T + F')) \simeq H^p(W, \mathcal{O}(T))$  for any  $p$ . Thus we have  $h^0(\omega_{S'}) = p_g$ ,  $h^1(\omega_{S'}) = 0$ . Then, since  $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S'})$ , we see that  $S'$  has only RDP. The canonical degenerate fiber  $D_s$  is in general a hyperelliptic curve of genus 3 as i) shows.

**9.6** We consider the singular fiber of type 2). Let  $P \in L$ . If it is not  $L_1 \cap L_2$ , then we can study  $S^*$  in a neighbourhood of  $P$  as in 9.5. So we assume  $P = L_1 \cap L_2$ . We can take a small open neighbourhood  $U$  of  $P$  in  $W$  with coordinates  $(t, x, y)$ , where  $t$  is as in 9.4 and  $(x, y)$  is an affine coordinate system on  $F_s \simeq \mathbf{P}^2$  such that  $L_1$  and  $L_2$  are defined by  $x = 0$  and  $y = 0$ , respectively. Then, on  $U$ , the equation (15) can be rewritten as

$$x^2y^2 + f_1xyt + \sum_{i=2}^{4c-(p_g-6)} f_i t^i = 0,$$

where the  $f_i = f_i(x, y)$  are polynomial functions in  $x, y$ .

Let  $\tau_1 : W_1 \rightarrow W$  be the blowing-up with center  $L_1$ . We denote  $F'$  and  $\mathcal{E}'$  the proper transform of  $F_s$  and the exceptional divisor of  $\tau_1$ , respectively. We let  $S_1$  be the proper transform of  $S^*$  by  $\tau_1$ . We cover  $\tau_1^{-1}(U)$  by open sets  $U_1, U_2$  with coordinates  $(u_1, v_1, y_1)$  and  $(u_2, v_2, y_2)$ , respectively, where

$$t = u_1v_1 = u_2, \quad x = u_1 = u_2v_2, \quad y = y_1 = y_2.$$

Then  $S_1$  is defined by

$$\begin{aligned} y_1^2 + f_1y_1v_1 + \sum_{i \geq 2} f_i u_1^{i-2} v_1^i &= 0 \quad \text{on } U_1, \\ y_2^2 v_2^2 + f_1y_2v_2 + \sum_{i \geq 2} f_i u_2^{i-2} &= 0 \quad \text{on } U_2. \end{aligned}$$

Let  $\tau_2 : W_2 \rightarrow W_1$  be the blowing-up with center the proper transform  $L'_2$  of  $L_2$  by  $\tau_1$ . Put  $\tau = \tau_1 \circ \tau_2$  and let  $\hat{F}$  and  $\mathcal{E}_1$  be the proper transform by  $\tau_2$  of  $F'$  and  $\mathcal{E}'$ , respectively. We also put  $\mathcal{E}_2 = \tau_2^{-1}(L'_2)$ . We remark that  $\mathcal{E}_1$  is  $\Sigma_1$  blown up at a point on  $\Delta_0$ , and  $\mathcal{E}_2$  is isomorphic to  $\Sigma_2$ . If  $S_2$  is the proper transform of  $S^*$  by  $\tau$ , then it is linearly equivalent to  $\tau^*S^* - 2\mathcal{E}_1 - 2\mathcal{E}_2$ . Let  $V$  be a open neighbourhood of  $Q \in F' \cap \mathcal{E}'$ . We can assume  $(u_1, v_1, y_1)$  forms a system of local coordinates on  $V$ . We cover  $\tau_2^{-1}(V)$  by two open sets  $V_1, V_2$  with coordinates  $(u_1, w_1, x_1)$  and  $(u_1, w_2, x_2)$ , respectively, where

$$v_1 = w_1x_1 = w_2, \quad y_1 = w_1 = w_2x_2.$$

Then  $S_2$  is defined by

$$\begin{aligned} 1 + f_1x_1 + \sum_{i \geq 2} f_i(u_1w_1)^{i-2}x_1^i &= 0 & \text{on } V_1, \\ x_2^2 + f_1x_2 + \sum_{i \geq 2} f_iu_1^{i-2}w_2^{i-2} &= 0 & \text{on } V_2. \end{aligned}$$

Thus it does not meet  $\hat{F}$ .  $S_2 \cap \mathcal{E}_1$  is defined by  $u_2 = y_2^2v_2^2 + f_1(0, y_2)y_2v_2 + f_2(0, y_2) = 0$  on  $U_2$ ,  $u_1 = 1 + f_1(0, w_1)x_1 + f_2(0, w_1)x_1^2 = 0$  on  $V_1$  and  $u_1 = x_2^2 + f_1(0, w_2x_2)x_2 + f_2(0, w_2x_2) = 0$  on  $V_2$ . Further,  $S_2 \cap \mathcal{E}_2$  is defined by  $w_1 = 1 + f_1(u_1, 0)x_1 + f_2(u_1, 0)x_1^2 = 0$  on  $V_1$  and  $w_2 = x_2^2 + f_1(u_1, 0)x_2 + f_2(u_1, 0) = 0$  on  $V_2$ . Since  $\hat{F} \sim \tau^*F - \mathcal{E}_1 - \mathcal{E}_2$ , and since it does not intersect with  $S_2$ , we have  $\omega_{S_2} = \mathcal{O}_{S_2}(\tau^*T)$ . Then we can calculate the invariants of  $S_2$  as in 9.5. We have  $\omega_{S_2}^2 = 3p_g - 6$ ,  $h^0(\omega_{S_2}) = p_g$  and  $h^1(\omega_{S_2}) = 0$ . Further,  $H^0(\omega_{S_2})$  is in bijection with  $H^0(W, \mathcal{O}(T))$ . In particular, we have  $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S_2})$ .

We assume that  $S_2$  is not normal and show that this leads us to a contradiction. By Lemma 9.2, any multiple curve is contained in  $\mathcal{E}_1 \cup \mathcal{E}_2$  because  $S_2$  does not meet  $\hat{F}$ . Recall that the singular points of  $C^*$  are not infinitely near. Since  $\tau^*T$  induces on each  $\mathcal{E}_i$  a fiber of the natural map  $\mathcal{E}_i \rightarrow \mathbf{P}^1$ , we see that the possible multiple curve is  $\Xi$  defined by  $u_2 = y_2 = 0$  on  $U_2$  and  $u_1 = x_2 = 0$  on  $V_2$ . This occurs when  $f_1$  and  $f_3$  vanish at  $x = y = 0$  and  $f_2$  vanishes twice at that point. Let  $\tau_3 : W_3 \rightarrow W_2$  be the blowing-up with center  $\Xi$ , and let  $S_3$  be the proper transform of  $S_2$  by  $\tau_3$ . We denote by  $\mathcal{E}_3$  the exceptional divisor. Then  $S_3$  is normal and  $\omega_{S_3} = \mathcal{O}(\tau_3^*\tau^*T - \mathcal{E}_3)$ . Thus  $H^0(\omega_{S_3})$  is in bijection with  $H^0(W, \mathcal{O}(T - \{x = y = 0\}))$ . It follows  $h^0(\omega_{S_3}) = p_g - 1$ . Since the minimal resolution of  $S_3$  is birational to  $S$ , this is impossible. Thus  $S_2$  is normal.

Let  $\hat{W}$  and  $S'$  be as in 9.4. Then  $\hat{W}$  is obtained from  $W_2$  by contracting the curve  $\Xi$  to the point  $\xi$ . If  $f_2(0, 0) \neq 0$ , then  $S_2$  does not meet  $\Xi$ . Thus  $S' \simeq S_2$  has only RDP as the above calculation shows. On the other hand, if  $f_2(0, 0) = 0$ ,  $S_2$  is obtained from  $S'$  by blowing up  $\xi$ . We show that  $\xi$  is a RDP on  $S'$ . Let  $\sigma : X \rightarrow S_2$  be the minimal resolution. Since  $X$  is birationally equivalent to  $S$ , we have  $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S)$ . Then we see that  $S_2$  has only RDP by  $\chi(\mathcal{O}_{S_2}) = \chi(\mathcal{O}_X)$ . We in particular have  $\omega_X = \sigma^*\omega_{S_2} = \mathcal{O}(\sigma^*\tau^*T)$ . Let  $\Xi'$  be the proper transform of  $\Xi$  by  $\sigma$ . Then

$$\omega_X \Xi' = (\sigma^*\omega_{S_2})\Xi' = (\sigma^*\omega_{S_2})(\sigma^*\Xi) = \omega_{S_2}\Xi = 0.$$

Thus  $\Xi'$  is a  $(-2)$ -curve on  $X$ . Then, since  $S_2$  has only RDP, each irreducible component of  $\sigma^*\Xi$  is a  $(-2)$ -curve. This implies that  $\xi$  is a RDP on  $S'$ . Thus  $S'$  has only RDP and  $S$  is the minimal resolution.

In order to describe the canonical degenerate fiber, we let  $\lambda : \mathcal{E}_1 \rightarrow \mathcal{E}' \simeq \Sigma_1$  be the restriction of  $\tau_2$  to  $\mathcal{E}_1$ . This is the blowing-up with center  $\mathcal{E}' \cap L'_2$  and its exceptional curve  $E$  can be identified with the intersection  $\mathcal{E}_1 \cap \mathcal{E}_2$ . With these notations,  $S_2|_{\mathcal{E}_1}$  is linearly equivalent to  $\lambda^*(2\Delta_0 + 4\Gamma) - 2E$ . Thus it is the proper transform of a curve of arithmetic genus 2 with a double point. Since  $S_2|_{\mathcal{E}_2} \sim 2\Delta_0 + 4\Gamma$  on  $\mathcal{E}_2 \simeq \Sigma_2$ , it is an elliptic curve.

Thus, in general, the canonical degenerate fiber consists of two elliptic curves meeting at two points. The self-intersection number of each curve is  $-2$ .

9.7 We suppose that  $S^*$  has the singular fiber of type 3). For any  $P \in L$ , we can take a small open neighbourhood  $U$  of  $P$  in  $W$  with coordinates  $(t, x, y)$ , where  $t$  is as in 9.4 and  $(x, y)$  is an affine coordinate system on  $F_s \simeq \mathbf{P}^2$  such that  $L$  is defined by  $x = 0$ . Then, on  $U$ , the equation (15) can be rewritten as

$$x^4 + f_1 x^2 t + \sum_{i=2}^{4c-(p_g-6)} f_i t^i = 0,$$

where the  $f_i = f_i(x, y)$  are functions in  $x, y$ . Since  $L$  is a double curve,  $f_2(0, y)$  is not identically zero.

Let  $\nu : \tilde{W} \rightarrow W$  and  $S'$  be as in 9.4. We denote by  $\mathcal{E}'$  and  $F'$  the exceptional divisor and the proper transform of  $F_s$ , respectively. We cover  $\nu^{-1}(U)$  by open sets  $U_1, U_2$  with coordinates  $(u_1, v_1, y_1)$  and  $(u_2, v_2, y_2)$ , respectively, where

$$t = u_1 v_1 = u_2, \quad x = u_1 = u_2 v_2, \quad y = y_1 = y_2.$$

Then  $S'$  is defined by

$$\begin{aligned} u_1^2 + f_1 u_1 v_1 + \sum_{i \geq 2} f_i u_1^{i-2} v_1^i &= 0 \quad \text{on } U_1, \\ u_2^2 v_2^4 + f_1 u_2 v_2^2 + \sum_{i \geq 2} f_i u_2^{i-2} &= 0 \quad \text{on } U_2. \end{aligned}$$

We let  $\tilde{\nu} : \tilde{W} \rightarrow \hat{W}$  be the blowing-up with center  $F' \cap \mathcal{E}'$  and put  $\tau = \nu \circ \tilde{\nu}$ . We denote by  $\tilde{F}$ ,  $\mathcal{E}_1$  and  $\mathcal{E}_2$  the proper transforms of  $F'$ ,  $\mathcal{E}'$  and the exceptional divisor of  $\tilde{\nu}$ , respectively. Then  $\mathcal{E}_1 \simeq \Sigma_1$  and  $\mathcal{E}_2 \simeq \Sigma_2$ . We have  $\mathcal{E}_2|_{\mathcal{E}_1} = \Delta_0$ ,  $\mathcal{E}_1|_{\mathcal{E}_2} \sim \Delta_0 + 2\Gamma$  and  $\tilde{F}|_{\mathcal{E}_2} = \Delta_0$ . Further, we can show  $\mathcal{E}_1|_{\mathcal{E}_1} \sim -2\Delta_0$  and  $\mathcal{E}_2|_{\mathcal{E}_2} \sim -\Delta_0 - \Gamma$ .

Let  $S''$  be the proper transform of  $S^*$  by  $\tau$ . Let  $V$  be a open neighbourhood of  $Q \in F' \cap \mathcal{E}'$ . We can assume  $(u_1, v_1, y_1)$  forms a system of local coordinates on  $V$ . We cover  $\tilde{\nu}^{-1}(V)$  by two open sets  $V_1, V_2$  with coordinates  $(w_1, x_1, y_1)$  and  $(w_2, x_2, y_1)$ , respectively, which satisfies

$$u_1 = w_1 x_1 = w_2, \quad v_1 = w_1 = w_2 x_2.$$

Then  $S''$  is defined by

$$\begin{aligned} x_1^2 + f_1 x_1 + f_2 + \sum_{i \geq 3} f_i w_1^{2i-4} x_1^{i-2} &= 0 \quad \text{on } V_1, \\ 1 + f_1 x_2 + f_2 x_2^2 + \sum_{i \geq 3} f_i w_2^{i-1} x_2^i &= 0 \quad \text{on } V_2. \end{aligned}$$

Thus  $S''$  does not meet  $\tilde{F}$ .  $S'' \cap \mathcal{E}_1$  is defined by  $u_2 = f_2(0, y_2) = 0$  on  $U_2$  and  $x_1 = f_2(0, y_1) = 0$  on  $V_1$ . Further,  $S'' \cap \mathcal{E}_2$  is defined by  $w_1 = x_1^2 + f_1(0, y_1)x_1 + f_2(0, y_1) = 0$  on  $V_1$  and  $w_2 = 1 + f_1(0, y_1)x_2 + f_2(0, y_1)x_2^2 = 0$  on  $V_2$ . Thus  $S'' \rightarrow S'$  is finite.

We have  $S'' \sim \tau^*(4T - (p_g - 6)F) - 2\mathcal{E}_1 - 4\mathcal{E}_2$ . Since  $\tilde{F} \sim \tau^*F - \mathcal{E}_1 - 2\mathcal{E}_2$  and since it does not intersect with  $S''$ , we get  $\omega_{S''} \sim \tau^*T|_{S''}$ . Then we can calculate the invariants of  $S''$  as in 9.5. We have  $\omega_{S''}^2 = 3p_g - 6$ ,  $h^0(\omega_{S''}) = p_g$  and  $h^1(\omega_{S''}) = 0$ . In particular, we have  $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S''})$ . We can show that  $S''$  has only isolated singular points as in 9.7. Thus  $S''$  has only RDP. Further, we see that  $S''$  is the normalization of  $S'$ . In particular, we have a holomorphic map  $S \rightarrow S''$ .

Since we have  $S''|_{\mathcal{E}_1} \sim 4\Gamma$  and  $S''|_{\mathcal{E}_2} \sim 2\Delta_0 + 4\Gamma$ , the canonical degenerate fiber  $D_s$  consists of an elliptic curve  $R$  and 4 rational curves  $l_i$ ,  $1 \leq i \leq 4$ , in general. Each  $l_i$  intersects with  $R$  at a point. Since  $\mathcal{E}_1^2 S'' = -8$  and  $\mathcal{E}_2^2 S'' = -2$ , each of them has self-intersection number  $-2$ .

**9.8** In order to show the existence of surfaces of type (s.3), we consider the surface  $S^*$  in  $\mathbf{P}_{a,b,c}$  defined by

$$\alpha_0^2 \mathcal{P} + \alpha_1 \mathcal{Q} = 0, \quad (16)$$

where

$$\begin{aligned} \alpha_0 &\in H^0(\mathbf{P}_{a,b,c}, \mathcal{O}(F)), \\ \alpha_1 &\in H^0(\mathbf{P}_{a,b,c}, \mathcal{O}(\epsilon F)), \\ \mathcal{P} &\in H^0(\mathbf{P}_{a,b,c}, \mathcal{O}(4T - (p_g - 4)F)), \\ \mathcal{Q} &\in H^0(\mathbf{P}_{a,b,c}, \mathcal{O}(2T - ((p_g - 6 + \epsilon)/2)F)), \end{aligned}$$

with  $\epsilon = 0$  or  $1$  according to  $p_g$  is even or odd. We assume that the  $\alpha_i$ ,  $\mathcal{P}$  and  $\mathcal{Q}$  are general. Let  $(X_0, X_1, X_2)$  be as in the proof of Lemma 9.3. We put  $\mathcal{P} = \sum p_{ij} X_0^{4-i-j} X_1^i X_2^j$  and  $\mathcal{Q} = \sum q_{ij} X_0^{2-i-j} X_1^i X_2^j$  as in (14).

By the argument found in [2, §2], we can show the following:

i) Consider the linear system  $|4T - (p_g - 4)F|$ . If  $2a + 1 \geq c$ , then its general member is irreducible and nonsingular. If  $a + c \leq 3b + 1$  and  $b \leq 2a + 1$ , then its general member is irreducible and has only RDP.

ii) Consider the linear system  $|2T - ((p_g - 6 + \epsilon)/2)F|$ . If  $b + c \leq 3a + 2$ , then it is free from base points. If  $a + c \leq 3b + 3$ , then its general member is irreducible and has at most singular points of type  $A_1$ .

In i) and ii), the singular points are on the intersection of the curve  $B$  defined by  $X_1 = X_2 = 0$  and the fibers defined by  $p_{01} = 0$  and  $q_{01} = 0$ , respectively. Further, we remark that the following holds.

iii) If  $a + c = 3b + 2$ ,  $3b + 3$ , then  $X_2$  is a fixed component of  $|4T - (p_g - 4)F|$ . In this case, we can write  $\mathcal{P} = X_2 \mathcal{P}'$  with  $\mathcal{P}' \in H^0(\mathcal{O}(3T - (p_g - 4 - c)F))$ . The linear system  $|3T - (p_g - 4 - c)F|$  is free from base points if  $b \leq 2a + 1$ . Thus we can assume that  $(\mathcal{P})$  is a divisor with simple normal crossings.

We show that  $S^*$  has only RDP except for the double curve  $C_s$  defined by  $\alpha_0 = \mathcal{Q} = 0$ , assuming that  $a + c \leq 3b + 3$  and  $b \leq 2a + 1$ . Let  $\xi$  be a singular point of  $S^* \setminus C_s$ . Then it is contained in the set of singular points of  $(\mathcal{P})$  on  $(\mathcal{Q})$ .

We first consider the case  $a + c \leq 3b + 1$ . Then, by i),  $(\mathcal{P})$  has only isolated singular points on  $B$ . If  $2a + 1 \geq c$ , then we can assume that  $(\mathcal{P})$  is nonsingular. If  $b + c \leq 3a + 2$ , then we can assume that  $(\mathcal{Q})$  does not pass through singular points of  $(\mathcal{P})$ . Thus  $S^*$  is nonsingular except for  $C_s$  in these cases. So we can assume that  $2a + 1 < c$  and  $b + c > 3a + 2$ . Then  $(\mathcal{P})$  and  $(\mathcal{Q})$  contain  $B$ . The singular point  $\xi$  is a zero of  $p_{01}$ . Put  $x_1 = X_1/X_0$  and  $x_2 = X_2/X_0$ . Then, in a neighbourhood of  $B$ ,  $(x_1, x_2)$  forms a system of affine coordinates on each fiber of  $\pi$ . We take a local parameter  $t$  of  $B$  at  $\xi$ . We can assume  $p_{01} = t$  in a neighbourhood of  $\xi$ . Then we can rewrite (16) locally as

$$\begin{aligned} & -x_2(\alpha_0^2 t + \cdots) \\ = & (\alpha_0^2 p_{20} + \alpha_1 q_{10}^2)x_1^2 + (\alpha_0^2 p_{30} + 2\alpha_1 q_{10} q_{20})x_1^3 + (\alpha_0^2 p_{40} + \alpha_1 q_{20}^2)x_1^4 \end{aligned}$$

Thus  $\xi$  is a RDP of type  $A_n$ , for some  $n \leq 3$ , if we choose  $\alpha_i, p_{ij}, q_{ij}$  general.

We next consider the case  $a + c \geq 3b + 2$ . Let  $\mathcal{P}'$  be as in iii). Since  $b \leq 2a + 1$ , we can assume that  $(\mathcal{P}')$  meets transversely with  $B$ . If  $a = b$ , then  $Bs|2T - ((p_g - 6 + \epsilon)/2)F| = \emptyset$  and we can assume that the divisor  $(X_2) + (\mathcal{P}') + (\mathcal{Q})$  has only normal crossings. Thus  $\xi$  is a  $A_1$ -singularity as (16) shows. If  $a < b$ , then  $q_{00} = q_{10} = 0$ . Thus  $(\mathcal{Q}) \cap (X_2) = B$  and  $(\mathcal{Q})$  contacts  $(X_2)$  at a generic point of  $B$ . Thus  $\xi \in B$ . Then a local study as in the previous case shows that it is a RDP of  $S^*$ .

**Remark 9.9** The singular fibers of type 3) are implicitly used in [1] to construct surfaces with pencils of nonhyperelliptic curves of genus 3. Note that we get a simple elliptic singularity of type  $\tilde{E}_7$  if we contract the curve  $R$  in 9.7.

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Deformations of a Complex Manifold near a Strongly Pseudo-Convex Real  
Hypersurface and a Realization of Kuranishi Family of Strongly  
Pseudo-Convex CR Structures

by

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## Introduction

Let  $(V, o)$  be a normal isolated singularity . If we cut  $V$  with a small real sphere centered at  $o$  , we have a real hypersurface  $M$  of  $V$  with a natural strongly pseudo-convex CR structure .

Deformations of strongly pseudo-convex CR structures on  $M$  with  $\dim_{\mathbb{R}} M \geq 5$  were considered first by M.Kuranishi (cf. [Ku3]) in order to treat deformations of  $(V, o)$  differential-geometrically, and were improved later by T.Akahori for  $M$  with  $\dim_{\mathbb{R}} M \geq 7$  (cf. [A1]) .

It is a natural question to ask whether the Kuranishi family of strongly pseudo-convex CR structures on  $M$  obtained in [A1] (also cf. [M1]) actually induces the versal family of deformations of  $(V, o)$  (cf. [G]) . In the case of  $\text{codh}(V, o) \geq 3$  , by [F] , the latter versal family is obtained as the maximal Stein completion of the versal family of deformations of its regular part  $V \setminus o$  .

On the other hand , realizations of deformations of abstract CR structures are one of basic interests of deformation theory of CR structures . Under some cohomological conditions , G.K.Kiremidjian has done such a realization as boundaries of deformations of the interior domain (cf. [Ki 1, 2, 3]) . But in order to approach the above problem , it is enough to consider embeddings of deformations of CR structures into deformations of a tubular neighbourhood of  $M$  .

So the main result of this paper is the following relative version of T.Ohsawa's embedding theorem of a strongly pseudo-convex CR structure (cf. [O]) :

**Theorem 2.** *Let  $M$  be a smooth strongly pseudo-convex compact real hypersurface of a complex manifold  $X$  with  $\dim_{\mathbb{C}} X \geq 4$ . Then the Kuranishi family of strongly pseudo-convex CR structures on  $M$  is realized as a real hypersurface of the formally versal convergent family of deformations of  $X$  near  $M$ .*

Hence, as stated above, we have

**Corollary.** *Let  $(V, o)$  be a normal isolated singularity with  $\dim_{\mathbb{C}}(V, o) \geq 4$  and  $\text{codh}(V, o) \geq 3$ . Let  $M = V \cap S$  with a sufficiently small real sphere  $S$  centered at  $o$ . Then the Kuranishi family of strongly pseudo-convex CR structures on  $M$  is realized as a real hypersurface of the versal family of deformations of  $(V, o)$ .*

The existence of the formally versal convergent family of deformations of  $X$  near  $M$  (Theorem 1) was essentially proved in [A2, 3]. We will prove Theorem 2 as an application of a formal deformation theory of  $X$  near  $M$ . That is, we will show that the versal map from the parameter space of the family in Theorem 1 into the one of the Kuranishi family of CR structures is formally isomorphic. The essential part of the proof is to construct the formal inverse map by showing that the Kuranishi family of CR structures on  $M$  is extendable to a formal family of complex structures on a tubular neighbourhood of  $M$ . This is a consequence of the isomorphisms  $H_{\partial}^i(\bar{\Omega}, T'X) \simeq H_{\partial_b}^i(M, T'X|_M)$  for  $i=1,2$ , which was obtained in [Y], where  $\Omega$  is a tubular neighbourhood of  $M$  in  $X$ .

The arrangement of this paper is as follows . In §1 , we will give a formulation of deformations of  $X$  near  $M$  . In §2 , we will adjust arguments in [A2], [A-K] and [Ku1, 2] to prove Theorem 1 . Though they treated only reduced deformations , we will consider non-reduced ones as well by slightly modifying their arguments . We will prove Theorem 2 in §3 .

In more general situations dealt with in [M2] , the method in this paper works well and Theorems 1 and 2 hold .

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# 1. Formulation of deformations of a complex manifold near a real hypersurface

Let  $X$  be a complex manifold with  $\dim_{\mathbb{C}} X = N \geq 4$  and  $M$  a strongly pseudo-convex compact real hypersurface of  $X$ . We will consider a functor of deformations of  $X$  near  $M$ :  $\mathcal{F}_M : (\text{Gan}) \rightarrow (\text{Sets})$ , where  $(\text{Gan})$  denotes the category of germs of complex spaces, given by

$$\mathcal{F}_M((T, o)) = \{ (\mathcal{X}, \pi) \mid \begin{array}{l} \text{(i) a complex space } \mathcal{X}, \\ \text{(ii) a smooth morphism } \pi: \mathcal{X} \rightarrow (T, o), \\ \text{(iii) } \pi^{-1}(o) \simeq \text{a neighbourhood of } M \text{ in } X \end{array} \} / \sim,$$

where  $(\mathcal{X}, \pi) \sim (\mathcal{X}', \pi')$  if there exist neighbourhoods  $\mathcal{X}_1$  and  $\mathcal{X}'_1$  of  $M \times o$  in  $\mathcal{X}$  and  $\mathcal{X}'$  respectively and a complex analytic isomorphism  $\chi: \mathcal{X}_1 \rightarrow \mathcal{X}'_1$  such that  $\pi' \circ \chi = \pi$ .

We may assume that there exists a smooth strictly plurisubharmonic function  $r: X \rightarrow (a_*, b^*)$ ,  $-\infty \leq a_* < 0 < b^* \leq +\infty$  such that  $dr \neq 0$  on  $X$ ,  $\Omega_{a,b} = \{a < r(x) < b\}$  is relatively compact in  $X$  for any  $a_* < a < b < b^*$  and  $M = \{r(x) = 0\}$ .

Because  $\mathcal{F}_M(\text{Spec } \mathbb{C}[t]/(t^2)) = \varinjlim_{U \supset M} H^1(U, \theta_U)$ , and by [A-G1],  $H^1(\Omega_{a,b}, \theta)$  measures the infinitesimal deformation space of  $\mathcal{F}_M$  for any  $a_* < a < 0 < b < b^*$ , where  $\theta$  denotes the sheaf of germs of holomorphic vector fields on  $X$ .

**Theorem 1.** Under the above assumption and if  $a_* < a < 0 < b < b^*$ , there exists a family  $\pi: \mathcal{X} \rightarrow T$  of complex manifolds with  $\pi^{-1}(o) \simeq \Omega_{a,b}$  and such that it gives a formally versal convergent family for  $\mathcal{F}_{M_c}$  for any  $a < c < b$ . Where we denote  $M_c = \{r(x) = c\}$ .

We will prove Theorem 1 in §2.

## 2. Proof of Theorem 1

We fix  $a_* < a_1 < a < 0 < b < b_1 < b^*$  and an hermitian metric on  $X$  which is real analytic on a neighbourhood of  $\bar{\Omega}_{a,b}$  and equals to the Levi metric near  $\partial\Omega_{a_1,b_1}$ . We denote  $\Omega = \Omega_{a,b}$  and  $\Omega_1 = \Omega_{a_1,b_1}$ .

In the first place, we will construct a versal family (in Kuranishi's sense) of complex structures of  $\mathcal{A}^m$ -class over  $\bar{\Omega}_1$ , by means of the method in [A2]. As shown in [A3], we are able to apply the method even to a (1,1)-convex-concave domain  $\Omega_1$ .

Let  $r = \dim_{\mathbb{C}} H^1(\bar{\Omega}_1, T^*X)$  and denote  $(t) = (t_1, \dots, t_r) \in \mathbb{C}^r$ . Let  $\|\cdot\|'_{(0,m)}$  be a norm, between the Sobolev  $(m+1)$ -norm and the tangential Sobolev  $(0,m)$ -norm, introduced in [A2]. In [A2], we obtained a convergent powerseries  $\varphi(t) \in \mathcal{A}_m^{0,1}\{t_1, \dots, t_r\}$ , where  $\mathcal{A}_m^{0,q}$  denotes the completion of  $A^{0,q}(\bar{\Omega}_1, T^*X)$  with respect to  $\|\cdot\|'_{(0,m)}$ -norm, such that

$$(2.1) \quad \varphi(0) = 0,$$

(2.2) if we denote by  $\varphi_1(t)$  the linear term of  $\varphi(t)$ , then  $\bar{\partial}\varphi_1(t) = 0$  and  $\varphi_1(t)$  spans  $H^{0,1}(\bar{\Omega}_1, T^*X)$ ,

$$(2.3) \quad \mathfrak{S}\varphi(t) = 0 \quad \text{on a neighbourhood of } \bar{\Omega},$$

$$(2.4) \quad \bar{\partial}\varphi(t) - (1/2)\bar{\partial}\mathfrak{N}[\varphi(t), \varphi(t)] = 0.$$

Where  $N$  is the Neumann operator for a new boundary condition introduced in [A2]. We recall the properties of  $N$  which will be needed in our arguments (cf. [A2], Theorem 5.1).

$$(2.5) \quad \psi = H\psi + (\bar{\partial}\mathfrak{S} + \mathfrak{S}\bar{\partial})N\psi,$$

$$(2.6) \quad \text{If } \bar{\partial}\psi = 0 \quad \text{and} \quad H\psi = 0 \quad \text{then} \quad \psi = \bar{\partial}\mathfrak{S}N\psi,$$

where  $H$  is the projection onto the harmonic space  $H^2$  with respect to the new boundary condition, and

(2.7)  $H^2$  is naturally isomorphic to  $H^{0,2}(\bar{\Omega}_1, T'X)$ .

Let  $h$  be a complex analytic map from a neighbourhood  $W$  of  $0 \in \mathbb{C}^r$  into  $H^2$  given by  $h(t) = H[\varphi(t), \varphi(t)]$ , and  $T = h^{-1}(0)$ .

Let  $P(\varphi(t)) = \bar{\partial}\varphi(t) - (1/2)[\varphi(t), \varphi(t)]$ .

**Proposition 2.1.**  $P(\varphi(t)) \in \mathcal{F}_{T,0} \otimes \mathcal{C}^{\mathcal{A}(0,m)}{}^{0,2} \{t_1, \dots, t_r\}$ , where  $\mathcal{C}^{\mathcal{A}(0,m)}{}^{0,2}$  denotes the completion of  $A^{0,2}(\bar{\Omega}_1, T'X)$  by tangential Sobolev  $(0, m)$ -norm.

The following Lemma implies Proposition 2.1 by the Grauert Division Theorem (cf. [G1]).

**Lemma 2.2.**  $P(\varphi(t)) \equiv 0 \pmod{(\mathfrak{m}^{\mu+1} + \mathcal{F}_{T,0}) \otimes \mathcal{C}^{\mathcal{A}(0,m)}{}^{0,2} \{t_1, \dots, t_m\}}$  for all  $\mu \geq 0$ .

*Proof.* We will prove Lemma 2.2 by the induction on  $\mu$ . For  $\mu=1$ , it is clear because  $P(\varphi(t)) \in \mathfrak{m}_t^2 \otimes \mathcal{C}^{\mathcal{A}(0,m)}{}^{0,2} \{t\}$ .

Assume that  $P(\varphi(t)) \equiv 0 \pmod{(\mathfrak{m}^{\mu+1} + \mathcal{F}_{T,0}) \otimes \mathcal{C}^{\mathcal{A}(0,m)}{}^{0,2} \{t\}}$  for  $\mu \geq 1$ .

By (2.4),  $P(\varphi(t)) = (1/2)\bar{\partial}\partial N[\varphi(t), \varphi(t)] - (1/2)[\varphi(t), \varphi(t)]$ .

If we denote by  $\psi(t)$  the canonical modulo of  $[\varphi(t), \varphi(t)] \pmod{(\mathfrak{m}^{\mu+2} + \mathcal{F}_{T,0}) \otimes \mathcal{C}^{\mathcal{A}(0,m)}{}^{0,2} \{t\}}$  by the Grauert division, then we have

$P(\varphi(t)) \equiv (1/2)\bar{\partial}\partial N\psi(t) - (1/2)\psi(t) \pmod{(\mathfrak{m}^{\mu+2} + \mathcal{F}_{T,0}) \otimes \mathcal{C}^{\mathcal{A}(0,m)}{}^{0,2} \{t\}}$ .

Hence  $H\psi(t) = 0$ . Since  $\bar{\partial}[\varphi(t), \varphi(t)] = 2[P(\varphi(t)), \varphi(t)]$ , we have  $\bar{\partial}\psi(t) = 0$  by the inductive assumption. Therefore we have

$\psi(t) = \bar{\partial}\partial N\psi(t)$  by (2.6). Q.E.D.

By (2.3) and (2.4) ,  $\varphi(t)$  satisfies an elliptic partial differential equation  $\sum_{i,j} (\partial^2 \varphi / \partial t_i \partial \bar{t}_j) + \square \varphi - (1/2) \mathfrak{D}[\varphi, \varphi] = 0$  and also satisfies  $\partial \varphi / \partial \bar{t} = 0$  , over a neighbourhood of  $\bar{\Omega} \times D$  , hence  $\varphi(t)$  is real analytic and depends complex analytically on  $t$  on a neighbourhood of  $\bar{\Omega} \times D$  where  $D$  is a neighbourhood of  $0$  in  $\mathbb{C}^r$  .

**Proposition 2.3.** *There exists a family of complex manifolds  $\pi: \mathcal{X} \rightarrow T$  with  $\pi^{-1}(o) \simeq \Omega$  and with a real analytic isomorphism  $G: \Omega \times T \rightarrow \mathcal{X}$  such that (i)  $G|_{\Omega \times o} = id_{\mathcal{X}|_{\Omega}}$  , (ii)  $G$  is a complex analytic map with respect to the complex structure on  $\Omega \times T$  defined by  $\varphi(t)$  , (iii)  $G$  is complex analytic on  $p \times T$  for each fixed  $p \in \Omega$  .*

*Proof.* By ideal-theoretically improvement of the argument in the proof of Proposition 7.3 of [Ku2] , we can obtain a collection of families of local charts . By the same argument in the proof of Proposition 1.3 of [Ku1] using the Grauert Division theorem instead of Lemma 1.3 of [Ku1] , we have an analytic space  $\mathcal{X}$  by patching these local charts together . Q.E.D.

Next we will show that this family  $\pi: \mathcal{X} \rightarrow (T, o)$  is a formally versal family for  $\mathcal{F}_{M_c}$  for any  $a < c < b$  .

Let  $a < a_2 < c < b_2 < b$  and  $\Omega_2 = \Omega_{a_2, b_2}$  . Then , by the same arguments in §3 of [A-K] , slightly modified with the Grauert Division Theorem (cf. [G]) and relying on the isomorphism  $H^i(\bar{\Omega}_1, T'X) \simeq H^i(\bar{\Omega}_2, T'X)$  for  $i=1,2$  (cf. [H] Theorem 3.4.8) , we have

**Proposition 2.4.** Let  $(T, o)$  and  $\varphi(t)$  be as constructed above. If  $\pi': \mathcal{X}' \rightarrow (S, o)$  be a family of complex manifolds with  $\pi'^{-1}(o) \simeq a$  neighbourhood of  $\bar{\Omega}_2$ . Suppose a morphism  $\alpha^\mu: S_\mu \rightarrow T_\mu$  and an embedding  $g^\mu: \bar{\Omega}_2 \times S_\mu \rightarrow \mathcal{X}' \times_S S_\mu$  are given for some  $\mu \geq 0$  satisfying

(1) <sub>$\mu$</sub>   $p'_2 \circ g^\mu = p_2$  and

(2) <sub>$\mu$</sub>   $(\bar{\partial} - \varphi(\alpha^\mu(s))) (\xi' \circ g^\mu(z, s)) \in (m_S^{\mu+1} + \mathcal{I}_{S, o}) \otimes_{\mathbb{C}} A^{0,1}(U \cap \bar{\Omega}_2, T'X)[[s]]$  for any charts  $(U, z)$  of  $X$  and  $(\mathcal{U}', (\xi', s))$  of  $\mathcal{X}'$ , where  $p'_2$  and  $p_2$  denote projections onto the second factors. Then there exist extensions  $\alpha^{\mu+1}: S_{\mu+1} \rightarrow T_{\mu+1}$  and  $g^{\mu+1}: \bar{\Omega}_2 \times S_{\mu+1} \rightarrow \mathcal{X}' \times_S S_{\mu+1}$  satisfying (1) <sub>$\mu+1$</sub>  and (2) <sub>$\mu+1$</sub> . Where by subscript  $\mu$  we denote the infinitesimal neighbourhood of order  $\mu$ .

We will now show that Proposition 2.4 implies the following formal versality of  $\pi: \mathcal{X} \rightarrow (T, o)$ .

**Proposition 2.5.** Let  $\pi: \mathcal{X} \rightarrow (T, o)$  be as in Proposition 2.3 and  $\pi': \mathcal{X}' \rightarrow (S, o)$  be a family of complex manifolds with  $\pi'^{-1}(o) \simeq a$  neighbourhood of  $M_c$ . If a morphism  $\alpha^\mu: S_\mu \rightarrow T_\mu$  and an isomorphism  $H^\mu: \mathcal{X}'_1 \times_S S_\mu \rightarrow \mathcal{X}_1 \times_T S_\mu$  are given for some  $\mu \geq 0$ , where  $\mathcal{X}'_1$  and  $\mathcal{X}_1$  are neighbourhoods of  $M_c \times o$  in  $\mathcal{X}'$  and  $\mathcal{X}$  respectively, then we have extensions  $\alpha^{\mu+1}: S_{\mu+1} \rightarrow T_{\mu+1}$  and  $H^{\mu+1}: \mathcal{X}'_2 \times_S S_{\mu+1} \rightarrow \mathcal{X}_2 \times_T S_{\mu+1}$ . Where  $\mathcal{X}'_2$  and  $\mathcal{X}_2$  are smaller neighbourhoods of  $M_c \times o$  in  $\mathcal{X}'$  and  $\mathcal{X}$  respectively.

*Proof.* We may assume that  $\bar{\Omega}_2 \times 0 \subset \mathfrak{X}_1$ . Let  $G^\mu = (H^\mu)^{-1}$  and  $g^\mu = G^\mu \circ (\alpha^\mu)^* (G|_{\bar{\Omega}_2 \times T})$ , where  $(\alpha^\mu)^* G$  is the real analytic isomorphism  $\Omega \times S_\mu \rightarrow \mathfrak{X} \times_T S_\mu$  induced from  $G$  in Proposition 2.3 via  $\alpha^\mu$ . Then, by Proposition 2.4, we have extensions  $\alpha^{\mu+1}: S_{\mu+1} \rightarrow T_{\mu+1}$  and  $g^{\mu+1}: \bar{\Omega}_2 \times S_{\mu+1} \rightarrow \mathfrak{X}' \times_S S_{\mu+1}$  satisfying (1) $_{\mu+1}$  and (2) $_{\mu+1}$  of Proposition 2.4.

Let  $\{\mathfrak{U}_i, (\xi_i, t)\}$ ,  $\{\mathfrak{U}'_i, (\xi'_i, t)\}$  and  $\{U_i, z_i\}$  be systems of local charts of  $\mathfrak{X}$ ,  $\mathfrak{X}'$  and  $X$  respectively and assume that  $\xi_i = f_{ij}(\xi_j, t)$  on  $\mathfrak{U}_i \cap \mathfrak{U}_j$ ,  $\xi'_i = h_{ij}(\xi'_j, s)$  on  $\mathfrak{U}'_i \cap \mathfrak{U}'_j$  and  $z_i = \bar{f}_{ij}(z_j)$  on  $U_i \cap U_j$ . We denote  $\xi_i \circ G(z_i, t) = \tilde{z}_i(z_i, t)$ .

Suppose that the isomorphism  $G^\mu$  is given by  $\xi'_i = G^\mu_i(\xi_i, s)$ . Let  $\chi_{i|\mu+1}(s)$  be a homogeneous polynomial of  $s$  of degree  $\mu+1$  given as the canonical modulo of  $g^{\mu+1}_i(s) - G^\mu_i(\tilde{z}_i(z_i, \alpha^{\mu+1}(s)), s)$  mod  $\mathfrak{m}_s^{\mu+2+\mathcal{G}}_{S,0}$  in  $A^{0,0}(U_i \cap \bar{\Omega}_2, T'X)[[s]]$ .

**Lemma 2.6.**  $\bar{\partial}\chi_{i|\mu+1}(s) = 0$ .

*Proof.*  $\bar{\partial}\chi_{i|\mu+1}(s) \equiv (\bar{\partial}-\varphi(\alpha^{\mu+1}(s)))g^{\mu+1}_i(s) - (\partial G^\mu_i/\partial \xi_i)(\bar{\partial}-\varphi(\alpha^{\mu+1}(s)))\tilde{z}_i(z_i, \alpha^{\mu+1}(s)) + \varphi(\alpha^{\mu+1}(s))\chi_{i|\mu+1}(s) \equiv 0 \pmod{\mathfrak{m}_s^{\mu+2+\mathcal{G}}_{S,0}}$ . Q.E.D.

Let  $G_{i|\mu+1}(s) = \chi_{i|\mu+1}(s)$  and  $G_i^{\mu+1}(s) = G^\mu_i(s) + G_{i|\mu+1}(s)$ .

**Lemma 2.7.**  $G^{\mu+1} = \{G_i^{\mu+1}(s)\}$  is a holomorphic map from  $\mathfrak{X}_2 \times_T S_{\mu+1}$  into  $\mathfrak{X}'_1 \times_S S_{\mu+1}$  where  $\mathfrak{X}_2$  is a smaller neighbourhood of  $M_c \times 0$ .

*Proof.* Let  $\Xi_{ij}^{\mu+1}(s)$  be the canonical modulo of

$$G_i^{\mu+1}(f_{ij}(\xi_j, \alpha^{\mu+1}(s)), s) - h_{ij}(G_j^{\mu+1}(\xi_j, s), s) \pmod{\mathfrak{m}_S^{\mu+2+\mathcal{G}}_{S,0}} \text{ in } C^0(U_i \cap U_j \cap \Omega_2, \theta)[[s]] .$$

$$\text{Then } \Xi_{ij}^{\mu+1}(\tilde{z}_j(z_j, \alpha^{\mu+1}(s)), s) \equiv g_i^{\mu+1}(z_j, s) - h_{ij}(g_j^{\mu+1}(z_j, s), s) \equiv 0 \pmod{\mathfrak{m}_S^{\mu+2+\mathcal{G}}_{S,0}} .$$

Let  $D = \{\text{ord}(h) \mid h \in \mathfrak{m}_S^{\mu+1+\mathcal{G}}_{S,0}\}$  . Then ,

since  $\Xi_{ij}^{\mu+1}(\xi_j, s) = \sum_{\nu \notin D} \xi_{ij}^{(\nu)}(\xi_j) s^\nu$  with  $\xi_{ij}^{(\nu)}(z_j) \in C^1(\{U_i \cap \Omega_2\}, \theta)$  and

$$\sum_{\nu \notin D} \xi_{ij}^{(\nu)}(z_j + o(s)) s^\nu \equiv 0 \pmod{\mathfrak{m}_S^{\mu+2+\mathcal{G}}_{S,0}} ,$$

we have  $\xi_{ij}^{(\nu)}(z_j) = 0$  by induction on  $|\nu|$  , where  $o(s)$  is a term of order greater than zero with respect to  $s$  .

Hence  $\Xi_{ij}^{\mu+1}(\xi_j, s) \equiv 0 \pmod{\mathfrak{m}_S^{\mu+2+\mathcal{G}}_{S,0}}$  . Q.E.D.

Hence we have an extension  $G^{\mu+1}$  of  $G^\mu$  . It is clear that  $(G^{\mu+1})^{-1}$  is an extension of  $H^\mu$  .

Therefore Proposition 2.5 and Theorem 1 are proved .

### 3. Proof of Theorem 2

Let  $\pi: \mathfrak{X} \rightarrow (T, o)$  be a family of complex manifolds with  $\pi^{-1}(o) \simeq \Omega$  in §2 and  $(\omega(s), (S, o))$  the Kuranishi family of CR structures on  $M$  with  $\omega(o)=0$  obtained in [A1] or [M1]. Though we treated in [M1] only reduced parameters, we can improve the argument by the same way as in the proof of Proposition 2.1.

By the same argument in [A-M], improved with Grauert Division Theorem, we have a holomorphic map  $\beta: (T, o) \rightarrow (S, o)$  and a  $C^k$ -embedding  $g: M \times (T, o) \rightarrow \mathfrak{X}$ .

Hence Theorem 2 will be reduced to the following Proposition.

**Proposition 3.1.**  $\beta$  is an isomorphism.

*Proof.* In order to prove this proposition, it is sufficient to show that  $\beta$  induces a formal isomorphism.

On  $M$ , we have a direct sum decomposition as  $C^\infty$ -vector bundles;  $CTM = CF + {}^\circ T' + {}^\circ T''$  where  ${}^\circ T' = T'X|_M \cap CTM$ ,  ${}^\circ T'' = T''X|_M \cap CTM$  and  $F$  is a real line bundle. We denote  $CF + {}^\circ T'$  by  $T'$  and  $(\rho'|_{T'})^{-1}$  by  $\tau$  where  $\rho': CTX|_M \rightarrow T'X|_M$  is the projection.

We denote by the same symbol  $\tau$  the composition of the restriction map  $A^{0,q}(\bar{\Omega}_{a,b}, T'X) \rightarrow A_b^{0,q}(M, T'X|_M)$  and the map  $A_b^{0,q}(M, T'X|_M) \rightarrow A_b^{0,q}(M, T')$  induced from the above bundle isomorphism  $\tau$ , where  $a_* < a < 0 < b < b^*$  (cf. [Ak2] p. 319).

**Lemma 3.2.** For any  $a_* < a < 0 < b < b^*$ ,  $\tau$  induces an isomorphism

$$H^{0, q}(\bar{\Omega}_{a, b}, T'X) \simeq H_{\bar{\partial}_b}^q(M, T') \quad (1 \leq q \leq n-2) .$$

*Proof.* By the same arguments in pp. 81 and 82 of [Y], we have

$$H^{0, q}(\bar{\Omega}_{a, 0}, T'X) \simeq H_{\bar{\partial}_b}^q(M, T'X|_M) \quad (1 \leq q \leq n-2) .$$

Since the bundle isomorphism  $\tau: T'X|_M \rightarrow T'$  induces the isomorphisms of cohomologies  $H_{\bar{\partial}_b}^{0, q}(M, T'X|_M) \simeq H_{\bar{\partial}_b}^{0, q}(M, T')$ , we infer the lemma from Theorem 3.4.8 of [H]. Q.E.D.

We note that Lemma 3.2 implies that  $\beta|_{T_1}$  is an isomorphism.

Suppose that  $\beta^\mu = \beta|_{T_\mu}: T_\mu \rightarrow S_\mu$  is an isomorphism for  $\mu \geq 1$ .

We recall that  $g(t)$  is represented by a formal power series  $g_i(t) \in A^{0, 0}(U_i \cap M, T'X)[[t]]$  for each local charts  $(U_i, (z_i))$  of  $X$  and  $(\mathfrak{U}_i, (\xi_i, t))$  of  $\mathfrak{X}$  respectively (cf. [A-M] §3). Let  $g^\mu$  be the embedding  $M \times T_\mu \rightarrow \mathfrak{X} \times_{T_\mu} T_\mu$  induced from  $g$ .

Since the sheaf of differentiable sections of  $T'X$  vanishing on  $M$  is a fine sheaf, we can extend  $g^\mu$  to an embedding  $\tilde{g}^\mu: (a$  neighbourhood of  $\bar{\Omega}_3 = \bar{\Omega}_{a_3, b_3}) \times T_\mu \rightarrow \mathfrak{X} \times_{T_\mu} T_\mu$  for some  $a < a_3 < 0 < b_3 < b$ .

Let  $\theta^\mu(t)$  be a family of complex structures over the neighbourhood of  $\bar{\Omega}_3$  induced from the complex structure of  $\mathfrak{X}$  via the embedding  $\tilde{g}^\mu$ . Let  $\varphi^\mu(s)$  be a  $A^{0, 1}(\bar{\Omega}_3, T'X)$ -valued polynomial of  $s$  of degree  $\mu$  given by  $\varphi^\mu(s) \equiv \theta^\mu((\beta^\mu)^{-1}(s)) \pmod{\mathfrak{m}_s^{\mu+1}}$ . Then we have

$$(3.1)_\mu \quad P(\varphi^\mu(s)) \equiv 0 \pmod{\mathfrak{m}_s^{\mu+1} + \mathcal{F}_{S, 0}} \quad \text{in } A^{0, 2}(\bar{\Omega}_3, T'X)[[s]] , \quad \text{and}$$

$$(3.2)_\mu \quad \tau\varphi^\mu(s) \equiv \omega(s) \pmod{\mathfrak{m}_s^{\mu+1} + \mathcal{F}_{S, 0}} \quad \text{in } A_b^{0, 1}(M, T')[[s]] .$$

Let  $E_q$  be a subbundle of  $T' \otimes \Lambda^q({}^\circ T'')$  such that  
 $\Gamma(M, E_q) = \{\omega \in A_b^{0,q}(M, {}^\circ T') \mid \bar{\partial}_b \omega \in A_b^{0,q+1}(M, {}^\circ T')\}$  and  $\mathcal{E}_q = \{\varphi \in A^{0,q}(\bar{\Omega}_3, T'X) \mid \tau\varphi \in \Gamma(M, E_q)\}$  (cf. [Ak1] Proposition 2.1 and [Ak2] p.323 respectively).

**Sublemma 1.** Let  $\varphi \in A^{0,1}(\bar{\Omega}_3, T'X)$  and  $\omega$  an almost CR structure on  $M$  induced from the almost complex structure  $\varphi$  on  $\bar{\Omega}_3$ . Then  $\tau\varphi \in A_b^{0,1}({}^\circ T')$  if and only if  $\omega \in A_b^{0,1}({}^\circ T')$ . Moreover  $\tau\varphi = \omega$ .

*Proof.* Since  ${}^\omega T'' \subset {}^\varphi T''X|_M$ ,  $u - \omega(u) \in {}^\varphi T''X|_M$  for  $u \in {}^\circ T''$ . If we denote by  $\rho'$  and  $\rho''$  the projections from  $CTX|_M$  onto  $T'X|_M$  and  $T''X|_M$  respectively,  $\varphi(u - \rho''\omega(u)) = \rho'\omega(u)$ . Since  $\tau\rho'|_{T'} = id$ ,  $\tau\varphi \in A_b^{0,1}({}^\circ T')$  if and only if  $\omega \in A_b^{0,1}({}^\circ T')$ . If  $\omega \in A_b^{0,1}({}^\circ T')$ , then  $\varphi(u) = \omega(u)$  for  $u \in {}^\circ T''$ . This implies  $\tau\varphi = \omega$ . Q.E.D.

Let  $P(\varphi)$  and  $P_b(\omega)$  denote the integrability conditions of an almost complex structure  $\varphi$  on  $\bar{\Omega}_3$  and of an almost CR structure  $\omega$  on  $M$  respectively (cf. [A2] Definition 2.1 and [M1] (1.2) respectively).

In the followings, by a subscript  $T'X$  (resp.  $T'$ ,  ${}^\circ T''$  and  $CF$ ) we denote the projection onto  $T'X$  (resp.  $T'$ ,  ${}^\circ T''$  and  $CF$ ).

**Sublemma 2.** If  $\varphi \in \mathcal{E}_1$ , then  $\tau P(\varphi) = P_b(\tau\varphi)$ .

Proof. If  $\tau\varphi \in \Gamma(M, E_1)$  we have

$$\tau P(\varphi)(u, v) = (\tau \bar{\partial} \varphi)(u, v) + \tau([\varphi(u), \varphi(v)]_{T'X}) - \tau\varphi([u, \varphi(v)]_{T''X} + [\varphi(u), v]_{T''X})$$

for  $u, v \in \Gamma(M, {}^\circ T'')$ . (Cf. [Ak2] Definition 2.1.) Since  $\varphi(u)$

and  $\varphi(v)$  are in  $\Gamma(M, {}^\circ T')$  we have  $\tau([\varphi(u), \varphi(v)]_{T'X}) = [\varphi(u), \varphi(v)]$ .

On the other hand  $P_b(\tau\varphi)(u, v) = \bar{\partial}_b \tau\varphi(u, v) + [\tau\varphi(u), \tau\varphi(v)]_T$

$$- \tau\varphi([u, \tau\varphi(v)]_{T''} + [\tau\varphi(u), v]_{T''}) - \tau\varphi([\tau\varphi(u), \tau\varphi(v)]_{T''}) \text{ for } u, v$$

$\in \Gamma(M, {}^\circ T'')$ . Since  $\varphi(u)$  and  $\varphi(v)$  are in  $\Gamma(M, {}^\circ T')$  we have

$$P_b(\tau\varphi)(u, v) = \bar{\partial}_b \tau\varphi(u, v) + [\varphi(u), \varphi(v)] - \varphi([u, \varphi(v)]_{T''} + [\varphi(u), v]_{T''})$$

Now, since  $\tau\varphi \in A_b^{0,1}({}^\circ T')$  we have  $\bar{\partial}_b \tau\varphi(u, v) = [\varphi(u), v]_T + [u, \varphi(v)]_T$

$$- \varphi([u, v])$$

Hence  $[\varphi(u), v]_{T''X} + [u, \varphi(v)]_{T''X} = [\varphi(u), v]_{T''} + [u, \varphi(v)]_{T''}$ . Thus we

have  $\tau P(\varphi)(u, v) = P_b(\tau\varphi)(u, v)$  for  $u, v \in \Gamma(M, {}^\circ T'')$ . Q.E.D.

As was shown in the proof of Theorem 2.4 of [A1], for any  $\omega \in A_b^{0,q}(\mathbb{C}F)$ , we have a  $\theta \in A_b^{0,q-1}({}^\circ T')$  such that  $\omega = (\bar{\partial}_b \theta)_{\mathbb{C}F}$ . Hence we have

**Sublemma 3.** Let  $B'$  be a finite dimensional subspace of  $A_b^{0,q}(\mathbb{C}F)$ . Then we have linear maps  $\theta: B' \rightarrow A_b^{0,q-1}({}^\circ T')$  and  $\tilde{\theta}: B' \rightarrow \mathcal{F}_{q-1} = \{\varphi \in A_b^{0,q-1}(\bar{\Omega}_3, T'X) \mid \tau(\varphi) \in A_b^{0,q-1}({}^\circ T')\}$  satisfying (1)  $\omega = (\bar{\partial}_b \theta(\omega))_{\mathbb{C}F}$  and (2)  $\tau \tilde{\theta} = \theta$ .

**Sublemma 4.** Let  $B$  be a finite dimensional subspace of  $A_b^{0,q}(\bar{\Omega}_3, T'X)$ . Then there exist linear maps  $p: B \rightarrow \delta_q$  and  $p_b: \tau(B) \rightarrow \Gamma(M, E_q)$  satisfying (1)  $p|_{B \cap \delta_q} = id$ , (2)  $p_b|_{\tau(B) \cap \Gamma(M, E_q)} = id$  and (3)  $\tau p = p_b \tau$ .

*Proof.* Let  $B'_q = (\tau B)_{CF}$  and  $B'_{q+1} = (\bar{\partial}_b \tau B)_{CF}$ . Then, by sublemma 3, we have linear maps  $\tilde{\theta}_i: B'_i \rightarrow \mathcal{F}_{i-1}$  and  $\theta_i: B'_i \rightarrow A_b^{0, i-1}(\circ T')$  satisfying (1) and (2) of sublemma 3 ( $i=q, q+1$ ). If we set  $p(\varphi) = \varphi - \bar{\partial} \tilde{\theta}_q((\tau \varphi)_{CF}) - \tilde{\theta}_{q+1}((\bar{\partial}_b \tau \varphi)_{CF})$  and  $p_b(\omega) = \omega - \bar{\partial}_b \theta_q((\omega)_{CF}) - \theta_{q+1}((\bar{\partial}_b \omega)_{CF})$ , then it is clear that  $p(\varphi) \in \mathcal{E}_q$ ,  $p_b(\omega) \in \Gamma(M, E_q)$  and satisfy (1), (2) and (3). Q.E.D.

Let  $D = \{\text{ord}(h) \mid h \in \mathcal{F}_{S,0}\}$  and  $\Lambda$  be the reducing system of  $D$ . Then we have a system of generators  $\{h_\alpha\}_{\alpha \in \Lambda}$  of  $\mathcal{F}_{S,0}$  with the form  $h_\alpha(s) = s^{\alpha + g_\alpha}$   $\text{ord}(g_\alpha) > \alpha$ . By the Grauert Division Theorem (cf. [G]), for any  $f(s) \in A[[s]]$  there exists a unique  $r = \sum_{\nu \notin D} r_\nu s^\nu \in A[[s]]$  such that  $f(s) - r(s) \in \mathcal{F}_{S,0} \otimes A[[s]]$ , where  $A$  denotes a  $\mathbb{C}$ -module.

By (3.2) $_\mu$ , we have  $\tau \varphi^\mu(s) - \omega(s) = \sum_{\alpha \in \Lambda} a_\alpha(s) h_\alpha(s) + \sum_{|I| \geq \mu+1, I \notin D} \beta_I s^I$  where  $a_\alpha(s) \in A_b^{0,1}(T')[s]$  and  $\beta_I \in A_b^{0,1}(T')$ . Let  $B$  be a finite dimensional subspace of  $A^{0,1}(\bar{\Omega}_3, T'X)$  which contains all coefficients of  $\varphi^\mu(s)$  and such that  $\tau(B)$  is generated by the coefficients of  $\tau \varphi^\mu(s)$ ,  $\omega(s)$ ,  $a_\alpha(s)$  of degree less than  $\mu+2$  and  $\beta_I$  for  $|I| = \mu+1$ . By sublemma 4, we have linear maps  $p: B \rightarrow \mathcal{E}_1$  and  $p_b: \tau(B) \rightarrow \Gamma(M, E_1)$  satisfying (1), (2) and (3) of sublemma 4.

If we set  $\psi^\mu(s) = p(\varphi^\mu(s))$ , then  $\psi^\mu(s) \in \mathcal{E}_1[s]$  and satisfies (3.2) $_\mu$ :  $\tau \psi^\mu(s) - \omega(s) = \sum_{\alpha \in \Lambda} a'_\alpha(s) h_\alpha(s) - \sum_{|I| = \mu+1, I \notin D} \beta'_I s^I \in \pi_S^{\mu+2} \otimes A_b^{0,1}(T')[[s]]$ , where  $a'_\alpha(s) \in \Gamma(M, E_1)[s]$  and  $\beta'_I \in \Gamma(M, E_1)$ .

Also by (3.2)<sub>μ</sub>, we have  $(\tau\varphi^\mu(s))_{CF \in (\mathfrak{m}_S^{\mu+1} + \mathfrak{f}_{S,0}) \otimes A_b^{0,1}(CF)[[s]]}$  and  $(\bar{\partial}_b \tau\varphi^\mu(s))_{CF \in (\mathfrak{m}_S^{\mu+1} + \mathfrak{f}_{S,0}) \otimes A_b^{0,2}(CF)[[s]]}$ . Hence by the definition of  $p$  and  $p_b$ , we have  $\psi^\mu(s) - \varphi^\mu(s) \in (\mathfrak{m}_S^{\mu+1} + \mathfrak{f}_{S,0}) \otimes A^{0,1}(\bar{\Omega}_3, T'X)[[s]]$ . Hence by (3.1)<sub>μ</sub>, we have

$$P(\psi^\mu(s)) \equiv P(\varphi^\mu(s)) \equiv 0 \pmod{(\mathfrak{m}_S^{\mu+1} + \mathfrak{f}_{S,0}) \otimes A^{0,2}(\bar{\Omega}_3, T'X)[[s]]}.$$

Since  $P(\psi^\mu(s)) \in \delta_2[s]$ , by the same argument as above, we have

$$(3.1)'_\mu \quad P(\psi^\mu(s)) = \sum_{\alpha \in \Lambda} b'_\alpha(s) h_\alpha(s) + \sum_{|I|=\mu+1, I \notin D} \gamma'_I s^I \\ \in \mathfrak{m}_S^{\mu+2} \otimes A^{0,2}(\bar{\Omega}_3, T'X)[[s]], \text{ where } b'_\alpha(s) \in \delta_2[s] \text{ and } \gamma'_I \in \delta_2.$$

**Lemma 3.3.** *There exists an extension  $\psi^{\mu+1}(s)$  of  $\psi^\mu(s)$  satisfying (3.1)'<sub>μ+1</sub> and (3.2)'<sub>μ+1</sub>.*

*Proof.* Let  $\psi'_{\mu+1}(s)$  be an  $\delta_1$ -valued homogeneous polynomial of  $s$  of degree  $\mu+1$  such that  $\tau\psi'_{\mu+1}(s)$  is the canonical modulo of

$$\tau\psi^\mu(s) - \omega(s) \pmod{\mathfrak{m}_S^{\mu+2} + \mathfrak{f}_{S,0}}. \quad \text{Then by (3.2)'}_\mu \text{ we have}$$

$$\psi'_{\mu+1}(s) \in \delta_1[s].$$

Let  $p'_{\mu+1}(s)$  be the canonical modulo of  $P(\psi^\mu(s) + \psi'_{\mu+1}(s))$

$$\pmod{\mathfrak{m}_S^{\mu+2} + \mathfrak{f}_{S,0}}. \quad \text{Then, by (3.1)'}_\mu \text{ and since (the canonical modulo}$$

$$\text{of } P(\psi^\mu(s) + \psi'_{\mu+1}(s)) \pmod{\mathfrak{m}_S^{\mu+2} + \mathfrak{f}_{S,0}} = \text{(the canonical modulo of}$$

$$P(\psi^\mu(s)) + \bar{\partial}\psi'_{\mu+1}(s) \pmod{\mathfrak{m}_S^{\mu+2} + \mathfrak{f}_{S,0}} = \text{(The canonical modulo of } P(\psi^\mu(s))$$

$$\pmod{\mathfrak{m}_S^{\mu+2} + \mathfrak{f}_{S,0}} + \bar{\partial}\psi'_{\mu+1}(s), \text{ we have } p'_{\mu+1} \in \delta_2[s].$$

Since  $\bar{\partial}P(\psi^\mu(s) + \psi'_{\mu+1}(s)) = -[P(\psi^\mu(s) + \psi'_{\mu+1}(s)), \psi^\mu(s) + \psi'_{\mu+1}(s)] \equiv 0$   
 $\text{mod } \pi_S^{\mu+2+g} S_{,0}$  in  $A^{0,3}(\bar{\Omega}_3, TX)[[s]]$ , we have

$$(3.3) \quad \bar{\partial}p'_{\mu+1}(s) = 0 .$$

By Sublemma 2 and the definition of  $\psi'_{\mu+1}(s)$ , we have

$$\tau P(\psi^\mu(s) + \psi'_{\mu+1}(s)) \equiv P_b(\omega(s)) \text{ mod } \pi_S^{\mu+2+g} S_{,0} \text{ in } \Gamma(M, E_2)[[s]] .$$

Hence we have

$$(3.4) \quad \tau p'_{\mu+1}(s) = 0 ,$$

Now we consider a subcomplex  $(\mathcal{E}^{0, \cdot}, \bar{\partial})$  of  $TX$ -valued  $\bar{\partial}$ -complex. Let  $\mathcal{E}^{0, q} = \{\varphi \in A^{0, q}(\bar{\Omega}_3, TX) \mid \tau\varphi = 0 \text{ on } M\}$ .

Then Lemma 3.2 implies

$$(3.5) \quad H_{\bar{\partial}}^q(\mathcal{E}^{0, \cdot}) = 0 \text{ for } 2 \leq q \leq n-2 .$$

Hence, by (3.3), (3.4) and (3.5), there exists

$$\psi''_{\mu+1}(s) \in \mathcal{E}^{0, 1} \text{ such that } \bar{\partial}\psi''_{\mu+1}(s) = -p'_{\mu+1}(s) .$$

Hence, if we set  $\psi^{\mu+1}(s) = \psi^\mu(s) + \psi'_{\mu+1}(s) + \psi''_{\mu+1}(s)$  then  $\psi^{\mu+1}(s)$  satisfies (3.2)'\_{\mu+1} and

$$(3.6) \quad P(\psi^{\mu+1}(s)) \equiv p'_{\mu+1}(s) + \bar{\partial}\psi''_{\mu+1}(s) = 0 \text{ mod } \pi_S^{\mu+2+g} S_{,0} \text{ in } \mathcal{E}_2[[s]] .$$

Hence we infer (3.1)'\_{\mu+1} from (3.6). Q.E.D.

**Lemma 3.4.** *There exists a family of complex manifolds  $\pi': \mathcal{X}' \rightarrow S_{\mu+1}$  associated with the family  $\psi^{\mu+1}(s)$  of complex structures and such that  $\mathcal{X}' \times_{S_{\mu+1}} S_\mu \simeq \mathcal{X}_1 \times_{T_\mu} S_\mu$  where  $\mathcal{X}_1$  is a neighbourhood of  $M \times 0$  in  $\mathcal{X}$ .*

*Proof.* First, we will show the existence of families of complex charts. Let  $\{V_\nu\}$  be a refinement of  $\{U_i\}$  with  $M \subset \bigcup_\nu V_\nu \subset \Omega_3$  and such that  $V_\nu$  is holomorphically equivalent to a polydisc in a complex Euclidean space. Let  $z_i^\mu(s)$  be the polynomial part of  $\tilde{g}_i^\mu((\beta^\mu)^{-1}(s))$  of degree less than or equal to  $\mu$ . Since

$\varphi^\mu(s) \equiv \psi^\mu(s) \pmod{\mathfrak{m}_S^{\mu+1} + \mathcal{I}_{S,0}}$  in  $A^{0,1}(\bar{\Omega}_3, T'X)[[s]]$  we have

$$(3.7)_\mu \quad (\bar{\partial} - \psi^\mu(s))z_i^\mu(s) \equiv 0 \pmod{\mathfrak{m}_S^{\mu+1} + \mathcal{I}_{S,0}} \text{ in } A^{0,1}(U_i, T'X)[[s]].$$

Let  $\xi_{\mu+1}(s)$  be the canonical modulo of  $-(\bar{\partial} - \psi^{\mu+1}(s))z_i^\mu(s) \pmod{\mathfrak{m}_S^{\mu+2} + \mathcal{I}_{S,0}}$  in  $A^{0,1}(U_i, T'X)[[s]]$ . Then  $\xi_{\mu+1}(s)$  is a homogeneous polynomial of  $s$  of degree  $\mu+1$ , and  $\bar{\partial}\xi_{\mu+1}(s) = 0$  because  $\bar{\partial}(\bar{\partial} - \psi^{\mu+1}(s))z_i^\mu(s) \equiv P(\psi^{\mu+1}(s))z_i^\mu(s) \pmod{\mathfrak{m}_S^{\mu+2} + \mathcal{I}_{S,0}}$  by (3.7) $_\mu$ .

Since  $V_\nu$  is a polydisc, there exists  $z_{\nu|\mu+1}(s)$  such that  $\bar{\partial}z_{\nu|\mu+1}(s) = \xi_{\mu+1}(s)|_{V_\nu}$ .

If we set  $z_\nu^{\mu+1}(s) = z_i^\mu(s)|_{V_\nu} + z_{\nu|\mu+1}(s)$  then we have (3.7) $_{\mu+1}$ .

Next, we obtain a family  $\pi': \mathcal{X}' \rightarrow S_{\mu+1}$  associated with  $\psi^{\mu+1}(s)$  by the same argument as in the proof of Proposition 2.5.

$\mathcal{X}' \times_{S_{\mu+1}} S_\mu \simeq \mathcal{X}_1 \times_{T_\mu} S_\mu$  follows from  $\psi^\mu(s) \equiv \varphi^\mu(s) \pmod{\mathfrak{m}_S^{\mu+1} + \mathcal{I}_{S,0}}$  in  $A^{0,1}(\bar{\Omega}_3, T'X)[[s]]$ . Q.E.D.

By the formal versality of  $\pi: \mathcal{X} \rightarrow (T, o)$  for  $\mathcal{F}_M$ , we have an extension  $\gamma^{\mu+1}: S_{\mu+1} \rightarrow T_{\mu+1}$  of  $\gamma^\mu = (\beta^\mu)^{-1}$ . Since  $\gamma^{\mu+1} \cdot \beta^{\mu+1} \equiv \gamma^1 \cdot \beta^1 \equiv 1 \pmod{\mathfrak{m}_t^2}$  and  $\beta^{\mu+1} \cdot \gamma^{\mu+1} \equiv \beta^1 \cdot \gamma^1 \equiv 1 \pmod{\mathfrak{m}_S^2}$ ,  $\beta^{\mu+1}$  is isomorphic.

This completes the proofs of Proposition 3.1 and of Theorem 2.

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# Cohomological Criterion of Numerically Effectivity

by

Atsushi Moriwaki

Let  $X$  be a  $d$ -dimensional non-singular projective variety and  $L$  a divisor on  $X$ . Then, if  $L$  is nef and big, by Kodaira-Kawamata-Viehweg vanishing Theorem,  $H^i(X, \mathcal{O}_X(L + K_X)) = 0$  for all  $i > 0$ . In this note, we consider an inverse problem of the above Theorem.

We fix our situation. Let  $X$  and  $L$  be as above. We assume that  $L$  is big and there exists a very ample divisor  $A$  such that for any general member  $B \in |A|$ ,  $L|_B$  is nef on  $B$ . Then, we have

**Theorem** If  $\limsup_{m \rightarrow \infty} \frac{\dim H^1(X, \mathcal{O}_X(K_X + mL))}{m^d} = 0$ , then  $L$  is nef.

We start the following claim.

**Claim 1.**  $H^i(X, \mathcal{O}_X(K_X + mL)) = 0$ , for  $m > 0, i > 1$ .

We take a general member  $B$  of  $|A|$  such that  $L|_B$  is nef. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + mL + (j-1)B) \longrightarrow \mathcal{O}_X(K_X + mL + jB) \longrightarrow \mathcal{O}_B(K_B + mL + (j-1)B) \longrightarrow 0.$$

By applying Kodaira-Kawamata-Viehweg vanishing on  $B$ ,

$$H^i(X, \mathcal{O}_X(K_X + mL + (j-1)L)) \xrightarrow{\sim} H^i(X, \mathcal{O}_X(K_X + mL + jL)),$$

for  $i > 1, j > 0$ . Hence Serre vanishing implies our claim 1.

Let  $C$  be an irreducible curve on  $X$  and  $I$  the defining ideal of  $C$ . We must show  $(L \cdot C) \geq 0$ .

**Claim 2.** To show  $(L \cdot C) \geq 0$ , we may assume  $C$  is non-singular.

There is a sequence of blowing-ups

$$X_k \xrightarrow{f_k} X_{k-1} \xrightarrow{f_{k-1}} \dots \xrightarrow{f_1} X_0 = X$$

such that  $X_i \xrightarrow{f_i} X_{i-1}$  is a blowing-up at some point of  $X_{i-1}$  and proper transform  $C_k$  of  $C$  by  $f_k \cdots f_1$  is non-singular. Let  $C_i$  the proper transform of  $C$  by  $f_i \cdots f_1$ ,  $L_i$  a pull-back of  $L$  by  $f_i \cdots f_1$  and  $E_i$  the exceptional locus of  $f_i$ . Then, since  $K_{X_i} + mL_i = f_i^*(K_{X_{i-1}} + mL_{i-1}) + (d-1)E_i$  and  $R^j(f_i)_*(\mathcal{O}_{X_i}((d-1)E_i)) = 0$  for  $j > 0$ . Therefore, we get  $H^1(X_i, \mathcal{O}_{X_i}(K_{X_i} + mL_i)) \simeq H^1(X_{i-1}, \mathcal{O}_{X_{i-1}}(K_{X_{i-1}} + mL_{i-1}))$ , which implies claim 2.

We set  $C_a = \text{Spec}(\mathcal{O}_X/I^{a+1})$ . Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + mkL) \otimes I^m \longrightarrow \mathcal{O}_X(K_X + mkL) \longrightarrow \mathcal{O}_{C_{m-1}}(L_X + mkL) \longrightarrow 0$$

and claim 1, we get the inequality

$$\dim H^1(C_{m-1}, \mathcal{O}_{C_{m-1}}(L_X + mkL)) \leq \dim H^1(\mathcal{O}_X(K_X + mkL)) + \dim H^2(\mathcal{O}_X(K_X + mkL) \otimes I^m).$$

Hence,

$$\frac{\chi(C_{m-1}, \mathcal{O}_{C_{m-1}}(L_X + mkL))}{m^d} \leq \frac{\dim H^1(\mathcal{O}_X(K_X + mkL))}{m^d} + \frac{\dim H^2(\mathcal{O}_X(K_X + mkL) \otimes I^m)}{m^d}.$$

Taking superior limit of the above inequality, we obtain

$$-\lim_{m \rightarrow \infty} \frac{\chi(C_{m-1}, \mathcal{O}_{C_{m-1}}(L_X + mkL))}{m^d} \leq \limsup_{m \rightarrow \infty} \frac{\dim H^2(\mathcal{O}_X(K_X + mkL) \otimes I^m)}{m^d}.$$

**Claim 3.** 
$$\lim_{m \rightarrow \infty} \frac{\chi(C_{m-1}, \mathcal{O}_{C_{m-1}}(L_X + mkL))}{m^d} = \frac{k(L \cdot C)}{(d-1)!} + \frac{\deg(I/I^2)}{d!}.$$

Considering the exact sequence

$$0 \rightarrow \text{Sym}^a(I/I^2) \otimes \mathcal{O}_C(K_X + mkL) \rightarrow \mathcal{O}_{C_a}(K_X + mkL) \rightarrow \mathcal{O}_{C_{a-1}}(K_X + mkL) \rightarrow 0,$$

we have

$$\begin{aligned} \chi(C_{m-1}, \mathcal{O}_{C_{m-1}}(K_X + mkL)) &= \sum_{a=0}^{m-1} \chi(\text{Sym}^a(I/I^2) \otimes \mathcal{O}_C(K_X + mkL)) \\ &= \sum_{a=0}^{m-1} \{\text{rank}(\text{Sym}^a(I/I^2))(K_X + mkL \cdot C) + \deg(\text{Sym}^a(I/I^2))\} \\ &= \sum_{a=0}^{m-1} \left\{ \binom{a+d-2}{d-2} (K_X + mkL \cdot C) + \frac{a}{d-1} \binom{a+d-2}{d-2} \deg(I/I^2) \right\}. \end{aligned}$$

Hence, by easy calculation, we get claim 3.

**Claim 4.** There exists a constant  $M$  such that  $M$  is independent of  $k$  and

$$\dim H^2(\mathcal{O}_X(K_X + mkL) \otimes I^m) \leq M \cdot m^d.$$

Take a general member  $B$  of  $|A|$  such that  $\mathcal{O}_B \otimes I^m = p_1^m \cdots p_r^m$  for all  $m > 0$ , where  $p_1, \dots, p_r$  are maximal ideals of  $\mathcal{O}_B$ . Set  $P = p_1 \cdots p_r$ . Let  $g: B' \rightarrow B$  be a blowing-up by ideal  $P$  and  $l_i$ 's exceptional set of  $g$ . There is a constant  $T$  such that  $Tg^*(A) - \sum l_i$  is ample on  $B'$ . For  $i > T(m-d+2)$ , since

$$\begin{aligned} g^*(K_B + (i-1)A + mkL) - \sum ml_i &= K_{B'} + (m-d+2)(Tg^*(A) - \sum l_i) \\ &\quad + (i-1 - T(m-d+2))g^*(A) + mkL \end{aligned}$$

and

$$(m-d+2)(Tg^*(A) - \sum l_i) + (i-1 - T(m-d+2))g^*(A) + mkL$$

is ample, we get

$$H^1(B', \mathcal{O}_{B'}(g^*(K_B + (i-1)A + mkL) - \sum ml_i)) = 0.$$

Hence the above vanishing and  $Rg_*\mathcal{O}(-\sum ml_i) = P^m$  imply

$$H^1(B, \mathcal{O}_B(K_B + (i-1)A + mkL) \otimes P^m) = 0 \quad \text{for all } i > T(m-d+2).$$

From the exact sequence

$$0 \rightarrow \mathcal{O}_B(K_B + (i-1)A + mkL) \otimes P^m \rightarrow \mathcal{O}_B(K_B + (i-1)A + mkL) \rightarrow \mathcal{O}_{m(p_1 + \dots + p_r)} \rightarrow 0,$$

we obtain the surjective homomorphism  $H^0(\mathcal{O}_m(p_1 + \dots + p_r)) \rightarrow H^1(B, \mathcal{O}_B(K_B + (i-1)A + mkL) \otimes P^m)$  and  $H^2(B, \mathcal{O}_B(K_B + (i-1)A + mkL) \otimes P^m) = 0$  for  $i > 0$ . Therefore, the exact sequence

$$0 \rightarrow \mathcal{O}_X(K_X + (i-1)A + mkL) \otimes I^m \rightarrow \mathcal{O}_X(K_X + iA + mkL) \rightarrow \mathcal{O}_B(K_B + (i-1)A + mkL) \otimes P^m \rightarrow 0,$$

implies an inequality

$$\dim H^2(X, \mathcal{O}_X(K_X + (i-1)A + mkL) \otimes I^m) \leq \sum_{i=1}^{T(m-d+2)} \dim H^0(\mathcal{O}_{m(p_1 + \dots + p_r)}).$$

Thus, we get claim 4.

By claim 3 and 4, we have an inequality  $\frac{k(L \cdot C)}{(d-1)!} - \frac{\deg(I/I^2)}{d!} \leq M$ . Since  $k$  is arbitrary positive integer,  $(L \cdot C)$  must be non-negative, which proves our Theorem.

## On Finite Galois Covering Germs

Makoto Namba

Dedicated to Professor Shingo Murakami on his sixtieth birthday

Introduction. We denote by  $\mathbb{C}^n$  the  $n$ -th Cartesian product of the complex plane  $\mathbb{C}$ . Let  $W = (W, 0)$  be the germ of open balls in  $\mathbb{C}^n$  with the center  $0 = (0, \dots, 0)$ . A finite covering germ is, by definition, a germ  $\pi : X \longrightarrow W$  of surjective proper finite holomorphic mappings, where  $X = (X, p)$  is a germ of irreducible normal complex spaces.

Every normal singularity  $(X, p)$  has the structure of a finite covering germ  $\pi : X \longrightarrow W$ , (see Gunning-Rossi[4]).

Finite covering germs were discussed in Gunning[3] from the ring theoretic point of view.

In this paper, we introduce the notion of finite Galois covering germs and prove two basic theorems (Theorems 2 and 3 below) on it.

1. Some definitions. Let  $M$  be an  $n$ -dimensional (connected) complex manifold. A finite covering of  $M$  is, by definition, a surjective proper finite holomorphic mapping  $\pi : X \longrightarrow M$ , where  $X$  is an irreducible normal complex space. Let  $\pi : X \longrightarrow M$  and  $\mu : Y \longrightarrow M$  be finite coverings of  $M$ . A morphism (resp. an isomorphism) of  $\pi$  to  $\mu$  is, by definition, a surjective holomorphic (resp. biholomorphic) mapping  $\varphi : X \longrightarrow Y$  such that  $\mu\varphi = \pi$ . We denote by  $G_\pi$  the group of all automorphisms

of  $\pi$  and call it the automorphism group of  $\pi$ .  $G_\pi$  acts on each fiber of  $\pi$ .

A finite covering  $\pi : X \longrightarrow M$  is called a finite Galois covering if  $G_\pi$  acts transitively on every fiber of  $\pi$ . In this case, the quotient complex space  $X/G_\pi$  (see Cartan[1]) is biholomorphic to  $M$ .

For a finite covering  $\pi : X \longrightarrow M$ , put

$$R_\pi = \left\{ p \in X \mid \pi \text{ is not biholomorphic around } p \right\},$$

$$B_\pi = \pi(R_\pi).$$

They are hypersurfaces (i.e. codimension 1 at every point) of  $X$  and  $M$ , respectively and are called the ramification locus and the branch locus of  $\pi$ , respectively.

Let  $B$  be a hypersurface of  $M$ . A finite covering  $\pi : X \longrightarrow M$  is said to branch at most at  $B$  if the branch locus  $B_\pi$  of  $\pi$  is contained in  $B$ . In this case, the restriction

$$\pi' : X - \pi^{-1}(B) \longrightarrow M - B$$

of  $\pi$  is an unbranched covering. The mapping degree of  $\pi'$  is called the degree of  $\pi$  and is denoted by  $\deg \pi$ .

By a property of normal complex spaces, we have easily (see Namba[5])

Proposition 1. (1)  $G_\pi \simeq G_{\pi'}$  naturally. (2)  $\pi$  is a Galois covering if and only if  $\pi'$  is a Galois covering.

Corollary.  $\#G_\pi \leq \deg \pi$ , where  $\#G_\pi$  is the order of the group  $G_\pi$ . Moreover, the equality holds if and only if  $\pi$  is a Galois covering.

The following theorem is a deep one.

Theorem 1 (Grauert-Remmert[2]). If  $\pi' : X' \longrightarrow M - B$  is an unbranched finite covering, then there exists a unique (up to isomorphisms) finite covering  $\pi : X \longrightarrow M$  which extends  $\pi'$ .

Take a point  $q_0 \in M - B$  and fix it. We denote by  $\pi_1(M - B, q_0)$  the fundamental group of  $M - B$  with the reference point  $q_0$ .

Corollary. There is a one-to-one correspondence between isomorphism classes of finite (resp. Galois) coverings  $\pi : X \longrightarrow M$  which branches at most at  $B$  and the set of all conjugacy classes of subgroups (resp. normal subgroups)  $H$  of  $\pi_1(M - B, q_0)$  of finite index. If  $H$  is normal, then  $\pi$  corresponding to  $H$  satisfies

$$G_\pi \simeq \pi_1(M - B, q_0)/H.$$

Example 1. Put  $X = \mathbb{C}^n$ ,  $M = \mathbb{C}^n$  and

$$\pi : (x_1, \dots, x_n) \in \mathbb{C}^n \longmapsto (a_1, \dots, a_n) \in \mathbb{C}^n,$$

where

$$a_1 = -(x_1 + \dots + x_n),$$

$$a_2 = x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n$$

$$\dots$$

$$a_n = (-1)^n x_1 \dots x_n.$$

In other words,  $x_j$  ( $1 \leq j \leq n$ ) are the roots of the equation

$$x^n + a_1x^{n-1} + \dots + a_n = 0.$$

Then  $\pi$  is a Galois covering of  $M = \mathbb{C}^n$  such that (i)  $B_\pi = \Delta$  is the discriminant locus and (ii)  $G_\pi \simeq S_n$  (the  $n$ -th symmetric group).

We may identify  $G_\pi$  and  $S_n$  through the isomorphism.  $S_n$  is then regarded as a finite subgroup of the general linear group  $GL(n, \mathbb{C})$ .

Example 2. We regard  $S_n$  as a finite subgroup of  $GL(n, \mathbb{C})$  as in Example 1. Put  $Y = \mathbb{C}^n$ . Let  $G$  be a subgroup of  $S_n$ . The quotient space  $Y/G$  is an irreducible normal complex space and the canonical projection

$$\mu : Y \longrightarrow Y/G = N$$

is a holomorphic mapping. Let

$$\alpha : M \longrightarrow N$$

be a resolution of singularity of  $N$ . Then the finite Galois covering

$\pi : X \longrightarrow M$  of  $M$ , defined by the following diagram, satisfies  $G_\pi \cong G$ :

$$\begin{array}{ccccc} X & \xrightarrow{\mathcal{S}} & M \times_N Y & \longrightarrow & Y \\ \pi \downarrow & & \downarrow & & \downarrow \mu \\ M & \xrightarrow{\text{id}} & M & \xrightarrow{\alpha} & N. \end{array}$$

Here,  $M \times_N Y$  is the fiber product,  $\mathcal{S}$  is the normalization and  $\text{id}$  is the identity mapping.

2. Finite Galois covering germs. Now, let  $W = (W, 0)$  be the germ of open balls in  $\mathbb{C}^n$  with the center  $0 = (0, \dots, 0)$ . Let  $\pi : X \longrightarrow W$  be a finite covering germ (see Introduction). Every notion in §1 can be easily extended to finite covering germs. In particular, a finite covering germ  $\pi : X \longrightarrow W$  is called a finite Galois covering germ if  $G_\pi$  acts transitively on every fiber of  $\pi$ . Also, a similar assertion to Corollary to Theorem 1 holds in the case of finite covering germs, if  $\pi_1(M - B, q_0)$  is replaced by the local fundamental group  $\pi_{1, \text{loc}, 0}(W - B)$  of  $W - B$  at  $0$ .

Example 3. Let  $\pi_0 : X \longrightarrow W$  be the restriction of the covering  $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$  in Example 1 to  $W = (W, 0)$  and  $X = (X, 0) = \pi^{-1}(W)$ . Then  $\pi_0$  is a finite Galois covering germ such that  $G_{\pi_0} \cong S_n$ .

There exist a lot of finite Galois covering germs in the following sense:

Theorem 2. For  $n \geq 2$ , let  $W = (W, 0)$  be the germ of balls in  $\mathbb{C}^n$  with the center  $0$ . For every finite group  $G$ , there exists a finite Galois covering germ  $\pi : X \rightarrow W$  such that  $G_\pi \simeq G$ .

Proof. Case 1. We first prove the theorem for the case  $n = 2$ . Let  $W$  be a ball in  $\mathbb{C}^2$  with the center  $0$ . Let  $L_j$  ( $1 \leq j \leq s$ ) be mutually distinct (complex) lines in  $\mathbb{C}^2$  passing through  $0$ . Put  $D_j = L_j \cap W$  ( $1 \leq j \leq s$ ) and

$$B = D_1 \cap \cdots \cap D_s,$$

(see Figure 1).



Take a point  $q_0 \in M - B$  and fix it. Let  $\gamma_j$  be a loop in  $M - B$  starting from  $q_0$  and rounding  $D_j - 0$  once counterclockwisely as in Figure 2. We identify  $\gamma_j$  with its homotopy class.



Then, as is well known,  $\pi_1(W - B, q_0)$  is a group generated by  $\gamma_1, \dots, \gamma_s$  with the generating relations

$$\gamma_j \delta = \delta \gamma_j \quad (1 \leq j \leq s),$$

where  $\delta = \gamma_1 \cdots \gamma_s$ .

Let  $F_{s-1}$  be the free group of  $(s-1)$ -letters  $b_1, \dots, b_{s-1}$ . Put  $b_s = (b_1 \cdots b_{s-1})^{-1}$ . Then there is the surjective homomorphism

$$\Phi : \pi_1(W - B, q_0) \rightarrow F_{s-1}$$

defined by  $\Phi(\gamma_j) = b_j$  ( $1 \leq j \leq s$ ).

For any finite group  $G$ , there is a surjective homomorphism

$$\Psi : F_{s-1} \rightarrow G$$

for a sufficiently large  $s$ .

Now, the kernel  $K$  of the surjective homomorphism

$$\Psi\Phi : \pi_1(W - B, q_0) \longrightarrow G$$

has a finite index such that

$$\pi_1(W - B, q_0)/K \cong G.$$

The finite Galois covering  $\pi : X \longrightarrow W$  corresponding to  $K$  in Corollary to Theorem 1 satisfies  $G_\pi \cong G$ .

The finite Galois covering germ determined by  $\pi$  is a desired one.

Case 2. Next, we prove the theorem for the case  $n \geq 3$ . Let  $W$  be a ball in  $\mathbb{C}^n$  with the center  $0$ . Let  $P$  and  $Q$  be a 2-plane and an  $(n - 2)$ -plane in  $\mathbb{C}^n$ , respectively, passing through  $0$  such that  $P \cap Q = \{0\}$ . Let  $H_j$  ( $1 \leq j \leq s$ ) be mutually distinct hyperplanes in  $\mathbb{C}^n$  passing through  $0$  and containing  $Q$  (see Figure 3).

Put  $D_j = H_j \cap W$  ( $1 \leq j \leq s$ ) and

$$B = D_1 \cap \cdots \cap D_s.$$

Then  $W - B$  and  $W \cap P - B \cap P$  are homotopic. Hence, by Case 1, taking sufficiently large  $s$ , there exists a normal subgroup  $K$  of  $\pi_1(W - B, q_0)$  of finite index such that

$$\pi_1(W - B, q_0)/K \cong G.$$

The rest of the proof is similar to Case 1.

q.e.d.

Figure 3

Now, we give a method of concrete constructions of every finite Galois covering germ. Our method is suggested by Professor Enoki and is different from and simpler than Namba[6] in which finite Galois coverings of projective manifolds were treated.

Theorem 3. Let  $\pi : X \longrightarrow W$  be a finite Galois covering germ. Put  $m = \deg \pi$ . Then there exists a germ  $f : W \longrightarrow \mathbb{C}^m$  of holomorphic mappings and a finite subgroup  $G$  of  $S_m$  with  $G \cong G_\pi$  such that  $\pi$  is obtained

by the following commutative diagram:

$$\begin{array}{ccccc}
 X & \xrightarrow{\mathfrak{S}} & W \times_N Y & \longrightarrow & \mathbb{C}^m = Y \\
 \pi \downarrow & & \downarrow & & \downarrow \mu \\
 W & \xrightarrow{\text{id}} & W & \xrightarrow{f} & \mathbb{C}^m/G = N,
 \end{array}$$

where  $W \times_N Y$  is the fiber product,  $\mathfrak{S}$  is the normalization and  $\text{id}$  is the identity mapping. Here  $S_m$  is regarded as a finite subgroup of  $GL(m, \mathbb{C})$  as in Example 1.

Proof. We may assume that  $W$  is a small ball in  $\mathbb{C}^n$  with the center 0. Take a point  $q_0 \in W - B$  and put

$$\pi^{-1}(q_0) = \{p_1, \dots, p_m\}.$$

Put  $G_\pi = \{\sigma_1 = 1, \sigma_2, \dots, \sigma_m\}$ .

Note that  $X$  is a Stein space. Let  $h$  be a holomorphic function on  $X$  such that

$$h(p_j) \neq h(p_k) \quad \text{for } j \neq k \quad \dots(1)$$

Put

$$h_j = \sigma_j^* h = h \circ \sigma_j \quad (1 \leq j \leq m).$$

Let  $F : X \rightarrow \mathbb{C}^m$  be the holomorphic mapping defined by

$$F(p) = (h_1(p), \dots, h_m(p)).$$

Then, for  $\sigma \in G$ ,

$$\begin{aligned}
 (\sigma^* F)(p) &= F(\sigma(p)) = (h_1(\sigma(p)), \dots, h_m(\sigma(p))) \\
 &= (h(\sigma(p)), h(\sigma_2 \sigma(p)), \dots, h(\sigma_m \sigma(p))) \\
 &= (h_{k(1)}(p), h_{k(2)}(p), \dots, h_{k(m)}(p)) \quad \dots(2)
 \end{aligned}$$

Thus  $\sigma$  gives the permutation

$$R(\sigma) = \begin{pmatrix} 1 & 2 & \dots & m \\ k(1) & k(2) & \dots & k(m) \end{pmatrix}.$$

The correspondence

$$R : \sigma \mapsto R(\sigma)$$

is then an isomorphism of  $G_{\pi}$  onto a subgroup  $G$  of  $S_m$ . (2) can be rewritten as

$$\sigma^*F = R(\sigma)F \quad \text{for all } \sigma \in G. \quad (3)$$

Hence  $F$  induces a holomorphic mapping  $f : W \rightarrow \mathbb{C}^m/G = N$  such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{F} & \mathbb{C}^m = Y \\ \pi \downarrow & & \downarrow \mu \\ W & \xrightarrow{f} & \mathbb{C}^m/G = N. \end{array}$$

By the assumption (1), we can easily show that  $f$  has the following two properties:

(i)  $f(W) \not\subseteq \text{Fix } G$ , where  $\text{Fix } G$  is the union of the fixed points of all elements of  $G$  except the identity and

(ii)  $f$  is not decomposed as follows:

$$\begin{array}{ccc} W & \xrightarrow{f'} & \mathbb{C}^m/H \\ \text{id} \downarrow & & \downarrow \nu \\ W & \xrightarrow{f} & \mathbb{C}^m/G, \end{array}$$

where  $H (\neq G)$  is a subgroup of  $G$ ,  $\nu$  is the canonical projection and  $f'$  is a holomorphic mapping.

A holomorphic mapping  $f$  with the properties (i) and (ii) is said to be  $G$ -indecomposable (see Namba[6]). For such a mapping  $f$ , the fiber product  $W \times_N Y$  is irreducible and the finite Galois covering  $\pi_0 : X_0 \rightarrow W$  defined by the commutative diagram

$$\begin{array}{ccccc} X_0 & \xrightarrow{\mathcal{G}} & W \times_N Y & \longrightarrow & \mathbb{C}^m = Y \\ \pi_0 \downarrow & & \downarrow & & \downarrow \mu \\ W & \xrightarrow{\text{id}} & W & \xrightarrow{f} & \mathbb{C}^m/G = N \end{array}$$

satisfies  $G_{\pi_0} \simeq G$ . Now, we can easily show that  $\pi$  is isomorphic to  $\pi_0$ ,

(see Namba[6]).

q.e.d.

Remark. (1)  $f(0)$  is not necessarily equal to  $\mu(0)$ , where  $0$  is the origin of  $\mathbb{C}^m$ . (2) A similar theorem to Theorem 3 holds for finite Galois coverings of a Stein manifold.

Problem. Characterize normal singularities  $(X,p)$  which has the structure of a finite Galois covering germs  $\pi : X \longrightarrow W$ .

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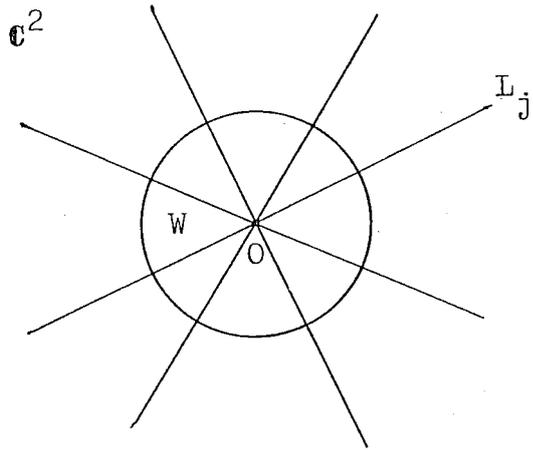


Figure 1

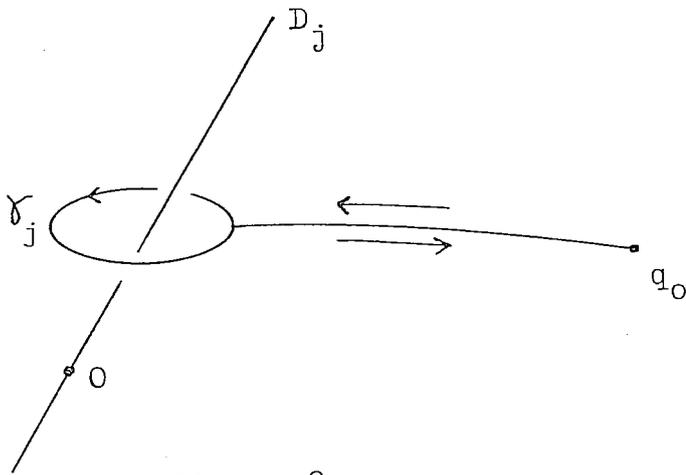


Figure 2

Finite Galois covering germs

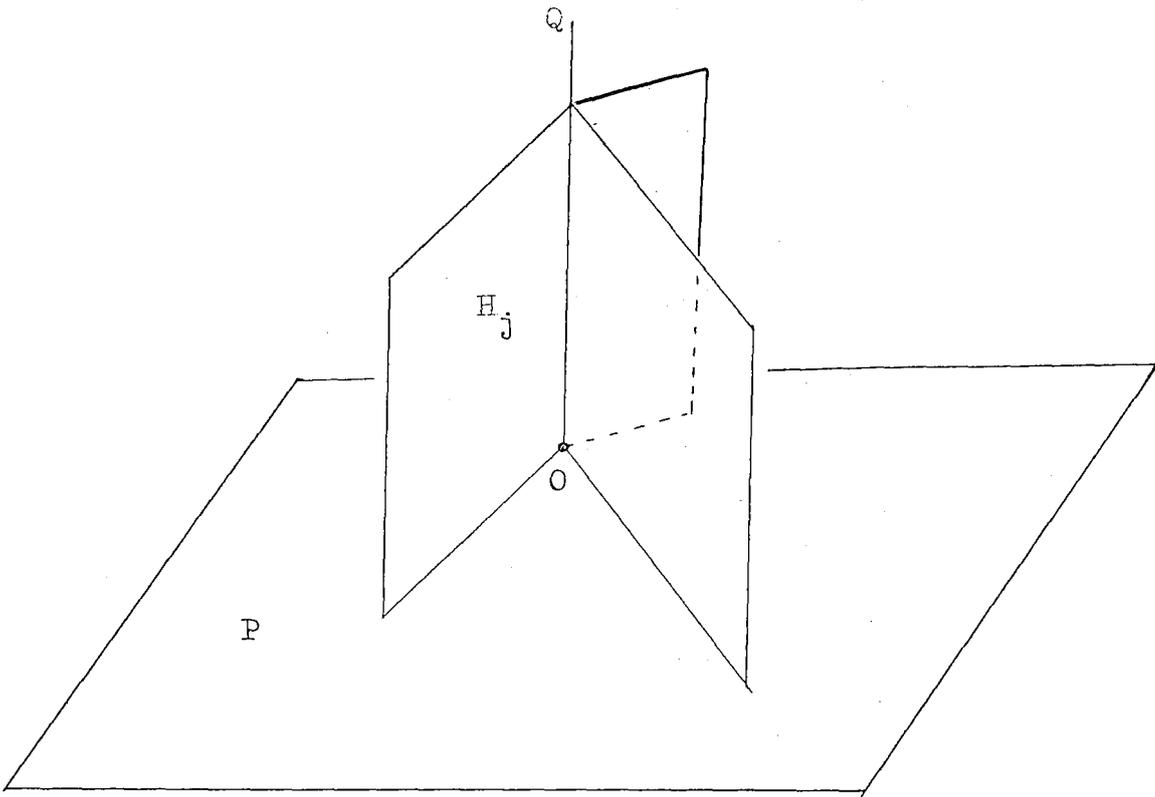


Figure 3

Finite Galois Covering Genus

# Zariski-decomposition Problem for Pseudo-effective Divisors

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**Introduction.** In this paper, we shall study some numerical properties of pseudo-effective divisors on projective complex manifolds. Here the "numerical properties" are the properties which depend only on the first Chern classes of the divisors. In the paper of [Z], Zariski found the so-called the "Zariski-decomposition"  $D = P + N$  for an effective divisor  $D$  on a projective smooth surface, where  $P$  is a nef  $\mathbb{Q}$ -divisor,  $N$  is an effective  $\mathbb{Q}$ -divisor with negative intersection matrix, and  $P \cdot N = 0$ . The construction of the decomposition depends only on the intersection numbers of  $D$  with irreducible curves on the surface. Fujita [F1] showed the similar decomposition exist for all pseudo-effective divisors on surfaces. The Zariski-decomposition is very useful to study the projective surfaces and open surfaces. In [F2], Fujita generalize the notion of Zariski-decomposition to higher dimensional case and conjectured the existence of the decomposition for pseudo-effective divisors. In higher dimensional case, one cannot obtain a desired decomposition (namely the positive part is nef) on the fixed manifold, in general. So we must blow-up the manifold. His conjecture is the existence of such a nice blowing-up. On the other hand, Cutkosky [C] found an example where the negative part of the Zariski-decomposition is not a  $\mathbb{Q}$ -divisor. Matsuda[Ma] also studied the Zariski-decomposition in Fujita's sense and considered the  $\sigma$ -decomposition (see §1). But in his paper, the negative part of the

$\sigma$ -decomposition may have infinitely many components. If the Zariski-decomposition of the canonical divisor of a projective manifold of general type exists, then the canonical ring is finitely generated over  $\mathbb{C}$ , by [K1]. Therefore, for example, the Flip Conjecture (the existence of the flip) in the minimal model theory (cf. [KMM]) follows from the existence of the relative Zariski-decomposition of the canonical divisor.

In §1, we shall define the  $\sigma$ -decomposition  $D = P_\sigma(D) + N_\sigma(D)$  for pseudo-effective divisors  $D$  on a projective complex manifold  $X$ . Here  $N_\sigma(D)$  is an effective  $\mathbb{R}$ -divisor determined by the first Chern class of  $D$ . We shall show the following properties ((1.9)):

- (a)  $c_1(P_\sigma(D)) \in \overline{Mv}(X)$ , and
- (b) if  $c_1(D - \Delta) \in \overline{Mv}(X)$  for some effective  $\mathbb{R}$ -divisor  $\Delta$ , then  $\Delta \geq N_\sigma(D)$ .

Here  $\overline{Mv}(X)$  is the movable cone of  $X$  which is the closure of the cone in  $H^2(X, \mathbb{R})$  generated by the first Chern classes of fixed part free effective divisors on  $X$  (see (1.8)). If  $X$  is a surface, then the  $\sigma$ -decomposition is nothing but the Zariski-decomposition, since the movable cone  $\overline{Mv}(X)$  is the nef cone in this case. We now formulate the Zariski-decomposition Problem as follows:

**Problem.** For a pseudo-effective  $\mathbb{R}$ -divisor  $D$  on  $X$ , does there exist a modification  $\mu : Y \longrightarrow X$  such that  $P_\sigma(\mu^*D)$  is nef?

If such a modification exists, then the decomposition  $\mu^*D = P_\sigma(\mu^*D) + N_\sigma(\mu^*D)$  is said to be the Zariski-decomposition of  $D$ . This is actually in the sense of Fujita [F2].

In §2, we define the  $\nu$ -decomposition  $D = P_\nu(D) + N_\nu(D)$  which satisfies the following two properties:

(c)  $c_1(P_\nu(D)) \in \text{NMv}(X)$ , and

(d) if  $c_1(D - \Delta) \in \text{NMv}(X)$  for some effective  $\mathbb{R}$ -divisor  $\Delta$ , then  $\Delta \geq N_\nu(D)$ .

Here  $\text{NMv}(X)$  is the cone consisting of the first Chern classes of pseudo-effective divisors whose restriction to any prime divisors on  $X$  are still pseudo-effective (cf. §2). The  $\sigma$ -decomposition and the  $\nu$ -decomposition of the given divisor are different in general, but the calculation of the  $\nu$ -decomposition is easier than that of the  $\sigma$ -decomposition. We obtain the Zariski-decomposition for suitable divisors using the  $\nu$ -decomposition in §4. §3 is devoted to the study of the relative version of  $\sigma$ -decomposition.

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## §0. Preliminaries.

We shall use the similar notation as in [KMM] and [N1]. Let  $X$

be an  $n$ -dimensional projective complex manifold. We denote by  $N^1(X)$  the real vector subspace in  $H^2(X, \mathbb{R})$  generated by the first Chern classes  $c_1(L)_{\mathbb{R}}$  of line bundles  $L$  on  $X$ . Note that  $c_1(L)_{\mathbb{R}} = 0$  if and only if  $L \cdot C = 0$  for any irreducible curves  $C$ . We denote the  $c_1(L)_{\mathbb{R}}$  by  $c_1(L)$ , simply. Let  $\text{Div}(X)$  be the divisor group of  $X$ . An  $\mathbb{R}$ -divisor  $D$  should be an element of  $\text{Div}(X) \otimes \mathbb{R}$ . Let  $D = \sum a_j \Gamma_j$  be the irreducible decomposition of an  $\mathbb{R}$ -divisor  $D$ .  $D$  is called effective, if  $a_j \geq 0$  for all  $j$ . Let  $\text{Eff}(X)$  be the cone in  $N^1(X)$  generated by the first Chern classes of effective  $\mathbb{R}$ -divisors and let us denote its closure by  $\text{PE}(X)$ , which is called the pseudo-effective cone. The interior of the cone  $\text{PE}(X)$  is denoted by  $\text{Big}(X)$ , which is called the big cone. An  $\mathbb{R}$ -divisor  $D$  is called pseudo-effective, (resp. big), if  $c_1(D) \in \text{PE}(X)$  (resp.  $c_1(D) \in \text{Big}(X)$ ). Then we have:

**Lemma (0.1).** *The following two conditions are equivalent:*

- (1)  $B$  is a big  $\mathbb{R}$ -divisor.
- (2)  $\limsup_{m \rightarrow \infty} h^0(X, [mB])/m^n > 0$ ,

where  $[mB]$  is the integral part of  $mB$ .

Let  $\text{Amp}(X)$  be the cone in  $N^1(X)$  generated by the first Chern classes of ample line bundles on  $X$ . Then  $\text{Amp}(X)$  is an open convex cone in  $N^1(X)$  which is called the ample cone of  $X$ . The nef cone  $\text{Nef}(X)$  should be the closure of the  $\text{Amp}(X)$ . An

$\mathbb{R}$ -divisor  $D$  is called ample (resp. nef), if  $c_1(D) \in \text{Amp}(X)$  (resp.  $c_1(D) \in \text{Nef}(X)$ ). Then we have:

**Lemma (0.2).** *Let  $A$  be an ample  $\mathbb{R}$ -divisor on  $X$ . Then  $A$  is numerically equivalent to  $\sum s_j H_j$ , for some ample  $\mathbb{Q}$ -divisors  $H_j$  and positive real numbers  $s_j$ .*

*Proof.* Let  $\Gamma_1, \Gamma_2, \dots, \Gamma_k$  be the irreducible components of  $A$  whose coefficients are irrational numbers. Since the ampleness is an open condition,  $[mA]$  and  $[mA] + k\Gamma_j$  ( $1 \leq j \leq k$ ) are ample for large integer  $m$ . We may assume that  $\langle mA \rangle := mA - [mA] = \sum_{j=1}^k t_j \Gamma_j$ , for  $0 < t_j < 1$ . Thus  $mA = \sum_{j=1}^k (t_j/k)([mA] + k\Gamma_j) + (1 - \sum_{j=1}^k (t_j/k))[mA]$ .  
Q.E.D.

### §1. $\sigma$ -decomposition.

Let  $\Gamma$  be a prime divisor of  $X$  and let  $B$  be a big  $\mathbb{R}$ -divisor on  $X$ . We define  $\sigma_\Gamma(B)$  (resp.  $\sigma_\Gamma(B)_\mathbb{Q}$ ) to be the following number:

$$\inf\{\text{mult}_\Gamma(\Delta) \mid \Delta \text{ is an effective } \mathbb{R}\text{-divisor} \\ \text{with } c_1(\Delta) = c_1(B) \text{ (resp. } \Delta \sim_{\mathbb{Q}} B)\},$$

where the symbol  $\sim_{\mathbb{Q}}$  means the  $\mathbb{Q}$ -linear equivalence relation.

Lemma (1.1). (1)  $\lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(B + \varepsilon A)_{\mathbb{Q}} = \sigma_{\Gamma}(B)_{\mathbb{Q}}$ , for any ample

$\mathbb{R}$ -divisor  $A$ .

$$(2) \quad \sigma_{\Gamma}(B)_{\mathbb{Q}} = \sigma_{\Gamma}(B).$$

(3)  $\sigma_{\Gamma}(B_1 + B_2) \leq \sigma_{\Gamma}(B_1) + \sigma_{\Gamma}(B_2)$ , for any big  $\mathbb{R}$ -divisors  $B_1$  and  $B_2$ .

*Proof.* (1). It is easy to see that if  $0 \leq \varepsilon_1 < \varepsilon_2$ , then

$$\sigma_{\Gamma}(B + \varepsilon_1 A)_{\mathbb{Q}} \geq \sigma_{\Gamma}(B + \varepsilon_2 A)_{\mathbb{Q}}. \quad \text{Thus} \quad \lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(B + \varepsilon A)_{\mathbb{Q}} \leq \sigma_{\Gamma}(B)_{\mathbb{Q}}.$$

On the other hand, by (0.1), there are an effective  $\mathbb{R}$ -divisor  $\Delta$  and a positive number  $\delta$  such that  $B \sim_{\mathbb{Q}} \delta A + \Delta$ . Therefore we have

$$(1 + \varepsilon)B \sim_{\mathbb{Q}} B + \varepsilon \delta A + \varepsilon \Delta. \quad \text{Thus the inequality}$$

$$(1 + \varepsilon)\sigma_{\Gamma}(B)_{\mathbb{Q}} \leq \sigma_{\Gamma}(B + \varepsilon \delta A)_{\mathbb{Q}} + \varepsilon \text{mult}_{\Gamma}(\Delta) \quad \text{holds. Taking } \varepsilon \downarrow 0, \text{ we}$$

$$\text{have } \sigma_{\Gamma}(B)_{\mathbb{Q}} \leq \lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(B + \varepsilon A)_{\mathbb{Q}}.$$

$$(2). \quad \text{By definition, we have only to prove } \sigma_{\Gamma}(B)_{\mathbb{Q}} \leq \sigma_{\Gamma}(B).$$

Let  $\Delta$  be an effective  $\mathbb{R}$ -divisor such that  $c_1(\Delta) = c_1(B)$ . Then

$B + A - \Delta$  is ample for any ample  $\mathbb{R}$ -divisor  $A$ . Thus

$$\sigma_{\Gamma}(B + A)_{\mathbb{Q}} \leq \text{mult}_{\Gamma}(\Delta). \quad \text{Taking } A \text{ very small, we have}$$

$$\sigma_{\Gamma}(B)_{\mathbb{Q}} \leq \text{mult}_{\Gamma}(\Delta) \quad \text{by (1). Therefore } \sigma_{\Gamma}(B)_{\mathbb{Q}} \leq \sigma_{\Gamma}(B).$$

(3) is obvious.

Q.E.D.

Let  $D$  be a pseudo-effective  $\mathbb{R}$ -divisor on  $X$ .

Lemma (1.2). (1)  $\lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(D + \varepsilon A) < +\infty$  for any ample  $\mathbb{R}$ -divisor  $A$ .

(2) The number  $\lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(D + \varepsilon A)$  does not depend on the choice of  $A$ .

*Proof.* (1). We have easily  $(D + \varepsilon A - \sigma_{\Gamma}(D + \varepsilon A)\Gamma) \cdot A^{n-1} \geq 0$  for any  $\varepsilon > 0$ . Hence  $\sigma_{\Gamma}(D + \varepsilon A) \leq (D + \varepsilon A) \cdot A^{n-1} / \Gamma \cdot A^{n-1}$ . Therefore  $\lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(D + \varepsilon A)$  is bounded.

(2). Let  $A'$  be another ample  $\mathbb{R}$ -divisor. Then there are an effective  $\mathbb{R}$ -divisor  $\Delta$  and a positive number  $\delta$  such that  $c_1(A') = c_1(\delta A + \Delta)$ . Hence  $\sigma_{\Gamma}(D + \varepsilon \delta A) + \varepsilon \text{mult}_{\Gamma}(\Delta) \geq \sigma_{\Gamma}(D + \varepsilon A')$ . Therefore  $\lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(D + \varepsilon A) \geq \lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(D + \varepsilon A')$ . Q.E.D.

In what follows, we denote the  $\lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(D + \varepsilon A)$  above by  $\sigma_{\Gamma}(D)$ . Note that  $\sigma_{\Gamma}(D) = \sigma_{\Gamma}(D')$ , if  $c_1(D) = c_1(D')$ .

**Lemma (1.3).** *Let  $\Gamma_1, \Gamma_2, \dots, \Gamma_{\ell}$  be mutually distinct prime divisors and let  $s_1, s_2, \dots, s_{\ell}$  be real numbers with*

*$0 \leq s_i \leq \sigma_{\Gamma_i}(D)$  for all  $i$ . Then  $\sigma_{\Gamma_i}(D - \sum_{j=1}^{\ell} s_j \Gamma_j) = \sigma_{\Gamma_i}(D) - s_i$  for all  $i$ .*

*Proof.* Take an ample  $\mathbb{R}$ -divisor  $A$  such that  $A - \sum \Gamma_j$  is also ample. By definition, for any positive number  $\varepsilon$ , there exists a positive number  $\delta$  such that  $0 \leq \sigma_{\Gamma_i}(D) - \sigma_{\Gamma_i}(D + \delta A) < \varepsilon$ , for

all  $i$ . Let  $E := D - \sum_{j=1}^{\ell} s_j \Gamma_j$ . Then we have

$$\begin{aligned} E + (\varepsilon + \delta)A &= D + (\varepsilon + \delta)A - \sum s_j \Gamma_j \\ &= (D + \delta A - \sum \sigma_{\Gamma_j}(D + \delta A)\Gamma_j) + \varepsilon(A - \sum \Gamma_j) \end{aligned}$$

$$\begin{aligned}
& + (\sum (\varepsilon - (\sigma_{\Gamma_j}(D) - \sigma_{\Gamma_j}(D + \delta A))) \Gamma_j) \\
& + (\sum (\sigma_{\Gamma_j}(D) - s_j) \Gamma_j) .
\end{aligned}$$

Therefore we have  $\varepsilon + \sigma_{\Gamma_j}(D + \delta A) - s_j \geq \sigma_{\Gamma_j}(E + (\varepsilon + \delta)A)$  . Taking

$\delta \leq \varepsilon$  , we get  $\varepsilon + \sigma_{\Gamma_j}(D) - s_j \geq \sigma_{\Gamma_j}(E + 2\varepsilon A)$  . Hence

$\sigma_{\Gamma_j}(D) - s_j \geq \sigma_{\Gamma_j}(E)$  . On the other hand, we have

$\sigma_{\Gamma_j}(E + \varepsilon A) + s_j \geq \sigma_{\Gamma_j}(D + \varepsilon A)$  , since  $E + \varepsilon A + \sum s_j \Gamma_j = D + \varepsilon A$  .

Therefore  $\sigma_{\Gamma_j}(D) - s_j = \sigma_{\Gamma_j}(E)$  . Q.E.D.

**Proposition (1.4).** *Let  $D$  be a pseudo-effective  $\mathbb{R}$ -divisor and let  $\Gamma_1, \Gamma_2, \dots, \Gamma_\ell$  be mutually distinct prime divisors on  $X$  such that  $\sigma_{\Gamma_i}(D) > 0$  for all  $i$  . Then  $c_1(\Gamma_1), c_1(\Gamma_2), \dots, c_1(\Gamma_\ell)$  are linearly independent in  $N^1(X)$  .*

*Proof.* Assume the contrary. Then we may assume

that  $c_1(\sum_{i=1}^s a_i \Gamma_i) = c_1(\sum_{j=s+1}^{\ell} b_j \Gamma_j)$  for some positive numbers  $a_i$  and

$b_j$  and for some  $1 \leq s < \ell$  . Take a positive number  $\varepsilon$  such that

$\sigma_{\Gamma_i}(D) > \varepsilon a_i$  for  $1 \leq i \leq s$  and that  $\sigma_{\Gamma_j}(D) > \varepsilon b_j$  for

$s+1 \leq j \leq \ell$  . Applying (1.3) to  $D - \varepsilon(\sum_{i=1}^s a_i \Gamma_i)$  and

$D - \varepsilon(\sum_{j=s+1}^{\ell} b_j \Gamma_j)$  , we have a contradiction.

Q.E.D.

**Corollary (1.5).** *For any pseudo-effective  $\mathbb{R}$ -divisor  $D$  , there exist*

at most finite numbers of prime divisors  $\Gamma$  on  $X$  such that  $\sigma_{\Gamma}(D) > 0$ .

**Definition (1.6).** For a pseudo-effective  $\mathbb{R}$ -divisor  $D$ , let  $N_{\sigma}(D)$  be the effective  $\mathbb{R}$ -divisor  $\sum \sigma_{\Gamma}(D)\Gamma$  and set  $P_{\sigma}(D) := D - N_{\sigma}(D)$ . The decomposition  $D = P_{\sigma}(D) + N_{\sigma}(D)$  is said to be the  $\sigma$ -decomposition of  $D$  and  $P_{\sigma}(D)$  and  $N_{\sigma}(D)$  are called the positive and negative parts of the  $\sigma$ -decomposition of  $D$ , respectively.

**Remark.** (1) We have  $N_{\sigma}(B) = \lim_{m \rightarrow \infty} (1/m) \cdot |[mB]|_{\text{fix}}$  for big divisors  $B$ .

(2) If  $P_{\sigma}(D)$  is nef, then the decomposition  $D = P_{\sigma}(D) + N_{\sigma}(D)$  is the Zariski decomposition in Fujita's sense ([F2]). Therefore we are interested in the following:

**Zariski-decomposition Problem.** For a pseudo-effective  $\mathbb{R}$ -divisor  $D$  on  $X$ , does there exist a modification  $\mu : Y \rightarrow X$  such that  $P_{\sigma}(\mu^*D)$  is nef?

**Proposition (1.7).** (1)  $\sigma_{\Gamma} : \text{PE}(X) \rightarrow \mathbb{R}_{\geq 0}$  is a lower semi-continuous function. (Therefore it is continuous on  $\text{Big}(X)$ ).

(2)  $\lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(D + \varepsilon E) = \sigma_{\Gamma}(D)$  for any pseudo-effective  $\mathbb{R}$ -divisors  $E$  on  $X$ .

(3) If  $\sigma_{\Gamma}(D) = 0$ , then for any ample  $\mathbb{R}$ -divisor  $A$ , there exists an effective  $\mathbb{R}$ -divisor  $\Delta$  with  $c_1(\Delta) = c_1(D + A)$  and

$\text{mult}_\Gamma(\Delta) = 0$  .

*Proof.* (1). Let  $\{D_n\}$  ( $n \in \mathbb{N}$ ) be a sequence in  $\text{PE}(X)$  which convergent to  $D$  . Assume that  $\sigma_\Gamma(D_n) \leq \lambda$  for any  $n$  . Take an ample  $\mathbb{R}$ -divisor  $A$  on  $X$  . Then for any  $\varepsilon > 0$  , there is a number  $n_0$  such that  $D - D_n + \varepsilon A$  is ample for  $n \geq n_0$  . Since  $D + \varepsilon A = (D - D_n + \varepsilon A) + D_n$  , we have  $\sigma_\Gamma(D + \varepsilon A) \leq \sigma_\Gamma(D_n) \leq \lambda$  .

(2). By (1), we have  $\liminf_{\varepsilon \downarrow 0} \sigma_\Gamma(D + \varepsilon E) \geq \sigma_\Gamma(D)$  . On the other hand,  $\sigma_\Gamma(D + \varepsilon E) \leq \sigma_\Gamma(D) + \varepsilon \sigma_\Gamma(E)$  . Therefore we are done.

(3). Let  $m$  be a positive number such that  $mA + \Gamma$  is ample. For any small  $\varepsilon > 0$ , there exist positive numbers  $\lambda$  and  $\delta$  and an effective  $\mathbb{R}$ -divisor  $B$  such that  $c_1(B + \delta\Gamma) = c_1(D + \lambda A)$ ,  $\text{mult}_\Gamma(B) = 0$  and that  $m\delta + \lambda < \varepsilon$  . Then  $c_1(B + \delta(mA + \Gamma)) = c_1(D + (m\delta + \lambda)A)$ . Thus we can find a desired effective divisor. Q.E.D.

The following definition is found in Kawamata[K2].

**Definition (1.8).** Let  $\overline{Mv}(X)$  be the cone in  $N^1(X)$  generated by the first Chern classes of fixed part free line bundles  $L$  (i.e.,  $|L|_{\text{fix}} = 0$ ). We denote its closure by  $\overline{Mv}(X)$  and denote by  $Mv(X)$  the interior of  $\overline{Mv}(X)$ . The cones  $\overline{Mv}(X)$  and  $Mv(X)$  are called the movable cone and the strictly movable cone, respectively.

**Proposition (1.9).** (1) For a pseudo-effective  $\mathbb{R}$ -divisor  $D$  ,  $N_\sigma(D) = 0$  if and only if  $c_1(D) \in \overline{Mv}(X)$  .

(2) For a pseudo-effective  $\mathbb{R}$ -divisor  $D$ , if there is an effective  $\mathbb{R}$ -divisor  $\Delta$  such that  $c_1(D - \Delta) \in \overline{Mv}(X)$ , then  $\Delta \geq N_\sigma(D)$ .

*Proof.* (1). Assume that  $N_\sigma(D) = 0$ . Then by the proof of (1.7, (3)), we see that  $c_1(D + A) \in \overline{Mv}(X)$ , for any ample  $\mathbb{R}$ -divisor  $A$ . Therefore  $c_1(D) \in \overline{Mv}(X)$ . The converse is derived from (1.7, (1)).

(2). By (1),  $N_\sigma(D - \Delta) = 0$ . Thus for any prime divisor  $\Gamma$ ,  $\sigma_\Gamma(D) \leq \sigma_\Gamma(D - \Delta) + \sigma_\Gamma(\Delta) \leq \text{mult}_\Gamma(\Delta)$ . Therefore  $N_\sigma(D) \leq \Delta$ . Q.E.D.

**Remark.** If  $X$  is a surface, then the movable cone is nothing but the nef cone. Therefore by (1.9), we see that the  $\sigma$ -decomposition is nothing but the usual Zariski decomposition (cf. [Z] and [F1]).

**Definition (1.10).** Let  $W \subset X$  be a subvariety with  $\text{codim } W \geq 2$ . For a pseudo-effective  $\mathbb{R}$ -divisor  $D$ , we define  $\sigma_W(D)$  to be the following number:

$$\liminf_{\varepsilon \downarrow 0} \{ \text{mult}_W(\Delta) \mid \Delta \text{ is an effective } \mathbb{R}\text{-divisor} \\ \text{with } c_1(\Delta) = c_1(D + \varepsilon A) \},$$

where  $A$  is an ample divisor.

From the following (1.11), we see that the number  $\sigma_W(D)$  does not depend on the choice of  $A$ .

Lemma (1.11). Let  $Q_W(X) \longrightarrow X$  be the blowing-up along  $W$  and let  $Y \longrightarrow Q_W(X)$  be a resolution of singularities. Then for the main exceptional divisor  $E_W$  on  $Y$ , we have  $\sigma_{E_W}(f^*D) = \sigma_W(D)$ , where  $f : Y \longrightarrow X$  is the composition.

*Proof.* Let  $\Delta$  be an effective  $\mathbb{R}$ -divisor on  $X$ . Then

$\text{mult}_W(\Delta) = \text{mult}_{E_W}(f^*\Delta)$ . Thus we have that

$$\sigma_W(D) = \lim_{\varepsilon \downarrow 0} \sigma_{E_W}(f^*(D + \varepsilon A)) = \sigma_{E_W}(f^*D) \quad \text{by (1.7, (2))}. \quad \text{Q.E.D.}$$

Lemma (1.12). (1)  $\sigma_W(D) \leq \sigma_x(D)$  for all  $x \in W$ .

(2) There is a countable union  $S$  of proper closed analytic subsets of  $W$  such that  $\sigma_W(D) = \sigma_x(D)$  for  $x \in W \setminus S$ .

(3) The function  $X \ni x \longmapsto \sigma_x(B)$  is upper semi-continuous for big  $\mathbb{R}$ -divisors  $B$ .

*Proof.* (1) and (2). Let  $\Delta$  be an effective  $\mathbb{R}$ -divisor and

$\Delta = \sum r_j \Gamma_j$  be the irreducible decomposition of  $\Delta$ . Then by

definition,  $\text{mult}_W(\Delta) = \sum r_j \text{mult}_W(\Gamma_j)$ . Therefore

$\text{mult}_x(\Delta) \geq \text{mult}_W(\Delta)$  and further there exists a Zariski-open subset

$U$  of  $W$  such that  $\text{mult}_x(\Delta) = \text{mult}_W(\Delta)$  for  $x \in U$ . Thus we are

done.

(3). We have  $\sigma_x(B) = \liminf_{\varepsilon \downarrow 0} \{\text{mult}_x(\Delta) \mid \Delta \geq 0 \text{ and}$

$c_1(\Delta) = c_1(B)\}$ , since  $B$  is big. Therefore the upper

semi-continuity of  $\sigma_x(B)$  is derived from the upper semi-continuity

of the function  $x \longmapsto \text{mult}_x(\Delta)$ .

Q.E.D.

Lemma (1.13). Let  $D$  be a pseudo-effective  $\mathbb{R}$ -divisor on  $X$ .

(1) If  $f : Y \longrightarrow X$  be a birational morphism from a projective manifold  $Y$ , then  $N_{\sigma}(f^*D) \geq f^*N_{\sigma}(D)$ .

(2)  $\sigma_X(D) = \sigma_X(P_{\sigma}(D)) + \text{mult}_X(N_{\sigma}(D))$  for  $x \in X$ .

(3) Let  $\mu : Q_X(X) \longrightarrow X$  be the blowing-up at a point  $x \in X$ . Then for any  $y \in \mu^{-1}(x)$ , we have  $\sigma_y(P_{\sigma}(\mu^*D)) \leq \sigma_x(P_{\sigma}(D))$ .

*Proof.* (1). Take an ample divisor  $A$  on  $X$ . If  $\Delta$  is an effective  $\mathbb{R}$ -divisor on  $Y$  such that  $c_1(\Delta) = c_1(f^*(D + \varepsilon A))$  for some positive  $\varepsilon$ , then  $\Delta = f^*(f_*\Delta)$  and  $c_1(f_*\Delta) = c_1(D + \varepsilon A)$ . Therefore we have  $N_{\sigma}(f^*(D + \varepsilon A)) \geq f^*N_{\sigma}(D + \varepsilon A)$ . By (1.7, (2)), we are done.

(2). Let  $E = \mu^{-1}(x)$  be the exceptional divisor for  $\mu$  (defined in (3)). Then by (1) and (1.3), we have  $\sigma_E(\mu^*D) = \sigma_E(\mu^*P_{\sigma}(D)) + \text{mult}_E(\mu^*N_{\sigma}(D))$ . This is just the desired formula by (1.11).

(3). By (1) and (1.7), it is enough to prove (3) in the case that  $c_1(D) \in \text{Mv}(X)$ . In this case, (3) is derived from the following fact: Let  $\Delta$  be an effective divisor on  $X$  and let  $\Delta'$  be the proper transform of  $\Delta$  by  $\mu$ . Then  $\text{mult}_y(\Delta') \leq \text{mult}_x(\Delta)$  for  $y \in \mu^{-1}(x)$ . Q.E.D.

Proposition (1.14). If  $c_1(D) \in \text{Mv}(X)$ , then there exist at most a finite number of subvarieties  $W$  of  $X$  with  $\sigma_W(D) > 0$  and  $\text{codim } W = 2$ .

*Proof.* Let  $Z$  be the intersection of all the supports of effective  $\mathbb{R}$ -divisors which are numerically equivalent to  $D$ . Then by (1.7, (3)),  $\text{codim } Z \geq 2$ . If  $\sigma_W(D) > 0$ , then  $W \subset Z$ . Q.E.D.

**Proposition (1.15).** *If  $\dim X = 3$ ,  $N_\sigma(D) = 0$  and  $D \cdot C_i < 0$  for some finitely many irreducible curves  $C_i$ , then there exists a bimeromorphic morphism  $\pi : X \longrightarrow Z$  onto a complex variety  $Z$  such that  $\pi(C_i)$  are points and  $\pi$  induces an isomorphism  $X \setminus \cup C_i \cong Z \setminus \cup \pi(C_i)$ .*

*Proof.* We may assume that  $c_1(D) \in \text{Mv}(X)$  and that  $D$  is a Cartier divisor. By the proof of (1.7 (3)), we have two effective Cartier divisors  $\Delta_1$  and  $\Delta_2$  such that  $\dim(\Delta_1 \cap \Delta_2) = 1$  and  $\Delta_i \in |mD|$  for some  $m > 0$ . Since  $D \cdot C_i < 0$ , by [N2, (1.4)], we can contract all the  $C_i$ . Q.E.D.

## §2. $\nu$ -decomposition.

We introduce another decomposition for a pseudo-effective  $\mathbb{R}$ -divisor  $D$ . We remark that for the positive part of the  $\sigma$ -decomposition  $P_\sigma(D)$  and for all the prime divisors  $\Gamma$ ,  $P_\sigma(D)|_\Gamma$  is pseudo-effective. So let us consider the set

$$\mathcal{G} := \{ \Delta \mid \text{an effective } \mathbb{R}\text{-divisor such that } (D - \Delta)|_\Gamma \text{ is pseudo-effective for all prime divisor } \Gamma \text{ on } X \}.$$

Then we have:

Lemma (2.1). The divisor  $N_{\nu}(D) := \sum_{\Gamma} \inf\{\text{mult}_{\Gamma}(\Delta) \mid \Delta \in \mathcal{G}\} \Gamma$  is also an element of  $\mathcal{G}$ .

*Proof.* For any prime divisor  $\Gamma$  and for any positive number  $\varepsilon$ , there is an effective  $\mathbb{R}$ -divisor  $\Delta \in \mathcal{G}$  such that

$\delta := \text{mult}_{\Gamma}(N_{\nu}(D)) - \text{mult}_{\Gamma}(\Delta) \leq \varepsilon$ . Thus

$(D - N_{\nu}(D))|_{\Gamma} + \delta\Gamma|_{\Gamma} = (D - \Delta)|_{\Gamma} + (\Delta' - N_{\nu}(D)')|_{\Gamma}$  is

pseudo-effective, where  $\Delta' := \Delta - \text{mult}_{\Gamma}(\Delta)\Gamma$  and

$N_{\nu}(D)' := N_{\nu}(D) - \text{mult}_{\Gamma}(N_{\nu}(D))\Gamma$ . Therefore  $N_{\nu}(D) \in \mathcal{G}$ . Q.E.D.

Since  $N_{\sigma}(D) \in \mathcal{G}$ , we have  $N_{\nu}(D) \leq N_{\sigma}(D)$ . Especially,

$P_{\nu}(D) := D - N_{\nu}(D)$  is a pseudo-effective  $\mathbb{R}$ -divisor.

Definition (2.2). The decomposition  $D = P_{\nu}(D) + N_{\nu}(D)$  is called the  $\nu$ -decomposition of  $D$ .  $P_{\nu}(D)$  and  $N_{\nu}(D)$  are called the positive part and the negative part of the  $\nu$ -decomposition of  $D$ , respectively.

Remark (2.3). Let  $\text{NMv}(X)$  be the cone generated by the first Chern classes of the pseudo-effective  $\mathbb{R}$ -divisors  $\Delta$  such that  $\Delta|_{\Gamma}$  are still pseudo-effective for any prime divisors  $\Gamma$ . Then it is easy to see that for pseudo-effective  $\mathbb{R}$ -divisors  $D$ ,  $c_1(P_{\nu}(D)) \in \text{NMv}(X)$  and that if  $c_1(D - \Delta) \in \text{NMv}(X)$  for some effective  $\mathbb{R}$ -divisor  $\Delta$ , then  $\Delta \geq N_{\nu}(D)$ .

Remark (2.4). We can calculate the  $\nu$ -decomposition of given  $D$  as follows: Let  $\mathcal{D}_1 = \{\Gamma_1, \Gamma_2, \dots, \Gamma_{m_1}\}$  be the set of prime divisors  $\Gamma$  such that  $D|_{\Gamma}$  is not pseudo-effective. If  $\mathcal{D}_1$  is empty, we stop here. Otherwise, the set

$$\mathfrak{X}_1 := \left\{ (r_i)_{1 \leq i \leq m_1} \mid 0 \leq r_j \in \mathbb{R} \text{ and } (D - \sum_{i=1}^{m_1} r_i \Gamma_i) |_{\Gamma_j} \text{ is pseudo-effective for all } j \right\}$$

is non-empty and we have  $(t_i^{(1)}) \in \mathfrak{X}_1$ , where  $t_j^{(1)} := \inf\{t \geq 0 \mid t = r_j \text{ for some } (r_i) \in \mathfrak{X}_1\}$ , by the same argument as in the proof of (2.1). Here we set

$$D^{(1)} := D - \sum_{i=1}^{m_1} t_i^{(1)} \Gamma_i.$$

$D^{(1)}$  is also pseudo-effective. Next we consider the set

$\mathcal{D}_2 = \{\Gamma_{m_1+1}, \Gamma_{m_1+2}, \dots, \Gamma_{m_2}\}$  consisting of all the prime divisors  $\Gamma$  such that  $D^{(1)}|_{\Gamma}$  is not pseudo-effective. If  $\mathcal{D}_2$  is empty, we stop here. Otherwise let  $\mathfrak{X}_2$  be the set  $\{(r_i)_{1 \leq i \leq m_2} \mid 0 \leq r_j \in \mathbb{R}$

and  $(D - \sum_{i=1}^{m_2} r_i \Gamma_i) |_{\Gamma_j}$  is pseudo-effective for all  $0 \leq j \leq m_2\}$ .

Then  $(t_i^{(2)}) \in \mathfrak{X}_2$ , where  $t_j := \inf\{t \geq 0 \mid t = r_j \text{ for some}$

$(r_i) \in \mathfrak{X}_2\}$  and we set  $D^{(2)} := D - \sum_{i=1}^{m_1} t_i^{(2)} \Gamma_i$ . Similarly, if we get

the sets  $\mathcal{D}_k$ ,  $\mathfrak{X}_k$  and the divisor  $D^{(k+1)}$ . Since the prime

divisors contained in some  $D_k$  are components of  $N_\sigma(D)$ , this process must be terminated. The last divisor  $D^{(k)}$  should be the positive part  $P_\nu(D)$  of the  $\nu$ -decomposition of  $D$ .

Remark (2.5). (1) Zariski [Z] and Fujita [F1] have constructed the Zariski-decomposition on surfaces just by the same way as above. But in their case,  $t_i^{(1)}, t_i^{(2)}, \dots$ , are calculated by linear equations.

(2) If  $P_\nu(D) \in \overline{Mv}(X)$ , then this  $\nu$ -decomposition is nothing but the  $\sigma$ -decomposition by (1.9) and (2.3).

(3) In general,  $N_\sigma(D) \neq N_\nu(D)$ . For example, let us consider the point blow-up  $f : Y \longrightarrow X$ . Then  $N_\nu(f^*D) = f^*N_\nu(D)$ , but  $N_\sigma(f^*D) \neq f^*N_\sigma(D)$ , if  $\sigma_X(D) > 0$ .

Proposition (2.6). *If  $\dim X = 3$ ,  $N_\nu(D) = 0$  and  $D \cdot C < 0$  for an irreducible curve  $C$ , then there exists a proper bimeromorphic morphism  $\pi : X \longrightarrow Z$  such that  $\pi(C)$  is a point and  $\pi$  induces an isomorphism  $X \setminus C \simeq Z \setminus \pi(C)$ .*

*Proof.* We may assume that  $D$  is a big Cartier divisor. Let  $\Delta$  be an element of  $|mD|$  for some positive integer  $m$ . Since  $\Delta \cdot C < 0$ , there is a component  $\Gamma$  of  $\Delta$  such that  $\Gamma \cdot C < 0$ . Thus  $\Gamma \supset C$ . By the assumption,  $\Delta|_\Gamma$  is a pseudo-effective divisor on  $\Gamma$ . Therefore there exists an effective Cartier divisor  $E$  on  $\Gamma$  such that  $(E \cdot C)_\Gamma < 0$ . Let  $\mathcal{I}$  be the defining ideal of  $E$  on  $X$ . Then we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-\Gamma) \otimes \mathcal{O}_C \longrightarrow \mathcal{I} \otimes \mathcal{O}_C \longrightarrow \mathcal{O}_\Gamma(-E) \otimes \mathcal{O}_C \longrightarrow 0 .$$

Therefore  $\mathcal{L} \otimes \mathcal{O}_C$  is an ample vector bundle. Hence by the contraction criterion [N2, (1.4)], we can contract the curve  $C$ . Q.E.D.

### §3. Relative version.

Let  $\pi : X \longrightarrow S$  be a projective surjective morphism from a complex manifold  $X$  onto a complex variety  $S$ . Fix a point  $P$  on  $S$ .

**Definition (3.1).** An  $\mathbb{R}$ -divisor  $D$  on  $X$  is called  $\pi$ -big over  $P$ , if for a  $\pi$ -ample divisor  $A$ , there exist a positive number  $\varepsilon$  and an effective  $\mathbb{R}$ -divisor  $\Delta$  such that  $D - \varepsilon A = \Delta$  in  $N^1(X/S; P)_{\mathbb{Q}}$  (cf. [N1 §4]). An  $\mathbb{R}$ -divisor  $D$  is called  $\pi$ -pseudo-effective over  $P$ , if  $D + \varepsilon A$  is  $\pi$ -big over  $P$  for any positive number  $\varepsilon$ .

Now we shall define the relative  $\sigma$ -decomposition. Let  $\Gamma$  be a prime divisor on  $X$  with  $\pi(\Gamma) \ni P$  and let  $B$  be an  $\mathbb{R}$ -divisor which is  $\pi$ -big over  $P$ . First of all, we define the following numbers:

$$\begin{aligned} \sigma_{\Gamma}(B; P)_{\mathbb{Q}} &:= \inf\{\text{mult}_{\Gamma}(\Delta) \mid \Delta \geq 0, \Delta = B \text{ in } N^1(X/S; P)_{\mathbb{Q}}\}, \\ \sigma_{\Gamma}(B; P) &:= \inf\{\text{mult}_{\Gamma}(\Delta) \mid \Delta \geq 0, \Delta = B \text{ in } N^1(X/S; P)\}. \end{aligned}$$

By the same argument as in (1.1), we have

$$\sigma_{\Gamma}(B; P)_{\mathbb{Q}} = \lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(B + \varepsilon A; P)_{\mathbb{Q}} \quad \text{for any } \pi\text{-ample divisor } A \text{ and}$$

$\sigma_{\Gamma}(B; P)_{\mathbb{Q}} = \sigma_{\Gamma}(B; P)$ . Let  $D$  be an  $\mathbb{R}$ -divisor on  $X$  which is  $\pi$ -pseudo-effective over  $P$ .

**Proposition (3.2).** (1). The limit  $\lim_{\varepsilon \downarrow 0} \sigma_{\Gamma}(D + \varepsilon A; P)$  (might be  $+\infty$ ) does not depend on the choice of a  $\pi$ -ample divisor  $A$ . (In what follows, we denote the limit by  $\sigma_{\Gamma}(D; P)$ ).

(2). If one of the following conditions are satisfied, then  $\sigma_{\Gamma}(D; P) < +\infty$  :

(a)  $\pi(\Gamma) = S$  ,

(b)  $\text{codim}_S(\pi(\Gamma)) = 1$  ,

(c) There exists an effective  $\mathbb{R}$ -divisor  $\Delta$  with  $\Delta = D$  in  $N^1(X/S; P)$ .

*Proof.* (1). This is derived from the same argument as in (1.2, (2)).

(2). (a). It is easily proved restricting  $D$  to "general" fibers of  $\pi$ . (c). Trivial. (b). We may assume that  $\pi$  is a projective fiber space and  $S$  is normal. Let  $\Gamma_0 := \Gamma$ ,  $\Gamma_1, \Gamma_2, \dots, \Gamma_{\ell}$  be all the prime divisors on  $X$  with  $\pi(\Gamma_i) = \pi(\Gamma)$ . Then there exist positive integers  $a_i$  ( $1 \leq i \leq \ell$ ), a reflexive sheaf  $\mathcal{L}$  of rank one on  $S$  and a Zariski open subset  $U$  of  $S$  such that  $\mathcal{L}|_U$  is invertible,  $\text{codim}_S(S \setminus U) \geq 2$  and

$$\pi^*(\mathcal{L}|_U) \simeq \mathcal{O}_X\left(\sum_{i=0}^{\ell} a_i \Gamma_i\right)|_{\pi^{-1}(U)} .$$

By taking a blowing-up of  $X$ , we may assume that

$\pi^* \pi_* \mathcal{O}_X(\sum_{i=0}^{\ell} a_i \Gamma_i) / \text{torsion}$  is invertible and isomorphic to  $\mathcal{O}_X(\sum_{i=0}^{\ell} a_i \Gamma_i - E)$  for some effective divisor  $E$  on  $X$  with  $\text{codim } \pi(E) \geq 2$ . Since  $\sum_{i=0}^{\ell} a_i \Gamma_i - E$  is  $\pi$ -nef, we have  $\sigma_{\Gamma_j}(\sum_{i=0}^{\ell} a_i \Gamma_i; P) \leq \sigma_{\Gamma_j}(E; P) = 0$ . Thus  $\sigma_{\Gamma_j}(D; P) = 0$  for some  $\Gamma_j$ . For any  $\varepsilon > 0$ ,  $(D + \varepsilon A - \sum_{i=0}^{\ell} \sigma_{\Gamma_i}(D + \varepsilon A; P) \Gamma_i)|_{\Gamma_j}$  is  $(\pi|_{\Gamma_j})$ -pseudo-effective over  $P$ . Hence if  $\pi(\Gamma_k \cap \Gamma_j) = \pi(\Gamma)$ , then  $\sigma_{\Gamma_k}(D; P) < +\infty$ . Since  $\pi$  is a fiber space, we have  $\sigma_{\Gamma}(D; P) < +\infty$ .

Q.E.D.

If  $\sigma_{\Gamma}(D; P) < +\infty$  for all prime divisor  $\Gamma$  with  $\pi(\Gamma) \ni P$ , then the similar results as in (1.3), (1.4) and (1.5) are obtained. Thus we can define the negative part  $N_{\sigma}(D; P)$  of the relative  $\sigma$ -decomposition to be the effective  $\mathbb{R}$ -divisor  $\sum \sigma_{\Gamma}(D; P) \Gamma$ . Also we can define the relative  $\nu$ -decomposition as in §2.

#### §4. Examples.

(I). Let  $f : X \longrightarrow Z$  be a proper bimeromorphic morphism from a 3-dimensional complex manifold  $X$  such that the  $f$ -exceptional set is just a compact smooth curve  $C$ . This morphism  $f$  is called the contraction of  $C$  and  $C$  is called an exceptional curve in  $X$  (cf. [N2]). Let  $P$  be the point  $f(C)$ . Now we shall consider the relative Zariski-decomposition Problem. Since  $N^1(X/Z; P)$  is one

dimensional, we treat a line bundle  $\mathcal{L}$  on  $X$  with  $\mathcal{L} \cdot C < 0$ .

Clearly  $N_{\sigma}(\mathcal{L}; P) = 0$ . To get the Zariski-decomposition of  $\mathcal{L}$ , we must consider the blow-up along  $C$ . We use the notation of [N2 §2]. Let  $\mu_1 : X_1 \longrightarrow X$  be the blowing-up along  $C$ , and let  $E_1$  be the exceptional divisor  $\mu_1^{-1}(C) \simeq \mathbb{P}_C(I_C/I_C^2)$ , where  $I_C$  is the defining ideal of  $C$  in  $X$ .

**Proposition (4.1).** *If the conormal bundle  $I_C/I_C^2$  is semi-stable, then we obtain  $N_{\nu}(\mu_1^*\mathcal{L}) = -2(\mathcal{L} \cdot C)/\deg(I_C/I_C^2)E_1$  and the positive part  $P_{\nu}(\mu_1^*\mathcal{L})$  is nef over  $Z$ . Namely, the relative Zariski-decomposition for  $\mathcal{L}$  exists.*

*Proof.* We must know when the divisor  $\Delta := (\mu_1^*\mathcal{L} - xE_1)|_{E_1}$  is pseudo-effective. Since  $I_C/I_C^2$  is semi-stable, every effective divisors are nef and  $\Delta$  is pseudo-effective, if  $\Delta^2 \geq 0$  and  $x > 0$  by [Mi (3.1)]. Thus  $N_{\nu}(\mu_1^*\mathcal{L}) = -2(\mathcal{L} \cdot C)/\deg(I_C/I_C^2)E_1$ . Q.E.D.

Assume that the conormal bundle  $I_C/I_C^2$  is not semi-stable. Then we have an exact sequence :  $0 \longrightarrow \mathcal{L}_0 \longrightarrow I_C/I_C^2 \longrightarrow \mathcal{M}_0 \longrightarrow 0$ , where  $\mathcal{L}_0$  and  $\mathcal{M}_0$  are line bundles on  $C$  with  $\deg(\mathcal{L}_0) > \deg(\mathcal{M}_0)$ . Thus there is the negative section  $C_1$  on the ruled surface  $E_1$  such that  $\mathcal{O}(C_1) \otimes_{\mathcal{O}_{C_1}} \simeq \mathcal{M}_0 \otimes \mathcal{L}_0^{-1}$ .

**Proposition (4.2).** *If  $2\deg(\mathcal{M}_0) \geq \deg(\mathcal{L}_0)$ , we have the relative Zariski-decomposition of  $\mathcal{L}$  over  $Z$ .*

*Proof.* Let  $\mu_2 : X_2 \longrightarrow X_1$  be the blowing-up along  $C_1$ ,  $E_2$  the  $\mu_2$ -exceptional divisor and let  $E_1'$  be the proper transform of  $E_1$ . By the exact sequence:

$$0 \longrightarrow \mathcal{O}(-E_1) \otimes \mathcal{O}_{C_1} \longrightarrow I_{C_1}/I_{C_1}^2 \longrightarrow \mathcal{O}_{C_1} \otimes_{\mathcal{O}_{E_1}} \mathcal{O}(-C_1) \longrightarrow 0,$$

if  $2\deg(\mathcal{M}_0) > \deg(\mathcal{L}_0)$ , then  $C_2 := E_1' \cap E_2$  is the negative section of  $E_2$ . If  $2\deg(\mathcal{M}_0) = \deg(\mathcal{L}_0)$ , then  $E_2$  is the ruled surface over  $C$  associated with a semi-stable vector bundle  $I_{C_1}/I_{C_1}^2$ . Therefore by [N2 (2.4)], we obtain a proper modification  $\phi : Y \longrightarrow X_2$  such that

- (1)  $\phi^{-1}(E_1' \cup E_2)$  is a union of ruled surfaces  $F_j$  ( $1 \leq j \leq k$ ) over  $C$  for some  $k \geq 2$ ,
- (2) there is just one surface  $F_k$  is a ruled surface associated to a semi-stable vector bundle on  $C$ , and that
- (3) the negative section of the other surface  $F_j$  ( $j < k$ ) is the complete intersection of  $F_j$  and other  $F_i$ .

Therefore if  $\Delta|_{F_j}$  is pseudo-effective for a divisor  $\Delta$  on  $Y$ , then  $\Delta|_{F_j}$  are nef. Thus the  $\nu$ -decomposition of the pull-back of  $\mathcal{L}$  gives the relative Zariski-decomposition. Q.E.D.

By the same argument as in (4.2), we have:

**Proposition (4.3).** *If there exist a vector bundle  $E$  of rank two on  $C$  and an open neighborhood  $U$  of the zero section of  $E$  such*

that  $U$  is isomorphic to  $X$ , then we can find the relative Zariski-decomposition of  $\mathcal{L}$ .

**Proposition (4.4).** *If there exist two irreducible divisors  $\Delta_1$  and  $\Delta_2$  such that  $\Delta_1 \cdot C < 0$ ,  $\Delta_2 \cdot C < 0$  and  $\Delta_1 \cap \Delta_2 = C$ , then we can find the Zariski-decomposition.*

*Proof.* Let  $m_1$  and  $m_2$  be positive integers such that  $m_1(\Delta_1 \cdot C_1) = m_2(\Delta_2 \cdot C_2)$ . Let  $f: V \rightarrow X$  be the normalization of the blowing-up of  $X$  along the ideal  $J := \mathcal{O}_X(-m_1\Delta_1) + \mathcal{O}_X(-m_2\Delta_2)$ . Since  $J \otimes \mathcal{O}_C \simeq \mathcal{O}_C(-m_1\Delta_1) \oplus \mathcal{O}_C(-m_2\Delta_2)$ ,  $f^{-1}(C)_{\text{red}} =: E$  is the ruled surface over  $C$  associated with the semi-stable vector bundle  $J \otimes \mathcal{O}_C$ . The divisor  $E$  is  $\mathbb{Q}$ -Cartier, since  $f^*J/\text{torsion}$  is an invertible sheaf and isomorphic to  $\mathcal{O}_V(-kE)$  for some  $k$ . Let us consider the  $\nu$ -decomposition of  $f^*\mathcal{L}$  on  $V$ . Since the divisor on  $E$  is pseudo-effective if and only if it is nef, the positive part of the  $\nu$ -decomposition is nef. Q.E.D.

**Remark (4.5).** (1). There is an example where the assumption of (4.4) is not satisfied: Let  $0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow \mathcal{O}_C \rightarrow 0$  be the non-trivial extension over an elliptic curve  $C$  and let  $E$  be the total space  $V(\mathcal{E} \otimes \mathcal{N})$  of  $\mathcal{E} \otimes \mathcal{N}$ , where  $\mathcal{N}$  is a negative line bundle on  $C$ . Then the zero-section of  $E$  is an exceptional curve, but there exist no such effective prime divisors  $\Delta_1, \Delta_2$  on any neighborhoods of the zero-section as in (4.4).

(2). If there is a proper bimeromorphic morphism  $X' \rightarrow Z$  which is isomorphic outside of  $P$  and is not isomorphic to the

original  $f$ , then the assumption of (4.4) is satisfied. But the converse does not hold in general. For example, let  $E$  be the total space  $V(\mathcal{O}_C \oplus \mathcal{M})$  of  $\mathcal{O}_C \oplus \mathcal{M}$  on an elliptic curve  $C$  such that  $\mathcal{M}$  has degree zero but is not a torsion element in  $\text{Pic}(C)$ . Then the relative Zariski-decomposition for a divisor  $L$  on  $X$  with  $L \cdot C < 0$  exists by (4.3), but its positive part is not relatively semi-ample over  $Z$ . Thus it is impossible to obtain the morphism  $X \longrightarrow Z$  above.

(II). Let  $\mathcal{E}$  be a locally free sheaf of rank  $r$  on a projective smooth curve  $C$ ,  $\pi : X := \mathbb{P}_C(\mathcal{E}) \longrightarrow C$  the associated projective bundle and let  $\mathcal{O}_{\mathcal{E}}(1)$  be the tautological line bundle on  $X$  such that  $\pi_* \mathcal{O}_{\mathcal{E}}(1) \simeq \mathcal{E}$ . We shall consider the Zariski-decomposition Problem for pseudo-effective divisors on  $X$ . Let  $F$  be a fiber of  $\pi$ . Then every  $\mathbb{R}$ -divisor is numerically equivalent to a linear combination of  $\mathcal{O}_{\mathcal{E}}(1)$  and  $F$ . By [HN], we have a unique filtration (which is called the Harder-Narasimhan filtration of  $\mathcal{E}$ ) by subbundles:  $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$ , which satisfies the following two conditions:

(1)  $\mathcal{E}_i/\mathcal{E}_{i-1}$  is a nonzero semi-stable vector bundle for all  $1 \leq i \leq \ell$ .

(2)  $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$  for  $1 \leq i \leq \ell-1$ , where  $\mu(\mathcal{E}) := \text{deg}(\mathcal{E})/\text{rank}(\mathcal{E})$ .

**Lemma (4.6).** *For a real number  $t$ , the divisor  $\mathcal{O}_{\mathcal{E}}(1) - tF$  is pseudo-effective if and only if  $t \leq \mu(\mathcal{E}_1)$ . Also the divisor*

$\mathcal{O}_{\mathcal{E}}(1) - tF$  is nef if and only if  $t \leq \mu(\mathcal{E}/\mathcal{E}_{\ell-1})$ .

*Proof.* First of all, we must mention that in the case of semi-stable  $\mathcal{E}$ , this lemma is proved in [Mi, (3.1)]. Therefore if  $\mathcal{Q} \ni t < \mu(\mathcal{E}_1)$ , then the divisor  $\mathcal{O}_{\mathcal{E}_1}(1) - tF$  on  $\mathbb{P}_C(\mathcal{E}_1)$  is big. Thus  $h^0(C, \text{Sym}^k(\mathcal{E}_1) \otimes \mathcal{A}^{tk}) \neq 0$  for  $k \gg 0$  with  $tk \in \mathbb{Z}$ , where  $\mathcal{A}$  is a line bundle of degree one on  $C$ . Hence  $h^0(C, \text{Sym}^k(\mathcal{E}) \otimes \mathcal{A}^{tk}) \neq 0$ . Therefore  $\mathcal{O}_{\mathcal{E}}(1) - tF$  is pseudo-effective for  $t \leq \mu(\mathcal{E}_1)$ . Conversely, assume that  $\mathcal{Q} \ni t > \mu(\mathcal{E}_1)$ . Then the divisors  $\mathcal{O}_{\mathcal{E}_i/\mathcal{E}_{i-1}}(1) - tF$  are not pseudo-effective on  $\mathbb{P}_C(\mathcal{E}_i/\mathcal{E}_{i-1})$  for  $1 \leq i \leq \ell$ , respectively. From the Harder-Narasimhan filtration, we see that  $h^0(C, \text{Sym}^k(\mathcal{E}) \otimes \mathcal{A}^{tk}) = 0$  for all  $k > 0$ . Thus  $\mathcal{O}_{\mathcal{E}}(1) - tF$  is not pseudo-effective for  $t > \mu(\mathcal{E}_1)$ . Next we assume that  $\mathcal{O}_{\mathcal{E}}(1) - tF$  is nef on  $X$ . Then  $\mathcal{O}_{\mathcal{E}/\mathcal{E}_{\ell-1}}(1) - tF$  on  $\mathbb{P}_C(\mathcal{E}/\mathcal{E}_{\ell-1}) \subset \mathbb{P}_C(\mathcal{E})$  is also nef. Therefore  $t \leq \mu(\mathcal{E}/\mathcal{E}_{\ell-1})$ . Conversely, suppose that  $\mathcal{O}_{\mathcal{E}}(1) - tF$  is not nef. Then there is an irreducible curve  $\Gamma$  in  $X$  with  $(\mathcal{O}_{\mathcal{E}}(1) - tF) \cdot \Gamma < 0$ . Let  $C' \rightarrow \Gamma$  be the normalization of  $\Gamma$  and let  $f : C' \rightarrow C$  be the composition with  $\Gamma \rightarrow C$ . Since the Harder-Narasimhan filtration of  $f^*\mathcal{E}$  is just the pull-back of  $\mathcal{E}_i$ 's, we may assume that  $\Gamma$  is a section of  $X \rightarrow C$ . Then  $\Gamma$  corresponds to a surjective homomorphism  $\mathcal{E} \rightarrow \mathcal{F}$  onto an invertible sheaf  $\mathcal{F}$  such that  $\deg(\mathcal{F}) < t$ . By a property of the Harder-Narasimhan filtration, we have  $\mu(\mathcal{E}/\mathcal{E}_{\ell-1}) \leq \deg(\mathcal{F})$ . Therefore  $t > \mu(\mathcal{E}/\mathcal{E}_{\ell-1})$ . Q.E.D.

**Proposition (4.7).** *We have the Zariski-decomposition of the divisor*

$$\mathcal{O}_{\mathcal{E}}(1) - \mu(\mathcal{E}_1)F .$$

*Proof.* If  $\mathcal{E}$  is semi-stable, we have nothing to prove. So we assume that  $\ell \geq 2$ . Let  $f : Y \longrightarrow X$  be the blowing-up along  $\mathbb{P}_{\mathbb{C}}(\mathcal{E}/\mathcal{E}_1)$ . Then we have a projective bundle structure  $\pi : Y \longrightarrow \mathbb{P}_{\mathbb{C}}(\mathcal{E}_1)$  such that the exceptional divisor  $E$  of  $f$  is isomorphic to the fiber product  $\mathbb{P}_{\mathbb{C}}(\mathcal{E}_1) \times_{\mathbb{C}} \mathbb{P}_{\mathbb{C}}(\mathcal{E}/\mathcal{E}_1)$ , where the restrictions of  $\pi$  and  $f$  to  $E$  correspond to the first and second projections, respectively. We need the following:

**Claim (4.8).** *The  $\mathbb{R}$ -divisor  $\Delta = x \text{pr}_1^* \mathcal{O}_{\mathcal{E}_1}(1) + y \text{pr}_2^* \mathcal{O}_{\mathcal{E}/\mathcal{E}_1}(1) - tF$  on  $\mathbb{P}_{\mathbb{C}}(\mathcal{E}_1) \times_{\mathbb{C}} \mathbb{P}_{\mathbb{C}}(\mathcal{E}/\mathcal{E}_1)$  is pseudo-effective if and only if  $x \geq 0$ ,  $y \geq 0$  and  $t \leq x\mu(\mathcal{E}_1) + y\mu(\mathcal{E}_2/\mathcal{E}_1)$ .*

*Proof.* First assume that  $x, y \geq 0$  and  $\lambda := x\mu(\mathcal{E}_1) + y\mu(\mathcal{E}_2/\mathcal{E}_1) - t \geq 0$ . Then  $\Delta = x \text{pr}_1^*(\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F) + y \text{pr}_2^*(\mathcal{O}_{\mathcal{E}/\mathcal{E}_1}(1) - \mu(\mathcal{E}_2/\mathcal{E}_1)F) + \lambda F$  is pseudo-effective by (4.6). Next suppose that  $\Delta$  is pseudo-effective. Since  $\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F$  is nef on  $\mathbb{P}_{\mathbb{C}}(\mathcal{E}_1)$ ,  $H_{\delta} := \mathcal{O}_{\mathcal{E}_1}(1) - (\mu(\mathcal{E}_1) - \delta)F$  is ample for any positive rational number  $\delta$ . Take a positive integer  $m$  so large that  $mH_{\delta}$  is a very ample Cartier divisor and take general members  $H_i \in |mH_{\delta}|$  for  $1 \leq i \leq \text{rank}(\mathcal{E}_1) - 1$  such that the intersection  $Z := \bigcap H_i$  is a smooth curve on  $X$ . Then the restriction of  $\Delta$  to the fiber product  $Z \times_{\mathbb{C}} \mathbb{P}_{\mathbb{C}}(\mathcal{E}/\mathcal{E}_1) \simeq \mathbb{P}_Z(\tau^*(\mathcal{E}/\mathcal{E}_1))$  is still pseudo-effective, where  $\tau$  is the morphism  $Z \longrightarrow \mathbb{C}$ . Now this restricted divisor is

numerically equivalent to:

$$y(\mathcal{O}_{\tau^*(\mathcal{E}/\mathcal{E}_1)}(1) - \mu(\tau^*(\mathcal{E}_2/\mathcal{E}_1))f) + \{x(\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F) \cdot Z + \lambda F \cdot Z\}f,$$

where  $f$  denotes the fiber class of  $\mathbb{P}_Z(\tau^*(\mathcal{E}/\mathcal{E}_1)) \longrightarrow Z$ . Thus by (4.6),  $y \geq 0$  and  $x(\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F) \cdot Z + \lambda F \cdot Z \geq 0$ . Hence

$$x(\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F) \cdot H_\delta^{\text{rank}(\mathcal{E}_1)-1} + \lambda F \cdot H_\delta^{\text{rank}(\mathcal{E}_1)-1} \geq 0. \text{ Taking } \delta \downarrow 0,$$

we have  $\lambda \geq 0$ . Similarly, if we take a general ample intersection on  $\mathbb{P}_C(\mathcal{E}/\mathcal{E}_1)$ , then the inequality  $x \geq 0$  is derived. Q.E.D.

*Proof of (4.7) continued:* We shall calculate the  $\nu$ -decomposition of  $f^*(\mathcal{O}_{\mathcal{E}}(1) - \mu(\mathcal{E}_1)F)$ . Since  $\pi^*\mathcal{O}_{\mathcal{E}_1}(1) \simeq f^*\mathcal{O}_{\mathcal{E}}(1) \otimes \mathcal{O}_Y(-E)$ ,

$$\mathcal{O}_E(-E) \simeq \text{pr}_1^*\mathcal{O}_{\mathcal{E}_1}(1) \otimes \text{pr}_2^*\mathcal{O}_{\mathcal{E}/\mathcal{E}_1}(-1). \text{ Therefore by (4.8),}$$

$(f^*(\mathcal{O}_{\mathcal{E}}(1) - \mu(\mathcal{E}_1)F) - \alpha E)|_E$  is pseudo-effective if and only if

$$0 \leq \alpha \leq 1 \text{ and } \mu(\mathcal{E}_1) \leq \alpha\mu(\mathcal{E}_1) + (1-\alpha)\mu(\mathcal{E}_2/\mathcal{E}_1). \text{ Since}$$

$\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1)$ , these inequalities hold if and only if  $\alpha = 1$ .

Therefore  $P_\nu(f^*(\mathcal{O}_{\mathcal{E}}(1) - \mu(\mathcal{E}_1)F)) = \pi^*(\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F)$ , which is

nef. Thus this is the Zariski-decomposition. Q.E.D.

Also by the same method as above, we can prove the following:

**Proposition (4.9).** *If  $\ell \leq 2$ , then the Zariski-decomposition exist for every pseudo-effective divisors on  $X$ .*

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## Rational curves on Weierstrass models

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### §1. Introduction

Recently, a lot of examples of Calabi-Yau threefolds (i.e. compact Kähler threefolds with trivial first Chern class and with finite fundamental group) have been constructed. In contrast with the case of K3 surfaces, it is observed that they have a large repertory of Euler numbers, in other words, they can not be connected by deformation in the proper sense. But some families of Calabi-Yau threefolds can be regarded as strata of other families by means of small resolutions. The following result was shown by H. Clemens and R. Friedman in [11],[21]: *A general quintic threefold has many mutually disjoint  $(-1, -1)$ -curves (i.e. rational curves with normal bundles isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ ) and the nodal variety obtained by the contraction of at least two of them is deformed to a 2-connected complex manifold  $V$  with  $K_V = 0$ . By the result of C.T.C. Wall [11] such a complex manifold  $V$  is diffeomorphic to a connected sum of  $N$  copies of  $S^3 \times S^3$ ,  $(S^3 \times S^3)^{\# N}$ , where  $N = 1/2 b_3(V)$ . Prompted by this result, M. Reid has suggested in [9] that all families of Calabi-Yau threefolds with  $\pi_1 = 0$  may be embedded in an irreducible moduli space as strata, and that a general point of this moduli space corresponds to a non-Kähler 2-connected threefold. For a more general and detailed statement, see [9].*

In this paper we restrict our attention to the smooth projective threefolds  $X$  with the following properties (\*)

(\*) 1)  $X$  has an elliptic fibration with a rational section over a smooth rational surface  $S$ .

2)  $P_g(X) = 1$  and  $\kappa(X) = 0$ .

Remark that many Calabi-Yau threefolds are contained in the above class. For example, the Calabi-Yau threefolds studied in [10] satisfy (\*).

We want to study the relationship between these  $X$  (exactly speaking, their birational classes) and 2-connected non-Kähler threefolds with trivial canonical bundle. My plan to this problem at this point (which might need some modification, and only one part of Step 1 has been completed) is divided into two steps:

Step 1. For  $X$  with the properties (\*), construct a flat family of threefolds  $f: \mathcal{X} \rightarrow \Delta$ : unit disk, such that

- 1) the central fibre  $\mathcal{X}_0$  is a (possibly singular) projective threefold birational to  $X$ ,
- 2) a general fibre  $\mathcal{X}_t$  is a smooth, simply-connected, projective threefold with *trivial canonical line bundle*,
- 3) there are sub-flat families  $\mathcal{E}_i$  ( $1 \leq i \leq m$ ) of curves of  $\mathcal{X}$ ,

$$\begin{array}{ccc}
 \mathcal{E}_i & \subset & \mathcal{X} \\
 \swarrow \text{flat} & & \downarrow \\
 & & \Delta
 \end{array}
 , \text{ where } \mathcal{E}_{i,t} \text{ is a curve on } \mathcal{X}_t$$

- 4)  $\mathcal{E}_i$  and  $\mathcal{E}_j$  ( $i \neq j$ ) do not intersect over  $\Delta^* = \Delta - \{0\}$
- 5) For  $t \neq 0$ ,  $\mathcal{E}_{i,t}$  ( $1 \leq i \leq m$ ) are mutually disjoint  $(-1, -1)$ -curves on  $\mathcal{X}_t$  (i.e. smooth rational curves with

normal bundles  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ ),

- 6) For  $t \neq 0$ , the complex analytic space  $\bar{\mathcal{X}}_t$  obtained from  $\mathcal{X}_t$  by contracting  $\mathcal{C}_{i,t}$ 's, can be deformed to a non-Kähler 2-connected complex manifold with trivial canonical line bundle.

Step 2: Observation of the relationship between  $\mathcal{X}_0$  and the non-Kähler manifold: For example, it is desired that there is a semi-stable degeneration  $g: \mathcal{Y} \rightarrow \Delta$  such that

- 1) one component of  $\mathcal{Y}_0$  is bimeromorphic to  $\mathcal{X}_0$  (hence, to  $X$  itself), and
- 2) a general fibre  $\mathcal{Y}_t$  is a non-Kähler 2-connected complex manifold with trivial canonical line bundle.

The purpose of this paper is to introduce a partial result with respect to Step 1. Our starting point towards Step 1 are the following, one of which is the result of N.Nakayama:

**Lemma** Let  $f: \mathcal{X} \rightarrow T$  be a flat projective morphism between algebraic varieties. Assume that a general fibre of  $f$  is a smooth threefold. Let  $t_0$  and  $t$  be two distinct point of  $T$ , and assume the following:

- a)  $\mathcal{X}_{t_0}$  is a (possibly singular) projective variety.
- b)  $\mathcal{X}_t$  is a smooth simply-connected threefold with trivial canonical line bundle.
- c) For  $\mathcal{X}_t$ , there are mutually disjoint  $(-1,-1)$ -curves  $\mathcal{C}_{i,t}$ 's on  $\mathcal{X}_t$ , and the complex analytic space  $\bar{\mathcal{X}}_t$  obtained by contracting these curves can be deformed to a non-Kähler 2-connected complex manifold with trivial canonical line bundle.

Then there is a flat family  $f: \mathcal{X} \rightarrow \Delta$ , for which  $\mathcal{X}_0 = \mathcal{X}_{t_0}$  and the properties 1)... 6) in Step 1 are satisfied.

*Proof*

Let  $D$  be an irreducible curve on  $T$  passing through  $t$  and  $t_0$ . If necessary, taking its normalization, we have a flat family of threefolds  $\mathcal{X} \rightarrow D$ , where  $D$  is a smooth curve.  $\text{Hilb}_{\mathcal{X}/D}$  is étale over  $D$  at  $[C_{i,t}]$  because  $N_{C_{i,t}/\mathcal{X}_t} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ . Let us denote by  $H_i$  the irreducible component of  $\text{Hilb}_{\mathcal{X}/D}$  containing  $[C_{i,t}]$ . For different  $i$  and  $j$ ,  $H_i$  and  $H_j$  may possibly coincide. But by taking a suitable finite cover of  $D$ , we may assume that  $H_i \neq H_j$  for  $i, j$  with  $i \neq j$ , and that each  $H_i$  is birational to  $D$ . In this situation, we restrict our attention to a sufficiently small neighbourhood  $\Delta$  of  $t_0$ .

From now on, we use the following notation:

$$\begin{array}{ccc} \mathcal{X} \times_D \Delta =: \mathcal{X}_\Delta \subset \mathcal{X} & & H_i \times_D \Delta =: H_{i,\Delta} \\ \downarrow & & \downarrow p_i \\ t_0 = 0 \in \Delta \subset D & & \Delta \end{array}$$

$p_i$  can be assumed to be an isomorphism over  $\Delta^*$ , because  $H_i$  is birational to  $D$ . Denote by  $\mathcal{H}_i$  the universal family over  $H_i$ . For a general point  $h \in H_i$ ,  $\mathcal{H}_i \times k(h)$  is isomorphic to  $\mathbb{P}^1$ . Therefore we may assume that  $\mathcal{H}_{i,\Delta} := \mathcal{H}_i \times_D \Delta$  is a  $\mathbb{P}^1$ -bundle over  $H_{i,\Delta}^*$ . Moreover we may assume that for a point  $h \in H_{i,\Delta}^*$ ,  $\mathcal{H}_i \times k(h)$  ( $1 \leq i \leq m$ ) are mutually disjoint  $(-1, -1)$ -curves on  $\mathcal{X}_{p_i(h)}$ , since  $C_{i,t}$  are mutually disjoint  $(-1, -1)$ -curves. Let  $\mathcal{E}_i$  be the image (with reduced structure) of  $\mathcal{H}_{i,\Delta}$  in  $\mathcal{X}_\Delta$ . Then  $\mathcal{E}_i$  is flat over  $\Delta^*$  because  $\mathcal{E}_i$  is irreducible, reduced and dominating 1-dimensional disk. By the construction,  $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$  for  $i, j$  ( $i \neq j$ ) over  $\Delta^*$ . Finally remark that 2) in Step 1. is a topological condition. As for 6) in Step 1. see [2].

**Theorem ([6,(3.4)1])** *Let  $X$  be a smooth projective threefold with (\*) . Then  $X$  is birational to a Weierstrass model  $W$  over  $\mathbb{P}^2$  or  $\Sigma_r$  ( $0 \leq r \leq 12$ ). (For the definition of a Weierstrass model, see (2.11).) Moreover  $W$  has only rational Gorenstein singularities.*

By the above results, we may only consider the Weierstrass models  $W$  over  $\mathbb{P}^2$  or  $\Sigma_r$  ( $0 \leq r \leq 12$ ). But the following lemma shows that the case of  $\Sigma_2$  can be excluded in our context.

**Lemma ( Jumping of the complex structure  $\Sigma_r$  )**

1) Let  $r$  be an even (resp. odd) integer.

Then for  $\Sigma_r$  ( $r \geq 2$ ), there is a smooth projective morphism  $f_r: \mathcal{Y} \rightarrow C_r$  such that a)  $C_r$  is a smooth curve ( not compact) with a distinguished point  $t_0$  , b)  $\mathcal{Y}_{t_0}$  is isomorphic to  $\Sigma_r$  , and c)  $\mathcal{Y}_t$  ( $t \neq t_0$ ) is isomorphic to  $\Sigma_0$  (resp.  $\Sigma_1$ ).

2) For an arbitrary  $n$ ,  $\chi(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = (2n + 1)^2$ .

If  $0 \leq r \leq 2$  , then  $h^1(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = h^2(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = 0$  for  $n \geq 1$ .

*Proof*

1) See [12] p.205 , Step M).

2) By the theorem of Riemann-Roch,  $\chi(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = (2n + 1)^2$ .

By the Serre duality, we have

$$h^2(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = h^0(\mathcal{O}_{\Sigma_r}(n+1)K_{\Sigma_r}),$$

$$h^1(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = h^1(\mathcal{O}_{\Sigma_r}(n+1)K_{\Sigma_r}).$$

We shall prove by induction that the right hand sides are zero for  $n \geq 1$ .

Since, for  $n = 1$ ,  $-K \sim 2C_0 + (2 + r)f$  ( $C_0$ : negative section,  $f$ : fibre), we can find a reduced element  $D$  in  $|-K|$ . Indeed, take as  $D$ ,  $C_0 + C + 2f$ , where  $C$  is a section of self-intersection number  $r$ . From the exact sequence

$$0 \longrightarrow \mathcal{O}(K) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we have

$$\begin{array}{ccccccc} H^0(\mathcal{O}) & \longrightarrow & H^0(\mathcal{O}_D) & \longrightarrow & H^1(\mathcal{O}(K)) & \longrightarrow & H^1(\mathcal{O}) = 0 \\ \parallel & & \parallel & & & & \\ \mathbb{C} & \simeq & \mathbb{C} & & & & \end{array}$$

This shows that  $H^1(\mathcal{O}(K)) = 0$ . For the case  $n \geq 2$ , from the exact sequence

$$0 \longrightarrow \mathcal{O}(nK) \longrightarrow \mathcal{O}((n-1)K) \longrightarrow \mathcal{O}_D((n-1)K) \longrightarrow 0,$$

we have

$$H^0(\mathcal{O}_D((n-1)K)) \longrightarrow H^1(\mathcal{O}(nK)) \longrightarrow H^1(\mathcal{O}((n-1)K)) = 0 \text{ (by induction).}$$

Since  $D = C_0 + C + 2f$ ,  $(C_0 \cdot (n-1)K) = (n-1)(r-2)$ ,  $(C \cdot (n-1)K) < 0$ ,  $(f \cdot (n-1)K) < 0$ , and  $(C_0 \cdot 2f) = 2$ , in the cases  $r \leq 2$   $H^0(\mathcal{O}_D((n-1)K)) = 0$ , which implies  $H^1(\mathcal{O}(nK)) = 0$ . Q.E.D.

Let  $W_2$  be a given Weierstrass model over  $\Sigma_2$ , and  $W_0$  a smooth Weierstrass model over  $\Sigma_0$ . Then, by the above lemma, we can connect two varieties by a flat deformation, and a general member of this flat deformation is a smooth Weierstrass model over  $\Sigma_0$ . Indeed, we may consider the following flat morphism (with the notation in Lemma):

$$\begin{array}{ccc} \mathbb{W} & \supset & W(K_{g_t}, a(t), b(t)) \\ \downarrow & & \downarrow \\ V_{C_2}(R^0 f_2 * \omega_g^{\otimes -4} / C_2) \times V_{C_2}(R^0 f_2 * \omega_g^{\otimes -6} / C_2) & \ni & (a(t), b(t)) \end{array}$$

, where  $t \in C_2$ ,

$$\begin{array}{l} a(t) \in H^0(\omega_{g_t}^{\otimes -4}), \\ b(t) \in H^0(\omega_{g_t}^{\otimes -6}), \end{array}$$

$4a^3 + 27b^2$  not identically  
zero.

Therefore we may only consider the Weierstrass models over  $\mathbb{P}^2$ ,  $\Sigma_r$  ( $0 \leq r \leq 12$ ,  $r \neq 2$ ). We are now in a position to state the main result of this paper.

### Theorem

Let  $W$  be a general smooth Weierstrass model over  $\mathbb{P}^1 \times \mathbb{P}^1$ . Then there are mutually disjoint rational curves  $C_1, \dots, C_4$  on  $W$  with the following properties:

- 1)  $H_{C_i/W} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  for each  $i$ .
- 2)  $C_1, \dots, C_4$  span  $H_2(W; \mathbb{Z})$ .

Moreover, if we denote by  $\varphi: W \longrightarrow \tilde{W}$  the contraction of  $C_i$ 's, then  $\tilde{W}$  can be deformed to a 2-connected compact complex manifold with  $K = 0$ .

The infinitesimal deformation spaces of Weierstrass models are very large. For example,  $h^1(W, \Theta_W) = 243$  in our case where  $W$  is a Weierstrass model over  $\mathbb{P}^1 \times \mathbb{P}^1$ .

We shall sketch the proof of the above result. First we relate a singular Weierstrass model  $W$  over  $\mathbb{P}^1 \times \mathbb{P}^1$  with a fibre product  $X = S \times_{\mathbb{P}^1} S^+$  of two rational elliptic surface with sections. The birational map between  $W$  and  $X$  is a composition of flops and divisorial contractions. The merit of considering  $X$  is the point that it is easy to study the rational curves on it. Next, we choose suitable isolated rational curves on  $X$ . It is desired that  $\alpha$ ) the birational map between  $X$  and  $W$  does not make any effect on these curves, and  $\beta$ ) these curves have different numerical classes. We need 2) to proceed with the argument along the idea [2]. Then, what kind of curves should we find on  $X$ ? We note that  $S$  has many

representations as blow-ups of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Take one of such representations and consider the linear system  $\Sigma$  on  $S$  coming from the rulings of  $\mathbb{P}^1 \times \mathbb{P}^1$ . A general member  $C$  of  $\Sigma$  is a nonsingular rational curve and is a double cover of  $\mathbb{P}^1$  by  $\pi: S \rightarrow \mathbb{P}^1$ . By Hurwitz formula,  $C$  has two ramification points  $\{P_1, P_2\}$ . Also for  $\lambda: S^+ \rightarrow \mathbb{P}^1$ , we consider similarly  $C^+ \in \Sigma^+$  and  $\{Q_1, Q_2\}$ . If  $\pi(P_1) = \lambda(Q_1)$  and  $\pi(P_2) = \lambda(Q_2)$ , then  $C \times_{\mathbb{P}^1} C^+ \subset X$  consists of two nonsingular rational curves  $D_1$  and  $D_2$  intersecting at two points transversely. Under the suitable conditions (which can be described by using the degree of a *ramification map* (Def.(2.5))),  $D_i$  is shown to be isolated. (Prop.(2.10)) The main object of this paper is a curve of such a type. Changing the representation of  $S$  (resp.  $S^+$ ) as a blow-up of  $\mathbb{P}^1 \times \mathbb{P}^1$ , and replacing  $\Sigma$  (resp.  $\Sigma^+$ ) by new one, we can find many isolated rational curves of different numerical types. We must take these curves mutually disjoint. For this purpose, in §3, we observe the degeneration of  $D_i$ . As a consequence, we will be able to choose suitable curves for which  $\alpha)$  and  $\beta)$  holds. These curves are regarded as the curves on  $W$ . The normal bundles of these curves in  $W$  are  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  or  $\mathcal{O} \oplus \mathcal{O}(-2)$ . In the latter case, we must deform the curve and  $W$  so that it splits up into a union of curves of the former type. Thus §4 deals with the deformation of rational curves on  $W$ . (Prop.(4.4)) In Appendix, we calculate  $H_2(W; \mathbb{Z})$  for a nonsingular Weierstrass model  $W$  and show that it is torsion-free, which will guarantee 2-connectedness of the smoothing of  $\tilde{W}$  in Theorem of §4. In the case where  $W$  is a Weierstrass models over  $\mathbb{P}^2$ , it seems that we must deal with the curves  $C$  of  $(1, -3)$  type (i.e.  $N_{C/W} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-3)$ ) and their deformation.

## § 2. Preliminaries

Let  $S$  and  $S^+$  be rational elliptic surfaces with sections, and we denote by  $\pi$  and  $\lambda$ , their fibrations over  $\mathbb{P}^1$ , respectively. We shall consider the case where the following are satisfied:

(2.1) Generic fibres of  $\pi$  and  $\lambda$  are not isogenous to each other.

(2.2) All singular fibres of  $\pi$  and  $\lambda$  are irreducible.

Taking the fibre product of  $S$  and  $S^+$  over  $\mathbb{P}^1$ , we obtain the following diagram (2.3):

$$\begin{array}{ccccc}
 & & S \times_{\mathbb{P}^1} S^+ & & \\
 & \swarrow p & \downarrow f & \searrow q & \\
 S & & & & S^+ \\
 & \searrow \pi & & \swarrow \lambda & \\
 & & \mathbb{P}^1 & & 
 \end{array}$$

From the assumption (2.2) we have the following commutative exact diagram:

$$\begin{array}{ccccc}
 \text{Pic } S \times \text{Pic } S^+ & \longrightarrow & \text{Pic } \pi^{-1}(\eta) \times \text{Pic } \lambda^{-1}(\eta) & \longrightarrow & 0 \\
 (2.4) \quad \downarrow p^* \otimes q^* & & \downarrow p^*(\eta) \otimes q^*(\eta) & & \\
 \mathbb{Z} f^* \mathcal{O}(1) \rightarrow \text{Pic } S \times_{\mathbb{P}^1} S^+ & \longrightarrow & \text{Pic } f^{-1}(\eta) & \longrightarrow & 0,
 \end{array}$$

where  $\eta$  is a generic point of  $\mathbb{P}^1$ .  $p^*(\eta) \otimes q^*(\eta)$  is an isomorphism by (2.1). Since  $f^* \mathcal{O}(1) \simeq p^* \pi^* \mathcal{O}(1)$ ,  $p^* \otimes q^*$  is surjective. Hence for a line bundle  $\mathcal{F}$  on  $X$ , we have  $\mathcal{F} = p^* L \otimes q^* M$ , where  $L$  (or  $M$ ) is a line bundle on  $S$  (or  $S^+$ , resp). By Künneth formula,  $f_* (\mathcal{F}) \simeq \pi_* L \otimes \lambda_* M$ , and if  $\mathcal{F}$  is effective, we may assume that  $L$  and  $M$  are effective. In fact, if  $\mathcal{F}$  is effective, then  $f_* \mathcal{F}$  has a non-zero section. Since  $\pi_* L$  and  $\lambda_* M$  are vector bundles on  $\mathbb{P}^1$ , they are direct sums of line bundles. Hence we conclude that for a suitable line bundle  $K$  on  $\mathbb{P}^1$ ,  $\pi_* L \otimes K$  and  $\lambda_* M \otimes K^{-1}$  have

non-zero sections. We can write  $\mathfrak{F} \simeq p^*(L \otimes \pi^*K) \otimes q^*(M \otimes \lambda^*K^{-1})$ . Since  $L \otimes \pi^*K$  and  $M \otimes \lambda^*K^{-1}$  are effective, we may assume that  $L$  and  $M$  are effective.

Remark Set  $X = S \times_{\mathbb{P}^1} S^+$ . Then  $X$  has at worst terminal singularities because the singular fibres of  $\pi$  and  $\lambda$  are of type  $I_1$  or  $II$ . Moreover  $X$  is  $\mathbb{Q}$ -factorial (i.e. for any Weil divisor  $D$  on  $X$ , some multiple  $mD$  is a Cartier divisor). This is shown as follows: If  $X$  is not  $\mathbb{Q}$ -factorial, then there is a small projective resolution  $r: Y \longrightarrow X$  for which  $Y$  is  $\mathbb{Q}$ -factorial. But then we have (2.4) for  $Y$  instead of  $X$  and the exceptional curve of  $r$  is numerically equivalent to 0, which contradicts the projectivity of  $Y$ . Hence  $X$  is  $\mathbb{Q}$ -factorial. On the other hand, since  $X$  is a Gorenstein 3-fold,  $X$  is factorial by [4, §5].

**Definition(2.5)** Let  $\pi: S \longrightarrow \mathbb{P}^1$  be a rational elliptic surface with sections. Let  $\Sigma$  be a linear system on  $S$  whose general member  $C$  is a nonsingular rational curve with  $C^2 = 0$ . Then the *ramification map*  $\varphi_{(S,\pi,\Sigma)}$  is defined as follows.

The members of  $\Sigma$  is parametrized by a projective line  $\mathbb{P}^1$ , because  $\dim \Sigma = 1$ . We shall denote by  $C_t$ , the member of  $\Sigma$  corresponding to  $t \in \mathbb{P}^1$ . For a general  $t \in \mathbb{P}^1$ ,  $\pi|_{C_t}: C_t \longrightarrow \mathbb{P}^1$  is a double cover with two ramification point  $P_t$  and  $Q_t$ . This correspondence from  $t$  to  $\{P_t, Q_t\}$  is extended to the map  $\varphi_{(S,\pi,\Sigma)}: \mathbb{P}^1 \longrightarrow \mathcal{G}$ , where  $\mathcal{G}$  is a symmetric product of two projective lines. We denote  $[\mathbb{C}(\mathbb{P}^1): \mathbb{C}(Im \varphi_{(S,\pi,\Sigma)})]$  by  $\deg \varphi_{(S,\pi,\Sigma)}$ .

From now, we shall consider the case where  $S$  is obtained by blowing-ups of  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $\Sigma$  is the pull back of  $[p_1^*\mathcal{O}(1)]$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ .

For example, if we assume that every singular fibre is irreducible, then we always have this case.

Let us identify the vector space  $H^0(\mathbb{P}^1 \times \mathbb{P}^1, p_1^* \mathcal{O}(2) \otimes p_2^* \mathcal{O}(2))$  with  $\mathbb{C}^9$ .

Let  $T := \{ (x, y) \in \mathbb{C}^9 \times \mathbb{C}^9 ; x = \alpha y \text{ or } y = \alpha x \text{ for some constant } \alpha \in \mathbb{C} \}$ . Then  $T$  is a closed subset of  $\mathbb{C}^9 \times \mathbb{C}^9$ . Consider the  $\mathbb{C}^*$ -action on  $\mathbb{C}^9 \times \mathbb{C}^9$  such that for  $\gamma \in \mathbb{C}^*$ ,  $\gamma(x, y) = (\gamma x, \gamma y)$ . Then  $T$  is invariant under this action, and hence we have the following quotient spaces with respect to this action.

$$\begin{array}{ccc} \mathbb{C}^9 \times \mathbb{C}^9 & \longrightarrow & \mathbb{P}^{17} \\ \cup & & \cup \\ \mathbb{C}^9 \times \mathbb{C}^9 - T & \longrightarrow & U \end{array}$$

By the identification of  $\mathbb{C}^9$  with  $H^0(\mathbb{P}^1 \times \mathbb{P}^1, p_1^* \mathcal{O}(2) \otimes p_2^* \mathcal{O}(2))$ , a point  $(x, y) \in \mathbb{C}^9 \times \mathbb{C}^9 - T$  corresponds to a pair  $(f, g)$  of sections of  $p_1^* \mathcal{O}(2) \otimes p_2^* \mathcal{O}(2)$ . Since  $(x, y)$  is not contained in  $T$ ,  $(f, g)$  determines a linear pencil  $\lambda f + \mu g$ ;  $(\lambda : \mu) \in \mathbb{P}^1$ . Thus we have

$$\begin{array}{ccc} (\lambda : \mu, [(f, g)]) & \in & \mathbb{P}^1 \times U \\ \downarrow & & \downarrow \\ [(f, g)] & \in & U \end{array}$$

If we replace  $U$  by a suitable open set of  $U$ , then each linear pencil  $\mathbb{P}^1 \times \{u\}$ ;  $u \in U$  has eight base points in number, and the blow-up of these points is a rational elliptic surface with sections  $S_u$ . For this  $S_u$  and the linear pencil  $\Sigma$  (the pull back of  $|p_1^* \mathcal{O}(1)|$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ ), the ramification map  $\varphi_u$  is defined. On a suitable open set  $V \subset U$ , there is a commutative diagram:

$$\begin{array}{ccc}
 \mathbb{P}^1 \times V & \xrightarrow{\varphi} & \mathcal{G} \times V \\
 & \searrow & \swarrow \\
 & V &
 \end{array}$$

such that  $\varphi|_u = \varphi_u$ .

We can find, by (2.6) below, at least one point  $u_0 \in V$  such that  $\deg \varphi_{u_0} = 1$ , which shows that for a general point  $u \in V$ ,  $\deg \varphi_u = 1$ .

Indeed, since  $\deg \varphi_{u_0} = 1$ , for a general point  $x$  on  $\mathbb{P}^1 \times \{u_0\}$   $\varphi_{u_0}$  is a closed immersion at  $x$ . On the other hand,  $\varphi$  is a finite morphism at  $x$ , which implies that  $\varphi$  is a closed immersion at  $x$  by Nakayama's lemma. This shows that for any point  $u \in V$  near  $u_0$ ,  $\deg \varphi_u = 1$ .

(2.6) Explicit description of  $\varphi$

$(T_0:T_1) \times (S_0:S_1) :=$  a bi-homogenous coordinate of  $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{aligned}
 f &= a_1 T_0^2 S_0^2 + a_2 T_0^2 S_0 S_1 + a_3 T_0^2 S_1^2 + a_4 T_0 T_1 S_0^2 + a_5 T_0 T_1 S_0 S_1 + \\
 &\quad a_6 T_0 T_1 S_1^2 + a_7 T_1^2 S_0^2 + a_8 T_1^2 S_0 S_1 + a_9 T_1^2 S_1^2 \\
 g &= b_1 T_0^2 S_0^2 + b_2 T_0^2 S_0 S_1 + b_3 T_0^2 S_1^2 + b_4 T_0 T_1 S_0^2 + b_5 T_0 T_1 S_0 S_1 + \\
 &\quad b_6 T_0 T_1 S_1^2 + b_7 T_1^2 S_0^2 + b_8 T_1^2 S_0 S_1 + b_9 T_1^2 S_1^2
 \end{aligned}$$

Consider the linear system  $\lambda f + \mu g$   $(\lambda:\mu) \in \mathbb{P}^1$ . If  $a_j$ 's and  $b_j$ 's are generally chosen, then this linear system has eight base points in number. Blowing up the base points, we have a rational elliptic surface with sections. Set  $\Sigma = |p_2^* \mathcal{O}(1)|$ . Then  $\Sigma$  is regarded as a linear system of the rational elliptic surface and we can consider the ramification map  $\varphi$  with respect to  $\Sigma$ .  $\Sigma$  is parametrized by  $(S_0:S_1) \in \mathbb{P}^1$ .

Put

$$\begin{aligned}
 F &= \{ \lambda (a_7 S_0^2 + a_8 S_0 S_1 + a_9 S_1^2) + \mu (b_7 S_0^2 + b_8 S_0 S_1 + b_9 S_1^2) \} T_1^2 + \\
 &\quad \{ \lambda (a_4 S_0^2 + a_5 S_0 S_1 + a_6 S_1^2) + \mu (b_4 S_0^2 + b_5 S_0 S_1 + b_6 S_1^2) \} T_1 T_0 + \\
 &\quad \{ \lambda (a_1 S_0^2 + a_2 S_0 S_1 + a_3 S_1^2) + \mu (b_1 S_0^2 + b_2 S_0 S_1 + b_3 S_1^2) \} T_0^2
 \end{aligned}$$

$$= \psi_1(\lambda, \mu, a, b, S_0, S_1) T_1^2 + \psi_2(\lambda, \mu, a, b, S_0, S_1) T_0 T_1 + \psi_3(\lambda, \mu, a, b, S_0, S_1) T_0^2.$$

Taking the discriminant  $D(F)$ , we write

$$D(F) = \psi_2^2 - 4\psi_1\psi_3 = \lambda^2 \phi_1(a, S_0, S_1) + \lambda\mu \phi_2(a, b, S_0, S_1) + \mu^2 \phi_3(b, S_0, S_1),$$

where  $\phi_1, \phi_2$  and  $\phi_3$  are homogenous polynomials of deg 2 with respect to  $(a, b)$ , and of deg 4 with respect to  $(S_0, S_1)$ .

Then  $\varphi$  is defined as follows:

$$\begin{array}{ccc} (S_0:S_1) \times (a, b) & \xrightarrow{\varphi} & (\phi_1 : \phi_2 : \phi_3) \times (a, b) \\ \uparrow \cap & \searrow & \uparrow \cap \\ \mathbb{P}^1 \times \mathbb{P}^{17} & & \mathbb{P}^2 \times \mathbb{P}^{17} \\ & \searrow & \swarrow \\ & (a, b) & \\ & \uparrow \cap & \\ & \mathbb{P}^{17} & \end{array}$$

where we identify the symmetric product  $\mathcal{G}$  of two projective lines with  $\mathbb{P}^2$ . If three polynomials  $\phi_1, \phi_2$  and  $\phi_3$  have common factors for a point  $u_0 = (a_0, b_0) \in \mathbb{P}^{17}$ , then  $\varphi$  is not a morphism above  $u_0$  ( $\varphi_{u_0}$  can be regarded as a morphism fibrewisely). This is the reason why we shrink  $U$  to  $V$  in (2.5).

For example if we put  $b_0 = (0, 0, 1/2, 1, 0, 1, 1, 0, 1/2)$  and  $a_0 = (0, 1, 0, 0, 1, 1, 0, 1, 1)$ , then  $\varphi$  is a morphism above  $u_0 = (a_0, b_0) \in \mathbb{P}^{17}$ . Moreover  $\varphi_{u_0}$  is described on  $U_0 := (S_0 \neq 0)$  as follows:

$$\begin{array}{ccc} \varphi_{u_0} : (s) & \longrightarrow & (s^4 - 2s^3 + (\text{lower terms}), -2s^3 + (\text{lower terms})), \\ || & & || \\ S_1/S_0 & & W_0/W_2 \quad \quad \quad W_1/W_2 \end{array}$$

where  $(W_0:W_1:W_2)$  is a homogenous coordinate of  $\mathbb{P}^2$ . Writing  $\alpha(s) = s^4 - 2s^3 + (\text{lower terms})$  and  $\beta(s) = -2s^3 + (\text{lower terms})$ , we have  $\mathbb{C}(s) = \mathbb{C}(\alpha(s), \beta(s))$ . Hence we have  $\deg \varphi_{u_0} = 1$ .

By (2.5) and (2.6) we have

Lemma(2.7) For a general triplet  $(S, \pi, \Sigma)$ ,  $\deg \varphi_{(S, \pi, \Sigma)} = 1$ .

Remark(2.8) Since  $S$  is a rational elliptic surface with sections,  $\pi$  is determined uniquely up to  $\text{Aut}(\mathbb{P}^1)$  for a fixed  $S$ . Thus the above property is independent of the choice of  $\pi$  if we fix  $S$  and  $\Sigma$ .

(2.9) Let  $(S, \Sigma, \pi)$  and  $(S^+, \Sigma^+, \lambda)$  be the same as above. We assume (2.1) and (2.2). We employ the following notation.

$$\begin{aligned} X &:= S \times_{\mathbb{P}^1} S^+ \\ \alpha: S &\longrightarrow \mathbb{P}^1 && \text{the morphism defined by } \Sigma \\ \beta: S^+ &\longrightarrow \mathbb{P}^1 && \text{the morphism defined by } \Sigma^+ \\ p: X &\longrightarrow S && \text{the first projection} \\ q: X &\longrightarrow S^+ && \text{the second projection} \\ \sigma &:= \alpha \circ p \\ \tau &:= \beta \circ q \\ \varphi &:= \varphi_{(S, \pi, \Sigma)} \\ \varphi^+ &:= \varphi_{(S^+, \lambda, \Sigma^+)} \end{aligned}$$

Remark that general fibres of  $\alpha$  and  $\beta$  are isomorphic to  $\mathbb{P}^1$ . Let  $F = \alpha^{-1}(x)$  (resp.  $F^+ = \beta^{-1}(y)$ ) be a general member of  $\Sigma$  (resp.  $\Sigma^+$ ). Then  $\sigma^{-1}(x) \xrightarrow{p} \alpha^{-1}(x) \simeq \mathbb{P}^1$  is a nonsingular elliptic K3 surface, and similarly for  $\tau^{-1}(y) \xrightarrow{q} \beta^{-1}(y) \simeq \mathbb{P}^1$ . Let  $\{P_1, P_2\} \in \mathbb{P}^1$  (resp.  $\{Q_1, Q_2\} \in \mathbb{P}^1$ ) be the ramification values with respect to  $\pi|_F: F \rightarrow \mathbb{P}^1$  (resp.  $\lambda|_{F^+}: F^+ \rightarrow \mathbb{P}^1$ ).

Proposition(2.10) Assume that 1)  $P_1 = Q_1$  and  $P_2 = Q_2$ , 2) (2.1) and (2.2) holds. Then  $D := F \times_{\mathbb{P}^1} F^+ \subset S \times_{\mathbb{P}^1} S^+ = X$  consists of two irreducible components  $D_1, D_2$  with the following properties:

- a)  $D_1$  and  $D_2$  are nonsingular rational curves with  $(D_1 \cdot D_2)_{\sigma^{-1}(x)} = 2$ ,  $(D_1 \cdot D_2)_{\tau^{-1}(y)} = 2$ . Moreover  $D_1$  intersects with  $D_2$  transversely.

b)  $N_{D_1/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  or  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ .

Moreover if we assume that  $\deg \varphi = 1$  or  $\deg \varphi^+ = 1$ , then  $D_i$  ( $1 \leq i \leq 2$ ) is an isolated rational curve.

*Proof.*

The proof of a) and b) is immediate from our assumption. Suppose that  $D_1$  moves in  $X$ . If  $N_{D_1/X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ , then  $D_1$  is always rigid, hence we may assume that  $N_{D_1/X} \simeq \mathcal{O} \oplus \mathcal{O}(-2)$ . Since  $D_1$  moves, the Hilbert scheme  $Hilb_X$  is smooth at  $[D_1]$ . Let  $u: \mathcal{H} \rightarrow H$  be the irreducible component of  $Hilb_X$  passing through  $[D_1]$  and the universal family over this component.  $\mathcal{H}$  is irreducible and there is a natural morphism from  $\mathcal{H}$  to  $X$ . We can regard its image as a Weil divisor on  $X$ . Since  $X$  is factorial by Remark preceding Definition (2.5), this is a Cartier divisor, which we denote by  $\mathcal{D}$ .

Then we have the following diagram:

$$\begin{array}{ccc}
 \mathcal{D} \subset X & & \\
 \sigma \times \tau \downarrow \Big|_{\mathcal{D}} & & \downarrow \sigma \times \tau \\
 G \subset \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\varphi \times \varphi^+} & \mathcal{G} \times \mathcal{G} \\
 i & & 
 \end{array}$$

, where  $G$  is the image of  $\mathcal{D}$  under  $\sigma \times \tau$  and  $\varphi$  (resp.  $\varphi^+$ ) is the ramification map with respect to  $(S, \pi, \Sigma)$  (resp.  $(S^+, \lambda, \Sigma^+)$ ) defined in §2. First,  $G$  is an irreducible curve because  $\mathcal{D}$  is the image of  $\mathcal{H}$  and  $D_1$  is contained in a fibre of  $\sigma \times \tau$ . Next, for a general point  $P \in G$ ,  $(\sigma \times \tau)^{-1}(P)$  is a singular fibre of type  $I_2$  (i.e. two nonsingular rational curves intersecting at two points transversely). This is shown as follows. A general fibre of  $\sigma$  (resp.  $\tau$ ) is an elliptic K3 surface.  $D_1$  deforms to a section  $s$  of a general fibre of  $\sigma$ , and  $s$  is, at the same time, a section of a general fibre

of  $\tau$ . Assume that  $s \in \sigma^{-1}(x')$ : general fibre and  $s \in \tau^{-1}(y')$ : general fibre. Then  $\alpha^{-1}(x') \xrightarrow{\pi} \mathbb{P}^1$  and  $\beta^{-1}(y') \xrightarrow{\lambda} \mathbb{P}^1$  are double covers of  $\mathbb{P}^1$ , respectively. Of course, both  $\alpha^{-1}(x')$  and  $\beta^{-1}(y')$  are nonsingular rational curves by definition. Let  $\{P_1', P_2'\} \in \mathbb{P}^1$  (resp.  $\{Q_1', Q_2'\}$ ) be the ramification value with respect to  $\pi$  (resp.  $\lambda$ ). If  $\{P_1', P_2'\}$  does not coincides with  $\{Q_1', Q_2'\}$ , then  $\alpha^{-1}(x') \times_{\mathbb{P}^1} \beta^{-1}(y') =: t$  is an irreducible curve on  $\sigma^{-1}(x')$  and  $p|_t: t \rightarrow \alpha^{-1}(x') \subset S$  is a double cover. Since  $t$  contains a section  $s$  of  $p|_{\sigma^{-1}(x')}: \sigma^{-1}(x') \rightarrow \alpha^{-1}(x')$ , this is a contradiction. Hence  $\{P_1', P_2'\}$  coincides with  $\{Q_1', Q_2'\}$ . In this case,  $t$  is a union of two sections  $s$  and  $s'$ .  $s$  and  $s'$  intersect at two points which corresponds  $\bigwedge_{\mathbb{P}^1}^{to} \{P_1' \times_{\mathbb{P}^1} Q_1', P_2' \times_{\mathbb{P}^1} Q_2'\}$ . Since  $t^2 = 0$ ,  $s^2 = -2$  and  $s'^2 = -2$  on the elliptic K3 surface  $\sigma^{-1}(x')$ ,  $s$  and  $s'$  intersect at these points transversely. Therefore, for a general point  $P \in G$ ,  $(\sigma \times \tau)^{-1}(P)$  is a singular fibre of type  $I_2$ . Furthermore, from the above observation, we conclude that  $(\varphi \times \varphi^+) \circ i$  factors through  $\Delta \subset \mathcal{G} \times \mathcal{G}$ .

Let us show that  $\mathcal{D}$  is a pull-back of a divisor on  $\mathbb{P}^1 \times \mathbb{P}^1$  by  $\sigma \times \tau$ . Let  $P$  be a general point of  $G$  and let us write  $(\sigma \times \tau)^{-1}(P) = C_1 + C_2$ . Then  $C_1 + C_2$  is contained in a smooth fibre  $F$  of  $\sigma$ , and  $\mathcal{D}|_F = aC_1 + bC_2 +$  (effective divisors on  $F$  disjoint from  $C_1$  and  $C_2$ ). Since (a general fibre of  $\sigma \times \tau \cdot \mathcal{D}) = 0$ , we have  $(C_1 + C_2 \cdot \mathcal{D}) = 0$ . Here we recall that  $X$  has two canonical projection  $p: X \rightarrow S$  and  $q: X \rightarrow S^+$ . Then  $p(C_i)$  (resp.  $q(C_i)$ ) coincides with a fibre of  $\alpha: S \rightarrow \mathbb{P}^1$  (resp.  $\beta: S^+ \rightarrow \mathbb{P}^1$ ) for  $i = 1, 2$ . On the other hand, by (2.4),  $\mathcal{O}(\mathcal{D}) \simeq p^*L \otimes q^*M$  for an effective divisor  $L$  on  $S$  and an effective divisor  $M$  on  $S^+$ , which implies that  $(\mathcal{D} \cdot C_1) \geq 0$ ,  $(\mathcal{D} \cdot C_2) \geq 0$  and consequently that  $(\mathcal{D} \cdot C_1) = 0$  and  $(\mathcal{D} \cdot C_2) = 0$ . Since  $C_i$ 's are  $(-2)$ -curves on the surface  $F$ , we have  $\mathcal{D}|_F = (aC_1 + aC_2; a \geq 0)$

+ (effective divisors disjoint from  $C_1$ 's). It is easy to see  $\mathcal{O}(\mathcal{D})|_{C_1 + C_2} \simeq \mathcal{O}_{C_1 + C_2}$ . Hence we have shown that except for finite fibres of  $\sigma \times \tau$ , the restriction of  $\mathcal{O}(\mathcal{D})$  to fibres of  $\sigma \times \tau$  are trivial. By the see-saw lemma,  $\mathcal{D}$  is a pull-back of a divisor by  $\sigma \times \tau$  in codimension one. Hence we have  $\mathcal{O}(\mathcal{D}) \simeq (\sigma \times \tau)^* \mathcal{O}(G)$ . Let us denote by  $p_1$  (resp.  $p_2$ ) the first projection (resp. the second projection) of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Write  $L = \mathcal{O}_S(F)$  and  $M = \mathcal{O}_S(F^+)$ . Then we have  $(\sigma \times \tau)^* p_1^* \mathcal{O}(1) = p^* L$  and  $(\sigma \times \tau)^* p_2^* \mathcal{O}(1) = q^* M$ . Thus  $\mathcal{O}(G) = p_1^* \mathcal{O}(m) \otimes p_2^* \mathcal{O}(n)$  and  $\mathcal{O}(\mathcal{D}) = p^* L^{\otimes m} \otimes q^* M^{\otimes n}$  for positive integers  $m$  and  $n$ . Since  $(\varphi \times \varphi^+) \circ i$  factors through  $\Delta \subset \mathcal{Y} \times \mathcal{Y}$  and  $\deg \varphi$  or  $\deg \varphi^+ = 1$ , we have  $m = 1$  or  $n = 1$ , and hence  $G \simeq \mathbb{P}^1$ .

We have the following commutative diagram:

$$\begin{array}{ccc}
 \tilde{\mathcal{D}} & \xrightarrow{\sim \text{normalization}} & \mathcal{D} \\
 \downarrow j & \searrow v & \downarrow \sigma \times \tau |_{\mathcal{D}} \\
 \tilde{G} & \xrightarrow{k} & G \simeq \mathbb{P}^1
 \end{array}$$

, where  $k \circ j$  is the Stein factorization of  $\tilde{\mathcal{D}} \rightarrow \mathbb{P}^1$ . Since a general fibre of  $\sigma \times \tau |_{\mathcal{D}}$  is of type  $I_2$ , a general fibre of  $j$  is isomorphic to  $\mathbb{P}^1$ . Hence we have  $\deg k = 2$ . Since every fibre of  $\sigma \times \tau$  is reduced, every fibre of  $\sigma \times \tau |_{\mathcal{D}}$  is also reduced. Let  $f$  be an arbitrary fibre of  $\sigma \times \tau |_{\mathcal{D}}$ . Then a general point  $P \in f$  is a smooth point of  $f$  because  $f$  is reduced. Since  $\sigma \times \tau |_{\mathcal{D}} : \mathcal{D} \rightarrow \mathbb{P}^1$  is a flat morphism, we conclude that  $\mathcal{D}$  is smooth at  $P$ , which implies that the normalization of  $\mathcal{D}$  makes effect only on finite points of every fibre. By Hurwitz formula,  $k$  is ramified over some points because  $G \simeq \mathbb{P}^1$ . Let  $P \in \mathbb{P}^1$  be one of these points. Then  $(k \circ j)^{-1}(P)$  is a multiple divisor on  $\tilde{\mathcal{D}}$ . On the other hand  $(k \circ j)^{-1}(P) = v^{-1} \circ (\sigma \times \tau |_{\mathcal{D}})^{-1}(P)$  and the right side is not a multiple divisor by the above

observation, which is a contradiction. Therefore,  $D_1$  is isolated under the assumption of the proposition. Q.E.D.

**Definition(2.11)** Let  $C$  be an nonsingular rational curve on a 3-fold  $X$ . Assume that  $X$  is smooth near  $C$  and that  $N_{C/X} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ . Blowing up  $X$  along  $C$ , we can produce a  $\mathbb{P}^1 \times \mathbb{P}^1$  as a exceptional divisor. Blowing down it to other directions, we have a threefold bimeromorphic to  $X$ . This process is called a *flop*. For the definition of a flop in more general situations, see [4],[8].

**Definition(2.12)** The Weierstrass model  $W(K_T, a, b)$  over a surface  $T$  is the divisor on  $\mathbb{P}_T(\mathcal{O}_T \oplus K_T^{\otimes 2} \oplus K_T^{\otimes 3})$  defined by the equation  $Y^2Z - (X^3 + aXZ^2 + bZ^3)$ , where  $X, Y$  and  $Z$  are natural injections

$$\begin{aligned} K_T^{\otimes 2} &\longrightarrow \mathcal{O}_T \oplus K_T^{\otimes 2} \oplus K_T^{\otimes 3} \\ K_T^{\otimes 3} &\longrightarrow \mathcal{O}_T \oplus K_T^{\otimes 2} \oplus K_T^{\otimes 3} \\ \mathcal{O}_T &\longrightarrow \mathcal{O}_T \oplus K_T^{\otimes 2} \oplus K_T^{\otimes 3} \end{aligned}$$

, respectively, and where  $a$  and  $b$  are sections of  $K_T^{\otimes -4}$  and  $K_T^{\otimes -6}$ , respectively. Moreover we assume that  $4a^3 + 27b^2$  is not identically zero. We shall denote  $W(K_T, a, b)$  simply by  $W$ .

### § 3. Rational curves

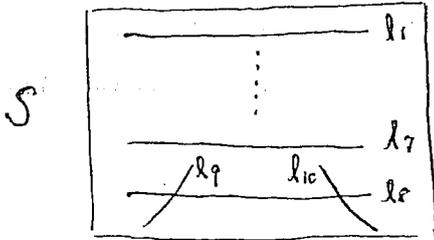
1. In this paragraph, we shall define the birational map  $\phi$  between  $X = S \times_{\mathbb{P}^1} S^+$  and a singular Weierstrass model  $\mu: W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ .

Let  $\pi: S \longrightarrow \mathbb{P}^1$  be a rational elliptic surface with sections. We assume that all singular fibres are irreducible. Then  $S$  has the representation as a nine points blow-up of  $\mathbb{P}^2$ . Thus there are (-1)-curves  $\ell_i$  ( $1 \leq i \leq 10$ ) with the following properties:

(1)  $\ell_i$ 's ( $i \neq 8$ ) are exceptional curves of the nine points blow-up of  $\mathbb{P}^2$ .

(2)  $(\ell_8 \cdot \ell_9) = (\ell_{10} \cdot \ell_8) = 1$ .  $(\ell_i \cdot \ell_8) = 0$  for other  $i$ .

The blow-down of  $\ell_1, \dots, \ell_8$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ .



Let  $\lambda: S^+ \longrightarrow \mathbb{P}^1$  be a rational elliptic surface with sections. We assume that all singular fibres are irreducible. Pick up one section  $m_1 \subset S^+$  of  $\lambda$  (which is a (-1)-curve on  $S^+$ ). The birational map  $\phi_{\ell, m_1}$  (which depends on  $\ell = (\ell_i)$  and  $m_1$ ) is defined as follows.

Let  $\gamma$  be the flop of the curves  $\ell_i \times_{\mathbb{P}^1} m_1$  ( $1 \leq i \leq 8$ ). Then we have the projective threefold  $X^+$  birational to  $X = S \times_{\mathbb{P}^1} S^+$ . Let  $E_i$  ( $1 \leq i \leq 8$ ) be the strict transform of  $\ell_i \times_{\mathbb{P}^1} S^+$  ( $1 \leq i \leq 8$ ) by  $\gamma$ . These  $E_i$ 's can be contracted to eight points and we obtain a singular Weierstrass model  $\mu: W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . We denote by  $\phi_{(\ell, m_1)}$  this birational map between  $X$  and  $W$ . The strict transform of  $S \times_{\mathbb{P}^1} m_1$  by  $\phi_{(\ell, m_1)}$  is the canonical section of  $\mu$ .

2. We shall explain the basic situation. Let  $\pi: S \longrightarrow \mathbb{P}^1$  and  $\lambda: S^+ \longrightarrow \mathbb{P}^1$  be rational elliptic surfaces with sections. We assume that every singular fibre of  $\pi$  and  $\lambda$  is irreducible. We denote by  $\ell$  and  $\ell'$  (resp.  $m$  and  $m'$ ) two  $(-1)$ -curves on  $S$  ( resp.  $S^+$  ) for which  $(\ell, \ell') = 1$  ( resp.  $(m, m') = 1$  ). Assume that

(3.1) generic fibre of  $\pi$  and  $\lambda$  are not isogenous, and

(3.2)  $\pi(P) = \lambda(Q)$ , where  $P = \ell \cap \ell'$  and  $Q = m \cap m'$ .

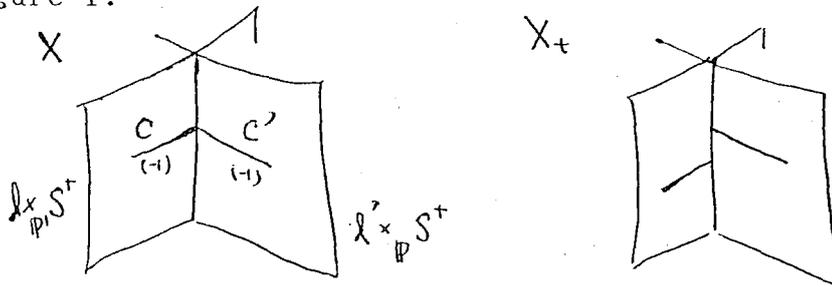
Let  $\Delta$  be a small disk around  $0 \in \mathbb{C}$ . Let  $f: \mathcal{G} \longrightarrow \Delta \times \mathbb{P}^1$  and  $g: \mathcal{G}^+ \longrightarrow \Delta \times \mathbb{P}^1$  be the deformation of  $\pi: S \longrightarrow \mathbb{P}^1$  and  $\lambda: S^+ \longrightarrow \mathbb{P}^1$ , respectively.

$$\begin{array}{ccc}
 \mathcal{G} & \longleftarrow & \mathcal{G}_0 = S \\
 f \downarrow & & f_0 \downarrow \\
 \Delta \times \mathbb{P}^1 & \longleftarrow & \{0\} \times \mathbb{P}^1
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{G}^+ & \longleftarrow & \mathcal{G}_0^+ = S^+ \\
 g \downarrow & & g_0 \downarrow \\
 \Delta \times \mathbb{P}^1 & \longleftarrow & \{0\} \times \mathbb{P}^1
 \end{array}$$

Since  $\ell$  and  $\ell'$  ( resp.  $m$  and  $m'$  ) are stable submanifolds in  $S$  ( resp.  $S^+$  ), they are deformed to  $(-1)$ -curves  $\ell_t$  and  $\ell'_t$  on  $\mathcal{G}_t$  ( resp.  $m_t$  and  $m'_t$  on  $\mathcal{G}_t^+$  );  $t \in \Delta - \{0\}$ . Here we impose the following general condition:

(3.3)  $f_t(P_t) \neq g_t(Q_t)$  for  $t \in \Delta - \{0\}$ , where  $P_t := \ell_t \cap \ell'_t$  and  $Q_t := m_t \cap m'_t$ .

Figure 1.



Let us employ the following notation:

$$C := \ell \times_{\mathbb{P}^1} m \subset X = S \times_{\mathbb{P}^1} S^+$$

$$C' := \ell' \times_{\mathbb{P}^1} m' \subset X$$

$C \cup C' :=$  the reduced subscheme of  $X$  whose support is  $C \cup C'$

For  $t \in \Delta - \{0\}$ ,

$$X_t := \mathbb{G}_t \times_{(\{t\} \times \mathbb{P}^1)} \mathbb{G}_t^+$$

$X_t^+ :=$  the threefold obtained from  $X_t$  by the flop  $\gamma_t$  of  $\ell_t' \times_{\mathbb{P}^1} m_t'$  on  $X_t$

$X^+ :=$  the threefold obtained from  $X$  by the flop  $\gamma$  of  $C' \subset X$

$D :=$  the strict transform of  $C$  by  $\gamma$

$E :=$  the strict transform of  $\ell \times_{\mathbb{P}^1} S^+$  by  $\gamma$

For  $t \in \Delta$ ,

$\alpha_t: \mathbb{G}_t \longrightarrow \mathbb{P}^1$ , the morphism defined by the linear system

$$|\ell_t^+ \ell_t'|$$

$\beta_t: \mathbb{G}_t^+ \longrightarrow \mathbb{P}^1$ , the morphism defined by the linear system

$$|m_t^+ m_t'|$$

$p_t: X_t \longrightarrow \mathbb{G}_t$ , the first projection

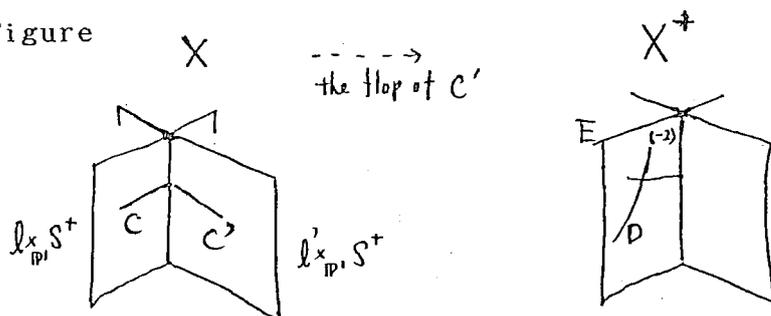
$q_t: X_t \longrightarrow \mathbb{G}_t^+$ , the second projection

$$\sigma_t := \alpha_t \circ p_t$$

$$\tau_t := \beta_t \circ q_t$$

*Remark.* General fibres of  $\alpha_t$  and  $\beta_t$  are isomorphic to  $\mathbb{P}^1$ , and general fibres of  $\sigma_t$  and  $\tau_t$  are elliptic  $K=3$  surfaces.

Figure



**Lemma(3.4)** *The following implication holds.*

- a)  $C \cup C'$  is isolated in  $X$ .
- $\rightarrow$  b)  $D \subset X^+$  is an isolated  $(-2)$ -curve (i.e.  $N_{D/X^+} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$  or  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ )

→c)  $D$  can be deformed to an isolated  $(-2)$ -curve  $D_t$  on  $X_t^+$  ( $t \in \Delta - \{0\}$ )  
 Furthermore  $\gamma_t^{-1}$  is an isomorphism in a neighbourhood of  $D_t$ .

*Proof.*

a) → b) Suppose that  $D \subset X^+$  moves. If  $N_{D/X^+} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ , then  $D$  is always isolated, which implies that  $N_{D/X^+} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$ . Let  $H$  be the germ of  $\text{Hilb}_{X^+/\mathbb{C}}$  at  $[D]$ . Then, since  $D$  moves in  $X^+$ ,  $\dim H = 1$  and  $H$  is isomorphic to  $(\mathbb{C}^1, 0)$ . Denoting by  $\mathcal{H}$  the universal family on  $H$ , we obtain the following diagram:

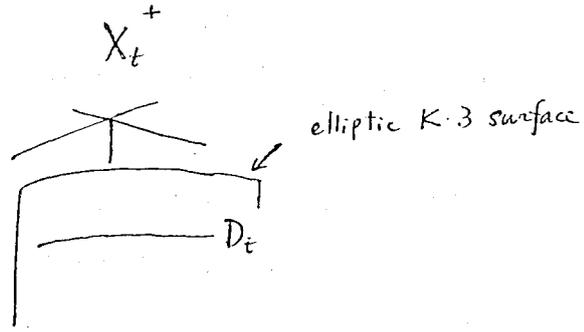
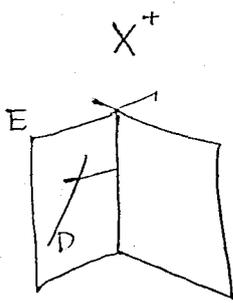
$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & X^+ \times H \xrightarrow{p_1} X^+ \\ & & p_2 \downarrow \\ & & H \end{array}$$

In the above diagram  $p_1|_{\mathcal{H}} : \mathcal{H} \longrightarrow X^+$  is unramified at each point on  $p_2^{-1}([D])$  because  $N_{D/X^+} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$ . Hence  $p_1(\mathcal{H})$  is smooth in a neighbourhood of  $D$ . Since  $h^0(N_{D/E}) = 0$ ,  $p_1(\mathcal{H})$  intersects with  $E$  transversely, and  $p_1(\mathcal{H})|_E = D$ . Indeed if  $p_1(\mathcal{H})|_E = nD$  for  $n \geq 2$ , then we have a non-trivial extension of  $D$  to the second order. This contradicts the fact that  $h^0(N_{D/E}) = 0$ . Moreover  $p_1(\mathcal{H})$  has the  $D$  as a ruling. Let  $\mathcal{B}$  be the strict transform of  $p_1(\mathcal{H})$  by  $\gamma^{-1}$ . Then  $\mathcal{B}$  contains  $C$  and  $C'$  as  $(-1)$ -curves, which implies that  $C \cup C'$  moves in  $X$ . This contradicts the condition a).

b) → c) The first part of c) follows from §2 of [2]. The latter part of c) follows from the assumption (3.3). Q.E.D.

*Remark.* In c),  $D_t$  can be regarded as an isolated  $(-2)$ -curve on  $X_t$  because  $\gamma_t^{-1}$  is an isomorphism near  $D_t$ . Hence by Lemma(3.4) if a) is satisfied, then we can find an isolated  $(-2)$ -curve on  $X_t$ .

Figure



We shall denote by  $\Sigma$  (resp.  $\Sigma^+$ ) the linear system on  $S$  (resp.  $S^+$ ) defined by  $l + l'$  (resp.  $m + m'$ ). Then the argument in (2.5) is applied to the triplet  $(S, \pi, \Sigma)$  and  $(S^+, \lambda, \Sigma^+)$ .

**Proposition(3.5)** *Let  $S$  and  $S^+$  be as above. Suppose that the triplet  $(S, \pi, \Sigma)$  is generally chosen such that Lemma(2.7) holds. Then  $C \cup C'$  is isolated in  $X$ .*

*Proof.*

Suppose that  $C \cup C'$  moves in  $X$ . Let us denote by  $N$ , the normal bundle  $N_{C \cup C' / X}$  and write  $P = C \cap C'$ ,  $j: C \rightarrow C \cup C'$  and  $j': C' \rightarrow C \cup C'$ . Then we have

$$0 \rightarrow N \rightarrow j_* (N|_C) \oplus j'_* (N|_{C'}) \xrightarrow{\Psi} \mathbb{C}(P)^{\oplus 2} \rightarrow 0$$

, where  $N|_C \simeq \mathcal{O}(-1) \oplus \mathcal{O}$  and  $N|_{C'} \simeq \mathcal{O}(-1) \oplus \mathcal{O}$ . Since  $\Phi: j_* (N|_C) \otimes \mathbb{C}(P) \rightarrow j_* (N|_C) \otimes \mathbb{C}(P) \oplus j'_* (N|_{C'}) \otimes \mathbb{C}(P) \xrightarrow{\Psi} \mathbb{C}(P)^{\oplus 2}$  and  $\Phi': j'_* (N|_{C'}) \otimes \mathbb{C}(P) \rightarrow \mathbb{C}(P)^{\oplus 2}$  are both isomorphisms, we have  $h^0(N) \leq 1$ . Since  $C \cup C'$  moves in  $X$ ,  $Hilb_{X/\mathbb{C}}$  is smooth of dim 1 at  $[C \cup C']$ . We note here that  $X$  has two fibrations  $\sigma_0$  and  $\tau_0$  whose general fibres are elliptic K3 surfaces. (See the notation above.)  $C \cup C'$  is contained in singular fibres of  $\sigma_0$  and  $\tau_0$ , respectively.

Since  $C \cup C'$  does not move in these singular fibre,  $C \cup C'$  is deformed to a section of a general fibre (an elliptic K3 surface) of  $\sigma_0$  (resp.  $\tau_0$ ). Let us denote by  $\alpha: \mathcal{H} \rightarrow H$ , the irreducible component of  $Hilb_{X/\mathbb{C}}$  passing through  $[C \cup C']$  and the universal family over this component. Then a general fibre of  $\alpha$  is isomorphic

to  $\mathbb{P}^1$ , which implies that  $\mathcal{H}$  is irreducible. There is a natural morphism from  $\mathcal{H}$  to  $X$ , and we can regard its image as a Weil divisor on  $X$ . Since  $X$  is factorial by Remark in §2, this is a Cartier divisor, which we denote by  $\mathcal{D}$ . Then we have the following diagram:

$$\begin{array}{ccc}
 \mathcal{D} \subset X & & \\
 \downarrow \sigma_0 \times \tau_0|_{\mathcal{D}} & & \downarrow \sigma_0 \times \tau_0 \\
 G \subset \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\varphi \times \varphi^+} & \mathcal{G} \times \mathcal{G} \\
 i & & 
 \end{array}$$

, where  $G$  is the image of  $\mathcal{D}$  under  $\sigma_0 \times \tau_0$  and  $\varphi$  (resp.  $\varphi^+$ ) is the ramification map with respect to  $(S, \pi, \Sigma)$  (resp.  $(S^+, \lambda, \Sigma^+)$ ) defined in §2. First,  $G$  is an irreducible curve because  $\mathcal{D}$  is the image of  $\mathcal{H}$  and  $C \cup C'$  is contained in a fibre of  $\sigma_0 \times \tau_0$ . Next, for a general point  $P \in G$ ,  $(\sigma_0 \times \tau_0)^{-1}(P)$  is a singular fibre of type  $I_2$  (i.e. two nonsingular rational curves intersecting at two points transversely). This is shown as follows. A general fibre of  $\sigma_0$  (resp.  $\tau_0$ ) is an elliptic K3 surface.  $C \cup C'$  deforms to a section  $s$  of a general fibre of  $\sigma_0$ , and  $s$  is, at the same time, a section of a general fibre of  $\tau_0$ . Hence we have the situation in the proof of Proposition(2.10), and consequently for a general point  $P \in G$ ,  $(\sigma_0 \times \tau_0)^{-1}(P)$  is a singular fibre of type  $I_2$ . From this, we can use the same argument of the proof of Proposition(2.10), which deduces the contradiction. Therefore,  $C \cup C'$  is isolated in  $X$ . Q.E.D.

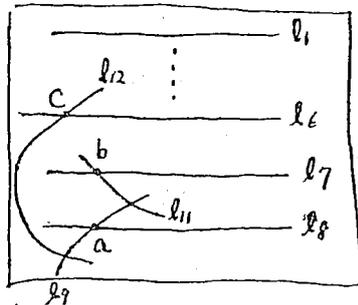
3. In the following we shall find four mutually disjoint, isolated  $(-2)$ -curves  $C_i$  ( $1 \leq i \leq 4$ ) on a singular Weierstrass model  $\mu: W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . These curves will have the following intersection numbers:

$$(3.6) \quad \begin{aligned} (\Sigma, C_1) &= 0 & (\mu^* p_1^* \mathcal{O}(1), C_1) &= 1 & (\mu^* p_2^* \mathcal{O}(1), C_1) &= 0 \\ (\Sigma, C_2) &= 0 & (\mu^* p_1^* \mathcal{O}(1), C_2) &= 0 & (\mu^* p_2^* \mathcal{O}(1), C_2) &= 1 \\ (\Sigma, C_3) &= 1 & (\mu^* p_1^* \mathcal{O}(1), C_3) &= 0 & (\mu^* p_2^* \mathcal{O}(1), C_3) &= 1 \\ (\Sigma, C_4) &= 1 & (\mu^* p_1^* \mathcal{O}(1), C_4) &= 1 & (\mu^* p_2^* \mathcal{O}(1), C_4) &= 0, \end{aligned}$$

where  $p_1$  (resp.  $p_2$ ) is the first (resp. second) projection of  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $\Sigma$  is the canonical section of  $\Sigma$ .

Let  $\pi: S \rightarrow \mathbb{P}^1$  be a rational elliptic surface with sections whose singular fibres are irreducible.  $S$  is a nine points blow-up of  $\mathbb{P}^2$  and a eight points blow-up of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let  $l_i$  ( $1 \leq i \leq 10$ ) be the same as in 1. of this section, and let  $l_i$  ( $i = 11, 12$ ) be the  $(-1)$ -curves on  $S$  for which  $(l_{11} \cdot l_9) = (l_{12} \cdot l_9) = (l_{11} \cdot l_7) = (l_{12} \cdot l_6) = 1$ ,  $(l_{11} \cdot l_j) = 0$  ( $j \neq 7, 9$ ) and  $(l_{12} \cdot l_j) = 0$  ( $j \neq 6, 9$ ). We write  $a = l_8 \cap l_9$ ,  $b = l_7 \cap l_{11}$  and  $c = l_6 \cap l_{12}$ . It is easy to see the existence of these curves.

Figure



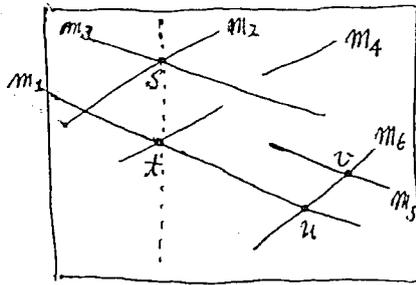
Let  $\lambda: S^+ \rightarrow \mathbb{P}^1$  be a rational surface with sections whose singular curves are irreducible. Suppose that  $m_i$  ( $1 \leq i \leq 6$ ) be the  $(-1)$ -curves on  $S^+$  for which

$$(3.7) \quad \begin{aligned} (m_i \cdot m_j) &= 0 \text{ if } (i, j) = (\text{odd}, \text{odd}) \text{ or } (\text{even}, \text{even}) \\ &\quad \text{with } i \neq j \\ (m_1 \cdot m_2) &= (m_1 \cdot m_4) = (m_1 \cdot m_6) = 1 \\ (m_3 \cdot m_2) &= 1 \quad (m_3 \cdot m_4) = (m_3 \cdot m_6) = 0 \\ (m_5 \cdot m_2) &= (m_5 \cdot m_4) = 0 \quad (m_5 \cdot m_6) = 1. \end{aligned}$$

Writing  $s = m_2 \cap m_3$ ,  $t = m_1 \cap m_4$ ,  $u = m_1 \cap m_6$  and  $v = m_5 \cap m_6$  we suppose one more condition:

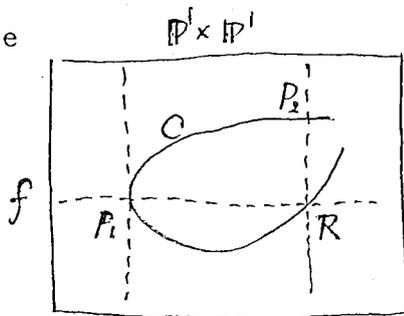
$$(3.8) \quad \lambda(s) = \lambda(t).$$

Figure



In order to construct the  $S^+$  satisfying (3.6) and (3.7), we must choose suitable eight points on  $\mathbb{P}^1 \times \mathbb{P}^1$  (which are, of course, the base points of certain linear pencil on  $\mathbb{P}^1 \times \mathbb{P}^1$ ) and blow up these points. Let us look at the following figure.

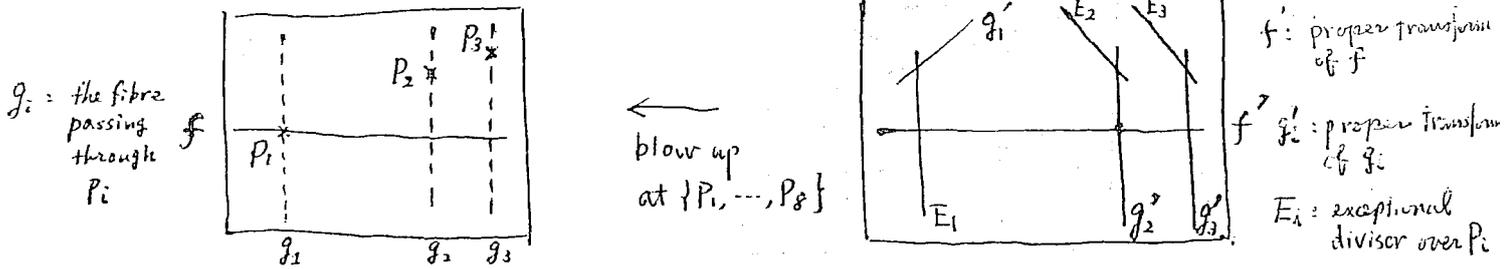
Figure



Here  $C$  is a nonsingular elliptic curve which is a divisor of  $(2,2)$  type in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let us denote by  $p_1$  (resp.  $p_2$ ) the first (resp. second) projection of  $\mathbb{P}^1 \times \mathbb{P}^1$ .  $C$  is a double cover of  $\mathbb{P}^1$  with respect to  $p_1$  and  $p_2$ . Let  $P_1$  be a ramification point of the double cover  $p_1|_C$ . Let  $f$  be the fibre of  $p_2$  passing through  $P_1$  and  $R$  a intersection of  $f$  and  $C$ . If we choose  $C$  generally, then  $C$  intersects with the fibre  $g$  of  $p_1$  passing through  $R$ , transversely. Write  $C \cap g = \{R, P_2\}$ . Take five points  $\{P_3, \dots, P_7\}$  on  $C$  disjoint from  $\{P_1, P_2, R\}$ . Then the divisor of  $(2,2)$  type passing through  $\{P_1, \dots, P_7\}$  forms a linear pencil  $\Sigma$ . Let  $\{C, C'\}$  be a basis of  $\Sigma$ . Then  $C \cap C' = \{P_1, \dots, P_7, P_8\}$ , where  $P_8$  coincides with none of  $\{P_1, \dots, P_7\}$ . Blowing up  $\{P_1, \dots, P_8\}$ , we have an elliptic fibration  $\lambda$  with sections. If we impose certain general conditions, then the

elliptic fibration obtained has only irreducible fibres.

Figure  $\mathbb{P}^1 \times \mathbb{P}^1$



Then we may set  $m_1 = f'$ ,  $m_2 = E_1$ ,  $m_3 = g'_1$ ,  $m_4 = g'_2$ ,  $m_5 = g'_3$  and  $m_6 = E_3$ . We assume the following condition (3.1), (3.2\*).

(3.1) Generic fibres of  $\pi$  and  $\lambda$  are not isogenous.

(3.2\*)  $\pi(a) = \lambda(s) (= \lambda(t))$ ,  $\pi(b) = \lambda(u)$  and  $\pi(c) = \lambda(v)$ .

Remark. By replacing  $\pi$  by  $\sigma \circ \pi$  for a suitable  $\sigma \in \text{Aut}(\mathbb{P}^1)$ , (3.2\*) is always satisfied.

Let  $f: \mathcal{G} \rightarrow \Delta \times \mathbb{P}^1$  and  $g: \mathcal{G}^+ \rightarrow \Delta \times \mathbb{P}^1$  be the deformations of  $\pi: S \rightarrow \mathbb{P}^1$  and  $\lambda: S^+ \rightarrow \mathbb{P}^1$ , respectively, for which (3.3) holds.

Then Lemma(3.4) and Proposition(3.5) is applied for each

$$\begin{aligned} D_1 &= (9, 2) \cup (8, 3) \\ D_2 &= (9, 4) \cup (8, 1) \\ D_3 &= (11, 6) \cup (7, 1) \\ D_4 &= (12, 6) \cup (6, 5), \end{aligned}$$

where  $(i, j)$  means  $\ell_i \times_{\mathbb{P}^1} m_j$ . Since  $D_i (1 \leq i \leq 4)$  are disjoint from each other, we have mutually disjoint, isolated  $(-2)$ -curves

$D_{i,t} (1 \leq i \leq 4)$  on  $X_t (t \neq 0)$ . As is explained in (§3, 1), there is

a birational map  $\phi_{(\ell_t, m_{1t})}$  between  $X_t$  and a singular Weierstrass

model  $W_t$  over  $\mathbb{P}^1 \times \mathbb{P}^1$ . But in a neighbourhood of each  $D_{i,t} (1 \leq i \leq 4)$

$\phi_{(\ell_t, m_{1t})}$  is an isomorphism. Therefore we can find mutually

disjoint  $(-2)$ -curves on  $W$ ;  $C_i = \phi_{(\ell_t, m_{1t})}(D_{i,t}) (1 \leq i \leq 4)$ . By the

construction we have (3.6).  $W$  can be deformed to a nonsingular

Weierstrass model over  $\mathbb{P}^1 \times \mathbb{P}^1$ , which we call  $W$  by abuse of language. Then the four curves constructed above is also deformed to isolated  $(-2)$ -curves on  $W$  (one curve may split up into disjoint several curves.). Hence we can find mutually disjoint, isolated  $(-2)$ -curves  $C_i$  ( $1 \leq i \leq 4$ ) on a nonsingular Weierstrass model  $W$ , and these curves satisfy (3.6).

#### §4. Deformation of rational curves

Let  $W$  be a nonsingular Weierstrass model over  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let  $C$  denote one of the isolated  $(-2)$ -curves constructed above. If  $N_{C/W} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$ , then we must deform  $(C, W)$  so that  $C$  splits into a disjoint union of  $(-1, -1)$ -curves to use Friedman's argument. In this section we observe the deformation of  $(C, W)$ .

**Lemma(4.1).** *Let  $W$  be a nonsingular Weierstrass model over  $\mathbb{P}^1 \times \mathbb{P}^1$ . Then  $W$  is unobstructed, that is, the moduli number  $m(W)$  of  $W$  can be defined and  $m(W) = h^1(W, \Theta)$ .*

*Proof.*

By the definition of a Weierstrass model,  $W$  is embedded in

$\mathbb{P} = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus K_{\mathbb{P}^1 \times \mathbb{P}^1}^{\otimes 2} \oplus K_{\mathbb{P}^1 \times \mathbb{P}^1}^{\otimes 3})$ . We shall denote  $K_{\mathbb{P}^1 \times \mathbb{P}^1}$  simply by  $K$ .

Let  $p$  denote the projection of  $\mathbb{P}$  to  $\mathbb{P}^1 \times \mathbb{P}^1$ . Taking cohomologies of the exact sequence

$$0 \longrightarrow \Theta_W \longrightarrow \Theta_{\mathbb{P}}|_W \longrightarrow \Theta_W(W) \longrightarrow 0,$$

we have the exact sequence

$$0 \longrightarrow H^0(W, \Theta_{\mathbb{P}}|_W) \xrightarrow{\psi} H^0(W, \Theta_W(W)) \xrightarrow{\delta} H^1(W, \Theta_W).$$

The injectivity of  $\psi$  follows from the fact that  $h^0(W, \Theta_W) = h^0(W, \Omega_W^2) = h^2(W, \Theta_W) = 0$ . First we shall show that  $\delta$  is surjective. Since

the Euler number  $e(W)$  of  $W$  is equal to  $-60c_1^2(\mathbb{P}^1 \times \mathbb{P}^1) = -480$  by [6, Th(3.10)] and  $h^{1,1}(W) = 3$ , we obtain  $h^1(W, \theta_W) = 243$ . Since  $\mathcal{O}_{\mathbb{P}}(W) = \mathcal{O}_{\mathbb{P}}(3) \otimes p^*(-6K)$ , a direct calculation using the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(W) \longrightarrow \mathcal{O}_W(W) \longrightarrow 0$$

shows that  $h^0(W, \mathcal{O}_W(W)) = 334$ . Consider the following exact sequences

$$(4.2) \quad 0 \longrightarrow \theta_{\mathbb{P}/\mathbb{P}^1 \times \mathbb{P}^1}|_W \longrightarrow \theta_{\mathbb{P}}|_W \longrightarrow p^*\theta_{\mathbb{P}^1 \times \mathbb{P}^1}|_W \longrightarrow 0$$

$$(4.3) \quad 0 \longrightarrow \mathcal{O}_W \longrightarrow p^*\varepsilon^{-1} \otimes \mathcal{O}_W(1) \longrightarrow \theta_{\mathbb{P}/\mathbb{P}^1 \times \mathbb{P}^1}|_W \longrightarrow 0$$

, where  $\varepsilon = \mathcal{O} \oplus K^{\otimes 2} \oplus K^{\otimes 3}$ . By (4.2) we have  $h^0(\theta_{\mathbb{P}}|_W) \leq h^0(\theta_{\mathbb{P}/\mathbb{P}^1 \times \mathbb{P}^1}|_W) + h^0(p^*\theta_{\mathbb{P}^1 \times \mathbb{P}^1}|_W)$ . On the other hand,  $h^0(\theta_{\mathbb{P}/\mathbb{P}^1 \times \mathbb{P}^1}|_W) = h^0(p^*\varepsilon^{-1} \otimes \mathcal{O}_W(1)) - 1$  by (4.3). An easy computation shows that  $h^0(\theta_{\mathbb{P}}|_W) \leq 91$ . As a consequence,  $\delta$  is surjective.

Next we shall show that  $h^1(\mathcal{O}_W(W)) = 0$ . Noting that  $\mathcal{O}_W(W) = \mathcal{O}_W(W) = \mathcal{O}_W(3) \otimes p^*(-6K)$  and applying Leray spectral sequence, we have

$$\begin{aligned} 0 &\longrightarrow H^1(p_*\mathcal{O}_W(3) \otimes \mathcal{O}(-6K)) \longrightarrow H^1(\mathcal{O}_W(W)) \\ &\longrightarrow H^0(R^1p_*\mathcal{O}_W(3) \otimes \mathcal{O}(-6K)). \end{aligned}$$

But the first term and the last term of the sequence are shown to be zero. Thus we conclude that  $H^1(\mathcal{O}_W(W)) = 0$ . Since  $H^1(\mathcal{O}_W(W)) = H^1(N_{W/\mathbb{P}}) = 0$ , the Hilbert scheme  $Hilb_{\mathbb{P}/k}$  is smooth at  $[W]$ , where  $[W]$  is the point on  $Hilb_{\mathbb{P}/k}$  which corresponds to  $W$ . This implies that the universal family over  $Hilb_{\mathbb{P}/k}$  is complete near  $[W]$  by [5] because  $\delta$  is surjective. Therefore, we conclude that  $m(W) = h^1(W, \theta_W)$ . Q.E.D.

Let  $\mu: W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  be a general nonsingular Weierstrass model

over  $\mathbb{P}^1 \times \mathbb{P}^1$ , and let  $C$  be the same as above. Then  $\mu(C)$  is a ruling  $f$  of  $\mathbb{P}^1 \times \mathbb{P}^1$ . If we write  $H = \mu^{-1}(f)$ , then  $C \subset H$ . In case  $N_{C/W} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$ , we shall use

**Proposition(4.4).** *Let  $\mu: W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  as above, and  $\Sigma$  a canonical section of  $\mu$ . Suppose that*

- 1)  $C$  is a contractible rational curve with  $N_{C/W} = \mathcal{O}(-2) \oplus \mathcal{O}$ ,
- 2)  $C \subset H = \mu^{-1}(f)$ , where  $f$  is a ruling of  $\mathbb{P}^1 \times \mathbb{P}^1$ , and  $C$  is a section of  $\mu|_H \longrightarrow f$ .
- 3)  $\langle C, \Sigma \rangle = 0$  or  $1$ .

Then  $W$  can be deformed so that  $C$  splits up into a disjoint union of rational curves with normal bundles  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ .

*Proof.*

According to [2, §4(c) Cor.(4.11)], it suffices to show that the natural map  $H^1(\theta_W) \longrightarrow H^1(\theta_{W|C})$  is surjective. If this is shown, then we can apply the same argument of [2, §4(c)(4.11)] with the aid of Lemma(4.1) to prove the existence of a desired deformation of  $W$ .

With the same notation of Lemma(4.1), we have

$$0 \longrightarrow \theta_W \longrightarrow \theta_{\mathbb{P}|W} \longrightarrow \theta_W(W) \longrightarrow 0.$$

Let  $I_C$  denote the defining ideal of  $C$  in  $W$ . Then from the above sequence we obtain

$$(4.5) \quad \begin{array}{ccccc} & & & & H^1(\theta_{\mathbb{P}|C}) \\ & & & & \downarrow \\ & & & & H^2(\theta_{\mathbb{P}|W} \otimes I_C) \\ & & H^2(\theta_W \otimes I_C) & \xrightarrow{\beta} & \\ & & \downarrow \alpha & & \downarrow \\ H^1(\theta_W(W)) & \longrightarrow & H^2(\theta_W) & \longrightarrow & H^2(\theta_{\mathbb{P}|W}) \\ & & & & \downarrow \\ & & & & H^2(\theta_{\mathbb{P}|C}) = 0 \end{array}$$

We have shown in the proof of Lemma(4.1) that  $H^1(\theta_W(W)) = 0$ . Since

$\dim C = 1$ ;  $H^2(\mathcal{O}_P|_C) = 0$ . We shall show that  $H^1(\mathcal{O}_P|_C) = 0$ . By Serre duality,  $h^1(\mathcal{O}_P|_C) = h^0(\Omega_P^1 \otimes \mathcal{O}_{P^1}(-2))$ , where we identify  $C$  with  $P^1$ . Consider the following exact sequences:

$$\begin{aligned} 0 &\longrightarrow p^*\Omega_{P^1 \times P^1}^1 \longrightarrow \Omega_P^1 \longrightarrow \Omega_{P/P^1 \times P^1}^1 \longrightarrow 0 \\ 0 &\longrightarrow \Omega_{P/P^1 \times P^1}^1 \longrightarrow p^*\varepsilon \otimes \mathcal{O}_P(-1) \longrightarrow \mathcal{O}_P \longrightarrow 0, \end{aligned}$$

where  $\varepsilon = \mathcal{O} \oplus K^{\otimes 2} \oplus K^{\otimes 3}$  and  $p$  is the projection of  $P$  to  $P^1 \times P^1$ .

From the first sequence, we obtain

$$\begin{aligned} 0 &\longrightarrow H^0(p^*\Omega_{P^1 \times P^1}^1 \otimes \mathcal{O}_{P^1}(-2)) \longrightarrow H^0(\Omega_P^1 \otimes \mathcal{O}_{P^1}(-2)) \\ &\longrightarrow H^0(\Omega_{P/P^1 \times P^1}^1 \otimes \mathcal{O}_{P^1}(-2)). \end{aligned}$$

Since  $H^0(p^*\Omega_{P^1 \times P^1}^1 \otimes \mathcal{O}_{P^1}(-2)) = 0$ , in order to show that

$H^0(\Omega_P^1 \otimes \mathcal{O}_{P^1}(-2)) = 0$  it suffices to prove that

$H^0(\Omega_{P/P^1 \times P^1}^1 \otimes \mathcal{O}_{P^1}(-2)) = 0$ . But by the second sequence, it is enough to show that  $h^0(p^*\varepsilon \otimes \mathcal{O}_P(-1) \otimes \mathcal{O}_{P^1}(-2)) = 0$ . If we write  $m = (C, \Sigma) \geq 0$ , then it follows from the assumption(2) that

$$p^*\varepsilon \otimes \mathcal{O}_P(-1) \otimes \mathcal{O}_{P^1}(-2) = \mathcal{O}_{P^1}(-3m-2) \oplus \mathcal{O}_{P^1}(-3m-6) \oplus \mathcal{O}_{P^1}(-3m-8).$$

Thus  $h^0(p^*\varepsilon \otimes \mathcal{O}_P(-1) \otimes \mathcal{O}_{P^1}(-2)) = 0$  follows, and consequently

$h^1(\mathcal{O}_P|_C) = 0$  is obtained. To prove the surjectivity of the map

$H^1(\mathcal{O}_W) \longrightarrow H^1(\mathcal{O}_W|_C)$ , we must show that  $\alpha$  in (4.5) is injective.

But since  $h^1(\mathcal{O}_P|_C) = 0$ , it is equivalent to show the injectivity of  $\beta$ .

Consider the following exact sequences:

$$(4.6) \quad \begin{array}{ccccccc} & & H^0(\mathcal{O}_W(W)) & & & & \\ & & \downarrow & & & & \\ & & H^0(\mathcal{O}_C(W)) & & & & \\ & & \downarrow & & & & \\ H^1(\mathcal{O}_P|_W \otimes I_C) & \xrightarrow{\gamma} & H^1(\mathcal{O}_W(W) \otimes I_C) & \longrightarrow & H^2(\mathcal{O}_W \otimes I_C) & \xrightarrow{\beta} & H^2(\mathcal{O}_P|_W \otimes I_C) \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

We may prove that  $\gamma$  is surjective. Let us write

$L = H^1(\mathcal{O}_W(W) \otimes I_C) = H^0(\mathcal{O}_C(W)) / H^0(\mathcal{O}_W(W))$ . Then we have the following commutative diagram:

$$\begin{array}{ccc} H^0(p^* \mathcal{E}^{-1} \otimes \mathcal{O}_C(1)) & \xrightarrow{\tau} & H^1(\mathcal{O}_{\mathbb{P}}|_W \otimes I_C) \\ \downarrow J & & \downarrow \\ H^0(\mathcal{O}_C(W)) & \longrightarrow & L \end{array}$$

Here  $\mathcal{O}_C(1)$  is the restriction of  $\mathcal{O}_{\mathbb{P}}(1)$  to  $C$ , and  $\tau$  is the composition of the three maps:  $H^0(p^* \mathcal{E}^{-1} \otimes \mathcal{O}_C(1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}/\mathbb{P}^1 \times \mathbb{P}^1}|_C)$ ,  $H^0(\mathcal{O}_{\mathbb{P}/\mathbb{P}^1 \times \mathbb{P}^1}|_C) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}}|_C)$  and  $H^0(\mathcal{O}_{\mathbb{P}}|_C) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}}|_W \otimes I_C)$ . (See (4.2) and (4.3).)  $J$  is the so called Jacobian map, and it is defined as follows:

Let  $F = Y^2Z - (X^3 + aXZ^2 + bZ^3)$  be the defining equation of  $W$  in  $\mathbb{P}$ . (See Definition.) We write

$$H^0(p^* \mathcal{E}^{-1} \otimes \mathcal{O}_{\mathbb{P}}(1)) = H^0(\mathcal{O}_C(1)) \oplus H^0(\mathcal{O}_C(1) \otimes p^* K^{-2}) \oplus H^0(\mathcal{O}_C(1) \otimes p^* K^{-3}).$$

Then  $J$  is given by

$$(\ell_1, \ell_2, \ell_3) \longrightarrow \ell_1(\partial F / \partial Z) + \ell_2(\partial F / \partial X) + \ell_3(\partial F / \partial Y) \\ \uparrow \\ H^0(\mathcal{O}_C(W)),$$

where

$$\begin{aligned} \ell_1 &\in H^0(\mathcal{O}_C(1)), \\ \ell_2 &\in H^0(\mathcal{O}_C(1) \otimes p^* K^{-2}), \\ \ell_3 &\in H^0(\mathcal{O}_C(1) \otimes p^* K^{-3}). \end{aligned}$$

First it follows that  $L = 0$  if  $(C, \Sigma) = 0$ . For, we have

$$\mathcal{O}_W(W) \cong \mathcal{O}_W(3) \otimes \mu^*(-6K) \cong \mathcal{O}_W(9\Sigma) \otimes \mu^*(-6K) \text{ and } \mathcal{O}_C(\Sigma) \cong \mathcal{O}_C.$$

Thus we have proved the proposition in this case. Next we consider the case

where  $(C, \Sigma) = 1$ . In the remainder we shall show that  $\text{im } J$  and

$\text{im } H^0(\mathcal{O}_W(W)) \subset H^0(\mathcal{O}_C(W))$  generate  $H^0(\mathcal{O}_C(W))$  in this case. Note that

$C \subset H \subset W$ , where  $H$  is the divisor in the assumption of the

proposition. In this situation,  $\{ \partial F / \partial Z|_H = 0 \}$  is a divisor on  $H$

which has no intersections with  $\Sigma|_H$ . Here  $\Sigma|_H$  denotes the restriction of  $\Sigma$  to  $H$ . Hence  $\{ \partial F / \partial Z|_C = 0 \}$  consists of 18 points ( which may contain multiple points. ). Let  $V_1$  denote the image of  $H^0(\mathcal{O}_C(1))$  in  $H^0(\mathcal{O}_C(W)) = H^0(\mathcal{O}_{\mathbb{P}^1}(21))$  under  $J$ . Then the linear system defined by  $V_1$  consists of 18 fixed points and 3 points which move freely. In particular,  $\dim V_1 = 4$ . On the other hand, we have

$\{ \partial F / \partial Y|_H = 0 \} = 3\Sigma|_H + \{ \text{effective divisors which have no intersections with } \Sigma|_H \}$ .

Let  $V_2$  denote the image of  $H^0(\mathcal{O}_C(1) \otimes p^*K^{-3})$  in  $H^0(\mathcal{O}_C(W))$  under  $J$ . Then  $\dim V_2 = 10$ . Finally let  $V_3$  be the subspace of  $\text{im } H^0(\mathcal{O}_W(W)) \subset H^0(\mathcal{O}_C(W))$  which defines the linear system on  $C$  of the following type:

$9\Sigma|_C + \{ 12 \text{ points which move freely } \}$ .

We have  $\dim V_3 = 13$ . We shall consider the intersection of  $V_2$  and  $V_3$ .  $\{ \partial F / \partial Y|_C = 0 \}$  consists of 12 points;  $3\Sigma|_C$  ( one point with multiplicity 3 ) and 9 points  $R_1, \dots, R_9$ , none of which lies on  $\Sigma$ . Hence every section  $s \in V_2 \subset H^0(\mathcal{O}_C(W))$  must be zero at these 12 points. On the other hand, we deduce from the definition of  $V_3$  that  $s^+ \in V_3 \subset H^0(\mathcal{O}_C(W))$  must be zero at  $\Sigma|_C$  and that its multiplicity is at least 9. Therefore, we conclude that for every section  $s \in V_2 \cap V_3$ ,  $\{ p \in C ; s \text{ is zero at } p \} = 9\Sigma|_C + R_1 + \dots + R_9 + \text{others}$ . This implies that  $\dim(V_2 \cap V_3) \leq 4$ . Since  $\dim V_2 = 10$  and  $\dim V_3 = 13$ , we have  $\dim(V_2 + V_3) \geq 19$ . But every section of  $V_2 + V_3$  must be zero at  $\Sigma|_C$  with the multiplicity  $\geq 3$ . Thus we obtain  $\dim(V_2 + V_3) \leq 19$ . Consequently,  $\dim(V_2 + V_3) = 19$  and  $\dim(V_2 \cap V_3) = 4$ . Next we shall consider the intersection of  $V_1$  and  $(V_2 + V_3)$ . Every section in  $V_1$  is zero at  $\{ \partial F / \partial Z|_C = 0 \}$  ( which

consist of 18 points different from  $\Sigma|_C$ ). On the other hand, every section in  $V_2 + V_3$  is zero at  $\Sigma|_C$  with the multiplicity  $\geq 3$ . Thus we conclude that  $\dim V_1 \cap (V_2 + V_3) = 1$ . From this we obtain

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim(V_2 + V_3) - \dim(V_1 \cap (V_2 + V_3)) \\ &= 4 + 19 - 1 \\ &= 22. \end{aligned}$$

Since  $\dim(V_1 + V_2 + V_3) = \dim H^0(\mathcal{O}_C(W)) = \dim H^0(\mathcal{O}_{\mathbb{P}^1}(21)) = 22$ ,  $V_1 + V_2 + V_3$  coincides with  $H^0(\mathcal{O}_C(W))$ . This implies that  $\text{im } J$  and  $\text{im}(H^0(\mathcal{O}_W(W)) \subset H^0(\mathcal{O}_C(W)))$  generate  $H^0(\mathcal{O}_C(W))$ . Therefore,  $\gamma$  in (4.6) is surjective and the proof of the proposition is completed.

We are now in a position to state the following theorem.

**Theorem.** *Let  $W$  be a general nonsingular Weierstrass model over  $\mathbb{P}^1 \times \mathbb{P}^1$ . Then there are mutually disjoint rational curves  $C_1, \dots, C_4$  on  $W$  with the following properties:*

- 1)  $N_{C_i/W} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  for each  $i$ .
- 2)  $C_1, \dots, C_4$  span  $H_2(W; \mathbb{Z})$ .

Moreover, if we denote by  $\varphi: W \longrightarrow \tilde{W}$  the contraction of  $C_i$ 's, then  $\tilde{W}$  can be deformed to a compact complex manifold  $W_t$  with  $b_2(W_t) = 0$  and  $K_{W_t} = \mathcal{O}_{W_t}$ .

*Proof.*

By §3, there are mutually disjoint, isolated  $(-2)$ -curves  $C_i$  ( $1 \leq i \leq 4$ ) on  $W$ . Let  $C$  be one of  $C_i$ 's. If  $C$  is a  $(-2, c)$ -curve, then by (4.1) and (4.4), there is a suitable deformation of  $(C, W)$  in  $\mathbb{P} = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus K_{\mathbb{P}^1 \times \mathbb{P}^1}^{\otimes 2} \oplus K_{\mathbb{P}^1 \times \mathbb{P}^1}^{\otimes 3})$  such that  $C$  splits up into  $(-1, -1)$ -curves. Remark that a deformation of  $W$  in  $\mathbb{P}$  is again a Weierstrass model over  $\mathbb{P}^1 \times \mathbb{P}^1$ . This is shown as follows. Consider

the following diagram:

$$\begin{array}{ccc} \mathbb{W} & \supset & \mathbb{W}_0 = W \\ f \downarrow & & \downarrow \\ T & \supset & t_0, \end{array}$$

where  $f$  is a flat deformation of  $W$ . We assume that  $T$  is a disk around  $t_0$ . Since  $H^2(\mathcal{O}_W) = 0$ , the line bundle  $\mathcal{O}_W(\Sigma)$  is a restriction of a line bundle  $\mathcal{L}$  on  $\mathbb{W}$ . Since  $h^1(\mathcal{O}_W(\Sigma)) = h^3(\mathcal{O}_W(\Sigma)) = 0$  ( $h^1 = 0$  follows from the spectral sequence of Leray for  $\mu: W \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  and  $h^3 = 0$  follows from Serre duality),  $h^0(\mathcal{L}_t)$  and  $h^2(\mathcal{L}_t)$  are constant around  $t_0$  by the invariance of  $\chi(\mathcal{L}_t)$ . In particular,  $R^0 f_* \mathcal{L}$  is a locally free sheaf near  $t_0$ , which implies that a non-zero section of  $\mathcal{L}_{t_0}$  extends to a non-zero section of  $\mathcal{L}_t$  for a  $t$  near  $t_0$ . Hence  $\Sigma$  extends to a section  $\Sigma_t$  of  $\mu_t: \mathbb{W}_t \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . If  $\mu_t$  has a reducible fibre, then some component  $F$  of this fibre does not intersect with  $\Sigma_t$ . This implies that  $\rho(\mathbb{W}_t) \geq 4$  because  $\Sigma_t, \mu_t^* p_1^* \mathcal{O}(1), \mu_t^* p_2^* \mathcal{O}(1)$  and a divisor  $D$  with  $(D, F) \neq 0$  span a subgroup of  $NS(\mathbb{W}_t)$  of rank 4. But since  $h^1(\mathcal{O}_W) = h^2(\mathcal{O}_W) = 0$ , the Picard number is invariant under deformation, that is,  $\rho(\mathbb{W}_t) = 3$ . This is a contradiction. Therefore,  $\mathbb{W}_t$  has only irreducible fibres and a section, which implies that  $\mathbb{W}_t$  is a Weierstrass model over  $\mathbb{P}^1 \times \mathbb{P}^1$ .

By the above observation, it follows from Proposition(4.4) that there are mutually disjoint rational curves  $C_1, \dots, C_4$  with  $N_{C_i/W} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  on a general Weierstrass model  $W$  over  $\mathbb{P}^1 \times \mathbb{P}^1$ . Then we have

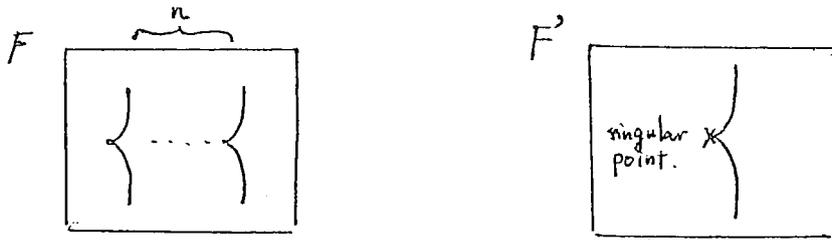
$$\begin{aligned}
& (\Sigma, C_1) = 0 \quad (\mu^* p_1^* \mathcal{O}(1), C_1) = 1 \quad (\mu^* p_2^* \mathcal{O}(1), C_1) = 0 \\
(4.7) \quad & (\Sigma, C_2) = 0 \quad (\mu^* p_1^* \mathcal{O}(1), C_2) = 0 \quad (\mu^* p_2^* \mathcal{O}(1), C_2) = 1 \\
& (\Sigma, C_3) = 1 \quad (\mu^* p_1^* \mathcal{O}(1), C_3) = 0 \quad (\mu^* p_2^* \mathcal{O}(1), C_3) = 1 \\
& (\Sigma, C_4) = 1 \quad (\mu^* p_1^* \mathcal{O}(1), C_4) = 1 \quad (\mu^* p_2^* \mathcal{O}(1), C_4) = 0.
\end{aligned}$$

Hence  $C_1, \dots, C_4$  generate  $H_2(W; \mathbb{Z})$  by Appendix. Moreover using (4.7) we can show that for  $\tilde{W}$  the assumption of Corollary(4.7) in [2] is satisfied. Therefore  $\tilde{W}$  is smoothable. Q.E.D.

**Appendix.** the calculation of  $H_2(W ; \mathbb{Z})$

Let  $\mu: W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  be a nonsingular Weierstrass model. We shall show that  $H_2(W ; \mathbb{Z})$  is torsion free and make explicit the generators of  $H_2(W ; \mathbb{Z})$ .

a) Let  $(T_0 : T_1) \times (S_0 : S_1)$  be the homogenous coordinate of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Set  $n = 12$  and  $b = T_1^n S_1^n + T_0^n S_1^n + T_1^n S_0^n + 2T_0^n S_0^n$ . Then  $b = 0$  defines a nonsingular curve  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ . Consider the Weierstrass model  $W$  defined by  $Y^2Z = X^3 + bZ^3$ , which is nonsingular. Let  $p_1$  be the first projection of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Then  $C$  is a  $n$ -cover of  $\mathbb{P}^1$ , and there are  $2n$  ramification points  $\{P_1, \dots, P_{2n}\} \in C$  for which the ramification index  $\delta(P_i) = n$  ( $1 \leq i \leq 2n$ ). Put  $\sigma = p_1 \circ \mu$ . Then  $\sigma$  is a  $K3$  fibration with  $n$  singular fibres. A general (resp. singular) fibre  $F$  (resp.  $F'$ ) has an elliptic fibration  $\alpha$  for which the singular fibres are  $n \times II$  (resp.  $1 \times II$ ).



b) Let  $F$  be a general fibre of  $\sigma$ . Then we have

$$H_1(F ; \mathbb{Z}) = 0$$

$$H_2(F ; \mathbb{Z}) = \mathbb{Z}e \oplus \mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)} \oplus \mathbb{Z}g ,$$

where the generators  $e$  and  $g$  are represented by the following 2-cycles on  $F$ .

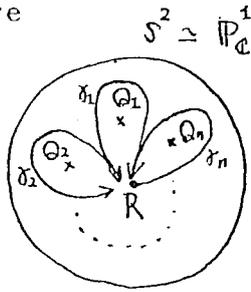
$e :=$  a smooth fibre of  $\alpha: F \longrightarrow \mathbb{P}^1$ , which is homeomorphic to  $S^1 \times S^1$

$g :=$  the canonical section  $\Sigma|_F$  of  $\alpha$ , which is homeomorphic to  $S^2$   
 $\mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)}$  is defined by two exact sequences:

$$\begin{aligned}
(1) \quad 0 &\longrightarrow Ker \longrightarrow \bigoplus_{i=1}^n H_1(S^1 \times S^1; \mathbb{Z}) \longrightarrow H_1(S^1 \times S^1; \mathbb{Z}) \\
&\qquad\qquad\qquad (a_1, \dots, a_n) \longrightarrow \sum_{i=1}^n a_i \\
(2) \quad 0 &\longrightarrow \mathbb{Z}^{\oplus 2} \longrightarrow Ker \longrightarrow \mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)} \longrightarrow 0 \\
&\qquad a = \vec{t}(a_1, a_2) \longrightarrow (a, Aa, \dots, A^{n-1}a)
\end{aligned}$$

A: monodromy matrix for a singular fibre

Figure



The fibration  $\alpha: F \rightarrow \mathbb{P}_C^1 \simeq S^2$  has singular fibres over  $\{Q_1, \dots, Q_n\}$ .

$\gamma_i: [0, 1] \rightarrow S^2$  is a circle around  $Q_i$  with  $\gamma_i(0) = R, \gamma_i(1) = R$ .

In the exact sequence (1), an element  $(a_1, \dots, a_n) \in \bigoplus_{i=1}^n H_1(S^1 \times S^1; \mathbb{Z})$  corresponds to a 2-chain  $C_1 + \dots + C_n$ ;  $C_i = \{C_i(t)\}_{t \in [0, 1]}$ ,  $a_i = C_i(1) - C_i(0)$ , where  $\{C_i(t)\}$  is a 1-cycle in  $S^1 \times S^1 \simeq \alpha^{-1}(\gamma_i(t))$ . Therefore an element in  $Ker$  is a 2-chain which goes to zero by the boundary map  $\partial$ .

$\mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)}$  is the quotient of  $Ker$  by the equivalence relation  $\sim$ ;  $(b_1, \dots, b_n) \sim (b_1^+, \dots, b_n^+) \leftrightarrow (b_1, \dots, b_n) = (b_1^+, \dots, b_n^+) + (a, Aa, \dots, A^{n-1}a)$  for  $a \in H_1(S^1 \times S^1; \mathbb{Z})$ .

Let  $\gamma$  be a circle around  $P_i \in \mathbb{P}^1$ . Then the monodromy action  $\gamma_*$  for the singular fibre  $\sigma^{-1}(P_i)$  is described as follows.

(3)

$$\begin{aligned}
\gamma_*|_{\mathbb{Z}e} &= id : \mathbb{Z}e \longrightarrow \mathbb{Z}e \\
\gamma_*|_{\mathbb{Z}g} &= id : \mathbb{Z}g \longrightarrow \mathbb{Z}g \\
(a_1, \dots, a_{n-1}, a_n) \in Ker &\longrightarrow \mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)} \\
\downarrow &\qquad\qquad\qquad \downarrow \qquad\qquad\qquad \gamma_* \downarrow \\
(Aa_n, Aa_{n-1}, \dots, Aa_1) \in Ker &\longrightarrow \mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)}
\end{aligned}$$

c) Let  $F'$  be a singular fibre of  $\sigma$ . Then we have

$$H_1(F'; \mathbb{Z}) = 0$$

$$H_2(F'; \mathbb{Z}) = \mathbb{Z}e \oplus \mathbb{Z}g,$$

where

$e :=$  a smooth fibre of  $\alpha: F' \rightarrow \mathbb{P}^1$ , which is homeomorphic to  $S^1 \times S^1$

$g :=$  the canonical section  $\Sigma|_{F'}$ , of  $\alpha$ , which is homeomorphic to  $S^2$

d) Consider  $\sigma: W \rightarrow \mathbb{P}^1(\mathbb{C}) \simeq S^2$ .  $S^2$  is covered by two open disks  $U$  and  $V$  with the following properties:  $P_i \in U$  ( $1 \leq i \leq 2n-1$ ),  $P_{2n} \in U$ ,  $P_{2n} \in V$ ,  $P_i \in V$  ( $1 \leq i \leq 2n-1$ ) and  $U \cap V \underset{\text{homeo}}{\simeq} S^1 \times (0,1)$ .

Moreover  $U$  is covered by  $(2n-1)$  open sets  $U_i$  ( $1 \leq i \leq 2n-1$ ) with the following properties: I)  $P_j \in U_i$  if and only if  $i = j$ . II) For  $i$  and  $j$  with  $i+1 \leq j$ ,  $U_i \cap U_j \neq \emptyset$  if and only if  $j = i+1$ . III)  $U_i \cap U_j$  is contractible to a point. By Mayer Vietoris sequence for  $\sigma^{-1}(U_i)$ 's, we have

$$H_1(\sigma^{-1}(U); \mathbb{Z}) = 0 \quad \text{and} \quad H_2(\sigma^{-1}(U); \mathbb{Z}) = \mathbb{Z}e \oplus \mathbb{Z}g,$$

where we use the same notation in b). Obviously,

$$H_1(\sigma^{-1}(V); \mathbb{Z}) = 0 \quad \text{and} \quad H_2(\sigma^{-1}(V); \mathbb{Z}) = \mathbb{Z}e \oplus \mathbb{Z}g.$$

Again by the Mayer Vietoris sequence for  $\sigma^{-1}(U)$  and  $\sigma^{-1}(V)$ , we have

$$\begin{aligned} H_2(\sigma^{-1}(U) \cap \sigma^{-1}(V); \mathbb{Z}) &\longrightarrow H_2(\sigma^{-1}(U); \mathbb{Z}) \oplus H_2(\sigma^{-1}(V); \mathbb{Z}) \\ \longrightarrow H_2(W; \mathbb{Z}) &\longrightarrow H_1(\sigma^{-1}(U) \cap \sigma^{-1}(V); \mathbb{Z}) \longrightarrow 0. \\ &\parallel \\ &\mathbb{Z} \end{aligned}$$

In the above sequence,  $H_2(\sigma^{-1}(U) \cap \sigma^{-1}(V); \mathbb{Z}) \simeq \mathbb{Z}e \oplus \mathbb{Z}g \oplus (\text{torsion})$  by (3) in b), which implies that  $H_2(W; \mathbb{Z}) \simeq \mathbb{Z}^{\oplus 3}$ . Since  $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^1$ ,  $\Sigma$  has two rulings, one of which coincides with  $g$ . We denote by  $h$  another one. Then  $H_2(W; \mathbb{Z}) \simeq \mathbb{Z}e \oplus \mathbb{Z}g \oplus \mathbb{Z}h$ , where  $e$  is a fibre of

$$\mu: W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

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