第１６回偏微分方程式論

札幌シンポジウム

（代表者 上見 練太郎）

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代表者　上見　練太郎

記

１．日　時　１９９１年８月８日（木）～８月１０日（土）

２．場　所　北海道大学理学部数学教室　４－５０８室

３．講　演

＊　この時間は講演者を囲んで自由な質問の時間とする予定です。

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Nonlinear Scattering for Long Range Interaction

Tohru OZAWA (RIMS, Kyoto University)

In the scattering theory for nonlinear waves, the basic idea is that for large times the solutions of nonlinear wave equations behave like the solutions of the corresponding free equations. This is possible only when we can take the point of view that the nonlinear interaction has no effect for large times, which in turn imposes restrictive conditions on the degree of nonlinearities and on the space dimensions in connection with decay rate in time of the free solutions. Even in the small data setting the conditions often exclude the possibility of scattering theory for many important equations especially in lower space dimensions such as the (modified) K-dV equation, the (derivative) nonlinear Schrödinger equation in 1+1 dimensions, the Klein-Gordon equation with cubic (resp. quadratic) nonlinearity in 1+1 (resp. 1+2) dimensions, and systems of quantum fields with Yukawa's coupling (Maxwell-Dirac, Klein-Gordon-Dirac, Klein-Gordon-Schrödinger, e.t.c.). In fact, most of these equations have no nontrivial solutions with the asymptotic form of the free solutions. In this talk, I present a new framework for the nonlinear scattering in the case where the degree of nonlinearities is not high enough to ensure asymptotically free solutions. The results given here, together with the recent papers [1][2], give an answer to the third problem of M. Reed.

We consider scattering for the nonlinear Schrödinger equation

$$i\partial_t u + (1/2)\Delta u = f(u)u. \tag{1}$$

Here u is a $\mathbb{C}$-valued function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, Δ is the Laplacian in $\mathbb{R}^n$, and f is an $\mathbb{R}$-valued function on $\mathbb{C}$. We treat the following two cases.

(I) The single power interaction in one space dimension:

$$f(u) = \lambda|u|^2,\ \lambda \in \mathbb{R}\setminus\{0\},\ (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2}$$

In this case, (1) is derived from the electromagnetic wave equation for the propagation of a laser beam in a nonlinear medium, from the Zakharov system for the propagation of the Langumuir waves in a plasma, from the Davey-Stewartson system for the propagation of surfaces of water waves, from isotropic Heisenberg equation for the evolution of classical spins, from the Ginsburg-Landau model for superconductivity, and so on.

(II) The Hartree type interaction in more than one space dimension:

$$f(u) = V*|u|^2,\ V(x) = \lambda|x|^{-1},\ (t,x) \in \mathbb{R}\times\mathbb{R}^n,\ n \ge 2, \tag{3}$$

where $*$ denotes the convolution in $\mathbb{R}^n$. In this case (1) is derived from a multibody Schrödinger equation in the self-consistent field approximation for a quantum system of bosons interacting through two body potential V. The associated time-independent version also arises in the quantum field theory, especially in the Hartree-Fock theory.

The above examples (2)-(3) have the following properties in common.

(a) Gauge invariance: $f(e^{i\theta}u) = f(u)$, $\theta \in \mathbb{R}$.

(b) Homogeneity: $D(t)^{-1}f(D(t)u) = t^{-1}f(u)$, $t > 0$,

where $D(t)$ is the dilation operator given by $(D(t)\psi)(x) = t^{-n/2}\psi(t^{-1}x)$. Property (a) leads to the conservation of the probability density which enables us to establish the well-posedness of the Cauchy problem for (1). More precisely, in both cases (I)-(II) it is proved that there is a unique group of nonlinear operators $\{S(t);\ t \in \mathbb{R}\}$ such that for any $k \in \mathbb{N}\cup\{0\}$

(1) $S(t)$ is a homeomorphism in the usual Sobolev space H^k and is an isometry in the L^2 norm for any $t \in \mathbb{R}$.

(2) $S(t+s) = S(t)S(s)$ for any $t, s \in \mathbb{R}$, $S(0) = \mathbb{1}$.

(3) For any $\phi \in H^k$, the map $t \longmapsto S(t)\phi$ is continuous from $\mathbb{R}$ to H^k.

(4) For any $t_0 \in \mathbb{R}$ and $\phi \in H^k$, $u(t) = S(t-t_0)\phi$ is a unique solution satisfying $u \in C(\mathbb{R};\ H^k) \cap \bigcap_{0\le\delta(q)<1} L^{2/\delta(q)}_{loc}(\mathbb{R};\ W^{k,q})$ and

$$u(t) = U(t-t_0)\phi - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))u(\tau)\, d\tau,\quad t \in \mathbb{R}, \tag{4}$$

where $U(t) = \exp(i(t/2)\Delta)$ and $\delta(q) = n/2-n/q$.

A crucial effect in scattering is given by the degree of nonlinear term $f(u)$ at $u = 0$, which is measured by the decay rate in t of the dilated potential $D(t)^{-1}f(D(t)u)$. In the usual scattering we compare solutions u to free solutions $U(t)\phi_\pm$ on the basis of the asymptotic conditions

$$\|u(t) - U(t)\phi_\pm\|_2 \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \pm\infty. \tag{5}$$

We would say that scattering theory for (1) had been possible only in the case where $D(t)^{-1}f(D(t)u) \sim t^{-\gamma}g(u)$ as $t \longrightarrow \infty$ for some function g and $\gamma > 1$. In n space dimensions, this corresponds $p > 1+2/n$ for $f(u) = \lambda|u|^{p-1}$ and $\gamma > 1$ for $f(u) = V*|u|^2$ with $V(x) = \lambda|x|^{-\gamma}$. On the other hand J. Ginibre (private communication) proved that (5) is impossible for any nontrivial solution when $D(t)^{-1}f(D(t)u) \sim t^{-\gamma}g(u)$ as $t \longrightarrow \infty$ for some g and $\gamma \le 1$. Property (b) therefore shows that the usual setting of scattering just fails for (2)-(3).

In order to state the main results we use the following notations. $W^{m,p}$ denotes the Sobolev space given by

$$W^{m,p} = \{\psi \in L^p;\ \|\psi\|_{W^{m,p}} = \sum_{|\alpha|\le m} \|\partial^\alpha \psi\|_p < \infty\},\quad m \in \mathbb{N}\cup\{0\},\ p \in [1,\infty].$$

Here $\|\cdot\|_p$ denotes the norm in $L^p = L^p(\mathbb{R}^n)$ and $\partial^\alpha = \Pi_{j=1}^n \partial_j^{\alpha_j}$, $\partial_j = \partial/\partial x_j$, for a multi-index α. $H^{m,s}$ denotes the weighted Sobolev space given by

$$H^{m,s} = \{\psi \in \mathcal{S}';\ \|\psi\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\psi\|_2 < \infty\},\quad m,\ s \in \mathbb{R}.$$

For $\phi_\pm \in H^{0,1}$ we define the phase functions $S^\pm$ by

$$S^\pm(t,x) = \mp \log|t|\cdot f(\hat{\phi}_\pm)(t^{-1}x),\ t \in \mathbb{R}\setminus\{0\},$$

where $\hat{}$ denotes the Fourier transform given by

$$\hat{\psi}(\xi) = (\mathcal{F}\psi)(\xi) = (2\pi)^{-n/2} \int \exp(-ix\cdot\xi)\psi(\xi)\ d\xi.$$

We define the unitary operators $\exp(iS_\pm(t))$ by

$$\exp(iS_\pm(t)) = \exp(iS^\pm(t,-it\nabla)) = \mathcal{F}^{-1}\exp(\mp i\log|t|\cdot f(\hat{\phi}_\pm))\mathcal{F}.$$

<u>Theorem 1.</u> Let f be as in (I) and let $k \in \mathbb{N}\cup\{0\}$. Then there is a constant $\varepsilon > 0$ with the following properties.

(1) For any $\phi_+ \in H^{k,2}\cap H^{0,k+2}$ with $\|\hat{\phi}_+\|_\infty \le \varepsilon$ there exists a unique $\phi \in H^{k,0}$ such that for any $\theta \in (1/2,1)$

$$\|S(t)\phi - \exp(iS_+(t))U(t)\phi_+\|_{k,0} = O(t^{-\theta}), \tag{6$_+$}$$

$$\left(\int_t^{+\infty} \|S(\tau)\phi - \exp(iS_+(\tau))U(\tau)\phi_+\|^4_{W^{4,\infty}}\ d\tau\right)^{1/4} = O(t^{-\theta/2}) \quad \text{as} \quad t \longrightarrow +\infty. \tag{7$_+$}$$

(2) For any $\phi_- \in H^{k,2}\cap H^{0,k+2}$ with $\|\hat{\phi}_-\|_\infty \le \varepsilon$ there exists a unique $\phi \in H^{k,0}$ such that for any $\theta \in (1/2,1)$

$$\|S(t)\phi - \exp(iS_-(t))U(t)\phi_-\|_{k,0} = O(|t|^{-\theta}), \tag{6$_-$}$$

$$\left(\int_{-\infty}^{t} \|S(\tau)\phi - \exp(iS_-(\tau))U(\tau)\phi_-\|^4_{W^{4,\infty}}\ d\tau\right)^{1/4} = O(|t|^{-\theta/2}) \quad \text{as} \quad t \longrightarrow -\infty. \tag{7$_-$}$$

<u>Theorem 2.</u> Let f be as in (II) and let $k \in \mathbb{N}\cup\{0\}$. Then there is a constant $\varepsilon > 0$ with the following properties.

(1) Let $\phi_+ \in H^{k,2}\cap H^{0,k+2}$ for $n \ge 3$ and $\phi_+ \in H^{k,2}\cap H^{0,k+3}$ for $n = 2$. Suppose that there is $\sigma \in (0,1/(n-1))$ such that $\|\hat{\phi}_+\|_{p(\sigma)}\|\hat{\phi}_+\|_{p(-\sigma)} \le \varepsilon$, where $p(\pm\sigma)$ is given by $p(\pm\sigma) = 2n/((1\pm\sigma)(n-1))$. Then there exists a unique $\phi \in H^{k,0}$ such that for any $\theta \in (1/2,1)$

$$\|S(t)\phi-\exp(iS_+(t))U(t)\phi_+\|_{k,0} = O(t^{-\theta}), \tag{8$_+$}$$

$$\left(\int_t^{+\infty} \|S(\tau)\phi-\exp(iS_+(\tau))U(\tau)\phi_+\|^4_{W^{4,\infty}}\, d\tau\right)^{1/4} = O(t^{-\theta/2}) \quad \text{as} \quad t \longrightarrow +\infty. \tag{9$_+$}$$

(2) Let $\phi_- \in H^{k,2}\cap H^{0,k+2}$ for $n \geq 3$ and $\phi_- \in H^{k,2}\cap H^{0,k+3}$ for $n = 2$. Suppose that there is $\sigma \in (0,1/(n-1))$ such that $\|\hat{\phi}_-\|_{p(\sigma)}\|\hat{\phi}_-\|_{p(-\sigma)} \leq \varepsilon$, then there exists a unique $\phi \in H^{k,0}$ such that for any $\theta \in (1/2,1)$

$$\|S(t)\phi-\exp(iS_-(t))U(t)\phi_-\|_{k,0} = O(|t|^{-\theta}), \tag{8$_-$}$$

$$\left(\int_{-\infty}^t \|S(\tau)\phi-\exp(iS_-(\tau))U(\tau)\phi_-\|^4_{W^{4,\infty}}\, d\tau\right)^{1/4} = O(|t|^{-\theta/2}) \quad \text{as} \quad t \longrightarrow -\infty. \tag{9$_-$}$$

By Theorems 1-2, the modified wave operators $W_\pm$ is defined as maps $\phi_\pm \longmapsto \phi$ from B^k to $H^{k,0}$, where B^k is the domain of $W_\pm$ given by $\{\psi \in H^{k,2}\cap H^{0,k+2};\ \|\hat{\psi}\|_\infty \leq \varepsilon\}$ in the case (I), for example. The Cauchy problem for (1) is solved so that the asymptotic behavior of solutions is described as $(6)_\pm$ or $(8)_\pm$ when the initial data are in the ranges of $W_\pm$. Moreover, we see: (A) $W_\pm$ are injective and isometries in the L^2 norm. (B) $W_\pm$ are continuous from B^k to $H^{k,0}$, with B^k topologized from the associated weighted Sobolev space. (C) Under the evolution $S(t)$, Range$(W_\pm)$ are asymptotically orthogonal to every bound state for (1). (D) $W_\pm$ have the intertwining properties: $S(t)W_\pm = W_\pm U(t)$ on $B^0\cap H^{2,0}$.

Our modified wave operators $W_\pm$ have some properties analogous to the modified wave operators of Dollard type for the Coulomb scattering. First, $W_\pm$ intertwine the interacting dynamics and the usual free dynamics as described in (D). Secondly, the modification of the wave operators has no contribution to the asymptotic behavior of the probability density both in the position and momentum space. Lastly, the asymptotic motion of solution of (1) is closely approximated by the solutions $w_\pm$ of

$$i\partial_t w_\pm + (1/2)\Delta w_\pm = f(\hat{\phi}_\pm)(-it\nabla)w_\pm.$$

In the scattering with long range potentials V, with the interacting dynamics given by the unitary propagator $\exp(-it(-(1/2)\Delta+V))$ we usually associate the modified free evolution given by the solution w of

$$i\partial_t w + (1/2)\Delta w = V(-it\nabla)w.$$

The substitution x by $-it\nabla$ in the potential term is common both to the linear and nonlinear case. Unfortunately, this is not enough for the present nonlinear case and it is our claim that the nonlinear potential $f(u)$ must be modified as $f(\hat{\phi}_\pm)(-it\nabla)$ through the introduction of $\hat{\phi}_\pm$.

It is a simple matter to see how the standard method breaks down in (I) and (II). The standard theory is carried out by solving the equations

$$u(t) = U(t)\phi_\pm + i\int_t^{\pm\infty} U(t-\tau)f(u(\tau))u(\tau)\, d\tau \tag{10}$$

for given $\phi_\pm$. If the procedure is to work, the integral in (10) should converge in L^2. But this is impossible since every nontrivial solution of (1) does not decay faster than the free solutions and the integrand decays like $O(|t|^{-1})$ at best. The very same situation happens in the Coulomb scattering, where Cook's integral diverges logarithmically.

Our method depends on solving another integral equations around modified free evolutions $v_\pm$ instead of $U(t)\phi_\pm$ in order that the equations could have convergent integrals. Rather than (10), we consider

$$u(t) = v_\pm(t) + i\int_t^{\pm\infty} U(t-\tau)(f(u(\tau))u(\tau) - (i\partial_\tau+(1/2)\Delta)v_\pm(\tau))\, d\tau \tag{11}$$

for suitable $v_\pm$ which give a nice cancellation of the divergent part of $f(u)u$. To this end we introduce the following approximate solutions $v_\pm$.

$$v_\pm(t) = \exp(iS^\pm(t))U(t)M(-t)\phi_\pm = i^{-n/2}\exp(iS^\pm(t))M(t)D(t)\hat{\phi}_\pm, \tag{12}$$

where $M(t) = \exp(i|x|^2/2t)$. $v_\pm$ turn out to satisfy (1) up to the rate $O(|t|^{-2}(\log|t|)^2)$ in L^2 as $t \longrightarrow \pm\infty$, because of the exact cancellation of the divergent terms $f(v_\pm)v_\pm$ and $|t|^{-1}f(\hat{\phi}_\pm)(t^{-1}x)v_\pm$ from $i\partial_t v_\pm$. Then, (11) are solvable near $t = \pm\infty$ by a contraction argument on the space defined as a closed ball centered at $v_\pm$. The space-time estimates of Strichartz type are essential in this step. The solution u, defined for large times, behaves like $v_\pm(t)$ as $t \longrightarrow \pm\infty$, and extends to all times by means of $S(t)$, and then ϕ in the theorems is given by $u(0) = \phi$. The rest of the statements of the theorems follow by proving $v_\pm(t) \sim \exp(iS_\pm(t))U(t)\phi_\pm$ as $t \longrightarrow \pm\infty$. Details will be given elsewhere.

References.

[1] N. Hayashi, T. Ozawa, Modified wave operators for the derivative nonlinear Schrödinger equation, preprint RIMS-746, 1991.

[2] T. Ozawa, Long range scattering for nonlinear Schrödinger equations in one space dimension, RIMS-731, 1990, Comm. Math. Phys, to appear.

Asymptotic formulas for the shock wave of the scalar conservation law with smooth initial data

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次のスカラー保存則方程式の初期値問題を考察する。

$$(1)\qquad \begin{cases} u_t + f(u)_x = 0 & t > 0,\ x \in R,\ u(x,t) \in R \\ u(x,0) = u_0(x) \end{cases}$$

ここで $f(u) \in C^2(R)$,　$f''(u) > \delta > 0$ とする。

エントロピー条件を課す。

$$(E)\qquad \exists E > 0\ ;\quad \frac{u(x+a,t) - u(x,t)}{a} \le \frac{E}{t} \quad a > 0, t > 0.$$

初期値 $u_0(x)$ が有界可測関数のとき (1)(E) を満たす弱解 $u(x,t)$ が一意的に存在する。さらに $u(x,t)$ は各 $t(>0)$ をとめる毎に x について局所的に有界変動である。(cf. Smoller[2])

一般に初期値が十分なめらかであっても、解は有限時間内で不連続となる。そこで $t>0$ に対し $u(x(t)-0,t) > u(x(t)+0,t)$ が成立する点 $x(t)$ を shock point 、曲線 $x = x(t)$ を shock curve と呼ぶ。(条件 (E) より、不連続点 $x(t)$ では常に $u(x(t)-0,t) > u(x(t)+0,t)$ となることが示される。)

$$t^* = \inf\{t > 0 |\ \exists x = x(t) \text{ such that } u(x(t)-0,t) > u(x(t)+0,t)\}$$

$$x^* = \lim_{t \to t^*+0} x(t)$$

とおく。

Lax[1] による解の表現式を応用することで、(y,z) 平面での 2 つのグラフ $z = u_0(y)$,　$z = (f')^{-1}\left(\frac{x-y}{t}\right)$ によって囲まれる領域の面積を比較し、$t \ge t^*$ での $x(t), u(x(t)-0,t), u(x(t)+0,t)$ の各値を図形的に求める方法が与えられる。この図形的方法と漸近解析とを組み合わせることで、次の各定理が導かれる。

定理１. (shock curve の発生)

次を仮定する。

(A1) $N \geq 5,\ u_0(x) \in C^{N+1}(R),\ f(u) \in C^{N+2}$

(A2) $\exists m;\ u_0'(x) > 0 \ \ x < m, \qquad u_0'(m) = 0 \qquad u_0'(x) < 0 \ \ x > m,$

$\exists \tau > m;\ (f' \circ u_0)''(x) < 0 \ \ m \leq x < \tau, \qquad (f' \circ u_0)''(\tau) = 0, \qquad (f' \circ u_0)''(x) > 0 \ \ \tau < x$

(A3) $(f' \circ u_0)^{(3)}(\tau) > 0.$

このとき shock curve は

$$(x^*, t^*) = \left(\tau - \frac{(f' \circ u_0)(\tau)}{(f' \circ u_0)'(\tau)},\ \frac{-1}{(f' \circ u_0)'(\tau)}\right)$$

より発生し、$t \longrightarrow t^* + 0$ において以下が成立する。

$$x(t) = x^* + \sum_{k=1}^{[\frac{N}{2}]} c_k (t - t^*)^k + O((t - t^*)^{\frac{N+1}{2}}),$$

$$u(x(t) - 0, t) = u_0(\tau) + \sum_{k=1}^{N-2} A_k (t - t^*)^{\frac{k}{2}} + O((t - t^*)^{\frac{N-1}{2}}),$$

$$u(x(t) + 0, t) = u_0(\tau) + \sum_{k=1}^{N-2} B_k (t - t^*)^{\frac{k}{2}} + O((t - t^*)^{\frac{N-1}{2}}).$$

ここで係数 c_k, A_k, B_k は $u_0(x)$ の $x = \tau$ での高階微係数と $f(u)$ の $u_0(\tau)$ での高階微係数より定められる。とくに、$A_k + B_k = 0$ (k:奇数), $A_k = B_k$ (k:偶数),

$$c_1 = f' \circ u_0(\tau), \qquad A_1 = \frac{\sqrt{6}\, u_0'(\tau)(f' \circ u_0)'(\tau)}{\sqrt{(f' \circ u_0)^{(3)}(\tau)}},$$

$$c_2 = \frac{-3\{(f' \circ u_0)'(\tau)\}^3 (f' \circ u_0)^{(4)}(\tau)}{10\{(f' \circ u_0)^{(3)}(\tau)\}^2} - \frac{4 f^{(3)}(u_0(\tau))\{u_0'(\tau)\}^2\{(f' \circ u_0)'(\tau)\}^2}{5(f' \circ u_0)^{(3)}(\tau)}.$$

この定理により shock curve は t^* まで有界な高階微係数を持つ一方、その両端の値 $u(x(t) - 0, t), u(x(t) + 0, t)$ は、(t, u) 平面では $t = t^*$ に上下から接しているという状況が示される。

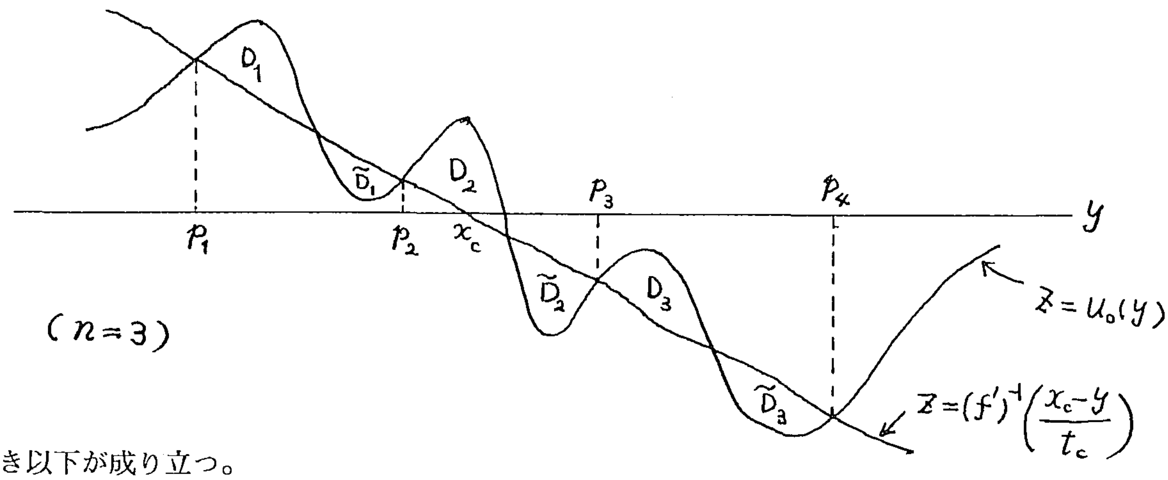
以下簡単のため $f(0) = f'(0) = 0$ とする。

定理２.(shock curve の相互作用)

$u_0(y) \in C^{N+1}(R),\ f(u) \in C^{N+2}(R)\ (N \geq 1)$ に対し次の条件を満たす点 (x_c, t_c) が存在すると仮定する。

(C1) (y,z) 平面で２つのグラフ $z = (f')^{-1}(\frac{x_c - y}{t_c})$ と $z = u_0(y)$ は、n+1 個の点 $y = p_1, p_2, ..., p_{n+1}$ $(p_1 < p_2 < \cdots < p_{n+1}, n \geq 2)$ で交わる。(下図参照)

(C2) D_i $\tilde{D}_i$ をそれぞれ $y \in [p_i, p_{i+1}]$ 上で２つのグラフによって囲まれる左と右の領域とすると、すべての $i; 1 \leq i \leq n$ に対し $| D_i |=| \tilde{D}_i |$ が成立する。(ここで $| D |$ は領域 D の面積を表す。)



このとき以下が成り立つ。

(1) n 本の shock curve $x_1(t),\ x_2(t), ..., x_n(t)\ (t \leq t_c)$ は１点 (x_c, t_c) で交わり、１本の shock curve $x(t)\ (t \geq t_c)$ となる。

(2) 各 shock curve 及びそこでの解の値は t_c の近傍で次の漸近表示を持つ。

$$x_i(t) = x_c + \sum_{k=1}^{N} c_k^i (t - t_c)^k + O((t - t_c)^{N+1}),$$

$$u(x_i(t) - 0, t) = u_0(p_i) + \sum_{k=1}^{N} A_k^i (t - t_c)^k + O((t - t_c)^{N+1}),$$

$$u(x_i(t) + 0, t) = u_0(p_{i+1}) + \sum_{k=1}^{N} B_k^i (t - t_c)^k + O((t - t_c)^{N+1}),$$

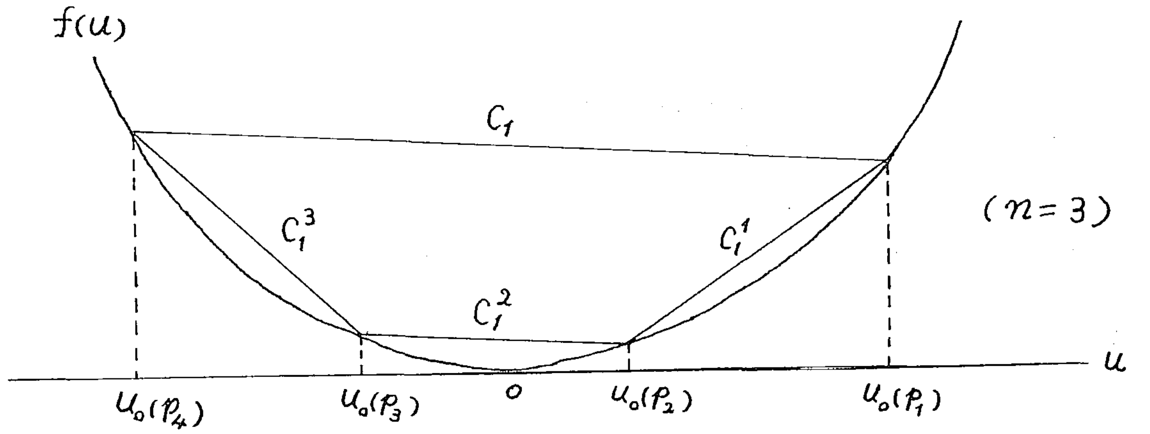
$$t \longrightarrow t_c - 0, \quad 1 \leq i \leq n.$$

$$x(t) = x_c + \sum_{k=1}^{N} c_k (t - t_c)^k + O((t - t_c)^{N+1}),$$

$$u(x(t) - 0, t) = u_0(p_1) + \sum_{k=1}^{N} A_k (t - t_c)^k + O((t - t_c)^{N+1}),$$

$$u(x(t)+0,t) = u_0(p_{n+1}) + \sum_{k=1}^{N} B_k (t-t_c)^k + O((t-t_c)^{N+1}),$$
$$t \longrightarrow t_c + 0.$$

係数 c_k^i, A_k^i, B_k^i は、$y = p_i,\ p_{i+1}$ での $u_0(y)$ の高階微係数と $u = u_0(p_i), u_0(p_{i+1})$ での $f(u)$ の高階微係数から決定される。さらに係数 c_k, A_k, B_k は、$y = p_1,\ p_{n+1}$ での $u_0(y)$ の高階微係数と $u = u_0(p_1), u_0(p_{n+1})$ での $f(u)$ の高階微係数から決定される。とくに各 shock curve の主要項の係数 c_1^i, c_1 は下図の各弦の傾きに等しい。



最後に初期値が compact support の関数としたときの時刻無限大での shock wave の挙動について調べる。

定理３.($t \to +\infty$ での shock wave)

次を仮定する。

(A1) Supp$u_0(y) = [c,d]$ かつ $u_0(y) > 0 \quad y \in (c,d)$

(A2) $u_0(y) \in C^{N+1}([c,d]),\ f(u) \in C^{N+2}(R)\ \ (N \geq 1)$, かつ $u_0'(c+0) > 0$。

$$M = \int_c^d u_0(y)dy$$

とおく。このとき十分大きな t に対し shock curve は１本となり、$t \longrightarrow +\infty$ において以下の漸近表示が得られる。

$$x(t) = \sum_{k=1}^{N} c_k\, t^{1-\frac{k}{2}} + O(t^{-\frac{N-1}{2}}),$$

$$u(x(t)-0,t)=\sum_{k=1}^{N} A_k\, t^{-\frac{k}{2}}+O(t^{-\frac{N+1}{2}}), \qquad u(x(t)+0,t)=0.$$

ここで係数 c_k, A_k, は $u_0(y)$ の $y=0$ での高階微係数と $f(u)$ の $u=0$ での高階微係数より定められる。とくに、

$$c_1=\sqrt{2f''(0)M}, \qquad A_1=\sqrt{\frac{2M}{f''(0)}},$$

$$c_2=c+\frac{f^{(3)}(0)M}{3f''(0)}, \qquad A_2=\frac{-2f^{(3)}(0)}{3(f''(0))^2}.$$

参考文献

[1] P. D. Lax, *Hyperbolic systems of conservation laws, II*, Comm. Pure Appl. Math., 10(1957), 537-566.

[2] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer Verlag, New York, 1983.

ON LEVI-TYPE CONDITIONS FOR HYPOELLIPTICITY OF CERTAIN DIFFERENTIAL OPERATORS

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The object of this talk is to give necessary and sufficient conditions for (C^∞–) hypoellipticity of operators of the form:

(1) $$P = D_x^2 + a(x)D_y^2 + b(x)D_y,$$

$$(D_x = \frac{1}{i}\frac{\partial}{\partial x},\ D_y = \frac{1}{i}\frac{\partial}{\partial y}),$$

in $\mathbb{R}^2_{xy}$, under the assumption

(2) (i) $a(x)$ and $b(x)$ are real-valued C^∞ functions,

(ii) $a(0) = 0$ and $a(x) > 0$ for $x \neq 0$.

It is known that P is hypoelliptic in the case where $b(x) \equiv 0$. Grušin [4] proved that if $a(x)$ vanishes to finite order on $x = 0$, and Fediĭ [1] proved that even though it vanishes to infinite order (in addition to (ii) of (2), he assumed that $a(x) = \phi(x)^2$ with $\phi(x) \in C^\infty$). Our interest here is devoted to considering conditions on $b(x)$ for the hypoellipticity when a function $a(x)$ satisfying (2) were given. Concerning this problem, several authors have studied. See for example, Grušin [4], Gilioli and Treves [3] , Menikoff [10], [11], Yamamoto [18] and Suzuki [17]. In particular, the following example was given by Menikoff [11] : In the case where $a(x) = x^{2k}$ and $b(x) = b\,x^{\ell}(b \neq 0$, real constant) it holds that (i) $\ell > k-1$ then P is hypoelliptic (ii) $\ell = k-1$ then P is hypoelliptic if and only if b avoids a certain discrete set of values (iii) $\ell < k-1$ then P is not hypoelliptic.

This example immediately suggests that the magnitude of $b(x)$ (compared with that of $a(x)$) governs the hypoellipticity (from such observation, the title of this talk is named as in the Cauchy problem). However, Menikoff's proof relys on the knowledge of the asymptotic behavior of transcendental functions, and so it seems that his method cannot be applicable to the case where both $a(x)$ and $b(x)$ vanish to infinite order on $x = 0$. The purpose of our work is to point out a principle behind Menikoff's example and, by applying it, we shall give some conditions for hypoellipticity of P with $a(x)$ and $b(x)$ being not assumed to vanish finitely.

In addition to (2), we assume that

(3) (i) $b(0) = 0$,

(ii) $a(x)$ and $|b(x)|$ are non-decreasing for $0 < x < \delta$.

Set

$$B(x) = \sup_{0<z<x} b(z)^2(x-z)^2.$$

Then we have the following:

Theorem 1. *Suppose (2) and (3). Moreover if*

$$\limsup_{x\downarrow 0} \frac{B(x)}{a(x)} = \infty, \tag{4}$$

then P is not hypoelliptic on $x = 0$.

The assumption (4) says that , in some sense, the magnitude of $b(x)$ is too large. Observe that the supremum of the function $b(z)^2(x-z)^2$ $(0 < z < x)$ attains in the point where $x - z = b(z)/b'(z)$. Thus the assumption (4) follows from

(4′) (i) $\displaystyle\lim_{z\downarrow 0}\frac{b(z)}{b'(z)} = 0,$

(ii) $\displaystyle\limsup_{z\downarrow 0}\frac{b(z)^4}{b'(z)^2\, a(z+b(z)/b'(z))} = \infty.$

This explains the case (iii) in Menikoff's example. Moreover we have

Example 1. Let $a(x) = \exp(-1/|x|^\sigma)$ and $b(x) = \exp(-1/|x|^\tau)$. Here σ and τ are positive constants. If $\sigma > \tau$, then (4′) is satisfied, and therefore P is not hypoelliptic. Theorem 2 below can show that P is hypoelliptic in the counter case (where $\sigma \le \tau$).

Example 2. Let $a(x) = \exp(-2/|x|^\sigma)$ and $b(x) = |x|^{-\tau}\exp(-1/|x|^\sigma)$. Easy computation shows that, if $\tau > \sigma + 1$, then (4′) is satisfied.

Next we give a sufficient condition for the hypoellipticity. Instead of (3), we assume here that

(5) (i) $b(0) = 0, b'(x) \neq 0$ for $x \neq 0$,

(ii) $a(x)/|b(x)|$ is non-decreasing for $0 < x < \delta$ and non-increasing for $-\delta < x < 0$.

Theorem 2. *We assume (2) and (5). Moreover suppose that there exists a constant C with $0 < C < 2^{-7}\cdot 3^{-2}$ such that*

(6) $$\frac{b(x)^4}{b'(x)^2} \le C\, a(x),$$

for $-\delta < x < 0$ and $0 < x < \delta$.

Then P is hypoelliptic.

The assumption (6) says that the magnitude of $b(x)$ is small(the form of (6) seems reasonable compared with that of $(4')$(ii)). Also Theorem 2 can be extended to a class of operators which depend on variable y, by introducing the notion of operator-valued pseudodifferential operators. Cf. Grušin [5], Sjöstrand [**16**] or theorem 3 in Hoshiro [**8**] (see also Morimoto [**13**]).

REFERENCES

[1] V.S. Fediǐ, On a criterion for hypoellipticity. Math. USSR Sb. **14**, 15-45 (1971)

[2] C. Fefferman, The uncertainty principle. Bull. Amer. Math. Soc. **9**, 129-206 (1983)

[3] A. Gilioli and F. Treves, An example in the solvability theory of linear PDE's. Amer. J. Math. **96**, 367-385 (1974)

[4] V.V. Grušin, On a class of hypoelliptic operators. Math. USSR Sb. **12**,458-479 (1970)

[5] V.V. Grušin, Hypoelliptic differential equations and pseudodifferential operators with operator-valued symbols. Math. USSR Sb. **17**, 497-514 (1972)

[6] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities 2nd ed..: Cambrige Univ. Press 1952

[7] T. Hoshiro, Microlocal energy method of Mizohata and hypoellipticity. J. Math. Kyoto Univ. **28**, 1-12 (1988)

[8] T. Hoshiro, Hypoellipticity for infinitely degenerate elliptic and parabolic operators of second order. J. Math. Kyoto Univ. **28**, 615-632 (1988)

[9] M. Koike, A note on hypoellipticity of degenerate elliptic operators. preprint

[10] A. Menikoff, Some example of hypoelliptic partial differential equations. Math. Ann. **221**, 167-181 (1976)

[11] A. Menikoff, Pseudo-differential operators with double characteristics. Math. Ann. **231**, 145-180 (1977)

[12] S. Mizohata, On the Cauchy problem.: Academic Press 1986

[13] Y. Morimoto, On the hypoellipticity for infinitely degenerate semi-elliptic operators. J. Math. Soc. Japan **30**, 327-358 (1978)

[14] O.A. Oleĭnik and E.V. Radkevič, Second order equations with non-negative characteristic form.: Plenum Press 1973

[15] L. Rothschild and E.M. Stein, Hypoelliptic differential operators and nilpotent groups. Acta Math. **137**, 247-320 (1976)

[16] J. Sjöstrand, Parametrices for pseudodifferential operators with multiple characteristics. Ark. Mat. **12**, 85-130 (1974)

[17] M. Suzuki, Hypoellipticity for class of degenerate elliptic operators of second order. preprint in Univ. of Tsukuba

[18] K. Yamamoto, Parametrices for pseudo-differential equations with double characteristics. Hokkaidō Math. J. **5**, 280-301 (1976)

The Global Weak Solutions on the Compressible Euler Equation with Spherical symmetry

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1 Introduction

The compressible 3-D Euler equation is given by

$$(1.1)\qquad \begin{aligned} &\rho_t + \nabla(\rho u) = 0\,, \\ &\rho(u_t + u\cdot\nabla u) + \nabla p = 0\,, \quad p = a^2\rho\,, \end{aligned}$$

where ρ, u and p are the density, the velocity and the pressure respectively, and a is a positive constant. If the solutions of (1.1) is spherically symmetry, (1.1) becomes

$$(1.2)\qquad \begin{aligned} &\rho_t + \frac{1}{r^2}(r^2\rho u)_r = 0\,, \\ &\rho(u_t + u\,u_r) + p_r = 0\,, \end{aligned}$$

where $r = |x|$, $\rho = \rho(r)$ and $u(x) = (x/|x|)\,u(r)$.

Let $\tilde\rho = r^2\rho$. Then we get

$$(1.3)\qquad \begin{aligned} &\tilde\rho_t + (\tilde\rho u)_r = 0\,, \\ &u_t + u\,u_r + \frac{a^2\tilde\rho_r}{\tilde\rho} = \frac{2a^2}{r}\,, \end{aligned}$$

from(1.2). We shall study (1.3) in the region

$$(1.4) \qquad r \geq 1 .$$

(1.3) can be reformulated in Lagrangean mass coordinates by using the transformation,

$$\tau = t , \quad \xi = \int_1^r \tilde{\rho}(r,t)\, dr .$$

We get

$$(1.5) \qquad \begin{aligned} \tilde{\rho}_\tau + \tilde{\rho}^2 u_\xi &= 0 , \\ u_\tau + a^2 \tilde{\rho}_\xi &= \frac{2a^2}{r} . \end{aligned}$$

Let $v = 1/\tilde{\rho}$. Then after changing τ to t,

$$(1.6) \qquad \begin{aligned} v_t - u_x &= 0 , \\ u_t + \left(\frac{a^2}{v}\right)_x &= \frac{2a^2}{1 + \int_0^x v(\zeta,t)d\zeta} . \end{aligned}$$

Here we shall consider in $t \geq 0$, $x \geq 0$ the initial boundary value problrm for (1.6) with the following boundary values.

$$(1.7) \qquad u(x,0) = u_0(x) , \quad v(x,0) = v_0(x) , \quad for \ x > 0,$$

$$(1.8) \qquad u(0,t) = 0 , \quad for \ t > 0,$$

where both $u_0(x)$ and $v_0(x)$ are bounded functions of bounded variation, and $v_0(x) \geq \delta > 0$. We shall show that (1.6), (1.7) and (1.8) have global weak solutions. We use the Glimm's difference scheme [1] and the ideas of [2] and [3].

2 The difference scheme

We shall use the difference scheme which was used in [2].

$$(2.1) \qquad \begin{aligned} Y &= \{ (m , n); \ m = 0, 2, 4, \cdots , n = 1, 2, 3, \cdots \} , \\ A &= \prod_{(m,n)\in Y} [(ml, (m+2)l) \times nh] \ m : even , \end{aligned}$$

where l/h will be determined later. We choose a point $\{a_{mn}\} \in A$ randomly, and let $a_{mn} = (c_{mn}, nh)$. We denote u^l and v^l the approximate solutions. Mesh lengths l and h are chosen so that $l/h > 1/(inf\ v^l)$, for any given $T > 0$. We shall show later that there exists a $\delta > 0$ such that $\inf v^l \geq \delta > 0$.

For $0 \leq t < h$, $ml \leq x < (m+2)l$, m : odd, we define

$$
\begin{aligned}
u^l(x,t) &= u_0^l(x,t) + U^l(x,t)t, \\
v^l(x,t) &= v_0^l(x,t),
\end{aligned} \tag{2.2}
$$

where u_0^l and v_0^l are the solutions of

$$
\begin{aligned}
v_t - u_x &= 0\,, \\
u_t + \left(\frac{a^2}{v}\right)_x &= 0,
\end{aligned} \tag{2.3}
$$

with initial data

$$
u_0(x) = \begin{cases} u_0(ml), \\ u_0((m+2)l), \end{cases} \quad v_0(x) = \begin{cases} v_0(ml), & x < (m+1)l, \\ v_0((m+2)l), & x > (m+1)l, \end{cases} \tag{2.4}
$$

and

$$
U^l(x,t) = \frac{2a^2}{1 + \sum_{j=1}^{\frac{m+1}{2}} v_0((2j-1)l) \times 2l}\,. \tag{2.5}
$$

For $0 \leq t < h$, $0 \leq x < l$, we define u^l and v^l as (2.2) where u_0^l and v_0^l are the solutions of (2.3) with initial boundary data

$$
u(x,0) = u_0(l), \quad v(x,0) = v(0,l), \quad x > 0, \tag{2.6}
$$

$$
u(0,t) = 0, \quad t > 0, \tag{2.7}
$$

and

$$
U^l(x,t) = 2a^2. \tag{2.8}
$$

Suppose that u^l and v^l are defined for $t < nh$. For $nh \leq t < (n+1)h$, $ml \leq x < (m+2)l$, m : odd, we define

$$
\begin{aligned}
u^l(x,t) &= u_0^l(x,t) + U^l(x,t) \cdot (t - nh), \\
v^l(x,t) &= v_0^l(x,t),
\end{aligned} \tag{2.9}
$$

where u_0^l and v_0^l are the solutions of (2.3) with initial data (t=nh)

$$(2.10)\qquad \begin{aligned} u_0(x) &= \begin{cases} u^l(c_{mn}, nh-0), & x < (m+1)l, \\ u^l(c_{m+2\,n}, nh-0), & x > (m+1)l, \end{cases} \\ v_0(x) &= \begin{cases} v^l(c_{mn}, nh-0), & x < (m+1)l, \\ v^l(c_{m+2\,n}, nh-0), & x > (m+1)l, \end{cases} \end{aligned}$$

and

$$(2.11)\qquad U^l(x,t) = \frac{2a^2}{1 + \sum_{j=1}^{\frac{m+1}{2}} v^l(c_{2j-1\,n}, nh-0) \times 2l}.$$

For $nh \le t < (n+1)h$, $0 \le x < l$, we define u^l and v^l as (2.9) where u_0^l and v_0^l are the solutions of (2.3) with initial (t=nh) boundary data

$$(2.12)\quad u(x,0) = u^l(c_{1n}, nh-0), \quad v(x,0) = v^l(c_{1n}, nh-0), \quad x > 0,$$

$$(2.13)\qquad u(0,t) = 0, \quad t > 0,$$

and $U^l(x,t)$ is as (2.8).

3 Main Results

By computing the variation of Riemann invariants, We obtain the following Theorem.

Theorem 3.1 *For any $T > 0$, the variation of u^l and v^l is bounded uniformly for h and $\{a_{mn}\}$. Their upper and lower bounds, especially positive lower bound of v^l, are also uniformly bounded.*

We also have the following Theorem.

Theorem 3.2 *For any interval $[x_1, x_2] \in [0.\infty)$, we get*

$$(3.1)\qquad \begin{aligned} &\int_{x_1}^{x_2} |\, u^l(x,t_2) - u^l(x,t_1)\,| + |\, v^l(x,t_2) - v^l(x,t_1)\,|\, dx \\ &\le M \cdot (\,|\, t_2 - t_1\,| + h\,), \quad 0 \le t_1, t_2 < T, \end{aligned}$$

where M depends on T, x_1 and x_2.

Let $h_n = T/n$ and $h_n/l_n = \tilde{\delta} < \delta$. Consider the sequence (u^{l_n}, v^{l_n}) $(n = 1, 2, \cdots)$. Then by Theorem 3.1 and Theorem 3.2, there exists a subsequence which converges in L^1_{loc} to functions (u,v). We can prove that u(x,t) and v(x,t) are the weak solutions of initial boundary value problem (1.6), (1.7) and (1.8) provided $\{a_{mn}\}$ is suitably chosen, namely, they satisfy the integral identity

$$\begin{aligned}(3.2)\qquad &\int_0^T \int_0^\infty u\phi_t + \left(\frac{a^2}{v}\right)\phi_x + \frac{2a^2}{1 + \int_0^x v(\zeta, t)d\zeta} \cdot \phi \, dxdt \\ &+ \int_0^\infty u_0(x)\phi(x, 0)dx = 0,\end{aligned}$$

$$(3.3)\qquad \int_0^T \int_0^\infty v\psi_t - u\psi_x \, dxdt + \int_0^\infty v_0(x)\psi(x, 0)dx = 0,$$

for any smooth functions ϕ and ψ with compact support and $\phi(0, t) = 0$.

Thus we get our main result.

Theorem 3.3 *Suppose $u_0(x)$ and $v_0(x)$ are bounded functions of bounded variation, and there exists a positive constant δ_0 such that $v_0(x) \geq \delta_0 > 0$. Then for any $T > 0$, (1.6), (1.7) and (1.8) have weak solutions in* $\{ (x, t);\ 0 < t < T,\ 0 < x < \infty \}$.

References

[1] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations,* Comm. Pure Appl. Math., 18, (1965), 697-715.

[2] T. Nishida, *Global solutions for an initial boundary value problem of a quasilinear hyperbolic system,* Proc. Japan Acad., 44, (1968), 642-646.

[3] L. A. Ying and C. H. Wang, *Global solutions of the Cauchy problem for a nonhomogeneous quasilinear hyperbolic system,* Comm. Pure Appl. Math., 33, (1980), 579-597.

界面化学反応の数理モデルについて

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§1. Introduction

In the present paper we investigate the asymptotic behavior of solutions for parabolic systems closely related with a chemical interfacial reaction model which is considered in Yamada and Yotsutani [5]. Let I and $\bar{I}$ denote the intervals $(0,1)$ and $[0,1]$ respectively. Consider the following problem for unknown functions $u_i = u_i(x,z)$ $(i = 1,2,3$ and $(x,z) \in \bar{I} \times [0,\infty))$:

$$
\begin{aligned}
&(1.1) \quad a_i(x)\frac{\partial u_i}{\partial z} = \frac{\partial^2 u_i}{\partial x^2}, && (x,z) \in \bar{I} \times (0,\infty),\\
&(1.2) \quad \frac{\partial u_i}{\partial x}(0,z) = R_i(u_1(0,z), u_2(0,z), u_3(0,z)), && z \in (0,\infty),\\
&(1.3) \quad \frac{\partial u_i}{\partial x}(1,z) = 0, && z \in (0,\infty),\\
&(1.4) \quad u_i(x,0) = \phi_i(x) \geq 0, && x \in I,
\end{aligned}
$$

where

$$
\begin{aligned}
R_1(u_1,u_2,u_3) &= k_1 R_0(u_1,u_2,u_3),\\
R_2(u_1,u_2,u_3) &= k_2 R_0(u_1,u_2,u_3),\\
R_3(u_1,u_2,u_3) &= -k_3 R_0(u_1,u_2,u_3),\\
R_0(u_1,u_2,u_3) &= ({u_1}^{n_1}{u_2}^{n_2} - {u_3}^{n_3})\beta(u_1,u_2,u_3)
\end{aligned}
$$

with positive constants k_i $(i = 1,2,3)$ and positive integers n_i $(i = 1,2,3)$, $a_i(x)$ $(i = 1,2,3)$ are given functions, and ϕ_i $(i = 1,2,3)$ are given nonnegative initial data.

The above problem, which we call (P), models chemical reactions on interfaces. Such a model has been proposed by Kawano et al. [3]. They put

$$\begin{aligned} &a_i(x) = c_i(1-x^2) \qquad (i = 1,2,3), \\ &R_0(u_1,u_2,u_3) = \frac{u_1u_2 - u_3}{1+u_2+u_1u_2}, \\ &\text{i.e., } \ \beta(u_1,u_2,u_3) = \frac{1}{1+u_2+u_1u_2}, \end{aligned}$$

where c_i $(i = 1,2,3)$ are positive constants. As to the derivation of (P), see also Appendix of [5].

Taking account of the chemical background, we impose the following conditions on a_i and R_i $(i = 1,2,3)$:

(A) $\quad a_i \in C^\infty(\bar{I}),\ \ a_i \geq 0$ on $\bar{I}$ and $a_i > 0$ on $[0,1)$.

(R.1) There exist an open subset U of R^3 and a positive constant δ_R such that

$$\begin{aligned} &U \supset [-\delta_R, \delta_R]^3 \cup [0,\infty)^3, \\ &\beta(u_1,u_2,u_3) \in C^\infty(U) \ \text{ and } \ \beta(u_1,u_2,u_3) > 0 \ \text{ on } U. \end{aligned}$$

(R.2) There exists a positive constant C_R such that

$$-\sum_{i=1}^{3} R_i(u_1,u_2,u_3){u_i}^{2p-1} \leq C_R \sum_{i=1}^{3} {u_i}^{2p}$$

for all $(u_1,u_2,u_3) \in [0,\infty)^3$ and $p \in [1,\infty)$.

§2. Main results

Our first theorem is concerned with the existence, uniqueness, regularity and positivity of solutions of (P).

Theorem 1. *In addition to* (A), (R.1) *and* (R.2), *assume that* $\phi = (\phi_1, \phi_2, \phi_3)$ *satisfies*

$$\phi \in L^\infty(I)^3 \quad \text{and} \quad \phi \geq 0 \ \text{on } I.$$

Then there exists a unique solution $u = (u_1, u_2, u_3)$ *of* (P) *such that*

(i) $u \in C^\infty(\bar{I} \times (0, \infty))^3$,

(ii) (1.1) *is satisfied for every* $(x, z) \in I \times (0, \infty)$,

(iii) (1.2) *and* (1.3) *are satisfied for every* $z \in (0, \infty)$,

(iv) (1.4) *is satisfied in* L^2*-sense, i.e.* $\lim_{z \to 0} \|u(\cdot, z) - \phi(\cdot)\| = 0$.

Moreover u *has the following properties :*

(v) u *is nonnegative and bounded on* $\bar{I} \times [0, \infty)$.

(vi) u *satisfies the law of "mass conservation", i.e.,*

$$\frac{1}{k_i}\|u_i(\cdot, z)a_i\|_1 + \frac{1}{k_3}\|u_3(\cdot, z)a_3\|_1 = E_i, \quad z \in [0, \infty) \quad (i = 1, 2), \tag{2.1}$$

where

$$E_i = \frac{1}{k_i}\|\phi_i a_i\|_1 + \frac{1}{k_3}\|\phi_3 a_3\|_1 \qquad (i = 1, 2), \tag{2.2}$$

(vii) *the positivity of* u_1, u_2 *and* u_3 *is assured by that of* E_1 *and* E_2, *i.e.,*

$$\begin{array}{lll}
u_1 > 0, u_2 > 0 \text{ and } u_3 > 0 & \text{on } \bar{I} \times (0, \infty), & \text{if } E_1 > 0 \text{ and } E_2 > 0, \\
u_1 > 0 \text{ and } u_2 \equiv u_3 \equiv 0 & \text{on } \bar{I} \times (0, \infty), & \text{if } E_1 > 0 \text{ and } E_2 = 0, \\
u_2 > 0 \text{ and } u_1 \equiv u_3 \equiv 0 & \text{on } \bar{I} \times (0, \infty), & \text{if } E_1 = 0 \text{ and } E_2 > 0, \\
u_1 \equiv u_2 \equiv u_3 \equiv 0 & \text{on } \bar{I} \times (0, \infty), & \text{if } E_1 = 0 \text{ and } E_2 = 0.
\end{array}$$

In the study of asymptotic properties of (P), we should note (2.1). Therefore, the stationary problem (SP) associated with (P) will be written in the form of algebraic equations for $u^\infty = (u_1^\infty, u_2^\infty, u_3^\infty) \in \mathbf{R}^3$:

$$u^\infty \geq 0, \tag{2.3}$$

$$R_0(u^\infty) = 0, \tag{2.4}$$

$$\frac{\|a_i\|_1}{k_i} u_i^\infty + \frac{\|a_3\|_1}{k_3} u_3^\infty = E_i \qquad (i = 1, 2). \tag{2.5}$$

Concerning (SP), we get the following theorem.

Theorem 2. *Suppose that* (A) *and* (R.1) *hold. Then there exists a unique solution* $u^\infty = (u_1^\infty, u_2^\infty, u_3^\infty)$ *of* (SP). *More precisely,*

$$\begin{aligned} &u_1^\infty > 0, u_2^\infty > 0 \text{ and } u_3^\infty > 0, \\ &\qquad \text{if } E_1 > 0 \text{ and } E_2 > 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &u_1^\infty = \frac{\|\phi_1 a_1\|_1}{\|a_1\|_1} > 0 \text{ and } u_2^\infty = u_3^\infty = 0, \\ &\qquad \text{if } E_1 > 0 \text{ and } E_2 = 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &u_2^\infty = \frac{\|\phi_2 a_2\|_1}{\|a_2\|_1} > 0 \text{ and } u_1^\infty = u_3^\infty = 0, \\ &\qquad \text{if } E_1 = 0 \text{ and } E_2 > 0, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &u_1^\infty = u_2^\infty = u_3^\infty = 0, \\ &\qquad \text{if } E_1 = 0 \text{ and } E_2 = 0. \end{aligned} \tag{2.9}$$

Now we are ready to state results on the asymptotic behavior of solutions for (P) as $z \to \infty$.

Theorem 3. *Suppose that* (A), (R.1) *and* (R.2) *hold. It follows that*

$$\lim_{z \to \infty} \|u(\cdot, z) - u^{\infty}\|_{\infty} = 0.$$

Theorem 4. *Suppose that* (A), (R.1) *and* (R.2) *hold. It follows that*

$$\lim_{z \to \infty} \|D_z{}^i D_x{}^j u\|_{\infty} = 0$$

for all nonnegative integers i, j *with* $(i, j) \neq (0, 0)$.

Remark 2.1. The rates of the convergence stated in Theorems 3 and 4 are all exponential.

References

[1] A. Friedmann, *Remarks on the maximum principle for parabolic equations and its applications*, Pacific J. Math. 8 (1958), 201–211.

[2] M. Iida, Y. Yamada and S. Yotsutani, *Convergence of solutions of a chemical interfacial reaction model*, preprint.

[3] Y. Kawano, K. Kusano, K. Kondo and F. Nakashio, *Extraction rate of acetic acid by long-chain alkylamine in horizontal rectangular channel*, Kagaku Kogaku Ronbunshu 9 (1983), 44–51.

[4] F. Rothe, "Global Solutions of Reaction-Diffusion Systems," Lec. Notes in Math. Vol. 1072, Springer-Verlag, Berlin/New York, 1984.

[5] Y. Yamada and S. Yotsutani, *A mathematical model on chemical interfacial reactions*, Japan J. Appl. Math. 7 (1990), 369–398.

Viscosity solutions of nonlinear elliptic PDEs associated with impulse control problems

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§1. Introduction

In this article we consider the following nonlinear elliptic partial differential equation (PDE) with implicit obstacle:

$$(1.1) \qquad \begin{cases} \max\{-\Delta u + u - f, u - Mu\} = 0 & \text{in} \quad \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and M is the nonlocal operator defined as follows:

$$Mu(x) = 1 + \inf\{u(x+\xi) \mid \xi \in (\mathbb{R}^+)^N, x + \xi \in \overline{\Omega}\}.$$

The equation (1.1) is associated with the impluse control problem, whose state is governed by stochastic differential equation with implusive jumps and whose value function has implusive costs.

Concerning the existence and uniqueness of solutions of (1.1), see A. Bensoussan - J. L. Lions [1] and B. Perthame [5] etc. They obtained them by assuming that there exists a subsolution $\underline{u}$ satisfying $\underline{u} \leqq g \leqq M\underline{u}$ on $\partial\Omega$. Without this assumption B. Perthame [6] showed the existence and uniqueness of viscosity solutions.

Our main purpose is to get the comparison principle and existence of viscosity solutions of (1.1) by applying the results in H. Ishii - P. L. Lions [3] and M. G. Crandall - H. Ishii - P. L. Lions [2]. **We emphasize** that by these methods we can treat the nonlinear PDEs of the type (1.1) whose principle parts are in some classes of general (possibly degenerate) elliptic operators.

§2. Assumptions and Definitions

We make the following assumptions.

(A.1) $\Omega \in \mathbb{R}^N$ is a bounded and convex domain with smooth boundary $\partial\Omega$.

(A.2) $f,\ g \in C(\overline{\Omega})$ and $f,\ g \geqq 0$ on $\overline{\Omega}$.

Let $\mathcal{O}$ be a subset of $\mathbb{R}^N$. For any function $u : \mathcal{O} \to \mathbb{R} \cup \{-\infty, +\infty\}$, we define the function $u^*,\ u_* : \mathcal{O} \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$u^*(x) = \lim_{r \to 0} \sup\{u(y) \mid y \in \mathcal{O},\ |y - x| < r\}, \quad u_* = -(-u)^*$$

For each $x \in \mathcal{O}$, we set

$$J_{\mathcal{O}}^{2,+}u(x) = \left\{ (p, X) \in \mathbb{R}^N \times \$^N \;\middle|\; u(y) \leqq u(x) + \langle p, y - x\rangle \right.$$
$$\left. + \frac{1}{2}\langle X(y - x), y - x\rangle + o(|y - x|^2) \quad \text{as } \mathcal{O} \ni y \to x \right\}$$

and $J_{\mathcal{O}}^{2,-}u(x) = -J_{\mathcal{O}}^{2,+}(-u(x))$. Here $\$^N$ denotes the set of all $N \times N$ real symmetric matrices and $\langle \cdot, \cdot \rangle$ is the Euclidian inner product in $\mathbb{R}^N$. We denote by $\bar{J}_{\mathcal{O}}^{2,+}u(x)$ and $\bar{J}_{\mathcal{O}}^{2,-}u(x)$ the following sets:

$$\bar{J}_{\mathcal{O}}^{2,+}u(x) = \{(p, X) \in \mathbb{R}^N \times \$^N \mid {}^{\exists}(x_n, p_n, X_n) \in \Omega \times \mathbb{R}^N \times \$^N$$
$$\text{such that } (p_n, X_n) \in J_{\mathcal{O}}^{2,+}u(x_n) \text{ and}$$
$$(x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X) \text{ as } n \to +\infty\},$$

and $\qquad \bar{J}_{\mathcal{O}}^{2,-}u(x) = -\bar{J}_{\mathcal{O}}^{2,+}(-u(x)).$

We give the definition of viscosity solutions of the nonlinear degenerate elliptic PDEs including the nonlocal operator M:

$$F(x, u, Du, D^2u, Mu) = 0 \qquad \text{in} \quad \Omega, \tag{2.1}$$

where F is a continuous function on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \$^N \times \mathbb{R}$ satisfying the degenerate ellipticity condition:

$$F(x, r, p, X + Y, s) \leqq F(x, r, p, X, s) \qquad \text{for all } x \in \Omega,\ r,\ s \in \mathbb{R},$$
$$p \in \mathbb{R}^N,\ X, Y \in \$^N \text{ and } Y \geqq O.$$

Definition. *Let u be a function defined on $\overline{\Omega}$.*

(1) *u is a viscosity subsolution (resp., supersolution) of* (2.1) *if $u^*(x) < \infty$ (resp., $u_*(x) > -\infty$) on $\overline{\Omega}$ and*

$$F(x, u^*(x), p, X, Mu^*(x)) \leqq 0 \text{ (resp., } F(x, u_*(x), p, X, Mu_*(x)) \geqq 0 \text{)}$$

for all $x \in \Omega$, $(p, X) \in \bar{J}_{\Omega}^{2,+}u^*(x)$ (resp., $(p, X) \in \bar{J}_{\Omega}^{2,-}u_*(x)$).

(2) *u is a viscosity solution of* (2.1) *if u is a viscosity subsolution and supersolution of* (2.1).

§3. Main results

Our main results are stated as follows.

Theorem 3.1. *Assume* (A.1) *and* (A.2). *Let u and v be, respectively, a viscosity subsolution and a supersolution of* (1.1). *If u and v satisfy*

$$\max\{u^* - g, u^* - Mu^*\} \leqq 0 \text{ and } \max\{v_* - g, v_* - Mv_*\} \geqq 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

then $u^* \leqq v_*$ on $\overline{\Omega}$.

Theorem 3.2. *Assume* (A.1) *and* (A.2). *Then there exist a unique viscosity solution $u \in C(\overline{\Omega})$ of* (1.1) *satisfying* $\max\{u - g, u - Mu\} = 0$ *on $\partial\Omega$.*

In what follows we mention the sketch of the proofs of Theorems.

Proof of Theorem 3.1. We assume u (resp., v) is upper (resp., lower) semicontinuous on $\overline{\Omega}$. We use some perturbation of viscosity subsolution. For each $m \in \mathbb{N}$, $u_m = (1 - 1/m)u - 1/m$ is a viscosity subsolution of the nonlinear PDE:

$$\begin{cases} \max\{-\Delta u_m + u_m - f, u_m - Mu_m\} + \dfrac{1}{m} = 0 & \text{in } \Omega, \\ \max\{u_m - g, u_m - Mu_m\} + \dfrac{1}{m} \leqq 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

In order to prove $u_m \leqq v$ on $\overline{\Omega}$ for all $m \in \mathbb{N}$, we shall suppose the contrary, i.e., $\sup_{\overline{\Omega}}(u_{m_0} - v) = \theta > 0$ for some $m_0 \in \mathbb{N}$. We take $z \in \overline{\Omega}$ such that $\theta = u_{m_0}(z) - v(z)$. Then by (3.1) and (3.2), we may consider $z \in \Omega$.

Let $\Phi(x,y)$ be a function defined by

$$\Phi(x,y) = u_{m_0}(x) - |x-z|^4 - v(y) - \frac{1}{2\varepsilon}|x-y|^2 \qquad \text{on} \quad \overline{\Omega \times \Omega}$$

and let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega \times \Omega}$ be a maximum point of $\Phi(x,y)$. From the inequality $\theta \leqq \Phi(x_\varepsilon, y_\varepsilon)$ and the semicontinuity of u_{m_0}, v, we have the behaviors of x_ε, y_ε, $u_{m_0}(x_\varepsilon)$, and $v(y_\varepsilon)$ as $\varepsilon \to 0$:

$$x_\varepsilon, y_\varepsilon \to z, \quad u_{m_0}(x_\varepsilon) \to u_{m_0}(z), \quad v(y_\varepsilon) \to v(z), \quad \frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \to 0.$$

Thus we can consider x_ε, $y_\varepsilon \in \Omega$. Moreover there exist X_ε, $Y_\varepsilon \in \mathbb{S}^N$ such that

$$\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), X_\varepsilon\right) \in \bar{J}^{2,+}_\Omega(u_{m_0}(x_\varepsilon) - |x_\varepsilon - z|^4), \ \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), Y_\varepsilon\right) \in \bar{J}^{2,-}_\Omega v(y_\varepsilon),$$
$$-\frac{3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X_\varepsilon & O \\ O & -Y_\varepsilon \end{pmatrix} \leqq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \quad (\ I = \text{identity matrix}\ ).$$

We remark that $((x_\varepsilon - y_\varepsilon)/\varepsilon + 4|x_\varepsilon - z|^2(x_\varepsilon - z), X_\varepsilon + Z_\varepsilon) \in \bar{J}^{2,+}_\Omega u_{m_0}(x_\varepsilon)$ $(Z_\varepsilon = 4|x_\varepsilon - z|^2 I + 8(x_\varepsilon - z)\otimes(x_\varepsilon - z))$. Hence using the facts that u_{m_0} and v are viscosity subsolution of (3.2) and supersolution of (1.1), respectively, we obtain the following inequalities:

$$\max\{-tr(X_\varepsilon + Z_\varepsilon) + u_{m_0}(x_\varepsilon) - f(x_\varepsilon), u_{m_0}(x_\varepsilon) - Mu_{m_0}(x_\varepsilon)\} + \frac{1}{m_0} \leqq 0$$
$$\max\{-trY_\varepsilon + v(y_\varepsilon) - f(y_\varepsilon), v(y_\varepsilon) - Mv(y_\varepsilon)\} \geqq 0$$

From these inequalities, we get a contradiction. Therefore we have $u_m \leqq v$ on $\overline{\Omega}$ for all $m \in \mathbb{N}$. Letting $m \to \infty$, we obtain the result. ∎

Proof of Theorem 3.2. It is easily seen that there exist a viscosity subsolution $\underline{u}$ and supersolution $\overline{u}$ of (1.1) such that $\max\{\underline{u}^* - g, \underline{u}^* - M\underline{u}^*\} \leqq 0$ and $\max\{\overline{u}_* - g, \overline{u}_* - M\overline{u}_*\} \geqq 0$, respectively on $\partial\Omega$.

We define the set $\mathcal{S}$ and the function u as the following:

$$\mathcal{S} = \{u : \text{viscosity subsolution of } (1.1) \mid \max\{u^* - g, u^* - Mu^*\} \leqq 0 \text{ on } \partial\Omega\},$$

$$u(x) = \sup\{v(x) \mid v \in \mathcal{S}\} \quad (x \in \overline{\Omega}).$$

We observe that Perron's method can be used (cf. M. G. Crandall - H. Ishii - P. L. Lions [2]). Therefore we obtain that u is a viscosity solution of (1.1) satisfying

$$\max\{u^* - g, u^* - Mu^*\} \leqq 0 \qquad \text{on} \quad \partial\Omega. \tag{3.3}$$

On the other hand, using the barrier argument we get

$$\max\{u_* - g, u_* - Mu_*\} \geqq 0 \qquad \text{on} \quad \partial\Omega. \tag{3.4}$$

Hence it follows from Theorem 3.1 that $u^* = u = u_*$ on $\overline{\Omega}$ and thus $u \in C(\overline{\Omega})$. Then (3.3) and (3.4) yields $\max\{u - g, u - Mu\} = 0$ on $\partial\Omega$. Theorem 3.1 also implies the uniqueness of viscosity solutions of (1.1) satisfying the boundary condition. ∎

For the detail, see the author [4].

References

[1] Bensoussan, A. and Lions, J. L., *Contrôlle impulsionnel et Inéquations Quasi-Variationnelles*, Dunod, Paris, 1982.

[2] Crandall, M. G., Ishii, H. and Lions, P. L., User's guide to viscosity solutions of second order partial differential equations, preprint.

[3] Ishii, H. and Lions, P. L., Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential Equations, **83** (1990), 26–78.

[4] Ishii, K., Viscosity solutions of nonlinear second order elliptic PDEs associated with impulse control problems, to appear in Funkcial. Ekvac.

[5] Perthame, B., Quasi-variational inequalities and Hamilton-Jacobi-Bellman equations in a bounded region, Comm. Partial Differential Equations, **9** (1984), 561–595.

[6] ——————, Some remarks on quasi-variational inequalities and the associated impulsive control problems, Ann. Inst. Henri. Poincaré, **2** (1985), 237–260.

Generalized interface evolution with the Neumann boundary condition

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1. Introduction.

We are concerned with geometric evolution (e.g. motion by mean curvature) of interfaces in a smoothly bounded domain $\Omega(\subset \mathbb{R}^n)$ whose boundary $\partial\Omega$ perpendicularly intersects with interfaces. In [12] the second author extended a level set approach introduced by Chen-Giga-Goto [1] and Evans-Spruck [3] to this type of the Neumann problem and obtained a unique global weak solutions for the initial value problem provided that Ω is convex. This note reports that the convexity assumption of Ω can be removed. The details and proofs will appear elsewhere.

One of key ingredients is the comparison principle for the Neumann boundary value problem for singular degenerate parabolic equations. For the Neumann problem this principle is first established by Lions [10] for the Hamilton-Jacobi equations. For nonsingular degenerate elliptic equations the comparison principle is established by Ishii and Lions [9]. See also [8] for more general oblique boundary conditions. However, their argument does not apply to singular equations. In [12] the second author obtained the comparison principle for our problem assuming that Ω is convex. His method appeals to the idea of [6] by regarding $\partial\Omega$ as space infinity. Unfortunately, the choice of test functions does not apply to general domains. In this note we construct test functions by using local coordinate patches near $\partial\Omega$ so that they apply to general domains.

In [7] Huisken considers the interface intersecting perpendicularly with $\partial\Omega$ and moving by mean curvature. He constructed a global smooth evolution of interfaces when Ω is a cylindical domain $D \times \mathbb{R}$ and the initial interface is the graph of a smooth function on D, where D is bounded. Although our theory presented below assumes that Ω is bounded, it can be extended to cylindical domain $D \times \mathbb{R}$ provided D is bounded. The motion by mean curvature with right contact angle at $\partial\Omega$ arises as a singular limit of a reaction-diffusion equation with the Neumann condition [11].

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2. Comparison principle.

We here present a simple and typical version of our comparison principle rather than stating its general form to avoid technical complexity. We consider an evolution equation of the form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \Omega \tag{1}$$

$$\partial u/\partial \nu = 0 \quad \text{on} \quad S = (0,T] \times \partial\Omega, \tag{2}$$

where $\partial/\partial\nu$ denotes the outer normal derivative on $\partial\Omega$; $u_t = \partial u/\partial t$, $\nabla u = \text{grad } u$; $\nabla^2 u$ denotes the Hessian of u in the space variables. We list assumptions on F.

(F1) $F : (\mathbb{R}^n \setminus \{0\}) \times \mathbf{S}^n \to \mathbb{R}$ is continuous, where $\mathbf{S}^n$ denotes the space of $n \times n$ real symmetric matrices equipped with usual ordering.

(F2) F is degenerate elliptic, i.e., $F(p, X+Y) \leq F(p, X)$ for all $Y \geq 0$.

(F3) $-\infty < F_*(0, O) = F^*(0, O) < \infty$ where F_* and F^* are the lower and upper semicontinuous relaxation (envelope) of F on $\mathbb{R}^n \times \mathbf{S}^n$, respectively, i.e.,

$$F_*(p, X) = \liminf_{\varepsilon \downarrow 0} \{F(q, Y); q \neq 0, |p-q| \leq \varepsilon, |X - Y| \leq \varepsilon\}$$

and $F^* = -(-F)_*$. Here $|X|$ denotes the operator norm of X as a self adjoint operator on $\mathbb{R}^n$.

THEOREM 1. *Let Ω be a smoothly bounded domain in $\mathbb{R}^n$. Suppose that F satisfies (F1)-(F3). Let u and v be, respectively, viscosity sub- and supersolutions of (1)-(2). If $u^*(0,x) \leq v_*(0,x)$, then $u^* \leq v_*$ on $\overline{Q}$.*

A definition of a viscosity (sub) solution for the Neumann problem goes back to [10] where the Hamilton-Jacobi equation is studied. We recall a definition of viscosity subsolution of (1)-(2) for the reader's convenience. We refer to [2] and [8] for nonsingular equations. Any function $u : Q \cup S \to \mathbb{R}$ is called a ***viscosity subsolution*** of (1)-(2) if $u^* < \infty$ on $Q \cup S$ and if, whenever $\phi \in C^2(Q \cup S)$, $(t,x) \in Q \cup S$ and $(u^* - \phi)(t,x) = \max_{Q \cup S}(u^* - \phi)$, one of the following holds

$$\phi_t(t,x) + F_*(\nabla\phi(t,x), \nabla^2\phi(t,x)) \leq 0 \tag{3}$$

$$(\partial\phi/\partial\nu)(t,x) \leq 0 \quad \text{and} \quad x \in \partial\Omega. \tag{4}$$

For example a function $u(t,x) = -2t - |x|^2$ a viscosity subsolution (actually solution) of (1)-(2) with

$$F(p, X) = -\operatorname{trace}((I - p \otimes p/|p|^2)X) \tag{5}$$

on an annulus Ω in $\mathbb{R}^2$ although $\partial u/\partial\nu \leq 0$ may not hold on the inner circle of $\partial\Omega$ in usual sense. One should be careful the meaning of (2).

3. Test functions.

The basic strategy of the proof of Theorem 1 is to find a parabolic super 2-jet of

$$w(t,x,y) = u(t,x) - v(t,y)$$

at a point where $u^* > v_*$. This idea is the same as in [6] and we also apply the Crandall-Ishii lemma (see e.g. [2]). Since it is difficult to compare boundary condition (4), we take a barrier near the boundary to avoid to handle (4). This idea is found in [12].

For ε, δ, $\gamma > 0$ we set

$$\begin{aligned}
\Phi(t,x,y) &= w(t,x,y) - \Psi(t,x,y) \\
\Psi(t,x,y) &= \Xi(x,y)/\varepsilon + B(t,x,y) \\
B(t,x,y) &= \delta(\varphi(x) + \varphi(y) + 2\beta) + \gamma/(T-t).
\end{aligned}$$

Here $\varphi \in C^2(\overline{\Omega})$ is a 'barrier' function of $\partial\Omega$ satisfying:

$$\begin{aligned}
&-\beta \leqslant \varphi < 0 \quad \text{in} \quad \Omega, \quad \varphi = 0 \quad \text{on} \quad \partial\Omega \quad \text{with a constant} \quad \beta > 0 \\
&\nu(x) = \nabla\varphi(x)/|\nabla\varphi(x)| \quad \text{and} \quad |\nabla\varphi(x)| \geq 1 \quad \text{on} \quad \partial\Omega.
\end{aligned}$$

If $\Xi \in C^2(\overline{\Omega} \times \overline{\Omega})$ satisfies following conditions, the method of [6] applies to establish Theorem 1.

(C1) $\quad \Xi(x,y) \geq c_0|x-y|^4 \quad \text{with} \quad c_0 > 0.$

(C2) $\quad |\Xi_x + \Xi_y| \leq c_1|x-y|^4, |\Xi_x|, |\Xi_y| \leq c_2|x-y|^3.$

(C3) $\quad |\Xi_{xx} + \Xi_{xy} + \Xi_{yx} + \Xi_{yy}| \leq c_3|x-y|^4.$

(C4) $\quad |\Xi_{xx}|, |\Xi_{xy}|, |\Xi_{yx}|, |\Xi_{yy}| \leq c_4|x-y|.$

(C5) $\quad \langle \nu(x), \Xi_x(x,y)\rangle \geq 0 \quad \text{for} \quad x \in \partial\Omega, \ y \in \overline{\Omega}$

$\langle \nu(y), -\Xi_y(x,y)\rangle \leq 0 \quad \text{for} \quad y \in \partial\Omega, \ x \in \overline{\Omega}$

provided that $|x-y|$ is sufficiently small.

Here $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^n$. If Ω is convex, then $\Xi(x,y) = |x-y|^4$ satisfies (C1)-(C5). However, for nonconvex Ω, this choice of Ξ violates (C5).

LEMMA 2. *There exists Ξ satisfying (C1)-(C5).*

Sketch of the proof. For each $a \in \partial\Omega$ there is a local coordinate $\chi_a = (\chi^1, \cdots, \chi^n)$ such that $\chi^n(x) = \mathrm{dist}(x, \partial\Omega)$ for $x \in \overline{\Omega}$. Let ψ_a be a cut-off function supported near $a \in \partial\Omega$ so that $\partial\psi_a/\partial\nu = 0$ on $\partial\Omega$. We set

$$\Lambda_a(x,y) = \psi_a(x)\psi_a(y)|\chi_a(x) - \chi_a(y)|^4.$$

Let ψ_0 be a cut-off function supported outside the boundary. We set

$$\Lambda_0(x,y) = \psi_0(x)\psi_0(y)|x-y|^4.$$

One can take finitely many $\{a_k\}_{k=1}^{\ell}$ so that the sum $\sum_{k=0}^{\ell} \Lambda_k$ satisfies (C1)-(C5) provided that $|x-y|$ is sufficiently small. Here $\Lambda_k = \Lambda_a$ with $a = a_k$. We set

$$\Xi(x,y) = \rho(|x-y|)|x-y|^4 + (1-\rho(|x-y|))\sum_{k=0}^{\ell} \Lambda_k(x,y)$$

with a cut-off function $\rho(\sigma)$ supported away from $\sigma = 0$. One observes that Ξ satisfies (C1)-(C5).

4. Interface evolution.

We remark that the theory in [1] and [4] can be extended to the motion of interfaces intersecting perpendicularly with $\partial\Omega$. The next lemma is fundamental to establish global solution for the initial value problem of (1)-(2) by Perron's method.

LEMMA 3([12]). *Assume the hypotheses of Theorem 1 concerning F. Suppose that F is geometric. Then for $u_0 \in C(\overline{\Omega})$ there are viscosity sub-and supersolutions $u_-, u_+ \in C([0,T] \times \overline{\Omega})$ of (1)-(2) with $u_\pm(0,x) = u_0(x)$.*

Although our theory applies to general interface equations as in [4], we state our results only for the motion by mean curvature just for simplicity.

THEOREM 4. *Let D_0 be an open set in Ω. Let $u_0 \in C(\overline{\Omega})$ satisfies $D_0 = \{x; u_0(x) > 0\}$. There is a unique viscosity solution $u \in C([0,\infty) \times \overline{\Omega})$ for (1)-(2) with (5) for arbitrary $T > 0$ such that $u(0,x) = u_0(x)$. The set $D = \{(t,x); u(t,x) > 0\}$ is determined by D_0 and called a generalized evolution by mean curvature with initial data D_0 and the right angle boundary condition.*

REMARK 5: In [4] $D(t)$ is determined by D_0 and $\Gamma_0 = \{u_0(x) = 0\}$. It turns out $D(t)$ is completely determined by D_0 as shown in [5].

REMARK 6: If we take Ξ as sketched in §3, we need $C^{3,1}$ regularity of $\partial\Omega$. However, by taking χ more clever way, we only need C^2 regularity of $\partial\Omega$ to establish Lemma 2.

References

[1] Y.-G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geometry **33** (1991), 749-786.

[2] M.G. Crandall, H. Ishii and P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations.* (preprint).

[3] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature I*, J. Differential Geometry **33** (1991), 635-681.

[4] Y. Giga and S. Goto, *Motion of hypersurfaces and geometric equations*, J. Math. Soc. Japan **44** (1992) (to appear).

[5] Y. Giga, S. Goto and H. Ishii, *Global existence of weak solutions for interface equations coupled with diffusion equations.* (preprint).

[6] Y. Giga, S. Goto, H. Ishii and M.-H.Sato, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains*, Indiana Univ. Math. J. **40** (1991). (in press).

[7] G. Huisken, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equations **77** (1989), 369-378.

[8] H. Ishii, *Fully nonlinear oblique derivative problems for nonlinear second-order elliptic PDE's*, Duke Math. J **62** (1991), 633-661.

[9] H. Ishii and P.L. Lions, *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations*, J. Differential Equations **83** (1990), 26-78.

[10] P.L. Lions, *Neumann type boundary conditions for Hamilton-Jacobi equations*, Duke Math. J. **52** (1985), 793-820.

[11] J. Rubinstein, P. Sternberg and J.B. Keller, *Fast reaction, slow diffusion, and curve shortening*, SIAM J. appl. Math. **49** (1989), 116-133.

[12] M.-H. Sato, *Interface evolution with Neumann condition.* in preparation.

多次元多相ステファン問題の周期解の構造について

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0. 序

多次元多相ステファン問題はエンタルピー u を用いて次のように表現される:

$$u_t - \triangle\beta(u) = 0 \quad \text{in } Q = I \times \Omega. \tag{1}$$

ここで Ω は R^N の有界領域, その境界 $\partial\Omega = \Gamma$ は滑らかとし, I は区間を表すとする. 以下では T を与えられた正定数とし, T を周期とする周期問題を考える.

例えば, 境界 Γ が $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ となっているとき,

$$\beta(u) = g_0 \quad \text{on } \Sigma_0 = I \times \Gamma_0,$$

$$\frac{\partial\beta(u)}{\partial n} + p(x)\beta(u) = g_1 \quad \text{on } \Sigma_1 = I \times \Gamma_1$$

なる混合境界条件を考える. ここで g_0, g_1, p は与えられた関数で $p \geq 0$, $g_i(t+T,x) = g_i(t,x)$ $(i=0,1)$ を満たすとする. この場合には Damlamian-Kenmochi [2] によって周期解の存在, 漸近安定性などの結果が得られている.

また, $h(t+T,x) = h(t,x)$ のとき, 境界条件が非斉次ノイマン条件

$$\frac{\partial\beta(u)}{\partial n} = h \quad \text{on } \Sigma = I \times \Gamma \tag{2}$$

の場合には Haraux-Kenmochi [3] において, 劣微分作用素を用いて周期解の存在, 構造等の結果が述べられている.

本講演では, Aiki-Kenmochi-Shinoda [1], Shinoda [6] にしてがって, $g = g(t,x,\xi)$ が $g(t+T,x,\xi) = g(t,x,\xi)$ を満たすとき, 非線形ノイマン条件

$$\frac{\partial\beta(u)}{\partial n} + g(t,x,\beta(u)) = 0 \quad \text{on } \Sigma \tag{3}$$

に対する周期解について述べる. 以下では, システム $\{(1),(3)\}$ を問題 P と呼ぶことにする.

1. 仮定と準備

$\beta = \beta(r) : \mathbb{R} \to \mathbb{R}$ は次の条件を満たす.

(4) β は非減少, リプシッツ連続関数で $\beta(0)=0$ かつ $\liminf_{|r|\to\infty}\dfrac{\beta(r)}{r}>0$ を満たす.

また $g=g(t,x,\xi):\mathbb{R}\times\Gamma\times\mathbb{R}\to\mathbb{R}$ に対しては次の (5)~(9) を仮定する.

(5) 全ての $M\geq 0$ に対してある正数 $C_g(M)$ が存在して $|\xi|, |\xi'|\leq M$ を満たす全ての ξ, ξ' とほとんど全ての $(t,x)\in\mathbb{R}\times\Gamma$ に対して

$$|g(t,x,\xi)-g(t,x,\xi')|\leq C_g(M)|\xi-\xi'|.$$

(6) 全ての $\xi\in\mathbb{R}$ に対して $g(\cdot,\cdot,\xi)\in L^2_{loc}(\mathbb{R};L^2(\Gamma))$.

(7) $M_1\leq M_2$ なる定数 M_1, M_2 が存在してほとんど全ての $(t,x)\in\mathbb{R}\times\Gamma$ に対して

$$g(t,x,\beta(M_1))\leq 0,\quad g(t,x,\beta(M_2))\geq 0.$$

(8) ほとんど全ての $(t,x)\in\mathbb{R}\times\Gamma$ と全ての $\xi\in\mathbb{R}$ に対して

$$g(t+T,x,\xi)=g(t,x,\xi).$$

(9) ほとんど全ての $(t,x)\in\mathbb{R}\times\Gamma$ に対して $g(t,x,\xi)$ は ξ に関して単調増加である.

簡単のため記号の約束をする. $H=L^2(\Omega)$ とし, その内積とノルムをそれぞれ $(\cdot,\cdot)$, $|\cdot|$ で定義する. 更に $V=H^1(\Omega)$ に対してはその双対を $\langle\cdot,\cdot\rangle$ で定義する.

次に弱解の定義を述べる.

定義 1. $J\subset\mathbb{R}$ を有界閉区間とする. $u:J\to H$ が J 上問題 P の弱解であるとは次の (10), (11) を満たすことである.

(10) u は J から H への弱連続関数で $u\in L^\infty(J\times\Omega)\cap W^{1,2}(J;V')$ かつ $\beta(u)\in L^2(J;V)$.

(11) ほとんど全ての $t\in J$ と任意の $z\in V$ に対して

$$\langle u'(t),z\rangle+(\nabla\beta(u(t)),\nabla z)+\int_\Gamma g(t,\cdot,\beta(u(t)))zd\Gamma=0.$$

定義 2. J' を $\mathbb{R}$ の任意の区間とする. このとき, $u:J'\to H$ が J' 上問題 P の弱解であるとは $J\subset J'$ なる任意の有界閉区間 J に対して u が J 上問題 P の弱解になるときをいう.

定義 3. $J'=[t_0,t_1]$ または $[t_0,\infty)$ とし $u_0\in H$ とする. $u:J'\to H$ がコーシー問題 $CP(u_0)$ の解であるとは u が J' 上 P の弱解でありかつ $u(t_0)=u_0$ を満たすことをいう. また $u:\mathbb{R}\to H$ が T-周期解であるとは u が $\mathbb{R}$ 上の問題 P の弱解で任意の $t\in\mathbb{R}$ に対して $u(t+T)=u(t)$ を満たすことをいう. 問題 P に対する T-周期解の全体を $\mathcal{P}_T$ で表わす.

まず最初に $CP(u_0)$ に対する結果を述べる.

定理 1 (cf. [4,5,7]). 任意に $t_0 \in \mathbf{R}$ を与える. このとき (4)~(7) のもとで $M_1 \leq u_0 \leq M_2$ を満たす任意の $u_0 \in L^\infty$ に対して $J = [t_0, \infty)$ 上 $CP(u_0)$ の弱解 u が唯ひとつ存在し

$$M_1 \leq u \leq M_2 \quad \text{a.e. on } J \times \Omega.$$

また後のために次の定理を用意しておく.

定理 2 (cf. [1,3]). $h \in L^2_{loc}(\mathbf{R}; L^2(\Gamma))$ は $\mathbf{R} \times \Gamma$ 上 $h(t+T, x) = h(t, x)$ で、更に $\int_0^T \int_\Gamma h(t, x) d\Gamma dt = 0$ を満たすとする. このとき次の (i)~(iv) が成立する.

(i) 任意の $a_0 \in \mathbf{R}$ に対して T-周期解 ω が存在して $\int_\Omega \omega(0, x)dx = a_0$.

(ii) ω を $\mathbf{R}$ 上のシステム {(1),(2)} の弱解とする. このとき ω が T-周期解であるための必要十分条件は $\omega \in L^\infty(\mathbf{R}; H)$ となることである.

(iii) ω_1, ω_2 を $\int_\Omega \omega_1(0, x)dx \leq \int_\Omega \omega_2(0, x)dx \cdots (*)$ を満たす $\mathbf{R}$ 上の T-周期解とする. このとき

$$\beta(\omega_1) \leq \beta(\omega_2) \quad \text{a.e. on } \mathbf{R} \times \Omega.$$

特に (*) において等号が成立するときは $\int_\Omega \zeta(x)dx = 0$ を満たす $\zeta \in H$ が存在して

$$\omega_1(t, x) - \omega_2(t, x) = \zeta(x) \quad \text{a.e. on } \mathbf{R} \times \Omega.$$

(iv) システム {(1),(2)} の $[t_0, \infty)$ 上の任意の弱解 u に対して T-周期解 ω が存在して

$$u(t) - \omega(t) \to 0 \quad \text{in } H \text{as } n \to \infty,$$

$$\beta(u(nT + \cdot) \to \beta(\omega) \quad \text{in } L^2(0, T; V) \text{as } n \to \infty.$$

3. 結果

定理 3. (4)~(8) のもとで $\omega \in \mathcal{P}_T$ が存在して

$$M_1 \leq \omega \leq M_2 \quad \text{a.e. on } \mathbf{R} \times \Omega.$$

以下では (4)~(9) を仮定する. このとき以下のことが結論される.

定理 4. 全ての $\omega_1, \omega_2 \in \mathcal{P}_T$ に対して次の (i)~(iii) が成立する.

(i) $g(\cdot, \cdot, \beta(\omega_1)) = g(\cdot, \cdot, \beta(\omega_2))$ a.e. on $\mathbf{R} \times \Gamma$.

(ii) $\int_\Omega \omega_1(0, x)dx \leq \int_\Omega \omega_2(0, x)dx$ ならば $\beta(\omega_1) \leq \beta(\omega_2)$ a.e. on $\mathbf{R} \times \Omega$.

(iii) $\int_\Omega \omega_1(0, x)dx < \int_\Omega \omega_2(0, x)dx$ のとき $\int_\Omega \omega_1(0, x)dx < a_0 < \int_\Omega \omega_2(0, x)dx$ を満たす任意の $a_0 \in \mathbf{R}$ に対して $\omega \in \mathcal{P}_T$ が存在して $a_0 = \int_\Omega \omega(0, x)dx$ となる.

注意. (9) を除くともはや定理 4 は成立しない. また (9) を仮定したとしても一般には $\{\omega \in \mathcal{P}_T\}$ は全順序集合にはならない.

例 1. $\Omega = (0,2)$ とし

$$\beta(r) = \begin{cases} r-1 & r \geq 1, \\ 0 & 0 < r < 1, \\ r & r \leq 0, \end{cases}$$

$g(t,x,\xi) = 2\xi^3 - \xi$ とおくと $\omega_1(t,x) \equiv 0$, $\omega_2(t,x) = x + \chi_{\mathbb{R}_+}(x-1) - 1$ はそれぞれ T-周期解となるが定理 4 を満たさない.

例 2. β は例 *1* と同じとし $g(t,x,\xi) \equiv 0$ とする. このとき, 閉区間 $[0,1]$ に値をとる Ω 上の任意の可測関数 $\omega(t,x) \equiv \omega(x)$ は T-周期解になるが順序関係は持たない.

定理 5. $t_0 \in \mathbb{R}$ とし u を $[t_0,\infty)$ 上の問題 P の任意の弱解とする. このとき $\omega \in \mathcal{P}_T$ が存在して

$$\beta(u(nT+\cdot)) \to \beta(\omega) \quad \text{in } L^2(0,T;V) \text{ as } n \to \infty.$$

4. 証明の概略

定理 3 ～ 5 を証明するに当って次の二つの命題は重要である.

命題 1. $i = 1,2$ に対して $u_{0,i} \in L^\infty(\Omega)$ とし, 更に u_i を $CP(u_{0,i})$ に対する有界閉区間 $J = [t_0,t_1]$ 上の弱解とする. このとき $s \leq t$ なる任意の $s,t \in J$ に対して次が成立する.

$$|(u_1(t) - u_2(t))^+|_{L^1(\Omega)} \leq |(u_1(s) - u_2(s))^+|_{L^1(\Omega)}.$$

命題 2. $\{u_{0,n}\}$ は $L^\infty(\Omega)$ の有界点列とし $u_{0,n}$ はある u_0 に H で弱収束しているとする. このとき $J = [t_0,t_1]$ 上の $CP(u_{0,n})$ の弱解 u_n は J 上の $CP(u_0)$ の弱解 u に次の意味で収束する.

(1) u_n は u に $L^\infty(J \times \Omega)$ で汎弱収束し, $W^{1,2}(J;V')$ で弱収束する.

(2) $\beta(u_n)$ は $\beta(u)$ に $L^2(J;H) \cap L^2_{loc}((t_0,t_1];V)$ で収束し, $L^2(J;V)$ で弱収束する.

(3) $g(\cdot,\cdot,\beta(u_n))$ は $g(\cdot,\cdot,\beta(u))$ に $L^2(J;L^2(\Gamma))$ で収束する.

特に $\hat{\beta}(u_n)$ が $\hat{\beta}(u)$ に収束するときは $\beta(u_n)$ は $\beta(u)$ に $L^2(J;V)$ で収束する. ここで $\hat{\beta}(z) = \int_\Omega \int_0^{z(x)} \beta(r)drdx$, $z \in H$ である.

まず定理 3 は不動点定理を用いて証明することが出来る. しかし (1.1) が仮定されているならば次のような構成的な証明が可能である. 即ち M_1 を初期値として $[0,\infty)$ 上の弱解 u をつくる. $u(nT+\cdot)$ が $CP(u(nT))$ の解であることに注意すると定理 1 と命題 1 により $\omega(t) := \lim_{n\to\infty} u(nT+t)$ が存在し, この ω が実際周期解になっている.

定理 4 についてはまず上の証明からある $\omega_* \in \mathcal{P}_T$ が存在して $\omega_* \leq \min\{\omega_1, \omega_2\}$ となる. 定義 1 により, $i = 1, 2$ とほとんど全ての $t \in [0, T]$ に対して

$$\langle \omega_*'(t) - \omega_i'(t), 1 \rangle + \int_\Gamma (g(t, \cdot, \beta(\omega_*(t))) - g(t, \cdot, \beta(\omega_i(t)))) d\Gamma = 0$$

が成立する. $[0, T]$ で積分すれば $g(t, x, \cdot)$ と β の単調性により $\mathbb{R} \times \Gamma$ 上 $g(\cdot, \cdot, \beta(\omega_*)) = g(\cdot, \cdot, \beta(\omega_i))$. よって (1) が成立する. (2) については定理 2 よりわかる. また (3) については定理 2 と (1) により $h = -g(\cdot, \cdot, \beta(\omega_1))$ に対してシステム $\{(0.1),(0.2)\}$ の周期解 ω が存在して $a_0 = \int_\Omega \omega(0, x) dx$ となる. 再び, 定理 2 により $\mathbb{R} \times \Omega$ 上 $\beta(\omega_1) \leq \beta(\omega) \leq \beta(\omega_2)$ となり, それゆえ $\mathbb{R} \times \Gamma$ 上 $h = g(\cdot, \cdot, \beta(\omega))$ を得る. これにより $\omega \in \mathcal{P}_T$ となり (3) は証明された.

最後に定理 5 を証明する. まず最初に $a_0 := \lim_{n \to \infty} \int_\Omega u(nT, x) dx$ が存在することに注意する. 一方, 命題 2 により $\beta(u(nT + \cdot))$ は $L^2(0, T; V)$ で相対コンパクトであるから, 部分列 $\{u(n_k T + \cdot)\}$ と $w \in L^\infty(\Omega)$ が存在して $\beta(u(n_k T + \cdot))$ が $\beta(w)$ に $L^2(0, T; V)$ で収束する. w は $\mathbb{R}$ 上の問題 P の周期解となり $a_0 = \int_\Omega w(0, x) dx$ を満たす. 定理 3 の (2) により $\beta(w)$ は a_0 から一意的に決まるのでこの w が求めるものである.

参考文献

1. T. Aiki, N. Kenmochi & J. Shinoda, *Periodic stability for some degenerate parabolic equations with nonlinear flux conditions*, to appear in Nonlinear Anal. T.M.A.
2. A. Damlamian & N. Kenmochi, *Periodicity and almost periodicity of solutions to a multi-phase Stefan problem in several space variables*, Nonlinear Anal. T.M.A. **12** (1988), 921-943.
3. A. Haraux & N. Kenmochi, *Asymptotic behaviour of solutions to some degenerate parabolic equations*, Funk. Ekvac. **34** (1991), 19-38.
4. M. Niezgodka & I. Pawlow, *A generalized Stefan problem in several space variables*, Appl. Math. Optim. **9** (1983), 192-224.
5. M. Niezgodka, I. Pawlow & A. Visintin, *Remarks on the paper by A. Visintin "Sur le problème de Stefan avec flux non linéaire"*, Boll. U.M.I., Anal. Funz. Appl. Serie V **18** (1981), 87-88.
6. J. Shinoda, ある退化放物型方程式の周期解について, 1991 年度春期学会実関数論分科会.
7. A. Visintin, *Sur le problème de Stefan avec flux non linéaire*, Boll. U.M.I., Anal. Funz. Appl. Serie V **18** (1981), 63-86.

2 次元 Navier-Stokes 流の外部問題の解の減衰について

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§1 導入と結果.

Ω ($\subset \mathbb{R}^2$) は非有界領域でその境界$\partial\Omega$ は滑らかであるとする。$Q_T = \Omega \times (0, T)$ において次の初期値境界値問題を考える

$$
\text{(N.S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, \text{ in } Q_T, \\ \text{div } u = 0, \qquad \text{in } Q_T, \\ u = 0, \qquad \text{on } \partial\Omega \times (0, T), \\ u|_{t=0} = a, \end{cases}
$$

ここに速度ベクトル $u = (u_1(x,t), u_2(x,t))$ および圧力 $p = p(x,t)$ は未知函数、$a = (a_1(x,t), a_2(x,t))$ は与えられた初期値である。

ここでは $a \in L^2_\sigma(\Omega)$ に対する (N.S) の大域的強解の存在とその $t \to \infty$ での漸近挙動を調べたい。Ωが $\mathbb{R}^n$ $(n \geq 3)$ の外部領域の場合は弱解の L^2-norm および 強解の L^p-norm の代数巾による減衰が得られている（ Borchers-Miyakawa [1], [2]、Iwashita [8])。　$n = 2$ のときは $\|u(t)\|_{L^2} \to 0$ のみが知られている（ Masuda [11])。

定義. $a \in L^2_\sigma(\Omega)$ とする。u が $(0, T)$ 上の *(N.S)* の強解であるとは次の *(1),(2),(3)* の条件を満たすことである

(1) $u \in C([0,T); L^2_\sigma(\Omega)) \cap C^1((0,T); L^2_\sigma(\Omega))$
(2) $u(t) \in D(A)$ *for* $t > 0$, $Au \in C((0,T); L^2_\sigma(\Omega))$
(3) u は次の式を満たす。

$$
\text{(A-N.S)} \qquad \begin{cases} \dfrac{du}{dt} + Au + P(u \cdot \nabla u) = 0, \quad 0 < t < T, \\ u(0) = a. \end{cases}
$$

ここに Pは $L^2(\Omega)$ から $L^2_\sigma(\Omega)$ への直交射影, $A \equiv -P\Delta$, $(D(A) = \{u \in H^2(\Omega); u|_{\partial\Omega} = 0\} \cap L^2_\sigma)$ は Stokes 作用素を表わす。

定理. $a \in L^2_\sigma(\Omega)$ とする。このとき $(0,\infty)$ 上の *(N.S)* の強解 u が一意的に存在する。更に u は次の性質を満たす。

(1) *(smoothness)* $u(t) \in C^1((0,\infty); D(A^\alpha))$ ただし $0 \le \alpha < 1$ 。

(2) *(decay)*

$$\|u(t)\|_p = \begin{cases} o(t^{1/p-1/2}), & \text{for } 2 \le p < \infty, \\ o(t^{-1/2}\sqrt{\log t}), & \text{for } p = \infty; \end{cases} \tag{1.1}$$

$$\|A^\alpha u(t)\|_2 = \begin{cases} o(t^{-\alpha}), & 0 < \alpha < 1 \\ o(t^{-1}\sqrt{\log t}), & \alpha = 1; \end{cases} \tag{1.2}$$

$$\|\dot{u}(t)\|_p = \begin{cases} o(t^{1/p-3/2}\sqrt{\log t}), & 2 \le p < \infty, \\ o(t^{-3/2}\log t), & p = \infty; \end{cases} \tag{1.3}$$

$$\|A^\alpha \dot{u}(t)\|_2 = o(t^{-\alpha-1}\sqrt{\log t}), \qquad 0 < \alpha < 1, \tag{1.4}$$

as $t \to \infty$.

§2 準備.

定理の証明には以下の補題が重要である。

補題１. $\varepsilon > 0$ $0 < \delta < 1/2$. $u, v \in D(A^{1/2}) \cap L^\infty$とする。

$$\Longrightarrow \quad \|(A+\varepsilon)^{-\delta} P(u \cdot \nabla v)\|_2 \le C_\delta \|A^{1/2-\delta} u\|_2 \|A^{1/2} v\|_2$$

ただし C_δは ε, u, vによらない定数。

注意

Ωが外部のときは A が有界な逆を持たないことに注意する。

補題１により次のような双線型作用素 $F_\delta(\cdot,\cdot)$ が定義できる

$$F_\delta(u,v) = \text{w-}\lim_{\varepsilon \to 0} (A+\varepsilon)^{-\delta} P(u \cdot \nabla v) \quad u, v \in D(A^{1/2}) \cap L^\infty$$

この F_δを density を用いて $D(A^{1/2})$ 上に拡張したものに対して、補題１より以下が得られる。

補題2．

(1) $\|F_\delta(u,v)\|_2 \le C_\delta \|A^{1/2-\delta}u\|_2 \|A^{1/2}v\|_2, \quad u,v \in D(A^{1/2})$

(2) $(F_\delta(u,v), A^\delta\phi) = (P(u\cdot\nabla v),\phi)$ for $u,v \in D(A^{1/2})$, $\phi \in D(A^\delta)$

(3) $A^\delta F_\delta(u,v) = P(u\cdot\nabla v)$ for $u,v \in D(A^{1/2}) \cap L^\infty$

補題1の証明は正値自己共役作用素の分数巾に対する Heinz の不等式に注意すると、

$$\|(-\Delta+\varepsilon)^{-\delta}P(u\cdot\nabla v)\|_2 \le C_\delta \|(-\Delta)^{1/2-\delta}u\|_2 \|(-\Delta)^{1/2}v\|_2 \tag{2.1}$$

を得れば十分である (Kato-Fujita [9]参照)。(2.1) は$-\Delta+\lambda$の $\mathbb{R}^2$における 基本解の積分表示を用いて示される。

次の補題は u と$\dot{u}$の L^∞評価を得るのに用いられる。

補題3． $u \in D(A^{s/2})(1 < s \le 2)$ とする。このとき $2 < p < \infty$ に対して

$$\|u\|_\infty \le C_s p^{1/2-\beta/2s} \|A^{1/2}u\|_2^{1-\beta}(\|u\|_2 + \|A^{s/2}u\|_2)^\beta$$

(ここで $u \in D(A^{s/2})$, $\beta = 2s/(2+p(s-1))$) ただし C_sは s にのみよる定数。

補題3は $n=2$ における Gagliardo-Nirenberg の不等式

$$\|u\|_p \le Cp^{1/2}\|u\|_2^{2/p}\|\nabla u\|_2^{1-2/p} \quad u \in H_0^1(\Omega), \quad 2 \le p < \infty$$

と

$$\|u\|_\infty \le C_s \|u\|_2^{1-\alpha}\|u\|_{H^s}^\alpha \qquad u \in H^s(\Omega)$$

(ただし$\alpha = 2/(2+p(s-1))$) および

$$\|\nabla u\|_2 = \|A^{1/2}u\|_2 \qquad u \in D(A^{1/2})$$

により得られる。

3 定理の証明の概略

強解の存在を示すにはつぎの iteration scheme

$$\begin{cases} u_0(t) = e^{-tA}a, \\ u_{j+1}(t) = e^{-tA}a - \displaystyle\int_0^t A^{1-\gamma}e^{-(t-s)A}F_{1-\gamma}(u_j,u_j)(s)ds, \quad 1/2 < \gamma < 1 \end{cases}$$

に対して、A の分数巾、A^{α} $(0<\alpha<1)$ を作用させ、非線型項を補題 2 により評価する。

$$K_{j,\alpha} \equiv \sup_{0<t\le T} t^{\alpha}\|A^{\alpha}u_j(t)\|_2$$

とおけば、次を得る。

$$K_{j+1,\alpha} \le K_{0,\alpha} + C_{1-\gamma}B(\gamma-\alpha, 1-\gamma)K_{j,\gamma-1/2}K_{j,1/2}.$$

したがって

$$k_j(T) = \max\{K_{j,\gamma-1/2}(T), K_{j,1/2}(T)\} \quad (j=0,1,\cdots),$$
$$\beta_{\gamma} = C_{1-\gamma}\max\{B(1/2,1-\gamma), B(\gamma-1/2,1-\gamma)\}$$

とおけば

$$k_{j+1} \le k_0 + \beta_{\gamma}(k_j)^2,$$

を得て、k_0が小さければ k_jが有界列であることがわかる。ほぼ同様にして $u_{j+1}-u_j$を評価して u_j が収束列であることがわかり極限 u が解になることが示される。

さらにエネルギー等式

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau = \|a\|_2^2$$

と a priori 評価

$$\|A^{\varepsilon}u(t)\|_2^2 \le \|A^{\varepsilon}a\|_2^2 \exp(C_{\varepsilon}\|a\|_2^2) \qquad 0<\varepsilon<1/2$$

によりこの解が大域解となることが示せる。

解の減衰はまず Masuda [11]の結果より

$$\|u(t)\|_2 \to 0 \quad t\to 0$$

が得られることに注意する。それにより $\|A^{\alpha}u(t)\|_2 = O(t^{-\alpha})$ が得られ Gagliardo-Nirenberg の不等式より L^p 減衰が得られる。

次に解の L^{∞}評価は補題 3 に

$$\|A^{1/2}u(t)\|_2 = O(t^{-1/2})$$

を用い $p=\log t$ と選ぶことにより得られる。

References

[1] Borchers,W. and Miyakawa, T., *Algebraic L^2 decay for Navier-Stokes flows in exerior domains*, Acta Math. **165** (1990), 189–227.

[2] Borchers,W. and Miyakawa, T., *L^2 decay for the Navier-Stokes flows in unbounded domains with application to exterior stationary flows*, preprint.

[3] Borchers,W. and Sohr, H., *On the semigroup of the Stokes operator for exterior domains in L^q-spaces*, Math. Z. **196** (1987), 415–425.

[4] Fujita, H. and Kato, T., *On the Navier-Stokes initial value problem 1*, Arch. Rat. Mech. Anal. **46** (1964), 269–315.

[5] Giga, Y., *Domains of fractional powers of the Stokes operator in L_r spaces*, Arch. Rat. Mech. Anal. **89** (1985), 251–265.

[6] Giga, Y. and Miyakawa, T., *Solution in L_r of the Navier-Stokes initial value problem*, Arch. Rat. Mech. Anal. **89** (1985), 267–281.

[7] Giga,Y. and Sohr, H., *On the Stokes operator in exterior domains*, J. Fac. Sci. Univ. Tokyo, Ser. IA Math. **36** (1986), 103–130.

[8] Iwashita, H., *$L_q - L_r$ estimates for solutions of nonstationary in an exterior domain and the Navier-Stokes initial value problem in L_q spaces*, Math. Ann. **285** (1989), 265–288.

[9] Kato, T. and Fujita, H., *On the nonstationary Navier-Stokes system*, Rend. Sem. Math. Univ. Padova **32** (1962), 243–260.

[10] Masuda, K., *On the stability of incompressible viscous fluid motions past object*, J. Math. Soc. Japan **27** (1975), 294–327.

[11] Masuda, K., *Weak solutions of the Navier-Stokes equations*, Tohoku Math. J. **36** (1984), 623–646.

[12] Miyakawa, T., *On stationary solutions of the Navier-Stokes equations in an exterior domain*, Hiroshima Math. J. **12** (1982), 115–140.

[13] Miyakawa, T. and Sohr, H., *On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains*, Math. Z. **199** (1988), 455–478.

[14] Yao Jing-Qi, *Comportment à l'infini des solution d'une équation de Schrödinger nonlinéaires dans un domaine extérier*, C. R. Acad. Sci. Paris, **15** (1982), 163–166.

台がコンパクトでない初期 data をもつ半線型波動方程式の大域解の存在について

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次の初期値問題を考える。

$$(1)\quad \begin{cases} u_{tt} - \Delta u = |u|^p & \text{in } R^n \times [0, T), \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), & x \in R^n. \end{cases}$$

ここで、$n = 2$ 又は $n = 3$, $p > 2$ は定数，$T > 0$ 又は $T = \infty$, $f \in C^3(R^n)$, $g \in C^2(R^n)$。又、(1)の解としては、$u(x, t) \in C^2(R^n \times [0, T))$ をみたす関数のみを考える。大域解とは、$T = \infty$ のときのことである。$p_0(n)$ を2次方程式 $(n-1)p^2 - (n+1)p - 2 = 0$ の正根とする。i.e. $p_0(2) = (3 + \sqrt{17})/2$, $p_0(3) = 1 + \sqrt{2}$。f, g の台がコンパクトのときは、$p > p_0(n)$ ならば大域解が存在し、$p \leqq p_0(n)$ ならば存在しない（blow up する）ことが知られている。（f, g についての適当な仮定の下で）。

以下、f, g の台がコンパクトでない場合を考える。

(2) $\quad p > p_0(n)$

及び

$$(3)\quad \sum_{|\alpha| \leqq 3} |D_x^\alpha f(x)| + \sum_{|\alpha| \leqq 2} |D_x^\alpha g(x)| \leqq \frac{\varepsilon}{(1 + |x|)^{1+\kappa}}, \quad x \in R^n$$

を仮定する。ここで $\varepsilon > 0$, $\kappa > 0$ は定数。$n = 3$ のとき、Asakura [2] は、$\kappa > \frac{2}{p-1}$ かつ ε が十分小ならば(1)の大域解が存在し、

(4) $f(x)=0$, $g(x) \geqq \dfrac{\varepsilon}{(1+|x|)^{1+K}}$ かつ $K<\dfrac{2}{p-1}$ $(\varepsilon>0)$

ならば大域解は存在しないことを示した。$n=2$ の場合、(4)の下で解の非存在を Tsutaya [7] と Agemi and Takamura [1] が独立に証明し、存在については [2] と同じ仮定の下で Tsutaya [7] と筆者が独立に証明した。

この講演では、$n=2$ かつ

(5) $\begin{cases} K \geqq \dfrac{2}{p-1} & \text{for } p \neq 5 \\ K > \dfrac{2}{p-1} = \dfrac{1}{2} & \text{for } p=5 \end{cases}$

ならば、十分小さい ε に対して (1) の大域解が存在することを示す。($n=3$ のときも (5) で $p \neq 5$ を $p \neq 3$ におきかえれば同様の結果が得られる。)

(6) $n=2$, $p=5$ かつ $K=\dfrac{2}{p-1}=\dfrac{1}{2}$

のときは、大域解は存在しないと予想されるが open である。今、(1) の lifespan を $T(\varepsilon)$ とする (これは (1) の解が存在するような T の上限である)。(6) の下でも

(7) $T(\varepsilon) \geqq \exp\left(C\left(\dfrac{1}{\varepsilon}\right)^{4/5}\right)$ for $0<\varepsilon \leqq {}^{\exists}\varepsilon_0$

を示すことはできる。即ち、(3) の ε が小さくなると、(1) の解の存在時間は指数的にのびる。これは、f, g の台がコンパクトの場合の

(8) $T(\varepsilon) \geqq \exp\left(C\left(\dfrac{1}{\varepsilon}\right)^{p-1}\right)$ for $p=p_0(2)$, $0<\varepsilon \leqq {}^{\exists}\varepsilon_0$

と似ている ([1], Th. 1)。 実は、f, g がコンパクト support でなくても

$$K \geqq \frac{1}{2}+\frac{1}{p}\ , \ p=p_0(2)$$

であれば (8) を示すことができる。

以下、これらの事情を少しのべる。(1) の解が積分方程式

(8) $\quad u = u_0 + L(|u|^p)$

の解で与えられることはよく知られている(一意性については[5]を参照されたい)。ここで、$n=2$ のときは

(9) $$L(w)(x,t) = \frac{1}{2\pi}\int_0^t (t-\tau)d\tau \int_{|\xi|<1} \frac{w(x+(t-\tau)\xi, \tau)}{\sqrt{1-|\xi|^2}} d\xi 。$$

又、u_0 は (1) の方程式の右辺を 0 とおいたときの解。(8) の解の存在を示すには、次の形の *a priori estimate*

(10) $$\|L(|u|^p)\| \leqq C\|u\|^p \quad \text{for } u(x,t) \in C^0(R^n \times [0,T))$$

が本質的な役割をはたすことはよく知られている。norm $\| \ \|$ を適当に選んで定数 C を T に無関係にとることができれば大域解の存在を示すことができる。又、T に依存していても、(7) のように、$T(\varepsilon)$ を下から評価することができる。ちなみに (8) の解が存在するための十分条件の一つは

(11) $$C p 2^p \|u_0\|^{p-1} \leqq 1 \quad かつ \quad \|u_0\| \leqq \frac{1}{2}$$

である。ここで C は (10) における定数。u_0 については次の評価が成り立つ。

<u>Lemma 1.</u> K にのみ依存する定数 C_0 が存在して

(12) $$\sum_{|\alpha|\leqq 2} |D_x^\alpha u_0(x,t)| \leqq C_0 \varepsilon \Phi(r,t)$$

for $(x,t) \in R^2 \times [0,\infty)$。ここで $r=|x|$, $\Phi(r,t)$ は K に応じて五つの形をもつが、重要な場合は

$$\Phi(r,t) = (1+t+r)^{-\frac{1}{2}}(1+|t-r|)^{\frac{1}{2}-\kappa} \quad \text{for } \frac{1}{2}<\kappa<1,$$
$$= (1+t+r)^{-\kappa} \quad \text{for } 0<\kappa<\frac{1}{2},$$
$$= \frac{1}{\sqrt{1+t+r}}\left(1+\log\frac{1+t+r}{1+|t-r|}\right) \quad \text{for } \kappa=\frac{1}{2}$$

の三つである。この最後の形が (5) の除外点 $p=5$ に影響している。$\| \ \|$ のとり方としては、e.g.

(13) $$\frac{1}{2}<\kappa<\frac{1}{2}+\frac{1}{p} \quad (<1)$$

sup は $R^2\times[0,T)$ でとる。

の場合は $\|u\| = \sup(|u(x,t)|/\Phi(r,t))$ をとる。このとき

Lemma 2. p と κ にのみ依存する定数 C_1 が存在して

(14) $$|L(|u|^p)(x,t)| \leqq C_1\|u\|^p\,\Phi_1(r,t) \quad \text{for } (x,t)\in R^2\times[0,T)$$

が成り立つ ($p>2$)。ここで $\Phi_1(r,t)$ も五つの形をとるが、重要な場合は次の三つである。

$$\Phi_1(r,t) = (1+t+r)^{-\frac{1}{2}}(1+|t-r|)^{\frac{5}{2}-p\kappa} \quad \text{for } \frac{5}{2}<p\kappa<3,$$
$$= (1+t+r)^{2-p\kappa} \quad \text{for } 0<p\kappa<\frac{5}{2},$$
$$= \frac{1}{\sqrt{1+t+r}}\left(1+\log\frac{1+t+r}{1+|t-r|}\right) \quad \text{for } p\kappa=\frac{5}{2}.$$

最初の場合は、(14)より、

$$|L(|u|^p)(x,t)|\,\Phi(r,t)^{-1} \leqq C_1\|u\|^p(1+|t-r|)^{2-(p-1)\kappa}$$

だから、$\kappa \geqq \frac{2}{p-1}$ ならば $2-(p-1)\kappa \leqq 0$ となり、

$$\|L(|u|^p)\| \leqq C_1\|u\|^p$$

が得られる。ここで注意すべきことは、(2)より $\frac{2}{p-1}<\frac{1}{2}+\frac{1}{p}$ が従うことである。逆に、$p \leqq p_0(2)$ ならば、(5) をみたす κ は $\kappa \geqq \frac{1}{2}+\frac{1}{p}$ となり、(10) を T に依存しない C で成立させるような norm を見つけることができない。終。

References

[1] Agemi R. and Takamura H., The lifespan of classical solutions to nonlinear wave equations in two space dimendions, Hokkaido university preprint series in mathematics, 118(1991).

[2] Asakura F., Existence of a global solution to a semi-linear wave equation with slowly decreasing initial data in three space dimendions, Comm. in P.D.E., 11, 1459-1487(1986).

[3] Glassey R.T., Existence in the large for $u = F(u)$ in two space dimensions, Math. Z. 178, 233-261(1981).

[4] John F., Plane waves and spherical means applied to partial differential equations, Interscience, New York(1955).

[5] John F., Nonlinear wave equations, Formation of singulerities, Pitcher Lectures in the mathematical science, Lehigh University, University Lecture Series, American Math. Soc. Providence (1990).

[6] Kubota K., Existence of a global solution to a semi-linear wave equation with initial data of non-compact support in low space dimensions, In preparation.

[7] Tsutaya K., Global existence theorem for semilinear wave equations with non compact data in two space dimensions, Preprint(1991).