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第18回偏微分方程式論

札幌シンポジウム

(代表者 久保田 幸次)

予稿集

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代表者 久保田 幸次

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3. 講演

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Geometric Wave Equations

Jalal Shatah

Courant Institute

In this survey we will report on some recent developments on the Cauchy problem for some geometric wave equations. The model that we have in mind is the harmonic map problem, which is the study of maps from space-time into complete Riemannian manifolds. This problem was considered by many authors such as D. Christodoulou, J. Ginibre and G. Velo, M. Grillakis, A. Shadi Tahvildar-Zadeh, S. Klainerman and M. Machedon, T. Sideres and myself, to name a few.

In general for a complete Riemannian manifold (N, g) and a map $u: \mathbb{R}^{n+1} \rightarrow N$ we consider the Cauchy problem

$$\begin{cases} D_\alpha \partial_\alpha u^a \equiv \partial^\alpha \partial_\alpha u^a + \Gamma_{bc}^a(u) \partial_\alpha u^b \partial^\alpha u^c = 0 \\ u(0, x) = U_0(x) \\ \partial_t u(0, x) = U_1(x) . \end{cases} \quad (1)$$

where Γ are the Christoffel symbols of the metric g .

We will mainly discuss a reduced problem which is referred to as an equivariant map. These maps arise when the target N is rotationally symmetric product manifold defined by

$$N = [0, \phi^*) \times S^{k-1}$$

where $\phi^* \in \mathbb{R}^+ \cup \{+\infty\}$. On N we have polar coordinates $(\phi, \chi) \in [0, \phi^*) \times S^{k-1}$ and the metric in these coordinate systems take the form

$$ds^2 = d\phi^2 + G^2(\phi) d\chi^n$$

where $G(0) = 0$, $G'(0) = 1$, and $d\chi^2$ is the standard metric on S^{k-1} . Using polar coordinates (t, r, ω) on \mathbb{R}^{n+1} equivariant maps are given by

$$(t, r, \omega) \rightarrow (\phi(t, r), \chi(\omega))$$

where $\chi(\omega)$ is a harmonic polynomial of degree $\ell > 0$, i.e. the restriction of a map from $\mathbb{R} \rightarrow \mathbb{R}^k$ where each component is a harmonic homogeneous polynomial of degree R . For these maps the Cauchy problem is

$$\begin{cases} \phi_{tt} - \phi_{rr} - \frac{n-1}{r} \phi_r + \frac{k}{r^2} f(\phi) = 0 \\ \phi(0, r) = \phi_0(r) \\ \partial_t \phi(0, r) = \phi_1(r) \end{cases} \quad (2)$$

Theorem. *The Cauchy problem (2) with initial data*

$$\phi \in H^{n/2}, \quad \phi_1 \in H^{(n-2)/2}$$

has a unique local solution on $[0, T^)$ for some $T^* > 0$:*

$$\phi \in L^\infty([0, T_0], H^{n/2}) \cap L^q([0, T_0], W^{\sigma, q})$$

Theorem. ($n=2$). *Let the function G in equation (2) satisfy the condition*

$$G(s) + sG'(s) > 0 \quad \text{for } s > 0$$

Then the Cauchy problem with smooth initial data has smooth solution for all time.

Theorem. *Let ϕ be a smooth solution in two space dimensions, with finite energy initial data, E_0 , then*

$$|\phi(t, r)| \leq \frac{C(E_0)}{\sqrt{t}}$$

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Decay and Global Existence for Nonlinear Dissipative Wave Equations

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1 Introduction

In this lecture we treat some nonlinear wave equations with the so-called dissipative terms, and we present some results concerning the decay property and the global existence of solutions which are derived through the effect of the dissipation.

When we are interested in the global existence for the nonlinear evolution equations the following two spirits are generally very useful:

(1) Monotonicity of nonlinear term \implies global existence of *weak* solution.

(2) [Decay estimate]+[Small data] \implies global existence of *strong* or *smooth* solution.

For illustration let us consider a very simple and typical example.

Example 1.

$$\begin{cases} u_{tt} - \Delta u + u_t + |u|^\alpha u = 0 & \text{in } [0, \infty) \times \Omega, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & \text{and } u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a bounded domain in R^n .

By the monotonicity of $|u|^\alpha u$ it is well known that the problem admits a weak solution $u \in L^\infty([0, \infty); L^{\alpha+2} \cap H_1^0) \cap W^{1,\infty}([0, \infty); L^2)$ if $u_0 \in H_1^0 \cap L^{\alpha+2}$ (Strauss [38].) Further, if $0 < \alpha \leq 2/(N-2)$ and the initial data is smooth the solution is unique and smooth (cf. Sather[34], W.v.Wahl[38]). For the above results the dissipative term does not make any contribution.

What about the case $\alpha > 2/(N-2)$? To get a smooth solution for this case we can use the decay estimate:

$$E(t) \equiv \frac{1}{2}(\|u_t(t)\|^2 + \|\nabla u(t)\|^2) + \frac{1}{\alpha+2} \int_\Omega |u|^{\alpha+2} dx \leq CE(0)e^{-\lambda t}, \lambda > 0, \quad (1)$$

which is derived due to the effect of the term u_t . Indeed, this estimate together with a standard argument implies that the problem admits a global smooth solution if

the data (u_0, u_1) satisfy a certain smallness condition. Similar assertions hold for more general or stronger nonlinearities (cf.P.Rabinowitz [34], A.Matsumura [13], Y.Ebihara [3,4],S.Klainerman [11], Y.Shibata [38], J.Shatah [37], A.Milani [13], S.Kawashima and Y.Shibata [10] M.Nakao [20,25] etc.)

In this lecture we are interested in some nonlinear wave equations which have the dissipative terms in more delicate situations than the example stated above. The first equation we consider is the one with a nonlinear dissipative term and the second one is the case of degenerate dissipative term . These are initial boundary value problems in a bounded domain, while we treat in the final section the Cauchy problems in the whole space.

2 Nonlinear Dissipative Term

The first problem we consider is the following:

$$(P.1) \quad \begin{cases} u_{tt} - \Delta u + \rho(u_t) + f(u) = 0 & \text{in } [0, \infty) \times \Omega, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & \text{and } u|_{\partial\Omega} = 0, \end{cases}$$

where we take for simplicity $\rho(v) = |v|^r v$ and $f(u) = |u|^\alpha u$.

Since our dissipative term $\rho(u_t)$ is nonlinear and further $\rho'(0) = 0$ (if $r > 0$) the dissipative mechanism should be very delicate. In Amerio and Prouse[1] it is proved that $\lim_{t \rightarrow \infty} E(t) = 0$ in the case $f(u) \equiv 0$ under a certain restriction on r , while we have proved the following preciser result.

Theorem 1 *Let $f(u) \equiv 0$. Then, the problem has a unique solution $u(t)$ in the class $C([0, \infty); H_1^0) \cap C^1([0, \infty); L^2) \cap L^{r+2}([0, \infty); L^{r+2})$ for each $(u_0, u_1) \in H_1^0 \times L^2$, satisfying the decay property:*

(1) *If $0 < r \leq 2/(N-2)$, then*

$$E(t) \leq \{E(0)^{-r/2} + C(t-1)^+\}^{-2/r} \leq C_0(1+t)^{-2/r}. \quad (2)$$

(2) *If $2/(N-2) < r$ and $(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0$, then*

$$E(t) \leq C_1(1+t)^{-2/r}, \quad (3)$$

where C_1 denotes a constant depending on $\|u_0\|_{H_2}$ and $\|u_1\|_{H_1^0}$.

Further, we have that

(3) *if $-1 < r < 0$ and $(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0$, then*

$$E(t) = \begin{cases} C_1(1+t)^{(1+r)/r} & \text{if } -1 < r \leq (4-N)/(N-2) \text{ or } 1 \leq N \leq 4, \\ C_1(1+t)^{2/((N-2)r)} & \text{if } N \geq 5 \text{ and } (4-N)/(N-2) < r < 0. \end{cases} \quad (4)$$

The proofs of (1),(2) and (3) in the above theorem are given in Nakao[16],[24] and [25], respectively (see also[23]). From this theorem we know that the decay property of the solutions of the problem (P.1) is certainly more delicate than the case $\rho(v) = v$. For related works see A.Haraux and E.Zuazua[6], E.Zuazua [42].

For later convenience we explain briefly the idea of the proof.

Multiplying the equation by u_t and integrating we get

$$\int_t^{t+1} \int_{\Omega} |u_t|^{r+2} dx ds = E(t) - E(t+1) \equiv D(t)^2$$

Next, multiplying the equation by u and integrating we get

$$\int_{t_1}^{t_2} \|\nabla u(s)\|^2 ds = \pm(u_t(t_i), u(t_i)) + \int_{t_1}^{t_2} \|u_t\|^2 ds - \int_{t_1}^{t_2} \int_{\Omega} |u_t|^r u_t u dx ds$$

($t \leq t_1 \leq t_2 \leq t+1$), where $\|\cdot\|$ denotes the usual L^2 norm.

Combining these we can derive the following difference inequality for the case $0 < r \leq 2/(N-2)$:

$$\sup_{t \leq s \leq t+1} E(s)^{1+r/2} \leq C_0(E(t) - E(t+1)). \quad (5)$$

This implies the desired estimate (2). Here, we note that to derive the above (5) we have used the inequalities:

$$\int_t^{t+1} \int_{\Omega} |u_t|^{r+1} |u| dx ds \leq CD(t)^{2(r+1)/(r+2)} \left\{ \int_t^{t+1} \|u\|_{L^{r+2}}^{r+2} ds \right\}^{1/(r+2)},$$

$$\|u\|_{L^{r+2}} \leq C \|\nabla u\| \quad (6)$$

and

$$\|u_t\| \leq C \|u_t\|_{L^{r+2}}. \quad (7)$$

For the proofs of the estimates (2) and (3) we must use, instead of the inequalities (6) and (7), the following ones, respectively:

$$\|u\|_{L^{r+2}} \leq \|\nabla u\|^{1-\theta} \|\Delta u\|^\theta, \theta = (r(N-2) - 4)/2(r+2), \quad (8)$$

and

$$\|u_t\| \leq \|u_t\|_{L^{r+2}}^{1-\theta} \|\nabla u_t\|^\theta, \theta = (-Nr)/(4 - (N-2)r). \quad (9)$$

We could extend the above results for abstract nonlinear evolution equations (cf.[17]) and also discuss on the decay property of the problem (P.1) above with $\rho(v)$ replaced by a more general one $\rho(t, v)$ such that $k_0(1+t)^{\theta_1} |v|^{r+2} \leq \rho(t, v)v \leq k_1(1+t)^{\theta_2} |v|^{r+2}$. See Nakao [18,19] and Y.Yamada [41] where such a general case is treated when $0 < r < 4/(N-2)$. Now, using the idea of the proof of Theorem 1 we can prove the global existence of strong solutions of the problem (P.1) with $f(u) = |u|^\alpha u, \alpha > 2/(N-2)$ (note that the case $0 < \alpha \leq 2/(N-2)$ is much easier and no problem occurs concerning the global existence of strong solutions).

Theorem 2 Let $f(u) = |u|^\alpha u$.

(1) Assume that the following conditions are satisfied:

(a) $0 < r \leq 2/(N - 4)$ ($0 < r < \infty$ if $N = 3, 4$),

(b) $2/(N - 2) < \alpha < 2/(N - 4)$

and

(c) $(4 - N)\alpha + 2 > 2r$.

Then, there exists a certain unbounded open set $S \subset H_2 \cap H_1^0 \times H_1^0$ containing $(0, 0)$ such that for $(u_0, u_1) \in S$ the problem (P.1) admits a unique solution $u(t)$ in the class

$$W^{2,\infty}([0, \infty); L^2) \cap W^{1,\infty}([0, \infty); H_1^0) \cap L^\infty([0, \infty); H_2 \cap H_1^0) \cap W^{1,r+2}([0, \infty); L^{r+2}).$$

(2) Assume that the following conditions are satisfied:

(a) $-1 < r < 0$,

(b) $2/(N - 2) < \alpha < 2/(N - 4)$

and

(c) $(4 - N)\alpha + 2 > -r \max\{(N - 2), 2(1 + r)^{-1}\}$.

Then, the same assertion as in (1) just above holds.

For the proof of Theorem 2 see Nakao [28].

3 Degenerate Dissipative Term

In this section we treat the initial-boundary value problem for the quasilinear wave equation of the form:

$$(P.2) \begin{cases} u_{tt} - \operatorname{div}\left\{\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right\} + a(x)u_t = 0 & \text{in } [0, \infty) \times \Omega. \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & \text{and } u|_{\partial\Omega} = 0. \end{cases}$$

When $a(x) \geq \epsilon_0 > 0$ it is well known that if (u_0, u_1) belongs to $H_{m+1} \times H_m$, $m > N/2 + 1$ and satisfies a certain compatibility condition and a smallness condition,

then the problem (P.2) admits a unique global solution $u(t) \in C^{m+1-i}([0, \infty); H_i \cap H_1^o)$, $i = 1, \dots, m+1$. (See Matsumura [13], Shibata [37], T.Nishida [32], Milani [14] and T.Kato [8] etc.)

Here, we are interested in the case $a(x) \geq 0$ and vanishes somewhere in $\bar{\Omega}$. More precisely, we make the following assumption.

A. $a(x) \geq 0$, $a(x) > 0$ a.e. $x \in \bar{\Omega}$ and

$$\int_{\Omega} \frac{1}{|a(x)|^p} dx < \infty \quad \text{for some } p, 0 < p < 1. \quad (10)$$

Note that our assumption allows that $a(x)$ vanishes on any $N-1$ submanifolds in $\bar{\Omega}$. In our situation the decay property of $E(t)$ is very delicate. To see this let us consider the linear equation:

$$u_{tt} - \Delta u + a(x)u_t = 0. \quad (11)$$

Using the inequality

$$\|u_t(t)\|^2 \leq \left(\int_{\Omega} a^{-p} dx \right)^{\frac{1}{(p+1)}} \left(\int_{\Omega} a(x)|u_t|^2 dx \right)^{\frac{p}{(p+1)}} \|u_t(t)\|_{\infty}^{\frac{2}{(p+1)}}. \quad (12)$$

we can apply the idea of the proof of Theorem 1 to prove

$$E(t) \leq C_m(1+t)^{-2mp/N}, \quad (13)$$

where C_m is a constant depending on $\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m}$ (see Nakao[22,27]).

Thus, the decay rate of $E(t)$ depends on p as well as the regularity of the solution itself, and hence it is very delicate. Now, our result on the existence and decay for the quasilinear equation reads as follows.

Theorem 3 *Let $a \in C^m(\bar{\Omega})$ and let $(u_0, u_1) \in H_{m+1} \times H_m$ with m such that*

$$m > \frac{N}{2} + 1 \quad \text{and} \quad (2m - N - 1)p > N \quad (14)$$

and let the compatibility condition of the m -th order be satisfied.

Then, the problem admits a unique solution $u(t)$ in the class:

$$\bigcap_{k=0}^m C^k([0, \infty); H_{m+1-k}) \cap C^{m+1}([0, \infty); L^2),$$

satisfying the inequality (13) provided that (u_0, u_1) is small in a certain sense.

The proof of Theorem 3 is given in Nakao[29].

4 Cauchy Problems

In this section We consider some Cauchy problems . The first one is a typical semilinear equation with a linear dissipation:

$$(P.3) \begin{cases} u_{tt} - \Delta u + u_t + |u|^\alpha u = 0 & \text{in } [0, \infty) \times R^N, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \end{cases}$$

If $-\Delta u$ is replaced by $-\Delta u + \lambda u$, $\lambda > 0$, in the equation above it is easy to prove that the energy $E(t)$ decays exponentially as $t \rightarrow 0$ for the usual energy bounded solutions. But, for our problem the decay property of $E(t)$ is more delicate and hence interesting. Our result for the problem (P.3) reads as follows.

Theorem 4 (1) Let $(u_0, u_1) \in H_1 \times L^2$ and $0 < \alpha < 4/(N-2)$. Then, we have $E(t) \leq C_0(1+t)^{-1}$, where C_0 denotes a constant depending on $\|u_0\|_{H_1} + \|u_1\|_{L^2}$.

(2) Let $1 \leq N \leq 3$, $4/N < \alpha \leq 2/(N-2)$ ($4/N < \alpha < \infty$ if $N = 1, 2$) and $(u_0, u_1) \in H_1 \cap L^r \times L^2 \cap L^r$, $1 \leq r \leq 2$. Then, we have

$$E(t) \leq C_1(1+t)^{-1-N(1/r-1/2)/2},$$

where C_1 is a constant depending on $E(0)$ and $\|u_0\|_{L^r} + \|u_1\|_{L^r}$.

(3) Let $3 \leq N \leq 6$, $\max\{4/N, 2/(N-2)\} < \alpha < 4/(N-2)$ and $(u_0, u_1) \in H_2 \cap L^r \times H_1 \cap L^r$, $1 \leq r \leq 2$. Then, we have

$$E(t) \leq C_1(1+t)^{-1-N(1/r-1/2)/2},$$

where C_2 is a constant depending on $\|u_0\|_{H_2 \cap L^r} + \|u_1\|_{H_1 \cap L^r}$.

For the proof of existence and uniqueness under our condition on α see Strauss [37], P.Brenner and W.v.Wahl [40] and J.Ginible and G.Velo [5] etc. The proof of the above theorem is given by combining the $L^p - L^{p'}$ estimate for the linear equation with the energy method used in the proofs of Theorems 1-3. For details see S.Kawashima, M.Nakao and K.Ono [9]. See also R.Racke [35] for related estimates for the linear equations. Note that Theorem 4 does not require any smallness conditions on the initial data.

Finally we consider (P.3) with $|u|^\alpha u$ replaced by $-|u|^\alpha u$ or $\pm|u|^{\alpha+1}$; we call this problem as (P.3)'. For such a problem we can use an idea of 'modified potential well' to get the following:

Theorem 5 Consider (P.3)' \equiv (P.3) with $|u|^\alpha u$ replaced by $\pm|u|^{\alpha+1}$ or $-|u|^\alpha u$. Assume that $4/N \leq \alpha < 4/(N-2)$ and

$$K(u_0) \equiv \|\nabla u_0\|^2 - \|u_0\|_{\alpha+2}^{\alpha+2} \geq 0$$

and

$$I_0^{(2\alpha-N\alpha+4)/2} E(0)^{N\alpha-4} \leq \epsilon \ll 1.$$

Then, there exists a unique global solution $u(t)$ in $C([0, \infty); H^1) \cap C^1((0, \infty); L^2)$, satisfying

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \leq C(1+t)^{-1}$$

For a proof of Theorem 5 see M.Nakao and K.Ono [29]. When $4/N > \alpha > 0$ we can derive an estimate for the life span, this method being applicable also to the equations without dissipative term. If we make additional assumptions on (u_0, u_1) as in (2),(3) of Theorem 4 we can improve the decay estimates for (P.3)' as in Theorem 4. When $-\Delta u$ is replaced by $-\Delta u + \lambda(x)u$ with $\lambda(x) \geq K_0(1+|x|)^{-\theta}$ we can discuss on the decay and global existence of (P.3) or (P.3)' even if u_t is replaced by nonlinear term $\rho(u_t) = |u_t|^r u_t$. For details see M.Nakao [21] and M.Nakao and K.Ono [30]. For related works see also T.Motai [15] and Kakita, Nishihara and Tamamura [7].

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On the Regularity of Solutions for Some Semilinear Schrödinger Equations

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0. Introduction

The aim of this report is to explain the results of the study on the regularity of solutions for some semilinear Schrödinger equations with quadratically growing principal symbols. More precisely we prove the propagation of ‘conormal’ regularity and that of wave front sets with respect to the weight $\sqrt{1 + |x|^2 + |\xi|^2}$ for those equations. We also show that the decay of the initial values implies the smoothness as a corollary .

Now we will explain the background. Since the end of 70’s there have been a lot of works on propagation and interaction of (micro-local) singularities for nonlinear hyperbolic equations (see the references in [Be]). The first general results are due to Bony [Bo], who introduced the paradifferential operators (with respect to the weight $\sqrt{1 + |\xi|^2}$).

On the other hand it is known that (linear) Schrödinger equations corresponding to the weight $\sqrt{1 + |x|^2 + |\xi|^2}$ admit the similar treatment as the strictly hyperbolic equations (see, for example, [He]).

So it is natural to consider the regularity of solutions for such semilinear Schrödinger equations by developing the corresponding paradifferential calculus.

There are many works on the relationship between the regularity of solutions and the decay rate of the initial values for linear or semilinear dissipative equations ([Ka], [HNT], [Je], [Ya] etc).

1. Preliminaries

Notations. $t_+ = \max\{0, t\}$ ($t \in \mathbf{R}$). $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$.

$$z = (x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d, \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}, \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}, \langle z \rangle = (1 + |z|^2)^{\frac{1}{2}}.$$

$$D = (D_1, \dots, D_d), \partial = (\partial_1, \dots, \partial_d), D_j = \frac{1}{\sqrt{-1}} \partial_j.$$

For $k \in \mathbf{Z}_+, 0 < r < 1$: $C_b^k = \{u \in C^k(\mathbf{R}^d) : \partial^\alpha u \in L^\infty \text{ if } |\alpha| \leq k\}$,

$$C_b^r = \{u \in C_b^0 : \sup_{x, h \in \mathbf{R}^d} |u(x+h) - u(x)| / |h|^r < \infty\},$$

$$C_b^{k+r} = \{u \in C_b^k : \partial^\alpha u \in C_b^r \text{ if } |\alpha| = k\}.$$

$$L^2 = L^2(\mathbf{R}^d), (\cdot, \cdot) = (\cdot, \cdot)_{L^2}, \|\cdot\| = \|\cdot\|_{L^2}.$$

$B^s = \{u \in L^2 : \langle x \rangle^s u \in L^2, \langle D \rangle^s u \in L^2\}$ ($s \geq 0$): Hilbert spaces.

$B^s = (B^{-s})'$ ($s \leq 0$): dual spaces.

For locally convex topological vector spaces X, Y , $L(X, Y)$ is the space of all continuous linear operators from X to Y , and $L(X) = L(X, X)$.

For a Frechét space $\Gamma \subseteq C^\infty(\mathbf{R}^{2d})$ and an interval I , $C_w(I; \Gamma)$ is the set of all $p \in C(I; C^\infty(\mathbf{R}^{2d}))$ such that $\{p(t, \cdot, \cdot)\}_{t \in J}$ is bounded in Γ for every compact subset J of I .

Definition 1.1. Let $m, \rho, \delta \in \mathbf{R}$. For $p \in C^\infty(\mathbf{R}^{2d})$

$$p \in \Gamma_{\rho, \delta}^m \text{ iff for all } \alpha, \beta \quad |\partial_x^\beta \partial_\xi^\alpha p(z)| \leq C_{\alpha\beta} \langle z \rangle^{m - \rho|\alpha| + \delta|\beta|}, \quad z \in \mathbf{R}^{2d};$$

$$p \in \Gamma^m \text{ iff for all } \alpha, \beta \quad |\partial_x^\beta \partial_\xi^\alpha p(z)| \leq C_{\alpha\beta} \langle z \rangle^{m - |\alpha| - |\beta|}, \quad z \in \mathbf{R}^{2d};$$

$$p \in \Gamma_+^m \text{ iff for all } \alpha, \beta \quad |\partial_x^\beta \partial_\xi^\alpha p(z)| \leq C_{\alpha\beta} \langle z \rangle^{(m - |\alpha| - |\beta|)_+}, \quad z \in \mathbf{R}^{2d}.$$

Further put $\Gamma^{-\infty} = \bigcap_{m \in \mathbf{R}} \Gamma^m$. (cf. [He], [Sh], [Hö, Chapter 18].)

Remark. For $-1 < \rho \leq 1, -1 \leq \delta < 1, \delta \leq \rho$, $\Gamma_{\rho, \delta}^m$ is a good symbol class. We know the boundedness, the product formula and the adjoint formula.

Definition 1.2. Let $r \geq 0, p \in C^\infty(\mathbf{R}^{2d})$. $p \in \Gamma_{1,1}^m(r)$ if and only if for all α, β

$$|\partial_x^\beta \partial_\xi^\alpha p(z)| \leq C_{\alpha\beta} \langle z \rangle^{m - |\alpha| + (|\beta| - r)_+}, \quad z \in \mathbf{R}^{2d}.$$

2. Applications to semilinear Schrödinger equations

Consider the following semilinear Schrödinger equations:

$$(2.1) \quad (\partial_t + iH(t))u(t) = f(t, u(t)) \text{ in } \mathcal{D}'((0, T) \times \mathbf{R}^d)$$

with $u \in C([0, T]; \mathcal{S}')$ and $f(t, u(t)) \in C([0, T]; \mathcal{S}')$. We assume

$$H(t) = h(t, x, D), \quad h = h_0 + h_1 + h_2,$$

$$h_2 \in C_w([0, T]; \Gamma^2), \quad h_1 \in C_w([0, T]; \Gamma_+^1), \quad h_0 \in C_w([0, T]; \Gamma_+^0);$$

$$\text{i.e. } h_j \in C([0, T]; C^\infty(\mathbf{R}^{2d}))$$

$$|\partial_x^\beta \partial_\xi^\alpha h_2(t, z)| \leq C_{\alpha\beta} \langle z \rangle^{2-|\alpha|-|\beta|}, \quad z \in \mathbf{R}^{2d}, \quad 0 \leq t \leq T,$$

$$|\partial_x^\beta \partial_\xi^\alpha h_1(t, z)| \leq C_{\alpha\beta} \langle z \rangle^{(1-|\alpha|-|\beta|)_+}, \quad z \in \mathbf{R}^{2d}, \quad 0 \leq t \leq T,$$

$$|\partial_x^\beta \partial_\xi^\alpha h_2(t, z)| \leq C_{\alpha\beta}, \quad z \in \mathbf{R}^{2d}, \quad 0 \leq t \leq T,$$

h_2 and h_1 are real valued,

$$f(t, \zeta) \in C([0, T] \times \mathbf{C}) \text{ is holomorphic w.r.t. } \zeta, \quad f(t, 0) \equiv 0.$$

Let $\chi_{ts}(y, \eta) = (X(t, s, y, \eta), \Xi(t, s, y, \eta))$ ($t, s \in [0, T]$, $y, \eta \in \mathbf{R}^d$) be the solution of

$$(2.2) \quad \begin{cases} \dot{x}(t) = \nabla_\xi h_2(t, x, \xi), & x(s) = y \\ \dot{\xi}(t) = -\nabla_x h_2(t, x, \xi), & \xi(s) = \eta. \end{cases}$$

We fix a solution of (2.1) u satisfying

$$(2.3) \quad u \in C([0, T]; B^\mu), \quad \sup_{0 \leq t \leq T} \|u(t)\|_{C_t^r} < \infty, \quad \mu > 0, \quad r > 0$$

throughout this section (see Appendix A for the existence of such solutions). In this case $f(t, u(t)) \in C([0, T]; B^\mu)$ and so $u(t) \in C^1([0, T]; B^{\mu-2})$.

Remark. There is $p \in C_w([0, T]; \Gamma_{1,1}^0(r))$ depending on u such that $f(t, u(t)) = p(t, x, D)u(t)$, $0 \leq t \leq T$.

Theorem 2.1. Let $k \leq r$, $s \in [0, T]$ and $\{P_j\}_{j=1}^N \subset \text{Op} \Gamma^1$ and suppose $P^I u(s) \in B^\mu$ when $|I| \leq k$. Then

$$P(s, \cdot)^I u(\cdot) \in C([0, T]; B^\mu) \text{ when } |I| \leq k.$$

Here $I = (i_1, \dots, i_j)$, $|I| = j$, $i_1, \dots, i_j \in \{1, \dots, N\}$, $P^I = P_{i_1} \cdots P_{i_j}$, $P_i(s, t) = \sigma(P_i)(\chi_{st}(x, D))$.

Corollary 2.2. Let $s \in [0, T]$, $k \in \mathbb{N}$, $k \leq r$ and let $J \subset [0, T]$ be an interval. Suppose there exist $a_j, b_j, c \in C_w(J; \Gamma_+^0)$, $j = 1, \dots, d$ and $\lambda(t) \in C(J)$ such that

$$\lambda(t)\langle \eta \rangle = \sum_{j=1}^d (a_j(t, y, \eta)X_j(s, t, y, \eta) + b_j(t, y, \eta)y_j) + c(t, y, \eta) \quad y, \eta \in \mathbb{R}^d, t \in J.$$

If $\langle x \rangle^k u(s) \in B^\mu$, then $\left(\frac{\lambda(t)}{\langle x \rangle}\right)^k u(t) \in C(J; B^{\mu+k})$.

Corollary 2.3. (1) Let $s \in [0, T]$, let $J \subset [0, T]$ be an interval ($s \notin J$) and suppose

$$|X(s, t, y, \eta)| \geq C_1|\eta| - C_2|y| - C_3, \quad y, \eta \in \mathbb{R}^d, t \in J$$

with $C_j > 0$. Then Corollary 2.2 is applicable with $\lambda(t) = 1$.

(2) Let $s \in [0, T]$, let $J \subset [0, T]$ be an interval ($s \in J$) and suppose

$$|X(s, t, y, \eta) - y| \geq |t - s|(C_1|\eta| - C_2|y| - C_3), \quad y, \eta \in \mathbb{R}^d, t \in J$$

with $C_j > 0$. (If $|\nabla_\xi h_2(s, x, \xi)| \geq C|\xi| - C'|x| - C''$, then the condition is satisfied for small J). Then Corollary 2.2 is applicable with $\lambda(t) = t - s$.

Definition 2.4. For $v \in S'(\mathbb{R}^d)$ $WF_{B^s} v$ is the subset of $\mathbb{R}^{2d} \setminus \{0\}$ defined as follows: $z_0 = (x_0, \xi_0) \notin WF_{B^s} v$ if and only if there exists $p \in \Gamma^0$ such that $p = 1$ in a conic neighborhood of z_0 and that $p(x, D)v \in B^s$.

Theorem 2.5. Assume $h_2(\lambda z) = \lambda^2 h_2(z)$ for $\lambda \geq 1$, $|z| \geq 1$. Then for $\mu < \nu \leq \mu + r$

$$\chi'_{ts}(WF_{B^\nu} u(s)) = WF_{B^\nu} u(t), \quad t, s \in [0, T].$$

Here χ'_{ts} is defined similarly as χ_{ts} with h_2 modified to be homogeneous of degree two.

Appendix. Lemmas on semilinear Schrödinger equations

Assume for all $\alpha, \beta \quad \partial_x^\alpha \partial_y^\beta f(t, \zeta) \in C(\mathbf{R} \times \mathbf{C})$, $\zeta = x + iy$, $f(t, 0) \equiv 0$. Let $H(t)$ be the operator appearing in Section 2. Consider the Cauchy problem

$$(A.1) \quad \begin{cases} (\partial_t + iH(t))u(t) = f(t, u(t)) \text{ in } \mathcal{D}'((0, T) \times \mathbf{R}^d) \\ u(0) = u_0. \end{cases}$$

Lemma A.1. Let $s > \frac{d}{2}$. For every $u_0 \in B^s$ there is a solution $u \in C([0, T]; B^s)$ satisfying (A.1) with $T = T(\|u_0\|_{B^s}) > 0$.

Lemma A.2. Suppose $u \in C([0, T]; B^s)$, $s > -r$, $r \geq 0$, $\sup_{0 \leq t \leq T} \|u(t)\|_{C_r^s} < \infty$ and that u satisfies (A.1). If $u_0 \in B^\tau$, $\tau > s$, then $u \in C([0, T]; B^\tau)$.

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Low Energy Asymptotics for Schrödinger Operators with Slowly Decreasing Potentials

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我々は、次の形の Schrödinger 作用素を考える。

$$H = -\Delta + V(x) \quad \text{on } L^2(\mathbf{R}^d), \quad V(x) \sim \text{const.}|x|^{-\rho} \quad \text{as } |x| \rightarrow \infty.$$

このような作用素に対する一つの興味深い問題として、エネルギーが0に近い領域、つまり低エネルギーでの散乱の解析があり、解の局所減衰の問題と結び付いて研究されてきた。特に、 $\rho > 2$ の場合は、ポテンシャル $V(x)$ は very short range であると呼ばれ、 $H_0 = -\Delta$ の摂動として解析できることが知られている。(例えば、Jensen-Kato [JK] を見よ。)

一方、 $0 < \rho < 2$ の時はポテンシャル $V(x)$ は slowly decreasing であると呼ばれ、very short range の場合の解析の方法は全く適用できず、低エネルギーでの解の挙動が異なる事が予想される。Yafaev は、ポテンシャルが slowly decreasing であって、しかも正の場合を考察し、 H^{-1} が適当な重み付き L^2 -空間の間で有界になる事を用いて、 $(H - z)^{-1}$ の $z \rightarrow 0$ での漸近挙動を調べた ([Y1])。

ここでは、半古典極限のアイデアを用いることにより、Yafaev の結果より精密な結果が得られることを紹介する。また、ポテンシャルが負の場合にも、(適当な条件のもとでは、) 類似の結果が得られる。

ポテンシャルが正の場合: ポテンシャル $V(x)$ は、次のような仮定を満たすとす。

仮定 A (i) $V(x)$ は \mathbf{R}^d 上の実数値関数であって、 $0 < \rho < 2$ なる ρ が存在して、任意の多重指数 α に対し、

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha V(x) \right| \leq C_\alpha \langle x \rangle^{-\rho - |\alpha|}, \quad x \in \mathbf{R}^d,$$

を満たす。

(ii) 正の実数 δ が存在して、 $x \in \mathbf{R}^d$ に対して、 $V(x) \geq \delta \langle x \rangle^{-\rho}$ を満たす。

(iii) 正の実数 ε と R が存在して、 $|x| > R$ ならば、 $x \cdot \partial V / \partial x(x) \leq -\varepsilon |x|^{-\rho}$ を満たす。

仮定Aの下では、よく知られているように、レゾルベントの境界値が正の実軸上で存在する。すなわち、 $\gamma > 1/2$ ならば、

$$\langle x \rangle^{-\gamma} (H - \lambda \pm i0)^{-1} \langle x \rangle^{-\gamma} \equiv \lim_{\varepsilon \downarrow 0} \langle x \rangle^{-\gamma} (H - \lambda \pm i\varepsilon)^{-1} \langle x \rangle^{-\gamma} \in B(L^2(\mathbf{R}^d)).$$

定理1 $V(x)$ は仮定Aを満たし、 $\gamma > 1/2 + \rho/4$ とする。すると、

$$\sup_{0 < \lambda \leq 1} \left\| \langle x \rangle^{-\gamma} (H - \lambda \pm i0)^{-1} \langle x \rangle^{-\gamma} \right\| \leq C < \infty.$$

さらに、 $\langle x \rangle^{-\gamma} (H - \lambda \pm i0)^{-1} \langle x \rangle^{-\gamma}$ は $\lambda = 0$ の近傍で Hölder 連続である。

また、レゾルベントのべきについても、次のような結果が得られる。

定理2 $V(x)$ は仮定Aを満たし、 $k \geq 1$ 、 $\gamma > \max(k - 1/2, k(1/2 + \rho/4))$ とする。すると、

$$\sup_{0 < \lambda \leq 1} \left\| \langle x \rangle^{-\gamma} (H - \lambda \pm i0)^{-k} \langle x \rangle^{-\gamma} \right\| \leq C_k < \infty.$$

さらに、 $\langle x \rangle^{-\gamma} (H - \lambda \pm i0)^{-k} \langle x \rangle^{-\gamma}$ は Hölder 連続である。

さて、(少なくとも形式的には) $(H - \lambda \pm i0)^{-1}$ の k 階微分は $k!(H - \lambda \pm i0)^{-k-1}$ だから、上の結果は $(H - \lambda \pm i0)^{-1}$ の微分可能性を意味している。

系3 φ が急減少関数ならば、 $\varphi(H - \lambda \pm i0)^{-1}\varphi$ は、 λ に関し $B(L^2(\mathbf{R}^d))$ -値の C^∞ -級関数である。

注意1 H のスペクトル分解を $E(\lambda)$ と書くと、 $E'(\lambda) = (2\pi)^{-1} ((H - \lambda - i0)^{-1} - (H - \lambda + i0)^{-1})$ なので、系3より、 φ が急減少ならば $\varphi E'(\lambda)\varphi$ は λ に関して C^∞ -級関数であることがわかる。 H は非負だから、 $\lambda < 0$ の時は勿論 $E'(\lambda) = 0$ であり、任意の $k \geq 0$ に対して、 $(\frac{d}{d\lambda})^k \varphi E'(0)\varphi = 0$ となる。これは、 H のスペクトル関数は、 λ が0に近づく時急減少している事を示している。実はさらに、 $\exp(-\gamma\lambda^{-(1/\rho-1/2)})$ のオーダーで劣指数的に減少していることもわかる。

これらの結果は、Jensen-Mourre-Perry [JMP] の方法と組み合わせることにより、Schrödinger 方程式の解の局所減衰に関する次の結果を導く。

定理4 仮定Aの下で、任意の $\gamma > \beta > 0$ に対して、

$$\left\| \langle x \rangle^{-\gamma} e^{-itH} \langle x \rangle^{-\gamma} \right\| \leq C \langle t \rangle^{-\beta}, \quad t \in \mathbf{R}.$$

注意2 注意1の最後に述べた劣指数的評価に着目すると、局所減衰についても、同様の劣指数的評価が得られることが予想される。実際、熱方程式の解： e^{-tH} については、 $\gamma > 0$ が存在して、任意の $R > 0$ に対して、

$$\left\| F(|x| \leq R) e^{-tH} \right\| \leq C \exp(-\gamma t^{(2-\rho)/(2+\rho)}), \quad t > 0$$

となる事が判る。しかし、Schrödinger 方程式の解については、ポテンシャルについて、もっと強い仮定(解析性など)が必要と思われる。

証明には、次のようなアイデアを用いる。 $\lambda > 0$ を考えるエネルギーとする時、座標変換： $x = \lambda^{-1/\rho}y$ を行くと、(形式的には、)

$$(H - \lambda \pm i0)^{-1} = \lambda^{-1}(-g^2\Delta_y + V_\lambda(y) - 1 \pm i0)^{-1} = \lambda^{-1}(H_\lambda - 1 \pm i0)^{-1},$$

ただし、 $g = \lambda^{1/\rho-1/2}$ 、 $V_\lambda(y) = \lambda^{-1}V(\lambda^{-1/\rho}y)$ 、となる。 g をプランク定数とみなせば、 H_λ はプランク定数を含む Schrödinger 作用素の形をしている。 $\rho < 2$ なので、 $\lambda \rightarrow 0$ の時 $g \rightarrow 0$ となり、エネルギー0の近くでの問題が、エネルギー1での H_λ の半古典極限の問題に変換される。そこで、半古典極限のテクノロジーを用いる事によって、我々の定理を証明することができる。

ポテンシャルが負の場合：ポテンシャルが正の場合は、ポテンシャル力は、大体、粒子を外に押し出す方向に働くので、上に述べたような一様な局所減衰が得られるのは自然なことに思われる。一方、ポテンシャルが負の場合は、ポテンシャル力は粒子を押し止める方向に働くので、局所減衰はポテンシャルが無い場合よりも悪いことが予測される。しかし、レゾルベントの境界値の評価に関しては、以下のような仮定のもとで、ポテンシャルが正の場合と同様の評価が得られる。

ここでは、解析を易しくするためにプランク定数 $\hbar > 0$ を導入し、 \hbar が十分小さい、つまり半古典的領域で考える。このとき、Schrödinger 作用素は、 $H = -\hbar^2\Delta + V(x)$ の形を持つ。ポテンシャルは次の仮定を満たすとする。

仮定 B (i) V は \mathbf{R}^d 上の実数値関数であって、 $0 < \rho < 2$ なる ρ が存在して、任意の多重指数 α に対して、

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha V(x) \right| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \quad x \in \mathbf{R}^d$$

を満たす。

(ii) 正の実数 δ が存在して、任意の x に対して、 $V(x) \leq -\delta \langle x \rangle^{-\rho}$ が成立する。

(iii)

$$\sup_{x \in \mathbf{R}^d} |V(x)|^{-1} \left| x \cdot \frac{\partial V}{\partial x}(x) \right| \equiv \rho' < 2.$$

定理 5 ポテンシャル $V(x)$ が仮定 B を満たすならば、 $\hbar_0 > 0$ が存在して、 $0 < \hbar \leq \hbar_0$ しかも、 $\gamma > 1/2 + \rho/4$ ならば、

$$\sup_{0 < \lambda \leq 1} \left\| \langle x \rangle^{-\gamma} (H - \lambda \pm i0)^{-1} \langle x \rangle^{-\gamma} \right\| \leq C < \infty.$$

が成立する。さらに、

$$\langle x \rangle^{-\gamma} (H - 0 \pm i0)^{-1} \langle x \rangle^{-\gamma} \equiv \lim_{\lambda \downarrow 0} \langle x \rangle^{-\gamma} (H - \lambda \pm i0)^{-1} \langle x \rangle^{-\gamma} \in B(L^2(\mathbf{R}^d))$$

が存在する。

注意 3 定理 5 の最後の主張は、 $\langle x \rangle^{-\gamma} E'(+0) \langle x \rangle^{-\gamma} \equiv \lim_{\lambda \downarrow 0} \langle x \rangle^{-\gamma} E'(\lambda) \langle x \rangle^{-\gamma}$ が存在することを示している。一般には、 $E'(+0) = 0$ とは限らない。実際、1次元の場合には $E'(0) \neq 0$ である事が知られている ([Y2])。従って、一般には γ がいかに大きくても、 $\left\| \langle x \rangle^{-\gamma} e^{-itH} \langle x \rangle^{-\gamma} \right\|$ は高々 $O(t^{-1})$ でしか減衰しない。

証明のアイデアは、ポテンシャルが正の場合とは全く異なる。作用素 L_0 を $L_0 = (-V)^{-1/2} H_0 (-V)^{-1/2}$ とおけば、(形式的には、)

$$(H - 0 \pm i0)^{-1} = (H_0 - (-V) \pm i0)^{-1} = (-V)^{-1/2} (L_0 - 1 \pm i0)^{-1} (-V)^{-1/2}$$

となる事に着目し、 L_0 に対する散乱理論を構成することによって、 $(H - z)^{-1}$ の $z \sim 0$ での挙動を解析することができる。

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STABILITY OF STANDING WAVES FOR THE GENERALIZED DAVEY-STEWARTSON SYSTEM

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1. INTRODUCTION AND RESULT

In the present paper we consider the stability of standing waves for the following nonlinear Schrödinger equation:

$$(1.1) \quad iu_t + \Delta u + a|u|^{p-1}u + bE_1(|u|^2)u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n,$$

where $a, b \geq 0$, $1 < p < 1 + 4/(n-2)$, $n = 2$ or 3 and E_1 is the singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi \in \mathbb{R}^n$.

The equation (1.1) has its origin in fluid mechanics where, for $n = 2$ and $p = 3$, it describes the evolution of weakly nonlinear water waves that travel predominantly in one direction. More precisely, (1.1) is the n -dimensional extension of the generalized Davey-Stewartson system in the elliptic-elliptic case, namely

$$\begin{cases} iu_t + \lambda u_{xx} + u_{yy} + a|u|^{p-1}u + b_1 uv_x = 0, \\ v_{xx} + \mu v_{yy} = b_2(|u|^2)_x, \end{cases}$$

where $\lambda, \mu > 0$ (see [5]).

By a standing wave, we mean a solution of (1.1) with the form

$$u(t, x) = e^{i\omega t} \varphi_\omega(x),$$

where $\omega > 0$ and φ_ω is a ground state of the following stationary problem:

$$(1.2_\omega) \quad \begin{cases} -\Delta \psi + \omega \psi - a|\psi|^{p-1}\psi - bE_1(|\psi|^2)\psi = 0, \quad x \in \mathbb{R}^n, \\ \psi \in H^1(\mathbb{R}^n), \quad \psi \neq 0. \end{cases}$$

Before stating our result, we introduce some notations.

$$S_\omega(v) = \frac{1}{2}|\nabla v|_2^2 + \frac{\omega}{2}|v|_2^2 - \frac{a}{p+1}|v|_{p+1}^{p+1} - \frac{b}{4} \int |v|^2 E_1(|v|^2) dx,$$

\mathcal{X}_ω = the set of solutions for (1.2 $_\omega$)

$$= \{\psi \in H^1(\mathbb{R}^n) : S'_\omega(\psi) = 0, \quad \psi \neq 0\},$$

\mathcal{G}_ω = the set of ground states for (1.2 $_\omega$)

$$= \{\varphi \in \mathcal{X}_\omega : S_\omega(\varphi) \leq S_\omega(\psi) \text{ for all } \psi \in \mathcal{X}_\omega\}.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

Remark 1. Cipolatti [3] showed that if $a \geq 0$, $b > 0$, $1 < p < 1 + 4/(n - 2)$ and $n = 2$ or 3 , then \mathcal{G}_ω is not empty for any $\omega \in (0, \infty)$.

Assumption (H). We assume that there is a choice $\varphi_\omega \in \mathcal{G}_\omega$ such that $\omega \mapsto \varphi_\omega$ is a C^1 mapping from the interval $(0, \infty)$ into $H^1(\mathbb{R}^n)$. Moreover when the space dimensions $n = 2$, we assume that

$$(1.3) \quad |\varphi|_2 = |\varphi_\omega|_2 \quad \text{for any } \varphi \in \mathcal{G}_\omega.$$

Remark 2. Clearly, the condition (1.3) is satisfied if the ground state of (1.2 $_\omega$) is unique, that is, for the above φ_ω ,

$$(1.4) \quad \mathcal{G}_\omega = \{e^{i\theta} \tau_y \varphi_\omega : \theta \in \mathbb{R}, y \in \mathbb{R}^n\},$$

where $(\tau_y v)(x) = v(x - y)$. Here, we remark that due to the invariances under translation and multiplication by $e^{i\theta}$, the following relation always holds.

$$\{e^{i\theta} \tau_y \varphi_\omega : \theta \in \mathbb{R}, y \in \mathbb{R}^n\} \subset \mathcal{G}_\omega.$$

Remark 3. For the uniqueness of ground states, it is known that when $a > 0$ and $b = 0$ (pure power case), the ground state of (1.2 $_\omega$) is unique for $1 < p < 1 + 4/(n - 2)$ (see [7]). However when $b \neq 0$, we do not know whether the ground state is unique or not.

Definition. We shall say that the standing wave $u_\omega(t) = e^{i\omega t} \varphi_\omega$ is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: If $u_0 \in H^1(\mathbb{R}^n)$ and the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $\|u_0 - \varphi_\omega\|_{H^1} < \delta$, then

$$\sup_{0 \leq t < \infty} \inf_{\varphi \in \mathcal{G}_\omega} \|u(t) - \varphi\|_{H^1} < \varepsilon.$$

Otherwise, u_ω is said to be *unstable*.

Remark 4. The above definition of stability is the same as that in [2, Theorem II.2] but slightly different from that in [4]. If the condition (1.4) holds, those definitions are equal.

Remark 5. The unique local existence of H^1 solution for (1.1) was established by Ghidaglia and Saut [6]: If $a, b \geq 0$, $1 < p < 1 + 4/(n - 2)$ and $n = 2$ or 3 , then for any $u_0 \in H^1(\mathbb{R}^n)$ there exist $T > 0$ and a unique solution $u(\cdot) \in C([0, T]; H^1(\mathbb{R}^n))$ of (1.1) with $u(0) = u_0$. Furthermore, $u(t)$ satisfies:

$$(1.5) \quad |u(t)|_2 = |u_0|_2,$$

$$(1.6) \quad \mathcal{E}(u(t)) = \mathcal{E}(u_0),$$

for all $t \in [0, T)$, where \mathcal{E} is defined on $H^1(\mathbb{R}^n)$ by

$$\mathcal{E}(v) = \frac{1}{2} |\nabla v|_2^2 - \frac{a}{p+1} |v|_{p+1}^{p+1} - \frac{b}{4} \int |v|^2 E_1(|v|^2) dx.$$

Cipolatti [4] proved that if $a \geq 0$, $b > 0$, $p \geq 3$ and $n = 2$ or 3 , then φ_ω is unstable for any $\omega \in (0, \infty)$. However, to our knowledge, we are aware of no results concerning the existence of stable standing waves for (1.1). Our result is the following.

Theorem 1. Under Assumption (H), if $a, b > 0$, $1 < p < 1 + 4/n$ and $n = 2$ or 3 , then there exists a sequence (ω_k) such that $\omega_k > 0$, $\omega_k \rightarrow 0$ and φ_{ω_k} is stable.

Remark 6. When $a > 0$ and $b = 0$ (pure power case), it is well known that if $1 < p < 1 + 4/n$, then all standing waves are stable, and if $1 + 4/n \leq p < 1 + 4/(n - 2)$, then all standing waves are unstable (see [1, 2, 12]).

Remark 7. Recently, in [9] the author studied the double power nonlinear Schrödinger equation:

$$(1.7) \quad iu_t + \Delta u + |u|^{p-1}u + |u|^{q-1}u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n,$$

where $1 < p < q < 1 + 4/(n - 2)$ and $n \geq 2$, and showed the following result by the same argument as this paper: Let ϕ_ω be a ground state of

$$(1.8_\omega) \quad \begin{cases} -\Delta\psi + \omega\psi - |\psi|^{p-1}\psi - |\psi|^{q-1}\psi = 0, & x \in \mathbb{R}^n, \\ \psi \in H^1(\mathbb{R}^n), \quad \psi \neq 0. \end{cases}$$

If $p < 1 + 4/n$, then there exists a sequence (ω_k) such that $\omega_k > 0$, $\omega_k \rightarrow 0$ and ϕ_{ω_k} is stable, and if $q > 1 + 4/n$, then there exists a sequence $(\tilde{\omega}_k)$ such that $\tilde{\omega}_k \rightarrow \infty$ and $\phi_{\tilde{\omega}_k}$ is unstable.

Since E_1 is the singular integral operator of order zero, when $n = 3$, under the same assumption as Theorem 1.1, there is a possibility that there exists a sequence $(\tilde{\omega}_k)$ such that $\tilde{\omega}_k \rightarrow \infty$ and $\varphi_{\tilde{\omega}_k}$ is unstable.

On the other hand, in Theorem 1.1, we do not know whether there exists a positive constant ω_0 such that φ_ω is stable for any $\omega \in (0, \omega_0)$. For that matter, the author showed the following result in [10]. Consider the equation (1.7) in the case of $n = 1$, $q = 2p - 1$ and $3 < p < 5$. Let ϕ_ω be a solution for (1.8 $_\omega$). Then there exist positive constants ω_1 and ω_2 such that ϕ_ω is stable if $\omega \in (0, \omega_1)$, and unstable if $\omega \in (\omega_2, \infty)$.

This paper is organized as follows. In Section 2 we first state Theorem 2, which gives a sufficient condition for the stability. Next we prove Theorem 1 by using Theorem 2. We should mention that Theorem 2 is based on the ideas of Shatah [11], who studies the case of nonlinear Klein-Gordon equations with local nonlinearity within the framework of the radially symmetric functions. However, since (1.1) is anisotropic, we need to remove the restriction of radial symmetry in the argument by Shatah [11]. Therefore, we use the concentration compactness principle due to P.-L. Lions [8], following Cipolatti [3], instead of the compactness of radially symmetric functions. We often use the two conservation laws (1.5) and (1.6).

In what follows, we omit the integral variables with respect to the spatial variable x , and we omit the integral region when it is the whole space \mathbb{R}^n . We denote the norms of $L^q(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ by $|\cdot|_q$ and $\|\cdot\|_{H^1}$, respectively.

2. PROOF OF THEOREM 1

In this section we prove Theorem 1 by using the following Theorem 2.

Theorem 2. *Assume that $n = 2$ or 3 and (H) holds. Let $d(\omega) = S_\omega(\varphi_\omega)$, $0 < \omega < \infty$. If $d''(\omega_0) > 0$, then φ_{ω_0} is stable.*

We define the following functionals on $H^1(\mathbb{R}^n)$:

$$\begin{aligned} T(v) &= |\nabla v|_2^2, \\ V_\omega(v) &= \frac{a}{p+1}|v|_{p+1}^{p+1} + \frac{b}{4} \int |v|^2 E_1(|v|^2) dx - \frac{\omega}{2}|v|_2^2, \\ P_\omega(v) &= \left(\frac{1}{2} - \frac{1}{n}\right)T(v) - V_\omega(v). \end{aligned}$$

We note that

$$(2.1) \quad \begin{aligned} S_\omega(v) &= \frac{1}{2}T(v) - V_\omega(v) \\ &= \mathcal{E}(v) + \frac{\omega}{2}|v|_2^2, \end{aligned}$$

$$(2.2) \quad P_\omega(v) = S_\omega(v) - \frac{1}{n}T(v).$$

Lemma 3. *Assume that $n = 2$ or 3 and (H) holds.*

- (1) $P_\omega(\psi) = 0$ for all $\psi \in \mathcal{X}_\omega$ (Pohozaev's identity),
- (2) $d'(\omega) = \frac{1}{2}|\varphi_\omega|_2^2$,
- (3) $d(\omega) = \inf\{\frac{1}{n}T(\psi) : \psi \in H^1(\mathbb{R}^n), \psi \neq 0, P_\omega(\psi) \leq 0\}$.

Proof of Theorem 1. Let $\tilde{\varphi}_\omega$ be the ground state of

$$(2.3) \quad \begin{cases} -\Delta\psi + \omega\psi - a|\psi|^{p-1}\psi = 0, & x \in \mathbb{R}^n, \\ \psi \in H^1(\mathbb{R}^n), \psi \neq 0. \end{cases}$$

$$\text{Since} \quad \int |v|^2 E_1(|v|^2) dx = \int \sigma_1(\xi) |\mathcal{F}(|v|^2)|^2 d\xi \geq 0$$

for all $v \in H^1(\mathbb{R}^n)$, where \mathcal{F} is the Fourier transform on \mathbb{R}^n , we have

$$P_\omega(\tilde{\varphi}_\omega) \leq \left(\frac{1}{2} - \frac{1}{n}\right)T(\tilde{\varphi}_\omega) + \frac{\omega}{2}|\tilde{\varphi}_\omega|_2^2 - \frac{a}{p+1}|\tilde{\varphi}_\omega|_{p+1}^{p+1} = 0.$$

Here, we have used Pohozaev's identity for the equation (2.3). From Lemma 3 (3), we have $d(\omega) \leq \frac{1}{n}T(\tilde{\varphi}_\omega)$ for all $\omega \in (0, \infty)$.

Moreover, since $\tilde{\varphi}_\omega(x) = \omega^{1/(p-1)}\tilde{\varphi}_1(\sqrt{\omega}x)$, we have $T(\tilde{\varphi}_\omega) = \omega^\alpha T(\tilde{\varphi}_1)$, where $\alpha = \frac{2}{p-1} - \frac{n-2}{2} > 1$. Therefore, we have

$$(2.4) \quad d(\omega) \leq \frac{1}{n}\omega^\alpha T(\tilde{\varphi}_1) \text{ with } \alpha > 1 \text{ for all } \omega \in (0, \infty).$$

Here, if $d''(\omega) \leq 0$ in $(0, \hat{\omega})$ for some $\hat{\omega} > 0$, then $d'(\omega) \geq d'(\hat{\omega})$ in $(0, \hat{\omega})$. Furthermore, since it follows from (2.4) that $\lim_{\omega \rightarrow 0} d(\omega) = 0$, we have

$$(2.5) \quad d(\omega) = \int_0^\omega d'(s)ds \geq d'(\hat{\omega})\omega \text{ for any } \omega \in (0, \hat{\omega}).$$

We remark that $d'(\hat{\omega}) > 0$ from Lemma 3 (2). Thus, (2.5) contradicts (2.4). Therefore, there exists a sequence (ω_k) such that $\omega_k > 0$, $\omega_k \rightarrow 0$ and $d''(\omega_k) > 0$. Hence, Theorem 1 holds by Theorem 2. \square

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ON THE SOLVABILITY OF PHASE FIELD EQUATIONS WITH CONSTRAINTS

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1. Introduction

This paper is concerned with a nonlinear parabolic PDEs of the following forms (1.1)-(1.2):

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} - \Delta u = f(t, x) \quad \text{in } Q := (0, +\infty) \times \Omega, \quad (1.1)$$

$$\nu \frac{\partial w}{\partial t} - \kappa \Delta w + \beta(w) + g(t, x, u, w) \ni 0 \quad \text{in } Q \quad (1.2)$$

with lateral boundary conditions (1.3)-(1.5) and with initial conditions (1.6):

$$u = h_D(t, x) \quad \text{on } \Sigma_D := (0, +\infty) \times \Gamma_D, \quad (1.3)$$

$$\frac{\partial u}{\partial n} + \alpha_N(x)u = h_N(t, x) \quad \text{on } \Sigma_N := (0, +\infty) \times \Gamma_N, \quad (1.4)$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma := (0, +\infty) \times \Gamma, \quad (1.5)$$

$$u(0, \cdot) = u_0, \quad w(0, \cdot) = w_0 \quad \text{in } \Omega. \quad (1.6)$$

Here Ω is a bounded domain in R^N ($N \geq 1$) with smooth boundary $\Gamma := \partial\Omega$, Γ_D is a compact subset of Γ with positive surface measure and $\Gamma_N := \Gamma \setminus \Gamma_D$; α_N is a non-negative, bounded and measurable function on Γ ; $\nu > 0$ and $\kappa > 0$ are constants; β is a maximal monotone graph in $R \times R$ such that there exists a proper l.s.c. convex function $\hat{\beta}$ on R satisfying $\partial\hat{\beta} = \beta$ in $R \times R$ and

$$\hat{\beta}(r) \geq a_0|r|^2 \quad \text{for all } r \in R, \quad (1.7)$$

where $\partial\hat{\beta}$ denotes the subdifferential of $\hat{\beta}$ in R and a_0 is a positive constant; g is a function defined on $R_+ \times \Omega \times R \times R$ such that

(g1) $g = g(t, x, \xi, \eta)$ is Lipschitz continuous in (ξ, η) on $R \times R$ with uniform Lipschitz constant $L(g)$ with respect to $(t, x) \in R_+ \times \Omega$, i.e.

$$|g(t, x, \xi, \eta) - g(t, x, \bar{\xi}, \bar{\eta})| \leq L(g)\{|\xi - \bar{\xi}| + |\eta - \bar{\eta}|\}$$

for all $\xi, \bar{\xi}, \eta, \bar{\eta} \in R$ and a.e. $(t, x) \in Q$;

(g2) $g(\cdot, \cdot, \xi, \eta) \in L^2_{loc}(R_+; L^2(\Omega))$ for each $\xi, \eta \in R$.

Moreover, we suppose for the data h_D, h_N, f and α_N that

$$\begin{cases} h_D \in W^1_{loc}(R_+; H^{\frac{1}{2}}(\Gamma)), \quad h_N \in W^1_{loc}(R_+; L^2(\Gamma)), \quad f \in L^2_{loc}(R_+; L^2(\Omega)), \\ \alpha_N \in L^\infty(\Gamma), \quad \alpha_N \geq 0 \text{ a.e. on } \Gamma. \end{cases} \quad (1.8)$$

For simplicity we denote by (P) the system (1.1)-(1.5) and by (CP) ($=$ (CP; u_0, w_0)) the Cauchy problem for (P) with initial condition (1.6).

It is the purpose of this paper to discuss (CP), which is called the phase-field model (cf. [6,2,5]) with constraint, from a view-point of the theory (cf. [1,7]) of subdifferentials of convex functions in a Hilbert space.

This paper is a part of [4].

2. Subdifferentials associated with (P)

Let us consider a Hilbert space $V := \{z \in H^1(\Omega); z = 0 \text{ a.e. on } \Gamma_D\}$ with norm

$$|z|_V := \{|\nabla z|_{L^2(\Omega)}^2 + \int_{\Gamma} \alpha_N |z|^2 d\Gamma\}^{\frac{1}{2}},$$

and denote by V^* and $\langle \cdot, \cdot \rangle$ the dual space of V and the duality pairing between V^* and V , respectively. Then, identifying $L^2(\Omega)$ with its dual space by means of the usual inner product

$$(v, z) := \int_{\Omega} v z dx,$$

we see that

$$V \subset L^2(\Omega) \subset V^*$$

with compact injections.

Let F be the duality mapping from V onto V^* which is given by the formula:

$$\langle Fv, z \rangle = \int_{\Omega} \nabla v \cdot \nabla z dx + \int_{\Gamma} \alpha_N v z d\Gamma \quad \text{for any } v, z \in V,$$

It is easy to see that V^* becomes a Hilbert space with inner product $(\cdot, \cdot)_*$ given by

$$(v, z)_* := \langle v, F^{-1}z \rangle (= \langle z, F^{-1}v \rangle) \quad \text{for any } v, z \in V^*.$$

Now, consider the product space

$$X := V^* \times L^2(\Omega).$$

Then X becomes a Hilbert space with inner product $(\cdot, \cdot)_X$ given by

$$([e_1, w_1], [e_2, w_2])_X := (e_1, e_2)_* + \nu(w_1, w_2) \quad \text{for any } [e_i, w_i] \in X \ (i = 1, 2).$$

Next, given two boundary data h_D and h_N , choose $h : R_+ \longrightarrow H^1(\Omega)$ such that for each $t \geq 0$

$$\begin{cases} h(t) = h_D(t) & \text{a.e. on } \Gamma_D, \\ \int_{\Omega} \nabla h(t) \cdot \nabla z dx + \int_{\Gamma} \alpha_N h(t) z d\Gamma = \int_{\Gamma} h_N(t) z d\Gamma & \text{for all } z \in V. \end{cases}$$

Also, using h and $\hat{\beta}$, for each $t \geq 0$, introduce the following proper l.s.c. convex function φ^t on X :

$$\varphi^t(U) = \begin{cases} \frac{1}{2}|e - w|_{L^2(\Omega)}^2 + \frac{\kappa}{2}|\nabla w|_{L^2(\Omega)}^2 + \int_{\Omega} \hat{\beta}(w) dx - (h(t), e) \\ \quad \text{if } U = [e, w] \in L^2(\Omega) \times H^1(\Omega) \text{ with } \hat{\beta}(w) \in L^1(\Omega), \\ +\infty & \text{otherwise;} \end{cases}$$

denote by $\partial\varphi^t$ the subdifferential of φ^t in X .

Theorem 2.1. (Damlamian-Kenmochi-Sato [3]) *Let $t \geq 0$, $[e^*, w^*] \in X$ and $[e, w] \in D(\partial\varphi^t)$. Then $[e^*, w^*] \in \partial\varphi^t([e, w])$ if and only if condition (a) and (b) below are satisfied:*
(a) $e^* = F(e - w - h(t))$, that is, $e - w - h(t) \in V$ and

$$\langle e^*, z \rangle = \int_{\Omega} \nabla(e - w - h(t)) \cdot \nabla z dx + \int_{\Gamma} \alpha_N(e - w - h(t)) z d\Gamma \quad \text{for all } z \in V;$$

(b) there exists a function $\xi \in L^2(\Omega)$ such that $\xi \in \beta(w)$ a.e. on Ω ,

$$\nu(w^*, z) = \kappa \int_{\Omega} \nabla w \cdot \nabla z dx + (\xi - e + w, z) \quad \text{for all } z \in H^1(\Omega).$$

Moreover, for $U_i^* = [e_i^*, w_i^*] \in \partial\varphi^t(U_i)$ with $U_i = [e_i, w_i] \in D(\partial\varphi^t)$ ($i = 1, 2$)

$$\begin{aligned} (U_1^* - U_2^*, U_1 - U_2)_X &= |(e_1 - w_1) - (e_2 - w_2)|_{L^2(\Omega)}^2 + \kappa |\nabla(w_1 - w_2)|_{L^2(\Omega)}^2 \\ &\quad + (\xi_1 - \xi_2, w_1 - w_2) \end{aligned} \quad (2.1)$$

where $\xi_i \in L^2(\Omega)$ is as any function ξ in (b) for each $i = 1, 2$.

This theorem is originally due to Visintin [9]. According to it we see that (P) can be reformulated as an evolution equation in X of the following form:

$$(E) \quad U'(t) + \partial\varphi^t(U(t)) + G(t, U(t)) \ni 0, \quad \text{in } X, \quad t \geq 0,$$

where $U(t) = [e(t), w(t)]$ with $e(t) = u(t) + w(t)$, $U'(t) = \frac{d}{dt}U(t)$ and

$$G(t, U(t)) = [-f(t), \frac{1}{\nu}(e(t) - w(t) + g(t, \cdot, e(t) - w(t), w(t)))].$$

3. The case $g(t, x, u, w) = -u - l(t, x)$

In this section, let l be a function in $L_{loc}^2(R_+; L^2(\Omega))$. For simplicity, we denoted by (P_0) the system (1.1)-(1.5) with $g(t, x, u, w) = -u - l(t, x)$ and by (CP_0) ($= (CP_0; u_0, w_0)$) the Cauchy problem for (P_0) with initial conditions (1.6). Now, let us recall some results on (P_0) obtained in [3].

Theorem 3.1. (cf. [3; Theorem 2.2]) *Let $0 < T < +\infty$. Then a couple $\{u, w\}$ of functions $u : [0, T] \rightarrow V^*$ and $w : [0, T] \rightarrow L^2(\Omega)$ is a solution of (P_0) on $[0, T]$, if and only if the function $U := [u + w, w] : [0, T] \rightarrow X$ satisfies that*

$$U \in C([0, T]; X) \cap W_{loc}^{1,2}((0, T]; X)$$

$$\varphi^{(\cdot)}(U) \in L^1(0, T)$$

and

$$U'(t) + \partial\varphi^t(U(t)) \ni f^*(t) \quad \text{in } X \text{ for a.e. } t \in [0, T] \quad (3.1)$$

where $f^* := [f, \frac{1}{\nu}l] \in L^2(0, T; X)$.

The solvability of (CP_0) is stated as follows:

Theorem 3.2. (cf. [3;Theorem 3.1]) *Let $0 < T < +\infty$. Assume that (1.7)-(1.8) hold and $l \in L^2_{loc}(R_+; L^2(\Omega))$. Then we have the following statements:*

(1) *If the initial data satisfy*

$$u_0 \in V^*, \quad w_0 \in L^2(\Omega) \text{ with } w_0 \in \overline{D(\beta)} \text{ a.e. on } \Omega, \quad (3.2)$$

then $(CP_0) = (CP_0; f, l; h_D, h_N; u_0, w_0)$ admits one and only one solution $\{u, w\}$ on $[0, T]$ such that

$$t^{\frac{1}{2}}u' \in L^2(0, T; V^*), \quad t^{\frac{1}{2}}u \in L^2(0, T; H^1(\Omega)), \quad (3.3)$$

$$tu' \in L^2(0, T; L^2(\Omega)), \quad tu \in L^\infty(0, T; H^1(\Omega)), \quad (3.4)$$

$$t^{\frac{1}{2}}w' \in L^2(0, T; L^2(\Omega)), \quad t\hat{\beta}(w) \in L^\infty(0, T; L^1(\Omega)), \quad t^{\frac{1}{2}}\xi \in L^2(0, T; L^2(\Omega)), \quad (3.5)$$

where ξ is the function in condition $(w\mathfrak{B})$ and

$$t^{\frac{1}{2}}w \in L^\infty(0, T; H^1(\Omega)). \quad (3.6)$$

(2) *If the initial data satisfy that*

$$\begin{cases} u_0 - h(0) \in V, \\ w_0 \in H^1(\Omega) \text{ with } \hat{\beta}(w_0) \in L^1(\Omega), \end{cases} \quad (3.7)$$

then the solution $\{u, w\}$ of $(CP_0) = (CP_0; f, l; h_D, h_N; u_0, w_0)$ has the regularity properties

$$u' \in L^2(0, T; L^2(\Omega)), \quad u \in L^\infty(0, T; H^1(\Omega)), \quad (3.8)$$

$$w' \in L^2(0, T; L^2(\Omega)), \quad \hat{\beta}(w) \in L^\infty(0, T; L^1(\Omega)), \quad \xi \in L^2(0, T; L^2(\Omega)), \quad (3.9)$$

$$w \in L^\infty(0, T; H^1(\Omega)). \quad (3.10)$$

4. Phase field equations with constraints

Our main result is stated as follows.

Theorem 4.1. *Assume that (1.7), (1.8), (3.2) and (g1)-(g2) hold. Then, for any $T > 0$, (CP) admits one and only one solution $\{u, w\}$ on $[0, T]$ such that*

$$\begin{cases} t^{\frac{1}{2}}u' \in L^2(0, T; V^*), \quad t^{\frac{1}{2}}u \in L^2(0, T; H^1(\Omega)), \\ tu' \in L^2(0, T; L^2(\Omega)), \quad tu \in L^\infty(0, T; H^1(\Omega)), \end{cases} \quad (4.1)$$

$$t^{\frac{1}{2}}w' \in L^2(0, T; L^2(\Omega)), \quad t\hat{\beta}(w) \in L^\infty(0, T; L^1(\Omega)), \quad t^{\frac{1}{2}}w \in L^\infty(0, T; H^1(\Omega)), \quad (4.2)$$

$$t^{\frac{1}{2}}\xi \in L^2(0, T; L^2(\Omega)), \quad (4.3)$$

where ξ is the function in condition $(w\mathfrak{B})'$. In particular, if the initial data u_0 and w_0 satisfy (3.7), then

$$u' \in L^2(0, T; L^2(\Omega)), \quad u \in L^\infty(0, T; H^1(\Omega)), \quad (4.4)$$

$$w' \in L^2(0, T; L^2(\Omega)), \quad \hat{\beta}(w) \in L^\infty(0, T; L^1(\Omega)), \quad w \in L^\infty(0, T; H^1(\Omega)), \quad (4.5)$$

$$\xi \in L^2(0, T; L^2(\Omega)). \quad (4.6)$$

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REGULARITY FOR WEAK SOLUTIONS OF
ELLIPTIC PARTIAL DIFFERENTIAL
EQUATIONS OF SECOND ORDER

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1. INTRODUCTION AND MAIN RESULTS

We consider the following elliptic equation of second order:

$$(1) \quad Lu \equiv -\operatorname{div}(A(x)\nabla u(x)) + \mathbf{c}(x) \cdot \nabla u(x) + V(x)u(x) = f \quad \text{in } \Omega.$$

Here $A(x) = (a_{ij}(x))$ is real-valued, \mathbf{c} and V are complex-valued and $A(x)$ satisfies

$$(2) \quad a_{ij}(x) = a_{ji}(x), \quad \lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \quad x \in \Omega, \quad \xi \in \mathbf{R}^n$$

for some $\lambda \in (0, 1]$. We assume

$$(A) \quad |V|, |\mathbf{c}|^2, f \in K_n^{loc}(\Omega).$$

Here we say $V \in K_n^{loc}(\Omega)$ if $\lim_{r \rightarrow 0} \eta(V; r; \Omega_1) = 0$ for each compact subdomain Ω_1 with $\overline{\Omega_1} \subset \Omega$, where

$$(3) \quad \eta(f; r) = \sup_{x \in \mathbf{R}^n} \int_{B_r(x)} \frac{|f(y)|}{|x-y|^{n-2}} dy$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$

and $\eta(f; r; G) = \eta(f\chi_G; r)$ and χ_G is the characteristic function of G and $B_r(x) = \{y \in \mathbb{R}^n; |x - y| < r\}$ for $r > 0$. We say $u \in H_{loc}^1(\Omega) = \{u \in L_{loc}^2(\Omega); \nabla u \in L_{loc}^2(\Omega)\}$ is a weak solution of (1) in Ω , if u satisfies

$$(4) \quad \int_{\Omega} \sum_{i,j=1}^n a_{ij} \partial_i u \partial_j \phi + \mathbf{c} \cdot \nabla \phi + Vu\phi \, dx = \int_{\Omega} f\phi \, dx$$

for every $\phi \in C_0^\infty(\Omega)$.

Throughout this paper we denote by $C(n, \lambda, \eta)$ the constant depends only on n , λ , and the modulus of functions $\eta(|V|; \cdot)$ and $\eta(|\mathbf{c}|^2; \cdot)$. Since we are only concerned with local properties of weak solutions, we may assume that Ω is bounded and $\eta(|V|; r; \Omega)$, $\eta(|\mathbf{c}|^2; r; \Omega)$, and $\eta(f; r; \Omega)$ tend to zero as $r \rightarrow 0$. For simplicity we use the notation $\eta(g; r) = \eta(g; r; \Omega)$.

Theorem 1. *Suppose (2) and ASSUMPTION (A) and let u be a weak solution of (1). Then u is continuous in Ω and there exist constants $r_o = r_o(n, \lambda, \eta)$, $C = C(n, \lambda, \eta) > 0$ and non-decreasing functions $\omega(s)$ and $\omega_f(s)$ satisfying $\lim_{s \rightarrow 0} \omega(s) = 0$, $\lim_{s \rightarrow 0} \omega_f(s) = 0$ such that*

$$(5) \quad \begin{aligned} |u(x) - u(x_o)| &\leq C\omega\left(\frac{|x - x_o|}{r}\right) \|u\|_{L^\infty(B_{5r}(x_o))} \\ &\quad + C\omega_f\left(\frac{|x - x_o|}{r}\right) \end{aligned}$$

for every $0 < r < r_o$ with $B_{8r}(x_o) \subset \Omega$. Moreover $|\nabla u|^2 \in K_n^{loc}(\Omega)$ holds.

Theorem 2. *Suppose ASSUMPTION (A) and \mathbf{c}, V are real-valued. Then for nonnegative weak solution of $Lu = 0$ there exist constants $C, r_o > 0$ such that*

$$(6) \quad \max_{B_r} u \leq C \min_{B_r} u$$

for $0 < r < r_o$ with $B_{4r} \subset \Omega$.

We also consider the Schrödinger equation with singular magnetic fields:

$$(7) \quad Tu \equiv -(\nabla - i\mathbf{b}(x))^2 u(x) + V(x)u(x) = f \quad \text{in } \Omega,$$

where $i = \sqrt{-1}$, $\mathbf{b}(x) = (b_j(x))_{j=1}^n$ is real-valued and $V(x)$ is complex-valued. If we apply Theorem 1 to this Schrödinger equation we must impose

$\operatorname{div} \mathbf{b} \in K_n^{loc}(\Omega)$. However we can show local boundedness of weak solution of (7) without this additional condition. We say $u \in H_{loc}^1(\Omega; \mathbf{C}) = \{u \in L_{loc}^2(\Omega; \mathbf{C}); \nabla u \in L_{loc}^2(\Omega; \mathbf{C}^n)\}$ is a weak solution of (7) in Ω , if u satisfies

$$(8) \quad \int_{\Omega} (\nabla u - i\mathbf{b}u) \cdot \overline{(\nabla \phi - i\mathbf{b}\phi)} + Vu\bar{\phi} \, dx = \int_{\Omega} f\bar{\phi} \, dx$$

for every $\phi \in C_0^\infty(\Omega; \mathbf{C})$, where $\bar{\phi}$ is the complex conjugate of ϕ . We also write $H_{loc}^1 = H_{loc}^1(\Omega; \mathbf{C})$ for simplicity. We denote by V_R the real part of V and use $(V_R)^- = \max(-V_R, 0)$. We also use $\int_A f \, dx = \frac{1}{|A|} \int_A f(x) \, dx$, where $|A|$ is the Lebesgue measure of A . For local boundedness of weak solution of (7) we have

Theorem 3. *Suppose $|V|, |\mathbf{b}|^2, f \in K_n^{loc}(\Omega)$ and let u be a weak solution of (7). Then $u \in L_{loc}^\infty(\Omega)$ and there exist constants $r_o = r_o(n, \eta), C = C(n, \eta) > 0$ such that*

$$(9) \quad \|u\|_{L^\infty(B_{r/2}(x_o))} \leq C \left(\int_{B_r(x_o)} |u|^2 \, dx \right)^{1/2} + C\eta(f; 2r)$$

for every $0 < r < r_o$ with $B_{2r}(x_o) \subset \Omega$, where r_o depends on $n, p, \eta(|V|; \cdot; \Omega)$ and $\eta(|\mathbf{b}|^2; \cdot; \Omega)$ and C only on n, p and $\eta((V_R)^-; \cdot; \Omega)$. When $f = 0$, we have

$$(10) \quad \|u\|_{L^\infty(B_{r/2}(x_o))} \leq C \left(\int_{B_r(x_o)} |u|^p \, dx \right)^{1/p}$$

for every $0 < p < +\infty$.

Since $-(\nabla - i\mathbf{b})^2 u = -\Delta u + 2i\mathbf{b}(x) \cdot \nabla u + i \operatorname{div} \mathbf{b}(x)u + |\mathbf{b}(x)|^2 u$, as an immediate consequence of Theorem 1 we have

Corollary 4. *Suppose $V, |\mathbf{b}|^2, \operatorname{div} \mathbf{b}, f \in K_n^{loc}(\Omega)$ and let u be a weak solution of (7). Then u is continuous in Ω and $|\nabla u|^2 \in K_n^{loc}(\Omega)$.*

Remark 1. If we assume somewhat stronger condition $V, |\mathbf{c}|^2, f \in K_{n, \delta}^{loc}(\Omega)$ for some $\delta > 0$, then Theorem 1 yields Hölder continuity for weak solutions of (1). Here we say $g \in K_{n, \delta}^{loc}(\Omega)$ if

$$(11) \quad \lim_{r \rightarrow 0} \sup_{x \in \mathbf{R}^n} \int_{B_r(x) \cap \Omega'} \frac{|g(y)|}{|x - y|^{n-2+\delta}} \, dy = 0$$

for every compact subdomain Ω' of Ω . We remark that $K_{n,\delta}^{loc}(\Omega) \subset K_n^{loc}(\Omega)$ for $\delta > 0$ and that if $g \in L_{loc}^p(\Omega)$ for some $p > n/2$, then $g \in K_{n,\delta}^{loc}(\Omega)$ for some $\delta > 0$.

Theorem 1, 2 and Theorem 3 generalize the previous results of [LU], [AS], [CFG], [Si], [HK]. The property $|\nabla u|^2 \in K_n^{loc}(\Omega)$ in the statement of Theorem 1 was first shown by Donig [Do] for weak solutions u of $Lu = -\Delta u + Vu = 0, V \in K_n^{loc}(\Omega)$ and was generalized to general elliptic equations (1) in [Ku1,2].

Remark 2. If $\mathbf{b} \in C_{loc}^1(\Omega)$ and $V, f \in K_n^{loc}(\Omega)$, one can show local boundedness of a distributional solution of (7), that is $u, Vu \in L_{loc}^1(\Omega)$ and u satisfies (7) in the distributional sense. Because we can use Kato's inequality directly (see [Hi]). We prove Theorem 3 by using Kato's inequality, but for an approximated solution. Since a local bounded distributional solution u belongs to $H_{loc}^1(\Omega)$ (see e.g. [HS, Lemma 2.2]), u is a weak solution. Applying Corollary 4 we can conclude that u is continuous and $|\nabla u|^2 \in K_n^{loc}(\Omega)$ even for a distributional solution of (7).

Remark 3. In [HS] Hinz and Stolz proved the local boundedness of distributional solution u of (7) with $u \in L_{loc}^2(\Omega)$ and $\nabla u \in L_{loc}^{4/3}(\Omega)$ under the assumptions $V \in L_{loc}^1(\Omega), (V)_R^- \in K_n^{loc}(\Omega)$ and $|\mathbf{b}|^2, \operatorname{div} \mathbf{b} \in L_{loc}^2(\Omega)$. But we do not know continuity of solutions under this conditions.

Example 1. We cannot expect in general Hölder continuity under ASSUMPTION (A). Let $u(x) = 1/(\log \frac{1}{r})^\alpha, \alpha > 0, r = |x|$. Then u is a weak solution of $-\Delta u + Vu = f$ in $B_1 = B_1(O)$ with $V = \frac{\alpha(\alpha+1)}{r^2(\log \frac{1}{r})^2} \in K_n^{loc}(B_1), f = \frac{(n-2)\alpha}{r^2(\log \frac{1}{r})^{1+\alpha}} \in K_n^{loc}(B_1)$. u is continuous but not Hölder continuous.

Example 2. Let $V(x) = \frac{1}{r^2(\log \frac{1}{r})^2}$ and $\mathbf{b}(x) = b(r)\frac{x}{r}, r = |x|$ with $b(r) = \frac{1}{r \log \frac{1}{r}}$. Then $V, |\mathbf{b}|^2 \in K_n^{loc}(B_1)$. Hence Theorem 1 implies continuity of weak solution of $-\operatorname{div}(A(x)\nabla u) + \mathbf{b} \cdot \nabla u + Vu = f \in K_n^{loc}(B_1)$, Theorem 3 implies local boundedness of weak solution of $Tu = f$. However since $\operatorname{div} \mathbf{b} \notin K_n^{loc}(B_1), \operatorname{div} \mathbf{b} = b_r + b\frac{n-1}{r} = \frac{n-2}{r^2 \log \frac{1}{r}} - \frac{1}{r^2(\log \frac{1}{r})^2}$, we cannot apply Corollary 4. Note that since $|\mathbf{b}| = |b(r)| \notin K_{n+1}^{loc}(B_1)$, the result of [CZ] cannot be applied.

We prove Theorem 1 by using some global integrability of the Green function of L and the mollified Green function of $L_0 = -\operatorname{div}(A(x) \nabla \cdot)$ (Theorem 4, Lemma 4 below) as in [Ku1,2] (cf. [CFG]).

Finally we give comments on different approach on this regularity problem. There exists an approach due to Simader, which is simple in the sense of using the Green function of the principal part $L_0 = -\operatorname{div}(A(x)\nabla)$, but only give partial results on our problem in the case $\mathbf{b} \neq 0$. For this approach see [Ku2, Appendix]. For a probabilistic approach see [AS], [CFZ], [CZ]. Especially Cranston and Zhao [CZ] proved Harnack's inequality for $L = -\Delta + \mathbf{b} \cdot \nabla + V$ under $V, |\mathbf{b}|^2 \in K_n^{\text{loc}}(\Omega)$ and an additional assumption $|\mathbf{b}| \in K_{n+1}^{\text{loc}}(\Omega)$ (see Example 2).

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Destabilization of layer solutions in chemical reactions.

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Chemically reacting and diffusing systems have been providing us with a great deal of spatio-temporal pattern formation phenomena. Moreover, the recent years have seen a renewed interest in dissipative structures in chemical systems. This interest is mainly due to the development of open spatial reactors by groups in Texas and in Bordeaux.

We are, here, interested in nonlinear reaction-diffusion models based upon chlorite-iodide reactions performed in the Couette flow reactor with continuous flow stirred tank reactors (CSTRs) at the ends [1]. Following the works by DeKepper et.al.[2] [3], Elezgaray and Arneodo [1] proposed the following type of reaction-diffusion model

$$\begin{cases} \partial u / \partial t = D \partial^2 u / \partial x^2 + \epsilon^{-2} [v - F(u)] \\ \partial v / \partial t = D \partial^2 v / \partial x^2 - u + a \end{cases} \quad t > 0, \quad x \in (0, 1) \quad (1)$$

which exhibits characteristic features of the chlorite-iodide reaction. Boundary conditions are Dirichlet ones :

$$\begin{cases} u(t, 0) = \alpha_0, & v(t, 0) = \beta_0 = F(\alpha_0) \\ u(t, 1) = \alpha_1, & v(t, 1) = \beta_1 = F(\alpha_1). \end{cases} \quad (2)$$

The most important feature of the Couette flow reactor is that the diffusion rate D and the boundary values α_0, β_0 , etc. are controllable parameters. The diffusion rate D is controlled by changing the rotation speed of the inner cylinder of the Couette flow reactor, while the boundary values are controlled by carefully maintaining chemical compositions of the CSTRs. Elezgaray and Arneodo [1] display various numerical simulation results for different values of these parameters, which agree with experimental results. In their experimental and numerical results, steady states of sharp internal transition layers are observed for small $\epsilon > 0$ and these states destabilize as the diffusion rate D is varied, giving rise to oscillating layers. When the parameter D is further changed, the amplitude of the oscillations gets bigger and the oscillating manner becomes erratic.

The purpose of our work is to mathematically understand the destabilization mechanism of the layer solutions of (1)(2) near the onset. To set up a stage, we assume the following properties for the nonlinear function $F(u)$:

- (1) The function $F(u)$ is smooth and has a local maximum v_M at $u = u_M$ and a local minimum v_m at $u = u_m$ with $u_M < u_m$.
- (2) $F'(u) > 0$ for $u < u_M$ and $u > u_m$, and $F'(u) < 0$ for $u_M < u < u_m$.
- (3) Let $u = h_0(v)$ be the inverse of $v = F(u)$ for $u > u_m$, and $h_1(v)$ the inverse of $v = F(u)$ for $u < u_M$. For $v \in [v_m, v_M]$, define $J(v)$ by

$$J(v) = \int_{h_0(v)}^{h_1(v)} [v - F(u)] du. \quad (3)$$

Then there is a unique $v^* \in (v_m, v_M)$ such that $J(v^*) = 0$.

Note that $J'(v) = h_1(v) - h_0(v) < 0$.

The qualitative properties of the function F in the above are justified by the careful studies of DeKepper et.al. [2][3].

In order to fix the situation further, we choose the parameters a , α_0 , etc. in the following way :

$$\alpha_1 < h_1(v^*) < u_m < a < h_0(v^*) < \alpha_0. \quad (4)$$

Under those conditions in the above, we have

Theorem 1. *For $\epsilon > 0$ small, there exists a two parameter family of steady state solutions $\mathcal{U}(x; \epsilon, D)$ of (1)(2) for $D \in [D_0, D_1]$, which exhibits an internal transition layer of width $O(\epsilon)$ near a well-defined point $x = x^*(\epsilon, D) \in (0, 1)$.*

The stability analysis for $\mathcal{U}(x; \epsilon, D)$ is extremely delicate as the following theorem indicates.

Theorem 2. *There are a constant $\delta > 0$ and a critical value $D(\epsilon) \in (D_0, D_1)$ such that the linearization of (1)(2) around $\mathcal{U}(x; \epsilon, D)$ has a unique pair of (complex) eigenvalues $\rho(\epsilon, D) = \rho_R(\epsilon, D) \pm i\rho_I(\epsilon, D)$ with $\rho_R > -\delta$ satisfying*

$$(a) \rho_R(\epsilon, D) < 0 \text{ for } D > D(\epsilon), \rho_R(\epsilon, D(\epsilon)) = 0, \text{ and } \rho_R(\epsilon, D) > 0 \text{ for } D < D(\epsilon).$$

$$(b) \partial \rho_R(\epsilon, D) / \partial D|_{D=D(\epsilon)} = O(\epsilon^{-1}) < 0.$$

$$(c) \rho_I(\epsilon, D(\epsilon)) = O(\epsilon^{-i^3}).$$

Remarks. (1) The order estimates in Theorem 2 (b)(c) clearly show that the destabilization of $U(x; \epsilon, D)$ through a Hopf-bifurcation cannot be captured by in the singular limit $\epsilon \downarrow 0$. This situation is in sharp contrast to the case described in the remark (4) below.

(2) Even if the parameters α_0, α_1 etc. are chosen differently from (4), we still have results analogous to those in Theorems 1 and 2.

(3) The fact that the diffusion rates of u and v are exactly equal is not essential to obtain such results as in Theorems 1 and 2. One only needs to require that the diffusion rates of u and v are of the same order.

(4) Problems related to (1)(2) were analysed by Nishiura and Mimura [4]. Their model is described by

$$\begin{cases} \epsilon\tau\partial u/\partial t = \epsilon^2\partial^2 u/\partial x^2 + [v - F(u)] \\ \partial v/\partial t = \partial^2 v/\partial x^2 - u + a \end{cases} \quad t > 0, \quad x \in (0, 1) \quad (5)$$

where the parameter τ is of order $O(1)$ as $\epsilon \rightarrow 0$. The problem (5) has the feature that there are substantial differences both in diffusion and reaction rates. It was shown in [4] that the problem (5) has an ϵ -family of solutions with an internal transition layer and that the solutions undergo a Hopf-bifurcation as the parameter τ decreases. In our problem, however, $\tau = \epsilon$ and the bifurcation parameter is the diffusion rate.

(5) The problem (1)(2) has a wider range of applicability than it appears. For example, a combustion model on a sufficiently long rectangular domain can be, at least formally, brought to a problem of the same type as (1)(2). We are currently investigating this problem.

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