Title	第19回偏微分方程式論 札幌シンポジウム 予稿集
Author(s)	久保田, 幸次
Citation	Hokkaido University technical report series in mathematics, 31, 1
Issue Date	1994-01-01
DOI	https://doi.org/10.14943/5150
Doc URL	https://hdl.handle.net/2115/5465
Type	departmental bulletin paper
File Information	31.pdf

第１９回偏微分方程式論
札幌シンポジウム

（代表者 久保田　幸次）

予稿集

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札幌シンポジウム

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記

１．日　時　１９９４年７月２８日（木）～７月３０日（土）

２．場　所　北海道大学理学部数学教室　４－５０８室

３．講　演

７月２８日（木）　　　　ページ

９：３０～１０：３０　鈴木　貴（愛媛大理）
Suzuki, T.

Two-dimensional equation $-\Delta u = \lambda u e^{u^2}$:
radial and non-radial cases ・・・ 1

１１：００～１２：００　笹原康浩（名大理）
Sasahara, Y.

Nonstandard harmonic extensions for exterior
Plateau problem ・・・

１３：３０～１４：００　＊

１４：００～１４：３０　**Dohmen, C**（北大理/**Univ. Bonn**）

Existence and uniqueness of selfsimilar shrinking
curves for anisotropic curvature flow equations ・・・ 4

１５：００～１５：３０　山内和幸（北大理（院））
Yamauchi, K.

On the extinction time of the motion by
anisotropic mean curvature ・・・

１５：３０～１６：３０　＊

７月２９日（金）
９：３０～１０：３０　Kan, P.T. (Indiana Univ.)

Hyperbolic conservation laws systems with umbilic degeneracy ・・・ 9

１１：００～１２：００　鈴木龍一（都立航空高専）
Suzuki, R.
望月　清（都立大理）
Mochizuki, K.

Critical exponent and critical blow-up for quasilinear parabolic equations ・・・ 10

１３：３０～１４：００　*

１４：００～１５：００　谷山英志（東大数理科学（院））
Taniyama, H.

On the formation of singularities in the curvature-type flow ・・・ 15

１５：３０～１６：００　庵原隆雄（京大理（院））
Iohara, T.

Benard-Marangoni convection with a free surface ・・・

１６：３０～１７：００　鈴木宏昌（広大理（院））
Suzuki, H.

Asymptonic configuration and stability of stationary interfacial patterns for reaction diffusion systems in higher dimensional spaces ・・・ 20

１７：００～１７：３０　*

７月３０日（土）
９：３０～１０：３０　檀　和日子（筑波大数学系）
Dan, W.

On a local energy decay of solutions of a dissipative wave equation ・・・ 24

１１：００～１２：００　小薗英雄（名大工）
kozono, H.
山崎昌男（一橋大商）
Yamazaki, M.

Exterior problem for the Navier-Stokes equations ・・・ 29

１２：００～１３：００　*

*　この時間は講演者を囲んで自由な質問の時間とする予定です。

連絡先　北海道大学理学部数学教室
Tel. 011-716-2111 内線 2679（小林）

Two-dimensional equation $-\Delta u=\lambda u e^{u^2}$: radial and non-radial cases

愛媛大・理　鈴木貴

問題とするのは、2次元有界領域 Ω における非線形固有値問題

$$-\Delta u=\lambda u e^{u^2} \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega \tag{1}$$

の正の解 $u(x)$ である。これは、変分法を通して Trudinger-Moser の不等式

$$T_0 \equiv \sup\left\{\int_\Omega e^{v^2}dx \;\middle|\; \|\nabla v\|_2^2 \leqq 4\pi\right\} < +\infty \tag{2}$$

と深く関っており、特に Carleson-Chang の定理によって $\Omega=B\equiv\{|x|<1\}$ の場合に、(2) は達成されるのであるから、このとき、

$$0<{}^\exists\lambda_0<+\infty, \quad {}^\exists u=u(|x|) : (1) \text{ の解}, \quad \|\nabla u\|_2^2=4\pi$$

となる。この解を Carleson-Chang の解という。

(2)については、良く考察されているのに対し、(1)については、$\Omega=B$ のときでさえ、十分な検討がなされていない。これについて予想も交えて準備的な考察を加え、その応用として(2)を考え直そうというのが本講演の目的である。以下は小川卓克（名大・理）との共同研究である。

我々は (1) または (2) に対する次の研究の見直しを行なう：

(a) Carleson-Chang の定理に対する McLeod-Peletier の証明

T_0 が $\Omega=B$ に対して達成されないとすると、

$$\lambda\downarrow 0, \quad \|u\|_{L^\infty}\longrightarrow +\infty, \qquad E=\|\nabla u\|_2^2<4\pi \tag{3}$$

なる (1) の解の族 $\{(\lambda,u)\}$ が存在する。この族に対しては

$$J=\int_\Omega (e^{u^2}-1)\,dx \longrightarrow \pi e$$

この図は次の事柄を含んでいる。

1) (3)をみたす解の族は、それ自身存在しない。
このことは、Carleson-Chang の定理のより直接的で見通しの良い証明を与えることになる。

2) 同じことだが、Shaw の解は $\lambda_0 < \lambda < \lambda_1$ でのみ存在する。
このような現象は、従来あまり深く考えられたことがなかった。

この予想の解明には次の micro-asymptotics がいく分役立つ:

$$\lambda \downarrow 0, \quad \|u\|_{L^\infty} \longrightarrow +\infty, \quad E = O(1)$$

$\Longleftrightarrow$

$^\exists \tau \longrightarrow +\infty$ s.t. 部分列に対して

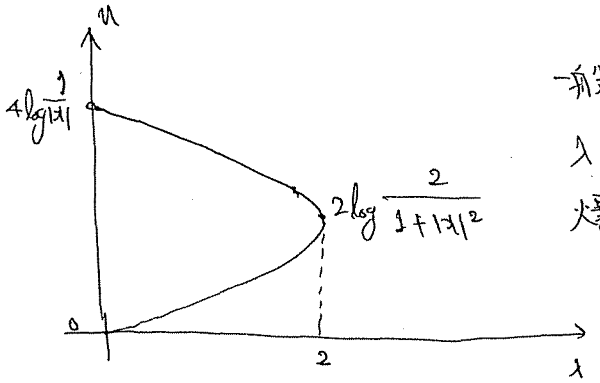
$$u^2(e^{-\tau/2}y) = u^2(e^{-\tau/2}) + 2\log\frac{2}{1+|y|^2} + o(1) \qquad (y \in \mathbb{R}^2 \text{ について広義一様})$$

$$\lambda u^2(e^{-\tau/2})\, e^{u^2(e^{-\tau/2}) - \tau} = 1 + o(1)$$

第1式 右辺第1項は、

$$-\Delta u = \lambda e^u \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B \tag{4}$$

の $\lambda = 2$ に対する解である。



一般の領域については、単連結の場合に、$\lambda \downarrow 0$ における $E = O(1)$ なる解の爆発点の有限性などを示すことができる。

となるので、　$T_0 \leqq \pi(1+e)$　が結論される。

ところが Carleson－Chang の見つけた関数によって、$T_0 > \pi(1+e)$ であるので矛盾である。

(b) Adimurthi の変分解

$$X = H_0^1(\Omega),\quad N_\lambda = \left\{ v \in X \setminus \{0\} \;\middle|\; \|\nabla v\|_2^2 = \lambda \int_\Omega v^2 e^{v^2} dx \right\}.$$

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \frac{\lambda}{2}\int_\Omega \left(e^{v^2} - 1\right) dx$$ に対して。

$0 < \lambda < \lambda_1 \equiv -\Delta$ の第1固有値 $\Longrightarrow$ $d_\lambda = \inf_{N_\lambda} J_\lambda$ は達成。(1)の古典解を与える。

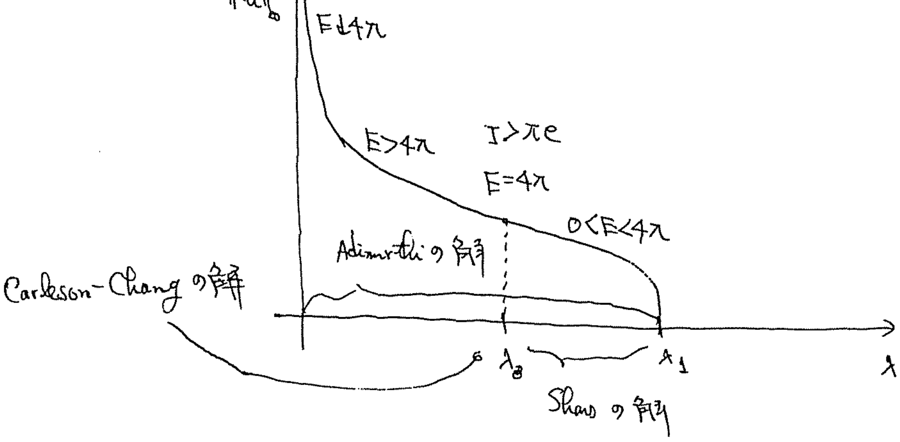
(c) Shaw の変分解

$$\mathcal{M}_\alpha = \left\{ v \in X \;\middle|\; \|\nabla v\|_2^2 < 4\pi,\ \int_\Omega \left(e^{v^2} - 1\right) dx = \alpha \right\}$$
$\ni \phi$

$\Longrightarrow$

$\inf\left\{ \|\nabla v\|_2^2 \;\middle|\; v \in \mathcal{M}_\alpha \right\}$ は達成。 ${}^\exists\lambda > 0$ に対し (1)の古典解を与える。

$\Omega = B$ の場合の (1) の解の大域分岐に関する我々の予想は以下のようなものである。



Existence and Uniqueness of Selfsimilar Shrinking Curves for Anisotropic Curvature Flow Equations

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This is a joint work with Prof. Yoshikazu Giga, Sapporo, and Dr. Noriko Mizoguchi, Tokyo.

We consider a simple looking ordinary differential equation of the form

$$u_{xx} + u - \frac{a(x)}{u} = 0 \quad \text{in } \mathbf{R} \tag{1}$$

with a given positive function a. This equation arises in describing a selfsimilar solution of anisotropic curvature flow equations. Since x is the argument of the normal n of the curve it is natural to impose 2π-periodicity for $a(x)$ in (1) and to ask for existence and uniqueness of 2π-periodic solutions.
To simplify the notation we notice that a 2π-periodic function can be regarded as a function on the flat torus $\mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$. thus we define

$$C^2_+(\mathbf{T}) = \{u \in C^2(\mathbf{R}) \,|\, u(x+2\pi) = u(x) \text{ for all } x \in \mathbf{R}, u > 0\}. \tag{2}$$

The physical background of the above problem is an evolution equation for embedded closed curves $\{\Gamma_t\}_{t>0}$ in $\mathbf{R}^2$ (see [Gu]):
Consider an equation for Γ_t of the form

$$V = a(x)k, \qquad a(x) = \beta(x)^{-1}(\gamma''(x) + \gamma(x)).$$

where β and $\gamma'' + \gamma$ are assumed to be positive, so that the equation is parabolic. γ is called the surface energy density and β is called the kinetic coefficient.
In case $a(x) = const.$ it is well known (see for instance [Ga1] , [Ga2] , [GH]) that any initial curve extincts in finite time, and that the type of skrinking is asymptotically similar to that of a shrinking circle $C_t = (t_* - t)^{1/2}\, C$, where C denotes the unit circle centered at the origin, the time t_* is the extinction time and λC denotes the dilatation of C with multiplier λ. In case of more general $a(x)$ it was shown in [Son] that solutions satisfying

$$\Gamma_t = (t_* - t)^{1/2}\, \Gamma$$

exist if $\beta(x)\gamma(x) = const.$ Then Γ defined as the boundary of the so-called Wulff-Shape W_γ, i.e.

$$W_\gamma := \left\{x \in \mathbf{R}^2 \,|\, x \cdot n(y) \le \gamma(y) \text{ for all } y \in \mathbf{R}\right\},$$

yields a solution Γ_t of the evolution problem.
Our existence result now shows that such selfsimilar solutions exist for arbitrary positive $a(x)$:

Main Existence Theorem. Assume that a is a positive, continuous function on $\mathbf{T}$. Then there is a function $u \in C^2_+(\mathbf{T})$ solving (1).

The proof is based on a-priori estimates and a continuity method.

Sketch of the proof.

Step 1: If one maximum is bounded from above, then all maxima are bounded from above and all minima are bounded from below by positive constants depending only on the data. Consider the following situation: u takes a local maximum in γ a local minimum in α and is monotone between α and γ. Estimating $a(x)$ by its maximal or minimal value, multiplying the resulting inequality by u_x and integrating yields

$$\begin{aligned} u(\gamma)^2 - 2A_2 \ln u(\gamma) &\leq u(\alpha)^2 - 2A_2 \ln u(\alpha) \qquad (3)\\ u(\alpha)^2 - 2A_1 \ln u(\alpha) &\leq u(\gamma)^2 - 2A_1 \ln u(\gamma). \qquad (4) \end{aligned}$$

Thus, if $u(\gamma) \leq M$, then (3) implies

$$-A_1 \ln u(\alpha) \leq M^2,$$

which bounds u from below in terms of the data. On the other hand $u(\alpha) \geq m$ and (4) lead to

$$u(\gamma)^2 - 2A_2 \ln u(\gamma) \leq A_2 - 2A_2 \ln m,$$

as a minimum of u can only take values less or equal $A_2^{1/2}$. So $u(\gamma)$ is bounded from above, as the quadratic term is dominant.

Step 2: Let $u \in C^2_+(I)$ be a solution of (1) on some open interval I. If u attains local minima in $\alpha, \beta \in I$, $\alpha < \beta$ and u_x changes its sign only once in (α, β), then there is a positive constant M_0, depending only on A_1, A_2, such that

$$u \leq M_0 \quad \text{in } (\alpha, \beta) \qquad (5)$$

provided that $\beta - \alpha \leq \pi$.
If we take the equation, multiply by u and integrate from α to β, we derive

$$\int_\alpha^\beta u_x^2 = \int_\alpha^\beta u^2 - \int_\alpha^\beta a(x). \qquad (6)$$

Defining $l(x)$ as the affine function intersecting with u in α and β and inserting $v(x) := u(x) - l(x)$ in the above equation we arrive at

$$\int_\alpha^\beta v_x^2 \leq \int_\alpha^\beta v^2 - J, \qquad (7)$$

where J is given by

$$J := \int_\alpha^\beta a(x) - 2\int_\alpha^\beta vl - \int_\alpha^\beta l^2 \geq A_1(\beta - \alpha) - 2\max\{u(\alpha), u(\beta)\} \int_\alpha^\beta u.$$

The Wirtinger inequality on the other hand gives

$$\int_\alpha^\beta v^2 \leq \int_\alpha^\beta v_x^2$$

provided $\beta - \alpha \leq \pi$. Thus we arrive at a contradiction, if J is positive. Otherwise

$$\frac{A_1(\beta - \alpha)}{\int_\alpha^\beta u} \leq 2\max\{u(\alpha), u(\beta)\}.$$

This implies $\max\{ u(\alpha), u(\beta)\}u(\gamma) \geq A_1/2$, where $u(\gamma)$ is the maximum of u in (α,β). However, this inequality and the first inequality derived in Step 1 can hold both only if either the minimum of u is bounded from below or the maximum of u is bounded from above by a constant only depending on the data; this can be shown easily by an indirect argument.

Step 3: Let $u \in C^2_+(\mathrm{T})$ be a singlepeak-solution of (1), i.e. the set of points not being local extrema consists of two connected components in T. Then there is a positive constant M_1, depending only on A_1, A_2, such that

$$u \leq M_1 \quad \textit{in } \mathrm{T}. \tag{8}$$

We assume without loss of generality strict extrema of u. Consider the maximal interval $[p,q]$, where u is concave. Multiplying the equation by $\sin(x-p)$ and integrating yields

$$0 = \int_p^q + \int_q^{p+2\pi} \frac{\sin(x-p)a(x)}{u(x)}dx =: I_0 + I_1.$$

One can prove that if the maximum $u(\gamma)$ in $[p.q]$ becomes large, the difference $q-p$ has to be close to π; moreover the value of the integral I_0 above becomes arbitrarily small.
On the other hand I_1 has roughly speaking the opposite sign in the sin −function, and u has a maximal possible value outside $[p,q]$ due to the convexity of u there. So I_1 has a certain minimal absolute value, which gives a contradiction, if $u(\gamma)$ is too large and thereby I_0 too small.

The a-priori bounds are now an immediate consequence of these three steps: If there exists at least one pair of local mimima with a distance less or equal π, then Step 2 estimates u, and due to Step 1 all extrema are estimated in terms of this maximum. The situation needed to apply Step 2 fails to exist only if u has exactly one local minimum, i.e. is a singlepeak solution. But in this case Step 3 yields the upper bound; thus in any case u is bounded a-priori by a constant only depending on the bounds of $a(x)$.

Step 4: Continuity Method.
Define

$$E := \left\{ v \in C^0_+(\mathrm{T}) \,|\, \frac{m}{2} \leq v \leq 2M \text{ in } \mathrm{T} \right\}. \tag{9}$$

and $S_\tau := S(.,\tau) := T \circ F(.,\tau)$, where $F : E \times [0,1] \to C^0_+(\mathrm{T})$ is given by

$$F(u,\tau) := 2u - \frac{\tau a(x) + (1-\tau)a_0}{u}, \quad a_0 \in [A_1, A_2] \tag{10}$$

and T denotes a linear compact operator from $C^0_+(\mathrm{T})$ into itself, given by $w = T(f)$, where w is the unique solution of

$$-w_{xx} + w = f \quad \text{in } \mathrm{T}.$$

A solution of our problem is a fixed point of S_τ with $\tau = 1$. As the a-priori bounds show that u has no fixed points on the boundary of E, the homotopy invariance yields

$$\deg\,(I - S_1, E, 0) = \deg\,(I - S_0, E, 0). \tag{11}$$

The mapping S_0, however, is known to have a unique fixed point $u_0 = a_0^{1/2}$, thus

$$\deg\,(I - S_0, E, 0) = \deg\,(I - S_0, B_\delta(u_0), 0) = -1$$

by a standard degree theory result.
This completes the existence proof.

Concerning uniqueness we unfortunately have to make an additional assumption on $a(x)$:

Uniqueness Theorem. Let $a(x)$ be a positive, continuous and $\pi-$periodic function in $\mathbf{R}$. Then the solution of (1) is unique.

The main tools in proving the result are the uniqueness of the Wulff-Shape as minimizer of a variational problem and a generalization of an isoperimetric inequality by Gage. The latter result requires the $\pi-$periodicity of $a(x)$.
Let us first introduce some notation: We denote the area of a set A by $m(A)$, the lenght of a curve Γ by L and its surface energy with respect to some surface energy density γ by

$$F_\gamma(\Gamma) = \int_0^L \gamma(s)\, ds.$$

Theorem: (see for instance [DP])
For any closed C^2-curve Γ and any $2\pi-$periodic surface energy density $f \in C^2$ the inequality

$$F_f(\Gamma)^2 - 4m(W_f)m(int\Gamma) \geq 0 \tag{12}$$

is valid. Moreover equality holds if and only if $\Gamma = \partial W_f$.

Proposition: (see [Ga3])
Let Γ be an arbitrary closed embedded C^2-curve represented by u and let the surface energy density f be in C^2 and be $\pi-$periodic. Then

$$\int_0^L \frac{u(s)^2}{f(s)} ds \geq \frac{m(W_f)}{m(int\Gamma)} F_f(\Gamma). \tag{13}$$

Moreover equality holds if and only if $\Gamma = \partial W_f$.

Proof of the Uniqueness Result.
Suppose there exist two different solutions f and u, the corresponding curves denoted by Γ_f and Γ_u, respectively. Regard f as the new surface energy density, allowing of course a Wulff-Shape W_f. As $a(x)$ has not changed, f and u are still solutions of the problem. Calculating the isoperimetric quantity in (12) for the solutions, in case of f we easily derive

$$m(int\Gamma_f) = m(W_f), \qquad F(\Gamma_f) = 2m(W_f).$$

Therefore $F(\Gamma_f)^2 - 4m(W_f)m(int\Gamma_f) = 0$ and so $int\Gamma_f = W_f$. In case of Γ_u the area is still the same, so the surface energy is

$$F(\Gamma_u) = \int_0^L \frac{u(s)^2}{f(s)} ds \geq \frac{m(W_f)}{m(int\Gamma_u)} F(\Gamma_u) = F(\Gamma_u)$$

by the above Proposition. Therefore $int\Gamma_u = W_f$, which immediately yields $f = u$.

Remark. The problem (1) was also studied in [Ga3] and [GL]. However, they have to assume that a is smooth in order to study a related parabolic partial differential equation. Our proof is more direct and requires only boundedness of a.

References

[DP] B. Dacorogna, C.E. Pfister, *Wulff-Theorem and Best Constant in Sobolev Inequality*, J. Math. Pure. Appl. **71** (1992), pp.97-118.

[Ga1] M. Gage, *An Isoperimetric Inequality With Application to Curve Shortening*, Duke M. J. **50** (1983), pp.1225-1229.

[Ga2] M. Gage, *Curve Shortening Makes Convex Curves Circular*, Inv. Math. **76** (1984), pp.357-364.

[Ga3] M. Gage, *Evolving Plane Curves by Curvature in Relative Geometries*, Duke M. J. **72** (1993), pp. 441-466.

[GH] M. Gage, R.S. Hamilton, *The Heat Equation Shrinking Convex Plane Curves*, J. Diff. Geometry **23** (1986), pp.69-96.

[GL] M. Gage, Yi Li, *Evolving Plane Curves by Curvature in Relative Geometries II*, Preprint No. 19 (1992), University of Rochester.

[Gu] M.E. Gurtin, *Thermodynamics of Evolving Phase Boundaries in the Plane*, Clarendon Press, Oxford (1993).

[Son] H.M. Soner, *Motion of a Set By the Curvature of Its Boundary*, J. Diff. Eq. **101** (1993), pp.313-392.

Hyperbolic Conservation Laws
Systems with Umbilic Degeneracy

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Abstract

We present a compactness framework theorem for the convergence of approximate solutions to general 2x2 systems of nonstrictly hyperbolic conservation laws. These systems are relevant in applications in multiphase flows in porous media, magnetohydrodynamics, and elasticity. We apply this framework theorem to a canonical class of quadratic 2x2 systems that are nonstrictly hyperbolic at isolated points in the state space. We prove the convergence approximate solusions generated by the vanishing viscosity method, the Godunov scheme, and the Lax Friedrichs scheme. As a direct consequence, we obtain the existence of global weak entropy solutions to these systems with arbitrarily large initial data in L^∞. Our proof uses compensated compactness and involves a very detailed and complicated analysis of the wave curve geometry and the singularities of solutions to a highly singular generalized Euler-Poisson-Darboux type equation.

Critical Exponent and Critical Blow-up for Quasilinear Parabolic Equations

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§1 Introduction

ΩをR^NまたはR^Nの外部領域(境界は十分滑らか)とし次の初期境界値問題を考える。

(1)　$\partial_t u = \Delta u^m + u^p$　　$(x,t)\in\Omega\times(0,T)$.

(2)　$u(x,0)=u_0(x)$　　$x\in\Omega$.

(3)　$u(x,t)=0$　　$(x,t)\in\partial\Omega\times(0,T)$.

ただし　$p>1$, $m\geq 1$, $u_0(x)$は非負有界な連続関数とする。

この時 $T>0$が十分小さければ(1)(2)(3)の非負連続で, xについて有界な弱解が一意的に存在する。Tを解の最大存在時間とすれば, $T=\infty$のとき解uは大域的, $T<\infty$のときuは有限時間で爆発するという。特に$T<\infty$のときは, 解の存在定理から次のことが言える。

(4)
$$\lim_{t\uparrow T}\ \sup_{x}\ u(x,t)=\infty.$$

$\Omega=R^N$のとき次のような結果がある。

(I) $1<p<m+\frac{2}{N}$のとき, (1)(2)の自明でない任意の非負解は有限時間で爆発する。

(II) $p>m+\frac{2}{N}$のとき, 初期値が十分小さければ(1)(2)大域解が存在する。

これらの結果は, $m=1$のときは Fujita, $m>1$のときは Galaktionov 等による。(I)の場合をblow-up case, (II)の場合を global existence case という。(I)と(II)の場合を分ける数

$$(5) \qquad p_m^* = m + \frac{2}{N}$$

をcritical exponent と呼ぶ。$p = p_m^*$ が blow-up case に含まれることは, $m=1$ のとき Hayakawa, Weissler, $m>1$ のとき最近 Galaktionov, Kawanago, Mochizuki-Suzuki によって示された。

Ω が $\mathbb{R}^N$ の外部領域のときは, $m=1$ のとき p_1^* が最近 critical exponent になることを Bandle-Levine が示した。しかし $p = p_1^*$ が blow-up case に含まれるかどうかは, まだわかっていない。$m>1$ になるとほとんど論文がないと思われる。我々は次の結果を得た。

Thm 1　Ω を外部領域, $N \geq 2$ とする。

(i) $1 \leq m < p < p_m^*$ のとき, (1)(2)(3)の自明でない任意の非負解は、有限時間で爆発する。

(ii) $p > p_m^*$ のとき、初期値が十分小さければ (1)(2)(3)の大域解が存在する。

Thm 2　Ω を外部領域, $N \geq 3$ とする。$p = p_m^*$ のとき (1)(2)(3)の自明でない任意の非負解は有限時間で爆発する。

§2 Proof of Thm 1

この§では Thm 1を示すが, (ii)は $\Omega = \mathbb{R}^N$ での結果と比較定理より, すぐ導びかれる. 以下(i)を示す。簡単にするために

$$(6) \qquad \Omega = E_R \equiv \{x \in \mathbb{R}^N \mid |x| > R\}$$

とする。まず, (1)(2)(3)の弱解 u のみたす積分等式のテスト関数

として次の関数を考える。

$$(7) \qquad S_\varepsilon(x) = \rho_R(|x|)e^{-\varepsilon(|x|-R)^2} \qquad (\varepsilon > 0).$$

ただし

$$\rho_R(r) = \begin{cases} \dfrac{r-R}{r}, & N \geqq 3 \\ \log r - \log R, & N = 2. \end{cases}$$

このとき　$\lambda = 2N+4$　とすれば、

$$\Delta S_\varepsilon(x) \geqq -\lambda\varepsilon S_\varepsilon(x) \quad \text{in} \quad E_R$$

より、積分等式は、

$$\int_{E_R} u(x,\tau)S_\varepsilon(x)dx - \int_{E_R} u_0(x)S_\varepsilon(x)dx$$

$$\geqq \int_0^\tau\int_{E_R}(-\lambda\varepsilon u^m + u^p)S_\varepsilon(x)dx$$

になる。故に Imai－Mochizuki の方法から　もし

$$(8) \quad \int_{E_R} u_0(x)S_\varepsilon(x)dx > \int_{E_R} S_\varepsilon(x)dx \times (\lambda\varepsilon)^{\frac{1}{p-m}}$$

が成り立つならば、uは有限時間で爆発する、ことが言える。従って

Lemma 1　$u(x,t)$を(1)(2)(3)の大域解であるとする。

$$(9) \quad \int_{E_R} u(x,t)\rho_R(|x|)e^{-\varepsilon(|x|-R)^2}dx \leqq \begin{cases} C(N)\varepsilon^{-\frac{N}{2}+\frac{1}{p-m}}\{1+o(1)\} \\ \qquad (N \geqq 3) \\ C(2)\varepsilon^{-1+\frac{1}{p-m}}\log(\varepsilon^{-\frac{1}{2}})\{1+o(1)\} \\ \qquad (N=2) \end{cases}$$

ただし　$C(N) = \pi^{\frac{N}{2}}\lambda^{\frac{1}{p-m}} = \pi^{\frac{N}{2}}(2N+4)^{\frac{1}{p-m}}$.

[証明] $\displaystyle\int_{E_R} S_\varepsilon(x)dx = \begin{cases} \pi^{\frac{N}{2}}\varepsilon^{-\frac{N}{2}}\{1+o(1)\}, & N\geq 3 \\ \pi\varepsilon^{-1}\log(\varepsilon^{-\frac{1}{2}})\{1+o(1)\}, & N=2 \end{cases}$

に注意する。各 $t>0$ に対し $u(x,t)$ を初期値 $u_0(x)$ であると考えると，u は大域解なので (8) が言えない，すなわち (9) が成り立つ □

[proof of Thm 1 (i)] $u(x,t)$ を (1)(2)(3) の大域解とする。$p < p_m^* = m + \frac{2}{N}$ であるので，$-\frac{N}{2} + \frac{1}{p-m} > 0$ が成り立つことに注意する。(9) で $\varepsilon \downarrow 0$ とすれば

$$\int_{E_R} u(x,t)\rho_R(|x|)dx \leq \liminf_{\varepsilon\downarrow 0}\int_{E_R} u(x,t)\rho_R(|x|)e^{-\varepsilon(|x|-R)^2}dx = 0$$

従って，$\rho_R(|x|)>0$ in E_R より $u(x,t)\equiv 0$ in E_R. □

§3 Proof of Thm 2.

$N\geq 3$，$p = p_m^* = m + \frac{2}{N}$；$u_0(x)\in C_0(\overline{E_R})$，$u_0(x)\not\equiv 0$；$u(x,t)$ を (1)(2)(3) の大域解とする。この時，$u(\cdot,t)\in L^1(E_R)$ $(t>0)$ であり，Lemma 1 で $\varepsilon\downarrow 0$ とすると

$$(10)\qquad \int_{E_R} u(x,t)\rho_R(|x|)\,dx \leq C(N) \qquad \text{for } t\geq 0.$$

弱解がみたす積分等式のテスト関数として $\rho_R(|x|)$ を考えると，$\Delta\rho_R\geq 0$ より

$$\int_{E_R} u(x,\tau)\rho_R(|x|)\,dx \geq \int_0^\tau\int_{E_R} u(x,t)^p\rho_R(|x|)dxdt + \int_{E_R} u_0(x)\rho_R(|x|)dx.$$

従って (10) より

$$(11)\qquad \int_0^\tau\int_{E_R} u(x,t)^p\rho_R(|x|)\,dxdt \leq C(N).$$

故に $u_k(x,t) = k^N u(kx, k^{\frac{N}{\ell}})$ $(\ell = (p_m^*-1)^{-1})$ とすると $p = p_m^*$ のとき u_k も (1) の大域解であるので

$$(12)\quad \int_0^\tau\int_{E_{R/k}} u_k^{\,p}\,\rho_{R/k}\,dx\,dt \le C(N) \quad \text{for any } \tau>0.$$

そこで $v(x,t)$ を次の問題の弱解とする。

$$\begin{cases} \partial_t v = \Delta v^m & (x,t)\in E_R\times(0,\infty) \\ v(x,0)=u_0(x) & x\in E_R \\ v(x,t)=0 & |x|=R \end{cases}$$

$v_k(x,t)=k^N v(kx, k^{\frac{N}{\alpha}}t)$ とすれば比較定理より $v_k\le u_k$。

(12) より

$$(13)\quad \int_0^\tau\int_{E_{R/k}} v_k^{\,p}\,\rho_{R/k}\,dx\,dt \le C(N) \quad \text{for } \tau>0.$$

<u>Lemma 2</u>　$N\geqq 3$, $u_0(x)\in C_0(\overline{E_R})$ のとき

$$v_k(x,t) \longrightarrow V_m(x,t,I_N) \quad (k\to\infty)$$

ただし，この収束は $\{\mathbb{R}^N\setminus\{0\}\}\times(0,\infty)$ の各コンパクト集合で一様収束，$V_m(x,t,L)$ は初期値が $L\delta(x)$ である方程式 $\partial_t v=\Delta v^m$ in $\mathbb{R}^N\times(0,\infty)$ の弱解，$I_N=\displaystyle\int_{E_R}\frac{|x|^{N-2}-R^{N-2}}{|x|^{N-2}}u_0(x)\,dx$ である。

従って (12) で　$k\to\infty$ とすると

$$\int_0^\tau\int_{\mathbb{R}^N} V_m^{\,p}\,dx\,dt \le C(N).$$

ところが，$p=p_m^*$ のとき，$I_N>0$ なら

$$\int_0^\tau\int_{\mathbb{R}^N} V_m^{\,p}\,dx\,dt = \infty.$$

これは矛盾であるので $u(x,t)$ は有限時間で爆発する。

ON THE FORMATION OF SINGULARITIES IN THE CURVATURE TYPE FLOW

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0. INTRODUCTION

二つの相を持つ物質が$\mathbb{R}^n$を満たしているとする。このとき、時刻tにおける二相の境界を$\Gamma(t)$であらわす。系全体のポテンシャルエネルギーが$H_{n-1}(\Gamma(t))$に比例するような問題について考えよう（ここで$H_{n-1}(\Gamma(t))$は$\Gamma(t)$の$n-1$次元ハウスドルフ測度）。$\Gamma(t)$が滑らかな超曲面のとき、ポテンシャルエネルギーを‘最も速く’縮めるのは、$\Gamma(t)$が

$$V = H \tag{0.1}$$

を満たすときである。ここでVは超曲面の法線方向の速度、Hは平均曲率。特に$n=2$、即ち境界が曲線で表されるときには満たすべき方程式は

$$V = k \tag{0.2}$$

となる。（この方程式は curve shortening equation と呼ばれる。）

この方程式の解については次のことが知られている。

Theorem 0.1. ([GaH][Gr1])

(1) 任意の滑らかな閉曲線に対してそれを初期曲線とする *(0.2)* の時間局所解が唯一つ存在する。
(2) 任意の滑らかな単純閉曲線は（古典解の範囲で）有限時間内に1点に縮む。1点に縮むまで解曲線は常に単純。
(3) *(2)* で1点に縮むときに解曲線の形は円に近づく。

一方、系に異方性がある場合、即ち、系のエネルギーが

$$\int_\gamma f(\theta)ds$$

で表される場合を考える。（ここでγは滑らかな境界、$\theta \in S^1$は接線方向の角度、αはS^1上で与えられたある関数。）これを最も速く減少させるのは、曲線が

$$V = (f(\theta) + f''(\theta))k \tag{0.3}$$

に従って動くときである。

他方、$\mathbb{R}^3$内の y軸に関して回転対称な (0.1) の解曲面と xy平面の交わりによってえられる曲線は

$$V = k + \frac{1}{x}\sin\theta$$

を満たす。

この2つの場合をふくむ一般化した方程式

$$V = F(x, y, \theta, k)$$

の解について Theorem 0.1 のような結果を導きたい。

1. 定式化

$F = F(x, y, \theta, k)$ は $\mathbb{R}^2 \times S^1 \times \mathbb{R}$ 上定義された関数で次を満たすものとする。

[F1] $\exists\lambda > 0 \qquad \lambda^{-1} \geqq F_k \geqq \lambda$
[F2] $\exists\mu > 0 \qquad |F(\cdot,\cdot,\cdot,0)| \geqq \mu$
[F3] $\exists\mu > 0 \qquad |F_x| + |F_y| + |kF_\theta| \geqq \mu(1 + k^2)$
[F4] $F(x, y, \theta, k) = -F(x, y, \theta + \pi, -k)$

この Fをもちいて (0.2) を一般化した方程式

$$V = F(x, y, \theta, k) \tag{1.1}$$

を得る。この方程式の解の性質を求めよう。

Remarks.

(1) (1.1) は放物型方程式の一種である。実際、$u = u(x, t)$ のグラフが (1.1) をみたせば u は

$$u_t = \sqrt{1 + u_x^2}F(x, u, \arctan u_x, u_{xx}/(1 + u_x^2)^{3/2})$$

をみたす。

(2) [F1] は（ある意味で）一様放物性を表している。

(3) [F4] は解が曲線の向き付けによらないことを意味している。即ち、ある曲線族が解なら、向きを逆にしてえられる曲線族も解である。

2. 交点の数の非増大性

上にみたように (1.1) は（少なくとも時間局所的には）周期境界条件を持った放物型方程式であり、その結果、最大値原理（の一種）から得られる次の重要な性質を持つ。

Lemma 2.1. (交点の数の非増大性) $\gamma_1(t), \gamma_2(t)$ を (1.1) の相異なる解とする。その交点の数を $z(t)$ とすると、以下がなりたつ。

(1) $t > 0$ に対して $z(t)$ は有限
(2) $z(t)$ は単調非増大
(3) $t = t_0$で $z(t)$ がへることと$\gamma_1(t_0), \gamma_2(t_0)$ が接点を持つことは同値。

この Lemma は次のかたちでつかうことが多い。

Corollary 2.2. $\gamma_1(t), \gamma_2(t)$ は端点を持つ曲線で *(1.1)* の解とする。

(1) $\gamma_1(0), \gamma_2(0)$ の交点はひとつ
(2) $\gamma_1(t), \gamma_2(t)$ は少なくとも1点で交わる

ならば、$\gamma_1(t)$ と$\gamma_2(t)$ は1点で正の角度で交わる。

このことから、(1.1) の特殊解から、一般解の性質を導くことができる。特に

$$0 = F(x, y, \theta, k)$$

の解、即ち、(1.1) で動かない曲線の性質を調べることが重要になる。

Curve shortening equation は、さまざまな非常によい性質を持っている。しかし、それらの性質を一般の方程式 (1.1) に拡張することは必ずしもたやすいことではない（[Gr2] 等)。そこで、その打開策として、ほしい性質を交点の数の議論に翻訳して一般の場合に拡張するという手段を使う。例えば

(1) 閉曲線γが凸⟶任意の直線とγの交点の数は2以下
(2) 曲線が x のグラフとして表される⟶任意の垂直な直線とその曲線の交点の数は1つ
(3) $|u_x| \leqq L$ ⟶傾きが$\pm L$ の直線と u のグラフとの交点の数は1つ

という具合にである。このような議論により [GaH] や [Gr1,2] のアイデアを一般の場合に拡張してやることができる。

3. 主結果その1

上の Lemma2.1 を駆使する事によって、Theorem 0.1(2) に対応する次の結果が得られる。

Theorem A. γ_0を $\mathbb{R}^2$内の任意の滑らかな単純閉曲線とする。このとき、γ_0 を初期曲線とする *(1.1)* の解は時間無限大まで存在するか、さもなければ有限時間内に1点に縮む。

Theorem 0.1(3) に対応する（弱い）結果として次がいえる。

Theorem B. 解が1点に縮むときその曲率は下からおさえられる。したがってその形は凸に近づく。

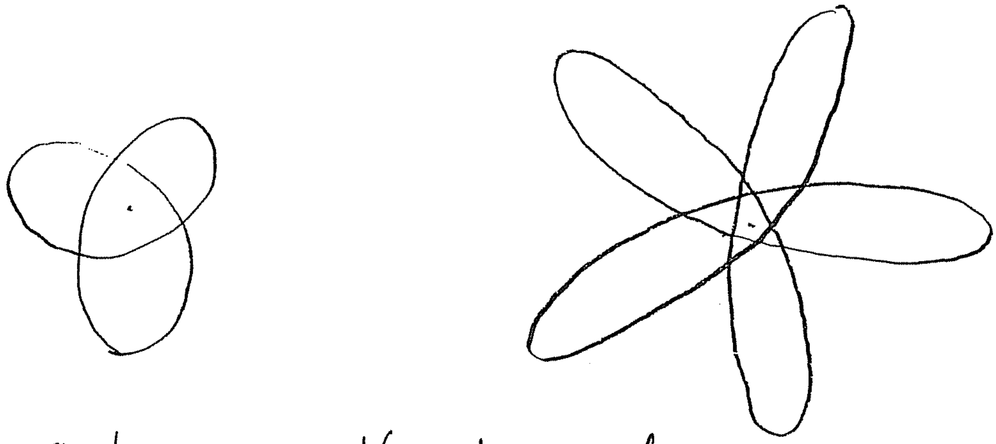
Fが特殊な形のときにはより強い結果がいえる。

Theorem C. $F = F(\theta, k)$ とする。このとき任意の滑らかな単純閉曲線は有限時間内に1点に縮み、途中で凸になる。

4. 解曲線の極限形

$$V = \alpha(\theta)k \tag{4.1}$$

の解が1点に縮むときの形について考える。解は原点に縮むとしておく。Curve shortening equation の場合には解の形は円に近づくことがわかっている (Theorem 0.1(3))。このことは方程式に異方性がないことのみからきているわけではない（例えば異方性の無い方程式 $V = k^{1/3}$は任意の楕円の形を変えないまま一点に縮める)。むしろ、curve shortening equation の selfsimilar solution が円しかないことによるのである。



Simple でない selfsimilar solution の例.

Definition. (selfsimilar solution) 1 点に縮むとき形を変えない (4.1) の解を selfsimilar solution とよぶ。

(4.1) の解$\gamma(t)$ をスケール変換することによって、$\widehat{\gamma}(\tau)$ を次のように定める。

$$\widehat{\gamma}(\tau) = \frac{1}{\sqrt{2(T-t)}}\gamma(t)$$

$$\tau = -\frac{1}{2}\log\frac{T-t}{T}$$

すると、$\widehat{\gamma}(\tau)$ は次の方程式の解になる。

$$V = \alpha(\theta)k - x\sin\theta + y\cos\theta. \tag{4.2}$$

selfsimilar solution は次のように特徴づけられる。

Lemma 4.1. *$\gamma(t)$ が (4.1) の selfsimilar solution であることと、$\widehat{\gamma}(t)$ が (4.2) の停留曲線であることとは同値である。*

一方、

Lemma 4.2. 閉停留曲線は十分たくさん存在する。

から、比較関数がたくさんあることになる。Lemma 2.1 をうまくつかってやることで次の評価をみちびくことができる。

Lemma 4.3. *任意の (4.2) の単純閉曲線解$\widehat{\gamma}(\tau)$ に対し、ある $M > m > 0$ が存在し、次がなりたつ。*

$$\widehat{\gamma}(\tau) \subset \{p \in \mathbb{R}^2 \mid m < |p| < M\}$$

このアプリオリ評価から一般論を使って、次の結果を得る。

Theorem D.

$$\omega(\widehat{\gamma}(0)) \subset \{(4.2)\text{ の単純閉停留曲線 }\}$$

このことは解曲線の極限形は閉単純 selfsimilar solution の集合に含まれることを意味している。

REFERENCES

[AbL] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Diff. Geom. **23** (1986), 175–196.

[AhI] K. Ahara and N. Ishimura, *On the mean curvature flow of "thin" doughnuts* (to appearin Lect. Notes Num. Appl. Anal.).

[An1] S. B. Angenent, *The zeroset of a solution of a parabolic equation*, J. für. die reine und angenandte Math **390** (1988), 79–96.

[An2] S. B. Angenent, *Parabolic equations for curves on surfaces part one*, Annals of Math. **132** (1990), 451–483.

[An3] S. B. Angenent, *Parabolic equations for curves on surfaces part two*, Annals of Math. **133** (1991), 171–215.

[DGM] C. Dohmen and Y. Giga and N. Mizoguchi, *Existence of selfsimilar shrinking curves for anisotropic curvature flow equations*, preprint.

[Ga] M. Gage, *Curve shortening makes convex curves circular*, Invent Math. **76** (1984), 357–364.

[GaH] M. Gage and R. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986), 69–96.

[GaL] M. Gage and Yi Li, *Evolving plane curves by curvature in relative geometries*, preprint.

[Gr1] M. Grayson, *The heat equation shrinks embedded plane curves to points*, J. Diff. Geom. **26** (1987), 285–314.

[Gr2] M. Grayson, *Shortening embedded curves*, Ann. of Math. **129** (1989), 71–111.

[GuA] M. Gurtin and S. B. Angenent, *Multiphase thermomechanics with an interfacial structure 2. Evolution of an isothermal interface*, Arch. Rat. Mech. and Anal. **108** (1989), 323–391.

[O] J. A. Oaks, *Singularities and self intersections of curves evolving on surfaces*, preprint.

ASYMPTOTIC CONFIGURATION AND STABILITY OF STATIONARY INTERFACIAL PATTERNS FOR REACTION DIFFUSION SYSTEMS IN HIGHER DIMENSIONAL SPACES

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Morphology of *final* patterns in phase transition are usually simple ones: only one phase dominates the whole domain (non-conserved) or it is decomposed into simple subdomains (conserved) after coarsening process. This is due to the tendency to minimize the area of interface. However, if there is a microscopic constraint to the system, the final pattern becomes much richer and has in general a variety of morphologies from lamellar to labyrinthine patterns. Block copolymer is one of such materials where two monomers (say, A and B) are connected at some point (constraint), and this is responsible for the formation of very fine and complicated structures depending on the ratio of composite monomers in the process of micro-phase separation [3][2][15]. Locally each monomer moves in a random way and tends to segregate each other (bistability), however connectivity does not allow them to form a large domain consisting of only one monomer (nonlocality). As one of the phenomenological models, it is known (for instance, [11]) that such fine patterns can be given by solving the following stationary problem:

$$(1.1)\qquad \begin{cases} 0 = \varepsilon^2 \Delta u + f(u,v) & \\ & \text{in } \Omega, \\ 0 = D\Delta v + u & \\ \dfrac{\partial u}{\partial n} = 0 = \dfrac{\partial v}{\partial n} & \text{on } \partial\Omega. \end{cases}$$

where u is the *order parameter* indicating A-rich or B-rich phase, v represents the nonlocal effect due to connectivity, $\varepsilon (\ll 1)$ corresponds to the interfacial thickness and $D(\gg 1)$ is proportional to the square of the polymerization index (namely, the length of block copolymer) which is usually quite large, and $f(u,v)$ is a cubic nonlinearity (typically of the form $u - u^3 - v$), and Ω is a smooth domain in $\mathbf{R}^N (N \geq 2)$. Note that the second equation of (1.1) can be solved uniquely with respect to v orthogonal to $\{constant\}$. Substituting this into the first equation, we have a scalar problem for u with nonlocal term. It is anticipated that many other phenomena could be described by similar models to (1.1), since the basic mechanism creating a variety of patterns is due to the competition between local dynamics and nonlocal effect. In fact similar patterns are observed in liquid crystal, magnetic thin film, and so on. The arguments in this note is valid to slightly more general system:

$$(1.2)\qquad \begin{cases} u_t = \varepsilon^2 \Delta u + f(u,v) & \\ & \text{in } \Omega, \\ \delta v_t = D\Delta v + g(u,v) & \\ \dfrac{\partial u}{\partial n} = 0 = \dfrac{\partial v}{\partial n} & \text{on } \partial\Omega. \end{cases}$$

where δ is a nonnegative constant. Note that the stationary problem of (1.2) includes that of (1.1). Although the precise assumptions for (f, g) are delegated to [8], they are qualitatively the same as (1.1). A naive approach to find nontrivial patterns of (1.2) is to consider the limiting case either $\varepsilon \downarrow 0$ or $D \uparrow \infty$. For the latter case it is known (see [6][4]) that the resulting equations become a scalar equation with a constraint of integral type and it is unlikely to have a stable complicated pattern for such a system, since there are no stable multi-layered solutions even in 1D case [7]. On the other hand we know very little about the former case in higher space dimensions, since it has been regarded to be extremely difficult to find the first approximate stationary solutions in the limit of $\varepsilon \downarrow 0$. Especiallly we are interested in the behavior of the asymptotic configuration of the interface Γ^ε. The aim of this note is to answer (at least partially) the following question.

Does (1.2) *has an ε-family of stationary layered solutions with smooth interface Γ^ε up to $\varepsilon = 0$?*

The answer is obviously affirmative, since we know planar and spherical layered solutions (see [14][12]). However those domains have very *special* geometries, i.e., rectangles and spheres and it is not a priori clear that such smooth interfaces persist up to $\varepsilon = 0$ for *generic* domains. It turns out that the answer is negative for generic ones under several hypotheses derived by the formal asymptotic analysis (matched asymptotic expansion (MAE) procedure). In fact we have the following result (partially announced in [9]).

MAIN THEOREM

(a) (Radial Symmetry) *Suppose that* (1.2) *has an ε-family of matched asymptotic solutions with compact smooth interfaces Γ^ε of dimension $N-1$ up to $\varepsilon = 0$ and that the C^1-matching condition holds, then Γ^0 must be a sphere.*

(b) (Non-existence) *Moreover assume the following Hypothesis:*

HYPOTHESIS *The nonlinear elliptic problem*

$$D\Delta v_0 + g(h_-(v_0), v_0) = 0 \qquad \textit{in } \mathbf{R}^N \backslash \Omega_0^+,$$

$$v_0 = v^* \ \textit{ and } \ \frac{\partial v_0}{\partial \nu} \equiv \textit{constant} \qquad \textit{on } \Gamma_0 = \partial\Omega_0^+$$

has only radially symmetric solutions, where $u = h_-(v)$ is one of the stable branches of $f(u,v) = 0$ and ν is unit normal vector to Γ^0.

Then (*a*) *implies that the reduced problem has no solutions for generic domains* Ω, *and hence there does not exist associated* ε*-family of matched asymptotic solutions.*

REMARK *Recently, the hypothesis in Main Theorem* (*a*) *was removed* ([5]).

One of the key ingredients for the proof of Main Theorem is the Serrin's result [13] (and its generalizations) for the over-determined Poisson equation. The above non-existence result is not a dissapoiting result and, in fact, it suggests an important thing about the behavior of the interface as $\varepsilon \downarrow 0$. Namely, if some stationary pattern of (1.2) exists up to $\varepsilon = 0$, but does not have a smooth limiting interface, then the configuration of the interface must become fine and complicated as $\varepsilon \downarrow 0$. In order to understand the morphology of the complicated patterns, it seems necessary to apply an appropriate *rescaling* to blow up the degenerate situation since there are no well-defined asymptotic limit of interfaces in the original framework. We shall also touch several points in this important direction. That is, we attempt to understand the Main Theorem from a positive point of view, applying MAE method to the rescaled system to (1.2) and its linearized eigenvalue problem.

References

[1] P.C.Fife, Semilinear elliptic boundary value problems with small parameters, Arch. Rational Mech. Anal., **52** (1973), 205-232.

[2] H.Hasegawa, H.Tanaka, K.Yamasaki, and T.Hashimoto, Macromolecules, **20**, 1651 (1987).

[3] T.Hashimoto, H.Tanaka, H.Hasegawa, (M.Nagasawa Ed.) *Molecular Conformation and Dynamics of Macromolecules in Condensed Systems*, Elsevier (1988).

[4] D.Hilhorst, Y.Nishiura, and M.Mimura, A free boundary problem arising in some reacting-diffusing system, Proc. Roy. Soc. Edinburgh, **118A** (1991), 355-378.

[5] Yi Li, private communication.

[6] Y.Nishiura, Global structure of bifurcating solutions of some reaction-diffusion systems, SIAM J. Math. Anal., **13** (1982), 555-593.

[7] Y.Nishiura, Coexistence of infinitely many stable solutions to reaction diffusion systems in the singular limit, Dynamics reported, Springer-Verlag.

[8] Y.Nishiura and H.Fujii, Stability of singularly perturbed solutions to systems of reaction diffusion equations, SIAM J. Math. Anal., **18** (1987), 1726-1770.

[9] Y.Nishiura and H.Suzuki, Asymptotic configuration of stationary interfacial patterns for reaction diffusion systems, to appear in the Proceedings of International Conference on Nonlinear Evolution PDE, Bejing, PR China, June 21-25, 1993.

[10] Y.Nishiura and H.Suzuki, in preparation.

[11] T.Ohta and K.Kawasaki, Macromolecules, **19**, 261 (1986).

[12] T.Ohta, M.Mimura, and R.Kobayashi, Higher-dimensional localized pattern in excitable media, Physics **34** D (1989), 115-144.

[13] J.Serrin A symmetry problem in potential theory, Arch. Rational Mech. Anal., **43** (1971), 304-318.

[14] M.Taniguchi and Y.Nishiura, Instability of planar interfaces in reaction diffusion systems, SIAM J. Math. Anal., **25** (1994) 99-134.

[15] E.L.Thomas, D.B.Alward, D.J.Kinning, D.C.Martin, D.L.Handlin, and L.J.Fetters, Macromolecules, **19**, 2197 (1986).

On a Local Energy Decay of Solutions of a Dissipative Wave Equation

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This study is concerned with a local energy decay property of solutions to the initial boundary value problem of the dissipative wave equation :

$$(D)\begin{cases} u_{tt}+u_t-\Delta u=0 & \text{in } \Omega \text{ and } t>0,\\ u=0 & \text{on } \Gamma \text{ and } t>0,\\ u(0,x)=u_0(x),\ \ u_t(0,x)=u_1(x) & \text{in } \Omega, \end{cases}$$

where Ω is an exterior domain in an $n-$ dimensional Euclidean space $\mathbb{R}^n$, whose boundary Γ is a C^∞ and compact hypersurface. Below, $r_0>0$ is a fixed constant such that $\Omega^c \subset B_{r_0}=\{x\in\mathbb{R}^n \mid |x|<r_0\}$. ($\Omega^c$ is the complement of Ω.)

In the case of usual wave equation, the local energy decays exponentially fast if n is odd and polynomially fast if n is even at least under the condition that Ω is non–trapping (cf. [9], [10], [11], [16]). This is reasonable from a physical point of view because the energy propagates along the wave fronts, so that the motion stops after time passes unless the wave front is trapped in a bounded set.

In the case of dissipative wave equation, the energy propagates again along the wave front. Moreover, the trapped energy also decreases in virtue of the dissipative term u_t, so that we can expect to get a local energy decay result without any geometrical condition on Ω. In fact, Shibata [14] proved the following theorem.

Theorem 1.1. *Assume that* $n\geq 3$. *Let* $R>r_0$ *and let* $u(t,x)$ *be a smooth solution of* (D) *such that* $\operatorname{supp} u(0,x)$, $\operatorname{supp} u_t(0,x)\subset \Omega_R=\{x\in\Omega \mid |x|<R\}$. *Then, there exists*

a constant $C > 0$ depending on n and R such that

$$\int_{\Omega_R} \Big\{ |u_t(t,x)|^2 + \sum_{|\alpha|\le 1} |\partial_x^\alpha u(t,x)|^2 \Big\} dx$$
$$\le C(1+t)^{-n} \Big\{ \sum_{|\alpha|\le 3} \int_\Omega |\partial_x^\alpha u_t(0,x)|^2 dx + \sum_{|\alpha|\le 4} \int_\Omega |\partial_x^\alpha u(0,x)|^2 dx \Big\},$$

where $\partial_x^\alpha v = \partial^{|\alpha|} v / \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The purpose of this study is to extend and improve the above result as follows.

Theorem 1.2. *Assume that $n \ge 2$. Let $R > r_0$ and $u_0 \in H^1_{0,R}(\Omega)$ and $u_1 \in L^2_R(\Omega)$, where*

$$L^2_R(\Omega) = \{ f \in L^2(\Omega) \mid \operatorname{supp} f \subset \Omega_R \},$$
$$H^1_{0,R}(\Omega) = \{ f \in H^1(\Omega) \mid \operatorname{supp} f \subset \Omega_R, \;\; f = 0 \text{ on } \Gamma \}.$$

Let $u(t,x)$ be a weak solution of (D). Then, there exists a constant C depending on n and R such that

$$\int_{\Omega_R} \{ |u_t(t,x)|^2 + \sum_{|\alpha|\le 1} |\partial_x^\alpha u(t,x)|^2 \} dx$$
$$\le C(1+t)^{-n} \Big\{ \int_\Omega |u_1(x)|^2 dx + \sum_{|\alpha|\le 1} \int_\Omega |\partial_x^\alpha u_0(x)|^2 dx \Big\}.$$

Compared with Theorem 1.1, our result removes the smoothness assumption on solutions of (D) and includes the case $n = 2$ as well as the case $n \ge 3$.

For the Cauchy problem of the dissipative wave equation (i.e. $\Omega = \mathbb{R}^n$), A. Matsumura [8] studied the decay rate of solutions. His argument was based on the concrete representation of solutions by use of the Fourier transform. When Ω is bounded it is well–known that the energy of solutions decays exponentially fast. Indeed, this fact is easily proved by a standard energy method combined with Poincaré's inequality. Since Ω is unbounded in our case, we cannot use Poincaré's inequality. And also, because of the boundary, we can not use the Fourier transform. Our method is based on a spectral analysis to the corresponding stationary problem with parameter λ.

Putting $u_t = v$, let us rewrite the problem (D) in the following form :

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = A\begin{bmatrix} u \\ v \end{bmatrix}.$$

To consider A to be dissipative, we introduce a space $H_D(\Omega)$. For any open set $\mathcal{O} \subset \mathbb{R}^n$, $C_0^\infty(\mathcal{O})$ denotes the space of all C^∞ functions on $\mathbb{R}^n$ whose support is compact and lies in $\mathcal{O}$ (in particular, such functions vanish near the boundary of $\mathcal{O}$), $L^2(\mathcal{O})$ a usual L^2 space on $\mathcal{O}$ with norm $\|\cdot\|_{\mathcal{O}}$ innerproduct $(\ ,\ )_{\mathcal{O}}$ and $H^s(\mathcal{O})$ a usual Sobolev space of order s on $\mathcal{O}$ with norm $\|\cdot\|_{s,\mathcal{O}}$. $\|\cdot\|_{k,\Omega}$ will be denoted simply by $\|\cdot\|_k$. Likewise for $\|\cdot\|_\Omega$ and $(\ ,\ )_\Omega$. Then, we put

$$\begin{aligned} H_D(\Omega) =& \{u \in H^1_{loc}(\Omega) \mid \nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) \in L^2(\Omega),\ \ u = 0 \quad \text{on } \Gamma, \\ & \exists\ \{u_n\} \subset C_0^\infty(\Omega) \text{ s.t. } \|\nabla(u_n - u)\| \to 0 \text{ as } n \to \infty\ \}, \end{aligned}$$

where $H^1_{loc}(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid u \in H^1(\Omega_R)\ \forall R > r_0\}$. Then, an underlying space for A is

$$\mathcal{H} = \left\{\begin{bmatrix} u \\ v \end{bmatrix} \mid u \in H_D(\Omega),\ \ v \in L^2(\Omega)\right\}.$$

We know that $\mathcal{H}$ is a Hilbert space equipped with the innerproduct

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix}\right)_{\mathcal{H}} = (u, w)_D + (v, z).$$

The domain of A is

$$\begin{aligned} D(A) =& \left\{\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \mid A\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H}\right\} \\ =& \left\{\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \mid v \in H_D(\Omega),\ \ \Delta u \in L^2(\Omega)\right\}. \end{aligned}$$

Then, A has the following properties.

Proposition. *(1) A is a closed operator.* *(2) A is a dissipative operator.* *(3) $\mathcal{R}(I - A) = \mathcal{H}$.* *(4) $D(A)$ is dense in $\mathcal{H}$.*

Lumer and Phillips theorem [13, Chapter 1, Theorem 4.3] implies that A generates a C^0 semigroup $\{T(t)\}$ on $\mathcal{H}$.

Since A is dissipative, $T(t)$ is a C_0 semigroup of contractions, so that $\|T(t)\| \leq 1$ $\forall t \geq 0$. Let α be a positive number. We have the following expression :

$$T(t)\mathbf{x} = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\alpha - i\omega}^{\alpha + i\omega} e^{\lambda t}(\lambda I - A)^{-1}\mathbf{x} d\lambda \quad \text{for } \mathbf{x} \in D(A^2).$$

(cf. [12, p.295] or [13, Chapter 1, Corollary 7.5]). By a lemma due to F. Huang in [4, §1, Lemma 1] (also see [7]), we have the following lemma.

Lemma. *For any* $\alpha > 0$ *and* $\mathbf{x} \in \mathcal{H}$, *put*

$$g(\omega) = \|((\alpha + i\omega)I - A)^{-1}\mathbf{x}\|_{\mathcal{H}}.$$

Then $g(\omega) \in L^2(\mathbb{R})$ *and*

$$\lim_{|\omega| \to \infty} g(\omega) = 0,$$
$$\int_{-\infty}^{\infty} g(\omega)^2 d\omega \leq \frac{\pi}{\alpha} \|\mathbf{x}\|_{\mathcal{H}}^2.$$

By virtue of the above lemma, we can deduce that the high frequency part decays sufficiently fast, so that the rate of decay is determined by the behavior of the resolvent in a neighborhood of $\lambda = 0$, which is investigated by use of a perturbation method. In fact when $n \geq 3$, by use of the fact that $(\lambda I - A)^{-1}$ in $\mathbb{R}^n$ is continuous up to $\lambda = 0$, we can construct a parametrix near $\lambda = 0$ as a compact perturbation from $(\lambda I - A)^{-1}$ in $\mathbb{R}^n$. But, when $n = 2$, $(\lambda I - A)^{-1}$ in $\mathbb{R}^2$ behaves like $\log \lambda$ near $\lambda = 0$, so that the idea in the case that $n \geq 3$ does not work at all. When $n = 2$, our strategy follows Borchers and Varnhorn [2] mainly. Expressing the resolvent of the stationary problem near $\lambda = 0$ by use of a single layer potential and a double layer potential and using the properties of Bessel functions we can investigate the behavior of the resolvent in a neighborhood of $\lambda = 0$. A key is that the integral equation on the boundary obtained by use of single and double layer potentials has no singular terms in the kernel with respect to λ. As a result we can deduce that $(\lambda I - A)^{-1}$ behaves like $C_1 + C_2/\log \lambda$.

REFERENCES

1. Abramowitz, M., Stegun, I.A., *Handbook of Mathematical Functions*, New York : Dover, 1970.
2. Borchers, W., Varnhorn, W., *On the boundedness of the Stokes semigroup in two–dimensional exterior domains*, Math. Z. **213** (1993), 275–299.
3. Chang, I–Dee, Finn, R., *On the Solutions of a class of equations occurring in continuum mechanics with application to the stokes paradox*, Arch. Rational Mech. **7** (1961), 389–401.
4. Huang, F.L., *Characteristic condition for exponential stability of linear dynamical systems in Hibert spaces*, Ann. of Diff. Eqns. **1(1)** (1985), 43–56.
5. Jaswon, M.A., Symm, G.T., *Integral equation methods in potential theory and elastostatics*, Academic Press inc., 24/28 Oval Road, London NW1, 1977.
6. Leis, R., *Initial boundary value problems in mathematical physics*, B.G. Teubner – Verlag Stuttgart; John Wiley & Sons, Chichester et al., 1986.
7. Liu, Z., Zheng, S., *Uniform exponential stability and approximation in control of thermoelastic system*, preprint (1991).
8. Matsumura, A., *On the asymptotic behaviour of solutions of semi–linear wave equations*, Publ. RIMS, Kyoto Univ. **13** (1976), 363–386.
9. Melrose, R.B., *Singularities and energy decay in acoustic scattering*, Duke Math. J. **46 (1)** (1979), 43–59.
10. Morawetz, C.S., *Decay of solutions of the exterior problem for the wave equation*, Commun.Pure Appl.Math. **28** (1975), 229–264.
11. Morawetz, C.S., Ralston, J.V., Strauss, W.A., *Decay of solution outside non–trapping obstacles*, ibid **30** (1977), 447–508.
12. Mizohata, S., *The theory of partial differential equations*, Cambridge University Press, Bentley House 200 Euston Road London NWI2DB, 1973.
13. Pazy, A., *Semigroups of linear operators and applications to partial differential equations*, Springer – Verlag, New York, 1983.
14. Shibata, Y., *On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain*, Tsukuba J. Math. **7** (1983), 1–68.
15. Shibata, Y., Soga, H., *Scattering theory for the elastic wave equation*, Publ. RIMS, Kyoto Univ. **25** (1989), 861–887.
16. Shibata, Y., Tsutsumi, Y., *On a global existence theorem of small amplitude solutions for nonlinear wave equations in an exterior domain*, Math. Z. **191** (1986), 165–199.
17. Varnhorn, W., *An explicit potential theory for the stokes resolvent boundary value problem in three dimensions*, Manus. Math. **70** (1991), 339–361.
18. Weck, N., Witsch, K.J., *Exterior Dirichlet problem for the reduced wave equation : asymptotic analysis of low frequencies*, Commun. P. D. E. **16 (2 & 3)** (1991), 173–195.

EXTERIOR PROBLEM FOR THE NAVIER-STOKES EQUATIONS

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Introduction.

Let Ω be an exterior domain in $\mathbb{R}^n (n \geq 3)$,i.e., a domain having a compact complement $\mathbb{R}^n \backslash \Omega$, and assume that the boundary $\partial\Omega$ is of class $C^{2+\mu}(0 < \mu < 1)$. The motion of the incompressible fluid occupying Ω is governed by the Navier-Stokes equations:

$$(N - S_0) \qquad \begin{cases} -\Delta w + w \cdot \nabla w + \nabla \pi = \operatorname{div} F & \text{in} \quad \Omega, \\ \operatorname{div} w = 0 \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \partial\Omega, \qquad w(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases}$$

where $w = w(x) = (w^1(x), \cdots, w^n(x))$ and $\pi = \pi(x)$ denote the unknown velocity vector and the unknown pressure of the fluid at point $x \in \Omega$, respectively, while $F = F(x) = (F^i_j(x))_{i,j=1,\cdots,n}$ is the given tensor with $\operatorname{div} F = (\sum_{j=1}^n \frac{\partial F^1_j}{\partial x_j}, \cdots, \sum_{j=1}^n \frac{\partial F^n_j}{\partial x_j})$ denoting the external force. The first purpose of this article is to show the existence, uniqueness and regularity of solutions to $(N - S_0)$ for a larger class of external forces. Our method is based on the functional analysis, i.e., the L^r-theory for the linearized equations to $(N - S_0)$, i.e., the Stokes equations:

$$(S) \qquad \begin{cases} -\Delta w + \nabla \pi = \operatorname{div} F & \text{in} \quad \Omega, \\ \operatorname{div} w = 0 \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \partial\Omega, \qquad w(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases}$$

plays an important role for our approach. To solve $(N - S_0)$ in the class $\nabla w \in L^r(\Omega)$, we have to restrict ourselves to the case $r = n/2$ which stems from the nonlinear structure $w \cdot \nabla w$. It should be noted that the norm $\|\nabla w\|_{L^{n/2}}$ is invariant under such change of scaling as $w_\lambda(x) = \lambda w(\lambda x)$ for each $\lambda > 0$. On the other hand it is known that (S) has a unique solution w with $\nabla w \in L^r(\Omega)$ if and only if r satisfies $n/(n-1) < r < n$. Unfortunately, the most physically relevant case $n = 3$ is excluded because $n/2 = n/(n-1) = 3/2$ is the critical power for the unique solvability of (S). Thus the method of linearization makes no contribution to solvability of the nonlinear equations $(N - S_0)$ in the class $\nabla w \in L^{n/2}(\Omega)$ for $n = 3$. To overcome this difficulty, we introduce a larger class $\nabla w \in L_{p,q}(\Omega)$ and show that (S) has a unique solution w in $L_{n,\infty}(\Omega)$ with $\nabla w \in L_{\frac{n}{2},\infty}(\Omega)$, where $L_{p,q}(\Omega)$ denotes the Lorentz space over Ω.

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The second purpose of this article is to show the stability in $L_{n,\infty}(\Omega)$ of our stationary solution w of $(N-S_0)$. If w is perturbed by a, then the perturbed flow $v(x,t)$ is governed by the following *non-stationary* Navier-Stokes equations:

$$(N-S_1) \qquad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + v \cdot \nabla v + \nabla q = \text{div } F \quad \text{in } \Omega, t > 0, \\ \text{div } v = 0 \quad \text{in} \quad \Omega, t > 0, \\ v = 0 \quad \text{on} \quad \partial\Omega, t > 0, \qquad v(x,t) \to 0 \quad \text{as} \quad |x| \to \infty, \\ v(x,0) = w(x) + a(x) \quad \text{for} \quad x \in \Omega. \end{cases}$$

We show that if the stationary flow w and the initial disturbance a are both small in the *common* space $L_{n,\infty}(\Omega)$, then there is a unique *global strong solution* v of $(N-S_1)$ such that $v(x,t) \to w(x)$ with *definite decay rates* as $t \to \infty$. Let w and v be solutions of $(N-S_0)$ and $(N-S_1)$, respectively. Then the pair of functions $u \equiv v - w, p \equiv q - \pi$ satisfies

$$(N-S') \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + w \cdot \nabla u + u \cdot \nabla w + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \Omega, t > 0, \\ \text{div } u = 0 \quad \text{in} \quad \Omega, t > 0, \\ u = 0 \quad \text{on} \quad \partial\Omega, t > 0, \qquad u(x,t) \to 0 \quad \text{as} \quad |x| \to \infty, \\ u|_{t=0} = a. \end{cases}$$

Thus our problem on the stability for $(N - S_0)$ can now be reduced to investigation into existence of global strong solutions to $(N - S')$ and their asymptotic behaviour. To solve $(N - S')$ globally in time, we need to establish the $L_{p,\infty} - L^r$-estimate for the semigroup $e^{-t\mathcal{L}_r}$, where $\mathcal{L}_r$ is an operator defined by

$$\mathcal{L}_r \equiv A_r + P_r(w \cdot \nabla u + u \cdot \nabla w).$$

Here P_r is the projection operator from $L^r(\Omega)$ onto $L^r_\sigma(\Omega)$ and $A_r = -P_r\Delta$ denotes the Stokes operator. To treat $\mathcal{L}_r$ as such perturbation of A_r as the $L_{p,\infty} - L^r$-estimate remains to be obtained, one required smallness of either $\|\nabla w\|_{\frac{n}{2}}$ or that of the quantity $\sup_{x\in\Omega} |x||w(x)| + \sup_{x\in\Omega} |x|^2|\nabla w(x)|$. Making use of the special structure $w \cdot \nabla u + u \cdot \nabla w$ of the perturbed convective term together with div u = div $w = 0$, we shall remove such an assumption on the derivatives ∇w. Indeed, we shall establish the $L_{p,\infty} - L^r$ estimates for $e^{-t\mathcal{L}_r}$ only by assuming that w is small in $L_{n,\infty}$.

§1 Results.

1.1. Existense of the stationary solution.

Let us introduce some function spaces. For $1 < p < \infty$, $\dot{H}^1_p(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla \cdot\|_p$; $\|\cdot\|_p$ is the norm of the usual L^p space. Since Ω is an exterior domain, $\dot{H}^1_p(\Omega)$ is larger than the usual Sobolev space $H^1_p(\Omega)$. By real interpolation, we define $\dot{H}^1_{p,q}(\Omega)$ by $\dot{H}^1_{p,q}(\Omega) \equiv (\dot{H}^1_{p_0}(\Omega), \dot{H}^1_{p_1}(\Omega))_{\theta,q}$, where $1 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ satisfy $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1 \le q \le \infty$. Note that for $1 \le q < \infty$, $\dot{H}^1_{p,q}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla \cdot\|_{p,q}$ of $L_{p,q}(\Omega)$; $(\cdot,\cdot)$ denotes the duality pairing between $L_{p,q}(\Omega)$ and $L_{p',q'}(\Omega)$, where $1/p + 1/p' = 1, 1/q + 1/q' = 1$. $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\dot{H}^1_{p',q'}(\Omega)^*$ and $\dot{H}^1_{p',q'}(\Omega)$. Our results read as follows:

Theorem 1. *Let p and q be as*

(i) $n' < p < n, 1 < q \leq \infty$,
(ii) $p = n', q = \infty$.

Then for every $\{f, g\} \in \dot{H}^1_{p',q'}(\Omega)^* \times L_{p,q}(\Omega)$, *there is a unique pair* $\{w, \pi\} \in \dot{H}^1_{p,q}(\Omega) \times L_{p,q}(\Omega)$ *such that*

$$\begin{cases} (\nabla w, \nabla \phi) - (\pi, \text{div } \phi) = \langle f, \phi \rangle \quad \text{for all} \quad \phi \in \dot{H}^1_{p',q'}(\Omega), \\ \text{div } w = g \quad \text{in} \quad \Omega, \end{cases}$$

holds.

Theorem 2. *(1)(existence) Let $n \geq 3$. There is a constant $\delta = \delta(n) > 0$ such that if $F \in L_{\frac{n}{2},\infty}(\Omega)$ satisfies $\|F\|_{\frac{n}{2},\infty} \leq \delta$, then there exists a pair $\{w, \pi\} \in \dot{H}^1_{\frac{n}{2},\infty}(\Omega) \times L_{\frac{n}{2},\infty}(\Omega)$ of the solution of $(N - S_0)$ in the following sense:*

$$\text{(1.1)} \qquad \begin{cases} (\nabla w, \nabla \phi) - (w \cdot \nabla \phi, w) - (\pi, \nabla \phi) = -(F, \nabla \phi) \quad \text{for all } \phi \in C_0^\infty(\Omega), \\ \text{div } w = 0 \quad \text{in } \Omega. \end{cases}$$

(2)(uniqueness) There is a constant $k = k(n)$ such that any pair $\{w, \pi\} \in \dot{H}^1_{\frac{n}{2},\infty}(\Omega) \times L_{\frac{n}{2},\infty}(\Omega)$ satisfying (1.1) with

$$\|w\|_{n,\infty} \leq k$$

is unique.

(3)(regularity) For $n'(= \frac{n}{n-1}) < r < \infty$, there is a constant $0 < \delta'(n, r) \leq \delta(n)$ such that if $F \in L_{\frac{n}{2},\infty}(\Omega) \cap L_{r,\infty}(\Omega)$ satisfies $\|F\|_{\frac{n}{2},\infty} \leq \delta'$, then the solution $\{w, \pi\}$ of $(N - S_0)$ given by the above (1.1) has the additional property

$$\nabla w \in L_{\frac{n}{2},\infty}(\Omega) \cap L_{r,\infty}(\Omega), \quad \pi \in L_{\frac{n}{2},\infty}(\Omega) \cap L_{r,\infty}(\Omega).$$

Remarks. *(1) For the assertion on uniqueness, we do not have to assume the smallness of $\|\nabla w\|_{\frac{n}{2},\infty}$ but of the weaker norm $\|w\|_{n,\infty}$.*

(2) For the assertion on regularity, the smallness for $\|F\|_{r,\infty}$ is not necessary, which is closely related to the invariance of the norm $\|\nabla w\|_{\frac{n}{2},\infty}$ under such change of scaling described as above.

1.2. Stability.

We shall next proceed to stability of the flow w obtained by Theorem 2. Let L^r_σ is the closure of $C^\infty_{0,\sigma}$, with respect to the L^r-norm $\| \quad \|_r$. L^r stands for the usual L^r-spaces over $\Omega, 1 < r < \infty$. Note that $L^r = L_{r,r}(\Omega)$. It is known that $L^r_\sigma = \{u \in L^r; \text{div } u = 0 \quad \text{in } \Omega, u \cdot \nu = 0 \text{ on } \partial\Omega\}$. Recall the Helmoltz decomposition:

$$L^r = L^r_\sigma \oplus G^r (\text{direct sum}), \quad 1 < r < \infty,$$

where $G^r = \{\nabla p \in L^r; p \in L^r_{loc}(\overline{\Omega})\}$. P_r denotes the projection operator from L^r onto L^r_σ along G^r. The Stokes operator A_r on L^r_σ is then defined by $A_r = -P_r \Delta$

with domain $D(A_r) = H^{2,r}(\Omega) \cap H^{1,r}_{0,\sigma}$, where $H^{1,r}_{0,\sigma}$ is the closure of $C^\infty_{0,\sigma}$ with respect to the $H^{1,r}$-norm $\|\phi\|_{H^{1,r}} = \|\phi\|_r + \|\nabla\phi\|_r$. By real interpolation, we define $L^{r,q}_\sigma$ by $L^{r,q}_\sigma \equiv (L^{r_0}_\sigma, L^{r_1}_\sigma)_{\theta,q}$, where $1 < r_0 < r < r_1 < \infty$ and $0 < \theta < 1$ satisfy $1/r = (1-\theta)/r_0 + \theta/r_1$ and $1 \le q \le \infty$. Then it follows that

$$L^{r,q}_\sigma = \{u \in L_{r,q}(\Omega); \operatorname{div} u = 0 \quad \text{in} \quad \Omega, u \cdot \nu = 0 \quad \text{on} \quad \partial\Omega\}$$

and that the Stokes operator A is also well-defined on $L^{r,q}_\sigma$ with the domain

$$D(A_{r,q}) = \{u \in L^{r,q}_\sigma; \nabla^j u \in L_{r,q}(\Omega), j = 1, 2, u|_{\partial\Omega} = 0\}.$$

On the stationary flow, we impose the following assumption:

Assumption 1. *w is a solenoidal vector field on $\overline{\Omega}$ with $w|_{\partial\Omega} = 0$ in the class*

$$w \in L^{n,\infty}_\sigma \cap L^\infty, \quad \nabla w \in L^{r_*}$$

for some r_* with $n < r_* < \infty$.

Remark. Take r_0 so that $r_* < r_0 < \infty$. If $F \in L_{\frac{n}{2},\infty}(\Omega) \cap L_{r_0,\infty}(\Omega)$ satisfies $\|F\|_{\frac{n}{2},\infty} \le \delta'(n, r_0)$, then Theorem 2(3) and the Sobolev embedding $H^{1,r_*} \subset L^\infty$ yields a solution w of $(N - S_0)$ satisfying Assumption 1.

Now we define an operator B_r on L^r_σ for $1 < r \le r_*$ by

$$B_r u \equiv P_r(w \cdot \nabla u + u \cdot \nabla w) \qquad \text{with the domain} \quad D(B_r) = H^{1,r}_{0,\sigma}.$$

It should be noted that if $\nabla w \in L^{r_*}$ and if $1 < r \le r_*$ then we have $u \cdot \nabla w \in L^r$ for $u \in H^{1,r}_{0,\sigma}$ and hence B_r is well-defined on $H^{1,r}_{0,\sigma}$. Then $\mathcal{L}_r$ is introduced by

$$\mathcal{L}_r = A_r + B_r, \quad 1 < r \le r_* \quad \text{with domain} \quad D(\mathcal{L}_r) = D(A_r).$$

On the initial disturbance a in $(N - S_1)$ we impose the following assumption.

Assumption 2. *The initial disturbance a is in $L^{n,\infty}_\sigma$.*

Our results on stability of w now read:

Theorem 3. *(1) Let w and a satisfy Assumption 1 and Assumption 2, respectively. There is a positive number $\kappa = \kappa(n, r_*)$ such that if*

$$\|w\|_{n,\infty} \le \kappa, \quad \|a\|_{n,\infty} \le \kappa, \tag{1.2}$$

then there exists a strong solution u of $(N - S')$ with the following properties.

(i) $u \in BC((0,\infty); L^{n,\infty}_\sigma) \cap C((0,\infty); D(A_{r_*})) \cap C^1((0,\infty); L^{r_*}_\sigma)$;
(ii) $du/dt + \mathcal{L}_{r_*} u + P_{r_*}(u \cdot \nabla u) = 0, \quad$ *in* $\quad L^{r_*}_\sigma, \quad t > 0$;
(iii) $u(t) \to a \quad$ *weakly* $*$ *in* $\quad L^{n,\infty}_\sigma \quad$ *as* $\quad t \downarrow +0$;
(iv) *(uniform estimate)*

$$\|u(t)\|_r \le C t^{-\frac{n}{2}(\frac{1}{n} - \frac{1}{r})}, \quad n < r \le r_*$$

for all $t > 0$ with a constant C depending only on n, r and r_*.

($BC(I;X)$:the set of bounded and continuous functions on the interval I with values in X)

(2)(uniqueness) There is a constant $k = k(n, r_*)$ such that any solution u of $(N-S')$ satisfying the above properties (i)-(iv) with

$$\limsup_{t\downarrow +0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r_*})}\|u(t)\|_{r_*} \le k \tag{1.3}$$

is unique.

The more rapidly spacial decay at infinity of the initial disturbance $a(x)$ is assumed, the sharper asymptotic behaviour for $u(t)$ as $t \to \infty$ is obtained:

Theorem 4. *For $n'(=\frac{n}{n-1}) < p < n$, there is a constant $\tilde{\kappa}(n, r_*, p) \le \kappa(n, r_*)$ ($\kappa(n, r_*)$: the constant in (1.2)) such that if $a \in L^{n,\infty}_\sigma \cap L^p_\sigma$ and if*

$$\|w\|_{n,\infty} \le \tilde{\kappa}, \quad \|a\|_{n,\infty} \le \tilde{\kappa}, \tag{1.4}$$

then the solution u given by Theorem 3 has the following additional properties:

$$u(\cdot) \quad \text{and} \quad t^{\frac{1}{2}}\nabla u(\cdot) \in BC([0,\infty); L^p);$$
$$\|u(t)\|_l = O(t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{l})}) \qquad \text{for} \quad p \le l \le r_* \quad \text{as} \quad t \to \infty.$$

Moreover, for every q with $p \le q < n$, there is a constant $\hat{\kappa} = \hat{\kappa}(n, r_*, p, q)$ such that if we assume in addition to (1.4) that

$$\|w\|_{n,\infty} \le \hat{\kappa}, \quad \|a\|_{n,\infty} \le \hat{\kappa},$$

then there holds

$$\|\nabla u(t)\|_l = O(t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{l})-\frac{1}{2}}) \qquad \text{for} \quad p \le l \le q \quad \text{as} \quad t \to \infty.$$

Remarks. *(1) Theorem 3 shows that $L^{n,\infty}_\sigma$ is the class of stable stationary flows and that it is the same class as that of initial disturbances. Up to the present, there is a similar result to the above one with the assumption that*

$$\sup_{x\in\Omega} |x||w(x)| + \sup_{x\in\Omega} |x|^2|\nabla w(x)|$$

is small enough. On the other hand, our results have clarified that the assumption on ∇w is superfluous. Moreover, the space $L^{n,\infty}_\sigma$ is larger than the class of functions such that $\sup_{x\in\Omega}|x||w(x)| < \infty$. Theorem 4 has improved also the results on L^2-stability for weak solutions with homogeneous boundary condition at infinity, i.e., $w(x) \to 0$ as $|x| \to \infty$.

(2) Since the semigroup $\{e^{-t\mathcal{L}}\}_{t\ge 0}$ is not strongly continuous in $L^{n,\infty}_\sigma$, we cannot verify whether our solution u satisfies

$$\lim_{t\downarrow +0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r_*})}\|u(t)\|_{r_*} = 0.$$

This is the reason why we impose the condition (1.3) on the uniqueness.