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# On higher order non-singular immersions of $RP^n$ in $CP^m$

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Haruo SUZUKI

## § 0. Introduction

We denote by  $RP^n$  the real projective space of dimension  $n$  and by  $CP^n$  the complex projective space of complex dimension  $n$ . We know many informations on higher order non-singular immersions of  $RP^n$  in  $RP^m$  or those of  $CP^n$  in  $CP^m$ . They can be seen in H. Suzuki [11, 12] and H. Ôike [8]. Results on higher order non-singular immersions of  $RP^n$  in euclidean spaces by H. Suzuki [10] and T. Kobayashi [7] are translated easily into those on higher order non-singular immersions of  $RP^n$  in  $RP^m$  or  $CP^n$  which are homotopically trivial. As for higher order non-singular immersions of  $RP^n$  in  $RP^m$  ( $n < m$ ), some results are announced in W.-L. Ting [13].

Let  $m, n$  be integers such that  $1 < n \leq 2m$ . The homotopy set  $[RP^n, CP^m]$  consists of just two elements. We consider in this article, higher order non-singular immersions of  $RP^n$  in  $CP^m$  which are not homotopically trivial, and also consider relations between the homotopy classes of higher order non-singular immersions of  $RP^n$  in  $CP^m$  and its order, under some additional conditions on dimensions  $n$  and  $m$ . Our main tools are Stiefel-Whitney classes of  $p$ th order normal (conormal) bundles. Mod 2 S-relations of stunted real projective spaces constructed from some independent cross-sections of a multiple of the canonical line bundle over  $RP^n$  can be used in quite restricted cases.

We state results in Section 1. We prove, in Section 2, the key lemma on the homotopically non-trivial maps of  $RP^n$  to  $CP^m$  and prove theorems of section 1 in section 3. Finally we give supplementary results on the higher order non-singular immersions which are homotopically trivial and results from the mod 2 S-relations of stunted projective spaces.

## § 1. Statement of results

Let  $\tau_p(RP^n)$  and  $\tau_p(CP^m)$  denote  $p$ th order tangent bundles of  $RP^n$  and  $CP^m$  respectively. The dimensions of their fibres are  $\nu(n, p) = \binom{n+p}{p} - 1$  and

$\nu(2m, p) = \binom{2m+p}{p} - 1$ . Let  $f: RP^n \rightarrow CP^m$  be a  $C^p$ -differentiable map and  $\tau_p(f): \tau_p(RP^n) \rightarrow \tau_p(CP^m)$  be the homomorphism of  $p$ th order tangent bundles induced by  $f$ . If we take a sequence of  $q$ th order dissections  $\{D^{(q)}\}$ ,  $q=1, 2, \dots, p-1$  on  $CP^m$ , we have a  $p$ th order osculating map of  $f$  with respect to  $\{D^{(q)}\}$ , which is a vector bundle homomorphism

$$D_p \cdot \tau_p(f) = D^{(1)} \dots D^{(p-1)} \cdot \tau_p(f): \tau_p(RP^n) \rightarrow \tau(CP^m)$$

covering  $f$ . If  $f$  is an immersion and  $D_p \cdot \tau_p(f)$  is of maximal rank on each fibre,  $f$  is called a  *$p$ th order non-singular immersion* with respect to the dissections. If  $p \geq 2$ ,  $2m \geq \nu(n, p)$  and  $D_p \cdot \tau_p(f)|_x (x \in RP^n)$  is not of maximal rank for a  $C^p$ -immersion  $f$ , one can say that  $x$  is an *inflexion point* with respect to the dissections, of order  $\leq p-1$ . (See W. F. Pohl [9] and E. A. Feldman [2, 3, 4, 5].)

The homotopy set  $[RP^n, CP^m]$  ( $1 < n \leq 2m$ ) are 1-1 correspondence with the cohomology group  $H^2(RP^n; Z) \cong Z_2$ . For a continuous map  $f: RP^n \rightarrow CP^m$  ( $1 < n \leq 2m$ ), we call the homotopy class  $\{f\} \in [RP^n, CP^m] \cong Z_2$  a *degree* of  $f$  and denote it by  $\deg(f)$ . This is expressed by an integer  $d \bmod 2$ . By computations of the Stiefel-Whitney classes of  $p$ th order normal (co-normal) bundle of the  $p$ th order non-singular immersion, we obtain the following results.

**THEOREM 1.1.** *Let  $p$  be an even positive integer, and let  $n, m$  be integers such that  $1 < n$  and  $\nu(n, p) - n < 2m < \nu(n, p) + n$ . Suppose there exists a  $p$ th order non-singular immersion  $f: RP^n \rightarrow CP^m$  with respect to dissections  $\{D^{(q)}\}$  on  $CP^m$  such that  $\deg(f) \neq 0$ . (a) If  $k = 2m - \nu(n, p) \geq 0$  then we have  $\left( \binom{n+p}{p} + k + 1 \right)_{k+i} = 0 \bmod 2$  and  $\left( \binom{n+p}{p} + k + i \right)_{k+i} = 0 \bmod 2$  for  $1 \leq i \leq n - k$ . (b)*

*If  $k = \nu(n, p) - 2m \geq 0$  then we have  $\left( \binom{n+p}{p} + 1 \right)_{k+i} = 0 \bmod 2$  and  $\left( \binom{n+p}{p} + i \right)_{k+i} = 0 \bmod 2$  for  $1 \leq i \leq n - k$ .*

**COROLLARY 1.2.** *Let  $n, m$  be positive integers,  $1 < n$  and let  $p$  be an even positive integer. We put  $s(n, p) = \max \{j | 0 < j \leq n, \left( \binom{n+p}{p} + j \right)_j \not\equiv 0 \bmod 2\}$  and  $d(n, p) = \max \{j | 0 < j \leq n, \left( \binom{n+p}{p} + 1 \right)_j \not\equiv 0 \bmod 2\}$ . If  $\nu(n, p) - d(n, p) < 2m < \nu(n, p) + s(n, p)$  then there is no homotopically non-trivial  $p$ th order non-singular immersion of  $RP^n$  in  $CP^m$  with respect to dissections on  $CP^m$ .*

We can also show easily the following theorem by Theorem 1.1.

**THEOREM 1.3.** Assume  $p=2^r(2q+1)$  where  $q, r$  are integers and  $r \geq 1$ , and assume  $n=2^l-1$  for an integer  $l > r$  and  $0 \leq 2m - \nu(n, p) < 2^{l-r} - 1$ . If  $RP^n$  can be immersed in  $CP^m$  by a homotopically non-trivial  $C^p$ -map  $f$ , then  $f$  has at least one inflexion point of order  $\leq p-1$  with respect to dissections  $\{D^{(q)}\}$  on  $CP^m$ .

As for relations between  $\deg(f)$  and the order  $p$ , we obtain some theorems. First we show an analogy of Theorem 1.1 of H. Suzuki [12] in a certain sense.

**THEOREM 1.4.** Suppose  $n, m$  and  $p$  are integers such that  $n > 1, p \geq 1$ ,  $\binom{n+p}{p} \not\equiv n \pmod{2}$  and suppose  $\nu(n, p) \leq 2m \leq \nu(n, p) + n - 2$  and  $\binom{\binom{n+p}{p} + n - 1}{n} \not\equiv 0 \pmod{2}$ . If  $f: RP^n \rightarrow CP^m$  is a  $p$ th order non-singular immersion with respect to dissections  $\{D^{(q)}\}$  on  $CP^m$  then we have  $\deg(f) \equiv p \pmod{2} = 0$ .

We can find the values of  $n, m$  and  $p$  satisfying conditions of Theorem 1.4. For examples, we have  $n=2, p=5$  and  $m=10$ ;  $n=2, p=8$  and  $m=22$ . Under these assumptions for  $n, m$  and  $p$ , a  $C^p$ -immersion  $f: RP^n \rightarrow CP^m$  has at least one inflexion point of order  $\leq p-1$  with respect to dissections  $\{D^{(q)}\}$  on  $CP^m$  if  $\deg(f) \neq 0$  or  $p \not\equiv 0 \pmod{2}$ .

In the above theorem, we assume  $\binom{n+p}{p} \not\equiv n \pmod{2}$  and  $\binom{\binom{n+p}{p} + n - 1}{n} \not\equiv 0 \pmod{2}$ . Without these assumptions, we can prove a result.

**THEOREM 1.5.** Suppose  $n, m$  and  $p$  are integers such that  $n > 1, p \geq 1$  and  $\nu(n, p) \leq 2m \leq \nu(n, p) + n - 1$ . If there exists a  $p$ th order non-singular immersion  $f: RP^n \rightarrow CP^m$  with respect to dissections  $\{D^{(q)}\}$  on  $CP^m$ , then one of the following cases takes place:  $\deg(f) \neq 0$  and  $p \equiv 0 \pmod{2}$ ;  $\deg(f) = 0$  and  $p \not\equiv 0 \pmod{2}$ ;  $\deg(f) = 0$  and  $p \equiv 0 \pmod{2}$ .

## § 2. Homotopically non-trivial map of $RP^n$ in $CP^m$ .

Let  $f: RP^n \rightarrow CP^m$  ( $1 < n \leq 2m$ ) be the homotopically non-trivial map. Let  $\eta$  be the canonical complex line bundle over  $CP^m$ . We denote by  $r$  the realification of a complex vector bundle. We shall compute the induced bundle  $f^*r\eta$  over  $RP^n$ . Let  $\xi$  be the canonical real line bundle over  $RP^n$ .

**LEMMA 2.1.** Under the above notations, we have

$$f^*r\eta = \xi \oplus \xi.$$

**PROOF.** Complex line bundles over  $RP^n$  are classified by the homotopy

set  $[RP^n, CP^m]$  since we assume  $n \leq 2m$ . The correspondence is given by taking the induced bundle. On the other hand, the set  $[RP^n, CP^m]$  are 1-1 correspondence with the cohomology group  $H^2(RP^n; Z)$ . This correspondence is given by  $c_1(f^! \eta) = f^*(c_1 \eta) \in H^2(RP^n; Z)$  where  $c_1$  is the first Chern class. By the assumption  $1 < n \leq 2m$ ,  $H^2(RP^n; Z)$  is isomorphic to  $Z_2$  and hence the set of isomorphism classes of complex line bundles over  $RP^n$  consists of just two elements which are the class of the trivial bundle and that of the non-trivial bundle. The induced complex line bundle  $f^! \eta$  is not trivial because  $f$  is not homotopically trivial.

Let  $w$  denote the total Stiefel-Whitney class, let  $a$  be the generator of  $H^1(RP^n; Z)$  and let  $c$  be the complexification of a real vector bundle. Then we have

$$rc\xi = \xi \oplus \xi,$$

and

$$w(rc\xi) = 1 + a^2.$$

This shows that the complex line bundle  $c\xi$  is not trivial, and it follows that

$$f^! \eta = c\xi.$$

Therefore, we get the relation

$$f^!(r\eta) = rf^! \eta = rc\xi = \xi \oplus \xi,$$

which completes the proof. (Cf. Lemma 7.1 of J. F. Adams [1].)

### § 3. Even order non-singular immersions of $RP^n$ in $CP^m$ .

Let  $f: RP^n \rightarrow CP^m$  ( $1 < n \leq 2m$ ) be a  $p$ th order non-singular immersion with respect to dissections  $\{D^{(q)}\}$  on  $CP^m$  and assume that  $\nu(n, p) - n < 2m < \nu(n, p) + n$ ,  $p > 1$ . Let  $\tau$  and  $\tau_p$  denote the tangent bundle and the  $p$ th order tangent bundle. Let  $\xi$  be the canonical line bundle over  $RP^n$  and  $\eta$  be the canonical complex line bundle over  $CP^m$ . Let  $c$  and  $r$  denote respectively the complexification and realification operator as in Section 2.

PROOF OF THEOREM 1.1. Let  $p$  be even and  $f$  be homotopically non-trivial. We prove first (a) If  $2m \geq \nu(n, p)$ , then we denote the  $p$ th order normal bundle (the cokernel of the  $p$ th order osculating map of  $f$  with respect to dissections  $\{D^{(q)}\}$ ) by  $\mu_p(f)$ . It follows that

$$\mu_p(f) \oplus \tau_p(RP^n) = f^! \tau(CP^m).$$

By taking the Whitney sum with the trivial 2-vector bundle and by making use the relation of H. Suzuki [10],

$$\tau_p(RP^n) \oplus 1 = \binom{n+p}{p} \xi^p,$$

we have

$$\mu_p(f) \oplus \left( \binom{n+p}{p} + 1 \right) = f'(m+1) r \eta.$$

We put  $k = 2m - \nu(n, p)$ . From Lemma 2.1, it follows that

$$(1) \quad \mu_p(f) \oplus \left( \binom{n+p}{p} + 1 \right) = \left( \binom{n+p}{p} + k + 1 \right) \xi.$$

The Stiefel-Whitney classes of  $\mu_p(f)$  are

$$\begin{aligned} w_j(\mu_p(f)) &= w_j \left( \left( \binom{n+p}{p} + k + 1 \right) \xi \right) \\ &= \binom{\left( \binom{n+p}{p} + k + 1 \right)}{j} a^j, \end{aligned}$$

where  $a \in H^1(RP^n; \mathbb{Z}_2)$  is the canonical generator, and they must be zero in dimensions  $j \geq k+1$ . Since we  $a^j \neq 0, 1 \leq j \leq n$ , it follows that

$$\binom{\left( \binom{n+p}{p} + k + 1 \right)}{k+i} \equiv 0 \pmod{2}, \quad i = 1, \dots, n-k.$$

We multiply the both sides of (1) by  $\xi$  and we get

$$(1') \quad \mu_p(f) \otimes \xi \oplus \left( \binom{n+p}{p} + 1 \right) \xi = \left( \binom{n+p}{p} + k + 1 \right) \xi$$

and in  $KO$ -theory we have

$$\mu_p(f) \otimes \xi = - \left( \binom{n+p}{p} + 1 \right) \xi + \left( \binom{n+p}{p} + k + 1 \right) \xi.$$

The Stiefel-Whitney classes of  $\mu_p(f) \otimes \xi$  are

$$w_j(\mu_p(f) \otimes \xi) = \binom{\left( \binom{n+p}{p} + k + 1 \right)}{j} a^j$$

and they must be zero in dimensions  $j \geq k+1$ . Since we have  $a^j \neq 0, 1 \leq j \leq n$ , it follows that

$$\binom{\left( \binom{n+p}{p} + k + 1 \right)}{k+i} \equiv 0 \pmod{2}, \quad i = 1, \dots, n-k.$$

Thus the proof of (a) is completed.

The proof of (b) is similar to that of (a) and is briefly indicated in the following. Let  $p$  be again even and assume that  $k = \nu(n, p) - 2m \geq 0$ . Let  $f$  be homotopically non-trivial. We denote the  $p$ th order conormal bundle (the kernel of the  $p$ th order osculating map of  $f$  with respect to dissections  $\{D^{(a)}\}$ ) by  $\mu'_p(f)$  and we have

$$\mu'_p(f) \oplus f^* \tau(CP^m) = \tau_p(RP^n),$$

and hence by Lemma 2.1,

$$(2) \quad \mu'_p(f) \oplus \left( \binom{n+p}{p} - k + 1 \right) \xi = \left( \binom{n+p}{p} + 1 \right) \xi.$$

By multiplying the both sides of (2) by  $\xi$ , we get

$$(2') \quad \mu'_p(f) \otimes \xi \oplus \left( \binom{n+p}{p} - k + 1 \right) \xi = \left( \binom{n+p}{p} + 1 \right) \xi.$$

From the vanishing of the Stiefel-Whitney classes  $w_j(\mu'_p(f) \otimes \xi)$  for  $j = k+1, \dots, n$ , it follows that

$$\binom{\binom{n+p}{p} + 1}{k+i} \equiv 0 \pmod{2}, \quad i = 1, \dots, n-k.$$

In  $K0$ -theory, we obtain from (2), the relation,

$$\mu'_p(f) = - \left( \binom{n+p}{p} - k + 1 \right) \xi + \left( \binom{n+p}{p} + 1 \right) \xi.$$

From the vanishing the Stiefel-Whitney classes  $w_j(\mu'_p(f))$  for  $j = k+1, \dots, n$ , it follows that

$$\binom{\binom{n+p}{p} + i}{k+i} \equiv 0 \pmod{2}, \quad i = 1, \dots, n-k.$$

Thus the proof of (b) is completed.

PROOF OF THEOREM 1.3. Under the assumption of Theorem 1.3,  $\binom{n+p}{p}$  is divisible by  $2^{l-r}$ . If we put  $k = 2m - \nu(n, p)$ , then we can see easily that

$$\binom{\binom{n+p}{p} + k + 1}{k+1} \not\equiv 0 \pmod{2}.$$

The required result follows immediately from Theorem 1.1.

#### § 4. Relations between $\deg(f)$ and order $p$

Now we consider relations between the homotopy class of the  $p$ th order non-singular immersion  $f: RP^n \rightarrow CP^m$  with respect to dissections  $\{D^{(a)}\}$  on  $CP^m$  and the order  $p$ , and prove Theorems 1.4, 1.5. We use the same notations as in section 3.

PROOF OF THEOREM 1.4. Suppose  $\binom{n+p}{p} \not\equiv n \pmod{2}$ . Since we have  $m \leq \frac{1}{2} \left( \binom{n+p}{p} + n - 3 \right) = m'$  by the assumption of our theorem, we can take the standard inclusion  $i: CP^m \rightarrow CP^{m'}$  which induces the injection of tangent bundles  $\tau(i): \tau(CP^m) \rightarrow \tau(CP^{m'})$ . We denote the composition homomorphism of vector bundles,

$$\tau_p(RP^n) \xrightarrow{D_p \cdot \tau_p(f)} \tau(CP^m) \xrightarrow{\tau(i)} \tau(CP^{m'})$$

by  $H$ .  $D_p \cdot \tau_p(f)$  is an injection by the assumption of our theorem and hence  $H$  is also an injection. Obviously, the map  $H$  covers the  $C^p$ -map  $h = i \cdot f: RP^n \rightarrow CP^m \rightarrow CP^{m'}$ . We denote the cokernel of  $H$  by  $\mu_p(H)$  which is a real vector bundle over  $RP^n$ . By the relation  $\tau_p(RP^n) + 1 = \binom{n+p}{p} \xi^p$  of H. Suzuki [10], we have, in  $K0(RP^n)$ ,

$$\begin{aligned} \mu_p(H) &= h^* \tau(CP^{m'}) - \tau_p(RP^n) \\ &= h^* (m' + 1) r \eta - \binom{n+p}{p} \xi^p - 1. \end{aligned}$$

Let  $d$  be an integer such that  $d \pmod{2} = \deg(f)$ . From Lemma 2.1, it follows that

$$\begin{aligned} \mu_p(H) &= 2(m' + 1) \xi^d - \binom{n+p}{p} \xi^p - 1 \\ &= \left( \binom{n+p}{p} + n - 1 \right) \xi^d - \binom{n+p}{p} \xi^p - 1. \end{aligned}$$

The total Stiefel-Whitney class of  $\mu_p(H)$  is given by

$$(3) \quad w(\mu_p(H)) = (1 + da)^{\binom{n+p}{p} + n - 1} (1 + pa)^{-\binom{n+p}{p}} \pmod{2}.$$

Since the dimension of  $\mu_p(H)$  is  $n - 2$ , it follows, therefore, that

$$w_n(\mu_p(H)) = w_{n-1}(\mu_p(H)) = 0.$$

By the formula (3), we get

$$w_n(\mu_p(H)) = \binom{\binom{n+p}{p} + n - 1}{n} (d - p)^n a^n = 0 \pmod{2}.$$

Since we have  $\binom{n+p}{p} + n - 1 \not\equiv 0 \pmod{2}$  by the assumption of our theorem and  $a^n \neq 0$ , we obtain

$$d \equiv p \pmod{2},$$

that is

$$\deg(f) = p \pmod{2}.$$

By this last relation and the formula (3), we have

$$w(\mu_p(H)) = (1 + da)^{n-1} \pmod{2},$$

and hence

$$w_{n-1}(\mu_p(H)) = d^{n-1} a^{n-1} \pmod{2}.$$

Since we have shown in the above that  $w_{n-1}(\mu_p(H)) = 0$  and have  $a^{n-1} \neq 0$ , it follows that

$$d \equiv 0 \pmod{2},$$

which completes the proof of theorem 1.4.

PROOF OF THEOREM 1.5. Let  $f: RP^n \rightarrow CP^m$  be a  $p$ th order non-singular immersion with respect to dissections  $\{D^{(a)}\}$  on  $CP^m$ , where  $n > 1$ ,  $p \geq 1$  and  $\nu(n, p) \leq 2m \leq \nu(n, p) + n - 1$ . Contrary to the consequence of our theorem, we suppose that  $\deg(f) \neq 0$  and  $p \not\equiv 0 \pmod{2}$ . Let  $\mu_p(f)$  be the  $p$ th order normal bundle of  $f$  and put  $k = 2m - \nu(n, p) \geq 0$ . By the formula  $\tau_p(RP^n) + 1 = \binom{n+p}{p} \xi^p$  and Lemma 2.1, we have, in  $K0(RP^n)$ ,

$$\begin{aligned} \mu_p(f) &= 2(m+1)\xi - \binom{n+p}{p} \xi - 1 \\ &= (k+1)\xi - 1. \end{aligned}$$

It follows that

$$w_{k+1}(\mu_p(f)) = a^{k+1}.$$

Since we have  $k+1 \leq n$  by the assumption of our theorem, we obtain  $w_{k+1}(\mu_p(f)) = a^{k+1} \neq 0$ . But the dimension of  $\mu_p(f)$  is  $k$ . This contradicts to  $w_{k+1}(\mu_p(f)) \neq 0$  and completes the proof of Theorem 1.5.

## § 5. Supplementary results.

1. Let  $f$  be a  $p$ th order non-singular immersion of  $RP^n$  in  $CP^m$  with respect to dissections  $\{D^{(a)}\}$  on  $CP^m$  where  $1 < n \leq 2m$ . We consider the isomorphism classes of the  $p$ th order normal bundle  $\mu_p(f)$  or the  $p$ th order conormal bundle  $\mu'_p(f)$ . The induced bundle  $f^! \tau(CP^m)$  is trivial if we have

$\deg(f)=0$ , and we see that the Whitney sum relations,

$$\mu_p(f) \oplus \tau_p(RP^n) = f^! \tau(CP^m) \cong 2m$$

and

$$\tau_p(RP^n) = \mu'_p(f) \oplus f^! \tau(CP^m) \cong \mu'_p(f) \oplus 2m$$

are just the same as the corresponding formulas for the  $p$ th order non-singular immersions of  $RP^n$  in the euclidean space  $R^{2m}$  with the standard dissections on  $R^{2m}$ . In the latter case, we know many results and they are translated into the former case. We illustrate two theorems below.

**THEOREM. 5.1.** *Let  $n, m$  and  $p$  be integers such that  $p$  is odd positive,  $1 < n \leq m$  and  $\nu(n, p) - n < 2m < \nu(n, p) + n$ . Suppose that there exists a  $p$ th order non-singular immersion  $f: RP^n \rightarrow CP^m$  with respect to dissections  $\{D^{(q)}\}$  such that  $\deg(f) = 0$ . (a) If  $0 \leq 2m - \nu(n, p) = k$ , then we have*

$$\left( \binom{n+p}{p} + k + i - 1 \right) \equiv 0 \pmod{2} \text{ and } \left( \binom{n+p}{p} + k \right) \equiv 0 \pmod{2} \text{ for } 1 \leq i \leq n - k.$$

$$(b) \text{ If } 0 \leq \nu(n, p) - 2m = k, \text{ then we have } \left( \binom{n+p}{p} \right) \equiv 0 \pmod{2} \text{ and } \left( \binom{n+p}{p} + i - 1 \right) \equiv 0 \pmod{2} \text{ for } 1 \leq i \leq n - k.$$

The proof of Theorem 5.1 is easy and omitted.

Let  $\xi$  be the canonical line bundle over  $RP^n$  and  $n \geq k > 0$ ,  $k' \geq 0$  be integers such that  $\binom{k+k'}{k} \not\equiv 0 \pmod{2}$ . According to §4 of T. Kobayashi [7], if  $(k+k')\xi$  has  $k'$  independent cross-sections, then stunted projective spaces  $RP^n/RP^{k-1}$  and  $RP^{n+k'}/RP^{k-1+k'}$  are mod 2 S-related. According to I. M. James [6],  $RP^n/RP^{k-1}$  is S-reducible if and only if  $(n+1) \equiv 0 \pmod{2^{\varphi(n-k)}}$ , where  $\varphi(s)$  is the number of integers  $i$  such that  $0 < i \leq s$  and  $i \equiv 0, 1, 2$  and  $4 \pmod{8}$ . We put  $n+1 = (2b+1)2^{c+4d}$  ( $0 \leq c \leq 3$ ) and  $j(n) = 2^c + 8d$ , where  $b, c$  and  $d$  are positive integers. Theorem 6.1 of T. Kobayashi [7] is interpreted as follows.

**THEOREM 5.2.** *Let  $p$  and  $r$  be integers such that  $p$  is odd positive,  $r > 3$  and  $2^r > p-1$ . If  $n = 2^r - 1$  and  $2m = \nu(n, p) + n - j(n)$ , then  $RP^n$  can not be immersed in  $CP^m$  by a homotopically trivial  $p$ th order non-singular map with respect to dissections  $\{D^{(q)}\}$  on  $CP^m$ .*

2. In general, we denote by  $\varphi(n, k)$  the number of integers  $i$  such that  $k < i \leq n$  and  $i \equiv 0, 1, 2$  and  $4 \pmod{8}$ . By making use of the mod 2 S-relations of stunted projective spaces and Adams operations  $\psi^3: \tilde{K}0(RP^{n+k}/RP^{k'-1+k}) \rightarrow \tilde{K}0(RP^{n+k}/RP^{k'-1+k})$ ,  $\tilde{K}0(S^r(RP^{n+k}/RP^{k'-1+k})) \rightarrow \tilde{K}0(S^r(RP^{n+k}/$

$RP^{k'-1+k})$ ), we have following analogies of Theorems 5.6, 5.8 of §5 [7].

Assume that  $\binom{n+p}{p}_k + k + 1 \not\equiv 0 \pmod{2}$  and  $\binom{n+p}{p} + 1 \equiv 0 \pmod{8}$ . If there exists a homotopically non-trivial  $p$ th order non-singular immersion  $f: RP^n \rightarrow CP^m$  with respect to dissections  $\{D^{(a)}\}$  on  $CP^m$ , where  $p$  is even positive and  $k = 2m - \nu(n, p) \geq 0$  then we have  $\binom{n+p}{p} + 1 \equiv 0 \pmod{2^{\varphi-1}}$ , where  $\varphi = \varphi(n, k-1)$  for  $k \not\equiv 0 \pmod{4}$  and  $\varphi = \varphi(n, k)$  for  $k \equiv 0 \pmod{4}$ .

Assume that  $\binom{n+p}{p}_k + 1 \not\equiv 0 \pmod{2}$  and  $\binom{n+p}{p} + 1 - k \equiv 0 \pmod{8}$ . If there exists a homotopically non-trivial  $p$ th order non-singular immersion  $f: RP^n \rightarrow CP^m$  with respect to dissections  $\{D^{(a)}\}$  on  $CP^m$ , where  $p$  is even positive and  $k = \nu(n, p) - 2m \geq 0$  then we have  $\binom{n+p}{p} + 1 - k \equiv 0 \pmod{2^{\varphi-1}}$ .

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