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# On a piecewise linear link isotopy group

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Kazuaki KOBAYASHI

## §1. Introduction

Throughout this paper we shall only be concerned with the piecewise linear category of polyhedra and piecewise linear maps (for the PL-category, [see 12]). The purpose of the paper makes a study of a group of order preserving ambient isotopy classes of $(n, k, \mu)$-*links* $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ in the metastable range.

In §2 we shall define a group $SI(n, k, \mu)$ of $(n, k, \mu)$-*link* isotopy classes in the strong sense under $n-k\geqq 3$ (Cor. for Th. 1) and we will introduce a concept of cobordism between $(n, k, \mu)$-*link* isotopy classes. Their cobordism classes form an abelian group $\mathscr{L}(n, k, \mu)$ provided $n-k\geq 3$ (Th. 1). In §3 we will introduce a homotopy on the group $SI(n, k, \mu)$. Then their homotopy classes form an abelian group $\mathscr{H}(n, k, \mu)$ (Th. 2). In §4 we introduce a group $LM(n, k, \mu)$ as follows, called the $(n, k, \mu)$-*link matrix group*. Let $LM(n, k, \mu)$ be a set of $(\mu\times\mu)$-matrixes of the form

$$\begin{bmatrix} e & \lambda_2^1 & \lambda_3^1 & \cdots & \lambda_\mu^1 \\ \lambda_1^2 & e & \lambda_3^2 & \cdots & \\ \cdots & & \ddots & \cdots & \\ \lambda_1^\mu & \lambda_2^\mu & \cdots & \lambda_{\mu-1}^\mu & e \end{bmatrix}$$

satisfying that 1) $\lambda_j^i$ is an element of $\pi_k(S^{n-k-1})$

2) $e$ is the identity element of $\pi_k(S^{n-k-1})$ and

3) $\lambda_j^i=(-1)^{n-k}\lambda_i^j$

Then $LM(n, k, \mu)\,(n\geqq k+3)$ becomes an abelian group under the usual sum operation of matrixes. We will study relations between $SI(n, k, \mu)$, $\mathscr{L}(n, k, \mu)$, $\mathscr{H}(n, k, \mu)$ and $LM(n, k, \mu)$ under some conditions.

$SI(n, k, \mu)\cong\mathscr{L}(n, k, \mu)$ provided $n-k\geq 3$ (Th. 4): $\mathscr{L}(n, k, \mu)\cong\mathscr{H}(n, k, \mu)$ provided $2n\geq 3k+4$ (Th. 5 and 6). Hence we shall obtain the following main result of the paper.

THEOREM 7. *$SI(n, k, \mu)$ is isomorphic to $LM(n, k, \mu)$ provided $2n\geq 3k+4$.*

## § 2. The strong link isotopy group and the link cobordism group.

In the paper a manifold, say $M$, is an orientable combinatorial manifold. Then $\partial M$ and $IntM$ stand for its boundary and its interior.

Let $S^n$, $D^n$ be standard $n$-dimensional combinatorial sphere and disk and $\overset{\mu}{\underset{j=1}{\cup}} S^k_j$, $\overset{\mu}{\underset{j=1}{\cup}} D^k_j$ be disjoint unions of $\mu$-spheres and disks of dimension $k$. Let $I=[-1, 1]$, $I_0=[0, 1]$ and $\Delta_k$ be a $k$-simplex. We shall always assign an orientation to a manifold and we shall consider a manifold pair $V=(M, N)$ (where the inclusion map $N\subset M$ is proper in the sense of Zeeman). Homeomorphisms will be orientation preserving unless otherwise stated. Set theoretic notations such as $\subset$, $=$, $\sigma$, Int when applied to oriented objects respect orientations.

DEFINITION 1. For the sake of convenience, a locally flat $(n, \mathrm{k})$-sphere pair $L=(S^n, \overset{\mu}{\underset{j=1}{\cup}} S^k_j)$ is called an *$(n, k, \mu)$-link* and a locally flat $(n, k)$-disk pair $B=(D^n, \overset{\mu}{\underset{j=1}{\cup}} D^k_j)$ is called an *$(n, k, \mu)$-braid* for $k\geqq 0$. As we consider links and braids, it is assumed that $n-k\geqq 2$ throughout this paper without mentions. If $B=(D^{n+1}, \overset{\mu}{\underset{j=1}{\cup}} D^{k+1}_j)$ is an *$(n+1, k+1, \mu)$-braid,* then it is obvious that $\partial B=\partial(D^{n+1}, \overset{\mu}{\underset{j=1}{\cup}} D^{k+1}_j)=(\partial D^{n+1}, \overset{\mu}{\underset{j=1}{\cup}} \partial D^{k+1}_j)$ is an *$(n, k, \mu)$-link.* When $\partial B=L$, we always assume $\partial D^k_j=S^k_j$, $j=1, 2, \cdots, \mu$.

DEFINITION 2. Let $L_i=(S^n, \overset{\mu}{\underset{j=1}{\cup}} S^k_{ij})$ be an *$(n, k, \mu)$-link,* $i=1, 2$. We say $L_1$ *isotopic to* $L_2$ if there is a level preserving homeomorphism, that is, an isotopy $H$ of $S^n\times I_0$ onto $S^n\times I_0$ satisfying

1) $H(x, 0)=(x, 0), x \varepsilon S^n$ and

2) $H(\overset{\mu}{\underset{j=1}{\cup}} S^k_{1j}, 1)=(\overset{\mu}{\underset{j=1}{\cup}} S^k_{2j}, 1)$, and written $L_1\approx L_2$.

The set of *$(n, k, \mu)$-links* is classified by this equivalence relation into classes which are called *$(n, k, \mu)$-link isotopy* classes. We denote the set of these classes $Iso(n, k, \mu)$.

We can define the same thing for braids and denote the set of $(n, k, \mu)$-*braid isotopy classes* $BIso(n, k, \mu)$.

DEFINITION 3. Let $L_1\approx L_2$. We say $L_1$ strong isotopic to $L_2$, written $L_1\overset{s}{\approx} L_2$, if the isotopy $H$ of Definition 2 satisfies the following condition 2)′ instead of the above 2);

2)′ $H(S^k_{1j}, 1)=(S^k_{2j}, 1)$, $j=1, 2, \cdots, \mu$.

The set of $(n, k, \mu)$-*links* is classified by this equivalence relation into classes which are called $(n, k, \mu)$-*link isotopy classes in the strong sense.* We denote the set of these classes $SI(n, k, \mu)$. We can define the same thing for braids and denote the set of $(n, k, \mu)$-*braid isotopy classes in the strong sense* $BSI(n, k, \mu)$.

DEFINITION 4. Let $L_1=(S^n, \bigcup_{j=1}^{\mu} S_{ij}^k)$ an $(n, k, \mu)$-*link*, $i=1, 2,.$

We say that $L_1$ is *equivalent to* $L_2$ if there is an orientation preserving homeomorphism $G$ of $S^n$ onto itself satisfying the conditions ;

1) $G(\bigcup_{j=1}^{\mu} S_{1j}^k)=\bigcup_{j=1}^{\mu} S_{2j}^k$

2) $G|\bigcup_{j=1}^{\mu} S_{1j}^k$ is orientation preserving.

Throughout this paper we will always assume that $G|\bigcup_{j=1}^{\mu} S_{1j}^k$ is orientation preserving (or reversing) provided that $G$ is orientation preserving (or reversing).

We can define the same thing for braids. The sets of $(n, k, \mu)$-*links* and *braids* are classified by this equivalence relation into classes which are called $(n, k, \mu)$-*link types* and *braid tpyes* respectively.

REMARK 1. Two links $L_1$ and $L_2$ belong to the same link type if and only if they belong to the same isotopy class by [2].

DEFINITION 5. If two links (or braids) say $L_1$ and $L_2$, belong to the same link type and if the homeomorphism $G$ of Definition 4 satisfies the following conslition 1)′ instead of 1) ;

1)′ $G(S_{1j}^k)=S_{2j}^k$, $j=1, 2, \cdots, \mu$,

we say $L_1$ *to be equivalent to* $L_2$ *in the strong sense.* The sets of $(n, k, \mu)$-*links* and *braids* are classified by this equivalence relation into classes which are called *strong* $(n, k, \mu)$-*link types* and *strong braid types* respectively.

REMARK 2. Two links belong to the same strong link type if and only if they belong to the same strong isotopy class by [2].

DEFINITION 6. The *standard* $(n, k, \mu)$-*pair* is the pair $\square=(D^n, \bigcup_{j=1}^{\mu} D_j^k)$ such that there is a commutative diagram

$$\begin{tikzcd} \bigcup_{j=1}^{\mu} D_j^k \arrow[r, "\beta"] \arrow[d, "\subset"] & \Delta^k \times I \arrow[d, "\times 0"] \\ D^n & (\Delta^k \times I) \times I^{n-k-1} \arrow[l, "s"] \end{tikzcd}$$

where $\beta$ is the proper embedding satisfying $\beta(D_j^k)=\Delta^k\times\left\{-1+\frac{2j}{\mu+1}\right\}$, $j=1, 2, \cdots, \mu$ and $s$ is a canonical homeomorphism.

An $(n, k, \mu)$-*link* $L$ is *trivial* if $L$ is equivalent (or equivalently equivalent in the strong sense) to the boundary of the standard $(n+1, k+1, \mu)$-*pair* and an $(n, k, \mu)$-*braid* $B$ is *trivial* if $B$ is equivalent (or equivalent in the strong sense) to the standard $(n, k, \mu)$-*pair*. We shall denote both the trivial $(n, k, \mu)$-*link type* and *braid type* by 0. Let $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ be an $(n, k, \mu)$-*link* and $B=(D^n, \bigcup_{j=1}^{\mu} D_j^k)$ be an $(n, k, \mu)$-*braid*. If $D_j^k\subset S_j^k$, $j=1, 2, \cdots, \mu$, $D^n\subset S^n$ and $D^n\cap S_j^k=D_j^k$, $j=1, 2, \cdots, \mu$, then we call $B$ is contained in $L$, denoted by $B\supset L$.

Similarly if $\tilde{B}=(\tilde{D}^n, \bigcup_{j=1}^{\mu} \tilde{D}_j^k)$ is another $(n, k, \mu)$-*braid*, then we say $B$ is contained in $\tilde{B}\,(B\subset\tilde{B})$ if $D_j^k\subset\operatorname{Int}\tilde{D}_j^k$, $j=1, 2, \cdots, \mu$, $D^n\subset\operatorname{Int}\tilde{D}^n$ and $D^n\cap\tilde{D}_j^k=D_j^k$, $j=1, 2, \cdots, \mu$. Let $L_i=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ be $(n, k, \mu)$-*links*, $i=1, 2$ and $B_3=(D^n, \bigcup_{j=1}^{\mu} D_{3j}^k)$ be an $(n, k, \mu)$-*braid*. We will denote $B_3=L_1\cap L_2$ if $D^n\subset S^n$, $S_{1j}^k\cap S_{2j}^k=D_{3j}^k$, $j=1, 2, \cdots, \mu$, $S_{1l}^k\cap S_{2j}^k=\phi$ $(j\neq l)$, $D^n\cap S_{1j}^k=D^n\cap S_{2j}^k=D_{3j}^k$ and $\bigcup_{j=1}^{\mu} S_{1j}^k$ induce orientations on $\bigcup_{j=1}^{\mu} D_{3j}^k$ which are opposite to one induced from $\bigcup_{j=1}^{\mu} S_{2j}^k$.

REMARK 3. If $B=(D^n, \bigcup_{j=1}^{\mu} D_j^k)$ is a trivial $(n, k, \mu)$-*braid*, then there exist an orientation preserving homeomorphism $H$ of $D^n$ and a commutative diagram same as Definition 6;

$$\begin{array}{ccccc}
\bigcup_{j=1}^{\mu} D_j^k & \xrightarrow{H} & \bigcup_{j=1}^{\mu} \tilde{D}_j^k & \xrightarrow{\beta} & \Delta^k\times I \\
\downarrow\subset & & \downarrow\subset & & \downarrow\times 0 \\
D^n & \xrightarrow{H} & D^n & \xleftarrow{s} & (\Delta^k\times I)\times I^{n-k-1}
\end{array}$$

where $(D^n, \bigcup_{j=1}^{\mu} \tilde{D}_j^k)$ is a standard pair. Joining $\beta(\bigcup_{j=1}^{\mu} \tilde{D}_j^k)$ each other by simple arc $\hat{\Delta}^k\times\left[-1+\frac{2}{\mu+1}, 1-\frac{2}{\mu+1}\right]$ in $\Delta^k\times I$ where $\hat{\Delta}^k$ is the barycenter of $\Delta^k$ and let the resultant complex be $X$. Then $\Delta^k\times I\searrow X$ and hence $(\Delta^k\times I)\times I^{n-k-1}\searrow X\times 0$. Hence $D^n$ is a regular neighborhood of $H^{-1}s(X\times 0)$.

LEMMA 0. *If $L=(S^n, \bigcup_{j=1}^{\mu} S_{1j}^k)$ and $B=(D^n, \bigcup_{j=1}^{\mu} D_j^k)$ are both trivial $(n, k, \mu)$-link and braid respectively such that $B\subset L$, then the complementary braid*

$\overline{L-B}=(S^n-\operatorname{Int} D^n, \bigcup_{j=1}^{\mu}(S_j^k-\operatorname{Int} D_j^k))$ *is also a trivial* $(n, k, \mu)$*-braid.*

Proof. Let $\square=(\tilde{D}^{n+1}, \bigcup_{j=1}^{\mu}\tilde{D}_j^{k+1})$ be a standard $(n+1, k+1, \mu)$*-pair* such that

$$\begin{tikzcd} \bigcup_{j=1}^{\mu}\tilde{D}_j^{k+1} \arrow[r, "\beta"] \arrow[d, "\subset"] & \Delta^{k+1}\times I \arrow[d, "\times 0"] \\ \tilde{D}^{n+1} & (\Delta^{k+1}\times I)\times I^{n-k-1} \arrow[l, "s"'] \end{tikzcd}$$

is commutative. Since $L$ is a trivial link, there exists an orientation preserving homeomorphism $H$ of $S^n$ onto $\partial\tilde{D}^{n+1}$ such that $H(S_j^k)=\partial\tilde{D}_j^{k+1}$, $j=1, 2, \cdots, \mu$.

Furthermore we may suppose that $\beta H|\bigcup_{j=1}^{\mu} D_j^k$ is properly embedded in $\Delta^k\times I$ where $\Delta^k$ is a face of $\partial\Delta^{k+1}$. We construct a complex $X$ in $\Delta^k\times I$ same as Remark 3. Since $B$ is a trivial braid, using Remark 3 $D^n$ and $H^{-1}s((\Delta^k\times I)\times I^{n-k-1})$ are regular neighborhoods of $H^{-1}s(X\times 0)$ mod $\bigcup_{j=1}^{\mu}(S_j^k-\operatorname{Int} D_j^k)$ in $S^n$. Then there is an ambient isotopy of $S^n$ moving $D^n$ onto $H^{-1}s(\Delta^k\times I)\times I^{n-k-1})$ keeping $H^{-1}s(X\times 0)\cup\bigcup_{j=1}^{\mu}(S_j^k-\operatorname{Int} D_j^k)$ fixed [6, Th. 1]. The end of isotopy throws $(S^n-\operatorname{Int} D^n, \bigcup_{j=1}^{\mu}(S_j^k-\operatorname{Int} D_j^k))$ onto $H^{-1}s(\beta(\bigcup_{j=1}^{\mu}\partial\tilde{\Delta}_j^{k+1})\times I^{n-k-1}, \beta(\bigcup_{j=1}^{\mu}\partial\tilde{\Delta}_j^{k+1})\times 0)$, which is trivial, where $\partial\tilde{\Delta}_j^{k+1}=\overline{\partial\Delta_j^{k+1}-H(D_j^k)}$. Hence $\overline{L-B}$ is trivial.

Lemma 1. *Let* $B_i=(D_i^n, \bigcup_{j=1}^{\mu}D_{ij}^k)$, $i=1, 2$ *be trivial* $(n, k, \mu)$*-braids. If* $B_3=B_1\cap B_2=\partial B_1\cap\partial B_2=(D_3^{n-1}, \bigcup_{j=1}^{\mu}D_{3j}^{k-1})$ *is a trivial* $(n-1, k-1, \mu)$*-braid such that* $D_1^n\cap D_2^n=\partial D_1^n\cap\partial D_2^n=D_3^{n-1}$, $D_{1j}^k\cap D_{2j}^k=\partial D_{1j}^k\cap\partial D_{2j}^k=D_{3j}^{k-1}$, $j=1, 2, \cdots, \mu$, *then* $B_1\cup B_2$ *is also a trivial* $(n, k, \mu)$*-braid.*

Proof. Since $B_i$, $i=1, 2, 3$ are trivial braids, there are homeomorphisms $H_i$, $i=1, 2, 3$ of $D_i$ and commutative diagrams

$$\begin{tikzcd} \bigcup_{j=1}^{\mu}D_{ij}^k \arrow[r, "H_i"] \arrow[d, "\subset"] & \bigcup_{j=1}^{\mu}\tilde{D}_{ij}^k \arrow[d, "\subset"] \\ D_i^n \arrow[r, "H_i"] & \tilde{D}_i^n \end{tikzcd} \quad i=1, 2 \text{ and } \quad \begin{tikzcd} \bigcup_{j=1}^{\mu}D_{3j}^{k-1} \arrow[r, "H_3"] \arrow[d, "\subset"] & \bigcup_{j=1}^{\mu}\tilde{D}_{3j}^{k-1} \arrow[d, "\subset"] \\ D_3^{n-1} \arrow[r, "H_3"] & \tilde{D}_3^{n-1} \end{tikzcd}$$

where $(\tilde{D}_i^n, \bigcup_{j=1}^{\mu}\tilde{D}_{ij}^k)$ $i=1, 2$ and $(\tilde{D}_3^{n-1}, \bigcup_{j=1}^{\mu}\tilde{D}_{3j}^{k-1})$ are standard pairs.

We may suppose that $\tilde{D}_3^{n-1}=\partial\tilde{D}_1^n\cap\partial\tilde{D}_2^n$ and $D_{3j}^{k-1}=\partial\tilde{D}_{1j}^k\cap\partial\tilde{D}_{2j}^k$, $j=1,2,\cdots,\mu$ and $H_3=H_1\cap H_2$. Hence we obtain the following commutative diagram,

$$(1)\qquad \begin{tikzcd}[column sep=large]
\bigcup\limits_{j=1}^{\mu}(D_{1j}^k \underset{D_{3j}^{k-1}}{\cup} D_{2j}^k) \arrow[r, "H_1\cup H_2"] \arrow[d, "\subset"] & \bigcup\limits_{j=1}^{\mu}(\tilde{D}_{1j}^k \underset{D_{3j}^{k-1}}{\cup} \tilde{D}_{2j}^k) \arrow[r, "\beta_1\cup\beta_2"] \arrow[d, "\subset"] & (\varDelta_1^k\times I)\cup(\varDelta_2^k\times I) \arrow[d, "\times 0"] \\
D^n \underset{D_3^{n-1}}{\cup} D_2^n \arrow[r, "H_1\cup H_2"] & \tilde{D}_1^n \underset{D_3^{n-1}}{\cup} \tilde{D}_2^n & (\varDelta_1^k\times I)\times I^{n-k-1}\cup(\varDelta_2^k\times I)\times I^{n-k-1}. \arrow[l, "s_1\cup s_2"']
\end{tikzcd}$$

If $\varDelta_1^k\cap\varDelta_2^k$ is a common $(k-1)$-*face* of the boundaries, there is a homeomorphism $\varDelta_1^k\cup\varDelta_2^k\longrightarrow\varDelta^k$, [11. Lemma 1].
So there is a commutative diagram

$$(2)\qquad \begin{tikzcd}[column sep=huge]
(\varDelta_1^k\times I)\cup(\varDelta_2^k\times I) \arrow[r, "a\times 1"] \arrow[d, "\times 0"] & \varDelta^k\times I \arrow[d, "\times 0"] \\
(\varDelta_1^k\times I)\times I^{n-k-1}\cup(\varDelta_2^k\times I)\times I^{n-k-1} & (\varDelta^k\times I)\times I^{n-k-1} \arrow[l, "(a^{-1}\times 1)\times 1"]
\end{tikzcd}$$

where $a:\varDelta_1^k\cup\varDelta_2^k\longrightarrow\varDelta^k$ is a natural homeomorphism and where 1 is the identity homeomorphism. From (1) and (2), $B_1\cup B_2$ is a trivial $(n, k, \mu)$-*braid*, complete the proof.

LEMMA 2. *Let* $B_i=(D_i^n, \bigcup\limits_{j=1}^{\mu}D_{ij}^k)$ *be* $(n, k, \mu)$-*braids such that* $B_1\cap B_2=\partial B_1\cap\partial B_2=(D_3^{n-1}, \bigcup\limits_{j=1}^{\mu}D_{3j}^{k-1})$ *is a trivial* $(n-1, k-1, \mu)$-*braid. If* $B_1$ *is trivial, then* $B_1\cup B_2$ *is equivalent to* $B_2$.

PROOF. Using [11] we may suppose that the homeomorphism $a$ of Lemma 1 is satisfying the condition

1) $a:\varDelta_1^k\cup\varDelta_2^k\longrightarrow\varDelta_2^k$
2) $a|\partial\varDelta_2^k-\mathrm{Int}\,(\varDelta_1^k\cap\varDelta_2^k)=$ *identity*.

Then by Lemma 1 we obtain the commutative diagram,

$$\begin{tikzcd}[column sep=large]
\bigcup\limits_{j=1}^{\mu}D_{2j}^k \arrow[r, "H_2"] \arrow[ddd, bend right=60, "\subset"'] & \bigcup\limits_{j=1}^{\mu}\tilde{D}_{2j}^k \arrow[r, "\beta_2"] & \varDelta_2^k\times I \\
\bigcup\limits_{j=1}^{\mu}(D_{1j}^k\cup D_{2j}^k) \arrow[r, "H_1\cup H_2"] & \bigcup\limits_{j=1}^{\mu}(\tilde{D}_{1j}^k\cup\tilde{D}_{2j}^k) \arrow[r, "\beta_1\cup\beta_2"] \arrow[d, "\subset"] & (\varDelta_1^k\times I)\cup(\varDelta_2^k\times I) \arrow[u, "a\times 1"'] \arrow[d, "\times 0"] \\
D_1^n\cup D_2^n \arrow[r, "H_1\cup H_2"] & \tilde{D}_1^n\cup\tilde{D}_2^n & (\varDelta_1^k\times I)\times I^{n-k-1}\cup(\varDelta_2^k\times I)\times I^{n-k-1} \arrow[l, "s_1\cup s_2"'] \arrow[d, "(a\times 1)\times 1"] \\
D_2^n \arrow[r, "H_2"] & \tilde{D}_2^n & (\varDelta_2^k\times I)\times I^{n-k-1} \arrow[l, "s_2"']
\end{tikzcd}$$

Hence we may define $H=H_2^{-1}s_2(a\times 1\times 1)(s_1\cup s_2)^{-1}(H_1\cup H_2)$.

Let $B=(D^n, \overset{\mu}{\underset{j=1}{\cup}} D_j^k)$ be an $(n, k, \mu)$-braid and $\bar{B}=(D^{n-1}, \overset{\mu}{\underset{j=1}{\cup}} D_j^{n-1})$ be a trivial $(n-1, k-1, \mu)$-*braid* such that $\bar{B}\subset\partial B$. Since $\bar{B}$ is a trivial braid there exist a homeomorphism $H$ of $D^{n-1}$ onto $D^{n-1}$ and a commutative diagram

$$\begin{tikzcd} \overset{\mu}{\underset{j=1}{\cup}} D_j^{k-1} \arrow[r, "H"] \arrow[d, "\subset"] & \overset{\mu}{\underset{j=1}{\cup}} \tilde{D}_j^{k-1} \arrow[r, "\bar{\beta}"] \arrow[d, "\subset"] & \Delta^{k-1}\times I \arrow[d, "\times 0"] \\ D^{n-1} \arrow[r, "H"] & \tilde{D}^{n-1} \arrow[r, "\bar{s}"] & (\Delta^{k-1}\times I)\times I^{n-k-1} \end{tikzcd}$$

We construct a complex $X$ in $\Delta^{k-1}\times I$ defined as Remark 3.

Let $N$ be the second derived neighborhood of $H^{-1}s(X\times 0)\cup\overset{\mu}{\underset{j=1}{\cup}} D_j^k$ in $D^n$. Since the complex $H^{-1}s(X\times 0)\cup\overset{\mu}{\underset{j=1}{\cup}} D_j^k$ is collapsible, $N$ is an $n$-ball [12. chap. 3]. And the inclusion $\overset{\mu}{\underset{j=1}{\cup}} D_j^k\subset N$ is proper.

LEMMA 3. *The braid* $C=(N, \overset{\mu}{\underset{j=1}{\cup}} D_j^k)$ *is trivial.*

PROOF. Let $N_j$ be the second derived neighborhood of $D_j^k$ in $D^n$, $j=1, 2, \cdots, \mu$. Then $N_j\subset N$ and $N_j\cap\overset{\mu}{\underset{j=1}{\cup}} D_j^k=D_j^k$. If $D^n\searrow D_0^n$ and $(D_0^n, D^k)$ is an unknotted ball pair for any $n$-balls $D^n, D_0^n$ and $k$-ball $D^k$, then $(D^n, D^k)$ is also an unknotted ball pair (see [5. Cor. 10]).

Since $N\searrow N_j$ and $(N_j, D_j^k)$ is an unknotted ball pair, $j=1, 2, \cdots, \mu$ [5, Cor. 10], $(N_j, D_j^k)$ is an unknotted ball pair for $j=1, 2, \cdots, \mu$.

And since it is obviously that $\overset{\mu}{\underset{j=1}{\cup}} D_j^k$ is split each other in $N$, $C=(N, \overset{\mu}{\underset{j=1}{\cup}} D_j^k)$ is a trivial braid.

LEMMA 4. *Let* $B_i=(D_i^n, \overset{\mu}{\underset{j=1}{\cup}} D_{ij}^k)$ *be* $(n, k, \mu)$-*braids*, $i=1, 2$, *and* $B_3=(D_3^{n-1}, \overset{\mu}{\underset{j=1}{\cup}} D_{3j}^{k-1})=B_1\cap B_2=\partial B_1\cap\partial B_2$ *be a trivial* $(n-1, k-1, \mu)$-*braid. If* $B_1$ *is trivial, then* $B_1\cup B_2$ *is equivalent to* $B_2$.

PROOF. Let $\square=(\tilde{D}^{n-1}, \overset{\mu}{\underset{j=1}{\cup}} \tilde{D}_j^{k-1})$ be a standard $(n-1, k-1, \mu)$-pair. Since $B_3$ is trivial there exist a homeomorphism $H: D_3^{n-1}\longrightarrow\tilde{D}^{n-1}$ and a commutative diagram,

$$\begin{tikzcd} \overset{\mu}{\underset{j=1}{\cup}} D_{3j}^{k-1} \arrow[r, "H"] \arrow[d, "\subset"] & \overset{\mu}{\underset{j=1}{\cup}} \tilde{D}_j^{k-1} \arrow[r, "\beta"] \arrow[d, "\subset"] & \Delta^{k-1}\times I \arrow[d, "\times 0"] \\ D_3^{n-1} \arrow[r, "H"] & \tilde{D}^{n-1} & (\Delta^{k-1}\times I)\times I^{n-k-1}. \arrow[l, "s"'] \end{tikzcd}$$

Let $X$ be a complex in $\Delta^{k-1}\times I$ defined as Remark 3.

Then $(\Delta^{k-1}\times I)\times I^{n-k-1}\searrow\Delta^{k-1}\times I\searrow X$. Hence $D_3^{n-1}\cup\bigcup_{j=1}^{\mu}D_{2j}^k\searrow H^{-1}s(X\times 0)\cup\bigcup_{j=1}^{\mu}D_{2j}^k$. Let $N$ be a regular neighborhood of $D_3^{n-1}\cup\bigcup_{j=1}^{\mu}D_{2j}^k$ in $D_2^n$. Then $N$ collapses to $X\cup\bigcup_{j=1}^{\mu}D_{2j}^k$. Hence $B=(N,\bigcup_{j=1}^{\mu}D_{2j}^k)$ is a trivial $(n,k,\mu)$-*braid* by Lemma 3. Then by Lemma 2, $B_1\cup B$ is equivalent to $B$ by homeomorphisms $G:D_1^n\cup N\longrightarrow N$ such that $G(D_{1j}^k\cup D_{2j}^k)=D_{2j}^k$, $j=1,2,\cdots,\mu$ and $G|\partial N-\operatorname{Int}D_3^{n-1}=identity$. Hence $B_1\cup B_2$ is equivalent to $B_2$ by extending to the homeomorphism $G:D_1^n\cup D_2^n\longrightarrow D_2^n$ by taking the identity on $D_2^n-\operatorname{Int}N$, completing the proof.

LEMMA 5. *Let* $L=(S^n,\bigcup_{j=1}^{\mu}S_j^k)$ *be an* $(n,k,\mu)$*-link and* $B_i=(D_i^n,\bigcup_{j=1}^{\mu}D_{ij}^k)$, $i=1,2,\cdots,m$ *be trivial* $(n,k,\mu)$*-braids. If each* $B_i$ *is contained in* $L$ *and disjoint from each other (that is* $D_{ij}^k\cap D_{pq}^k=\phi$, $D_i^n\cap D_p^n=\phi$ *for* $i\neq p$*) then there is a trivial* $(n,k,\mu)$*-braid* $B=(D^n,\bigcup_{j=1}^{\mu}D_j^k)$ *such that* $B_i\subset B\subset L$ *for each* $B_i$.

PROOF. Since $B_1,\cdots,B_m$ are disjoint from each other, there is a trivial $(n,k,\mu)$-*braid* $B_0\subset L$ such that $B_0,B_1,\cdots,B_m$ are disjoint from each other. Let $B_{m+1}=L-\operatorname{Int}B_0$, then $B_{m+1}$ is a locally flat pair. Since $B_0$ is the trivial braid, $\partial B_0=\partial B_{m+1}$ is a trivial $(n-1,k-1,\mu)$-*link*. We take a trivial $(n-1,k-1,\mu)$-*braid* $\tilde{B}=(D^{n-1},\bigcup_{j=1}^{\mu}D_j^{k-1})$ in $\partial B_0$ such that $D_j^{k-1}\subset\partial D_{0j}^k$, $j=1,2,\cdots,\mu$, $D^{n-1}\subset\partial D_0^n$. Then there are a homeomorphism $H$ of $D^n$ onto $\tilde{D}^n$ and a commutative diagram,

$$\begin{tikzcd}
\bigcup_{j=1}^{\mu}D_j^{k-1} \arrow[r, "H"] \arrow[d, "\subset"] & \bigcup_{j=1}^{\mu}\tilde{D}_j^{k-1} \arrow[r, "\beta"] \arrow[d, "\subset"] & \Delta^{k-1}\times I \arrow[d, "\times 0"] \\
D^{n-1} \arrow[r, "H"] & D^{n-1} & (\Delta^k\times I)\times I^{n-k-1} \arrow[l, "s"']
\end{tikzcd}$$

We construct a complex $X$ in $\Delta^{k-1}\times I$ as Remark 3 and let $N$ be a regular neighborhood of $H^{-1}s(X\times 0)\cup\bigcup_{j=1}^{\mu}D_{m+1j}^k$ in $D_{m+1}^n$.

Then the braid $B=(N,\bigcup_{j=1}^{\mu}D_{m+1j}^k)$ is a trivial $(n,k,\mu)$-*braid* by Lemma 3 which is the required braid.

LEMMA 6. *Let* $B_i$ *be trivial* $(n,k,\mu)$*-braids,* $i=0,1,2$, *such that* $B_1,B_2\subset B_0$, $B_i=(D_i^n,\bigcup_{j=1}^{\mu}D_{ij}^k)$. *Then there is an automorphism* $G$ *of* $D_0^n$ *such that* $G(D_1^n)=D_2^n$, $G(D_{0j}^k)$ *and* $G(D_{1j}^k)=D_{2j}^k$. *Furthermore* $G$ *is isotopic to the identity* *with an isotopy* $G_t$, $t\varepsilon[0, 1]$ *which keep* $\partial D_0^n$, $\bigcup_{j=1}^{\mu} \partial D_{0j}^k$ *fixed.*

PROOF. Since $B_0$ is trivial, there is a commutative diagram,

$$\begin{array}{ccccc} \bigcup_{j=1}^{\mu} D_{0j}^k & \xrightarrow{H} & \bigcup_{j=1}^{\mu} \tilde{D}_j^k & \xrightarrow{\beta} & \Delta^k \times I \\ \downarrow\subset & & \downarrow\subset & & \downarrow\times 0 \\ D_0^n & \xrightarrow{H} & \tilde{D}^n & \xleftarrow{s} & (\Delta^k \times I) \times I^{n-k-1}. \end{array}$$

Let $\Delta_j^{k+1}(\varepsilon)=\Delta^k \times\left[-1+\frac{2j}{\mu+1}-\varepsilon,\ -1+\frac{2j}{\mu+1}+\varepsilon\right]\subset\Delta^k\times I$, $j=1, 2, \cdots, \mu$, $0<\varepsilon \leq\frac{2}{3(\mu+1)}$. Since $\beta H(D_{ij}^k)\subset\beta(\operatorname{Int}\tilde{D}_j^k)\subset\operatorname{Int}\Delta_j^{k+1}(\varepsilon)$, $i=1, 2$, there is an automorphism $k_j$ of $\beta(D_j^k)$ such that $k_j\beta H(D_{1j}^k)=\beta H(D_{2j}^k)$ and $k_j$ is isotopic to the identity which keeps $\beta(\partial\tilde{D}_j^k)$ fixed, by [2. (I). Th. 3].

Then $k_j$ can obviously be extended to an automorphism $K_j$ over $\Delta_j^{k+1}(\varepsilon)$ so that $K_j$ is isotopic to the identity by an isotopy which keeps $\partial\Delta_j^{k+1}(\varepsilon)$ fixed. Hence there exists an automorphism $\tilde{K}$ over $\Delta^k\times I$ so that $\tilde{K}$ is isotopic to the identity by an isotopy which keeps $(\Delta_k\times I-\bigcup_{j=1}^{\mu}\operatorname{Int}\Delta_j^{k+1}(\varepsilon))$ fixed and $K|\operatorname{Int}\Delta_j^{k+1}(\varepsilon)=K_j$.

We can extend $K$ to an automorphism $\tilde{G}$ over $(\Delta^k\times I)\times I^{n-k-1}$ by $\tilde{G}(x, t) =(K(x), t)$ for $x\varepsilon\Delta^k\times I$, $t\varepsilon I^{n-k-1}$ which is isotopic to the identity and keeps $(\Delta^k\times I-\bigcup_{j=1}^{\mu}\operatorname{Int}\Delta_j^{k+1}(\varepsilon))\times I^{n-k-1}$ fixed. Let $G=H^{-1}s\tilde{G}$. Then $G$ is the required automorphism which is isopopic to the identity.

LEMMA 7. *Let* $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ *be an* $(n, k, \mu)$*-link and* $B_1, B_2$ *be trivial* $(n, k, \mu)$*-braids sucn that* $B_1, B_2\subset L$, $B_i=(D_i^n, \bigcup_{j=1}^{\mu} D_{ij}^k)$, $i=1, 2$. *Then there is an automorphism* $G: S^n\longrightarrow S^n$ *such that* $G(D_1^n)=D_2^n$ *and* $G(D_{1j}^k)=D_{2j}^k$, $j=1, 2, \cdots, \mu$. *Furthermore* $G$ *is isotopic to the identity by an isotopy* $G_t$.

PROOF. We may assume that $B_1$ and $B_2$ are disjoint, otherwise, using Lemma 5 and 6, we make them disjoint. Then by Lemma 5 there is a trivial $(n, k, \mu)$-*braid* $B_0=(D_0^n, \bigcup_{j=1}^{\mu} D_{0j}^k)\subset L$ such that $B_1, B_2\subset B_0$. By Lemma 6 there are an automorphism $G$ of $D_0^n$ and an isotopy $G_t$, which can be extended to the whole of $S^n$ which is identity outside of $D_0^n$.

The extension is denoted by also $G$, $G_t$ respectively. Then by the choice of $G$ and Lemma 6 it can be easily checked that they satisfy the conditions of Lemma.

LEMMA 8. *Let* $B_i=(D_i^n, \bigcup_{j=1}^{\mu} D_{ij}^k)$, $i=1, 2$, *be* $(n, k, \mu)$*-braids and* $B_3=(D_3^{n-1},$

$\bigcup_{j=1}^{\mu} D_{3j}^{k-1})=\partial B_1 \cap \partial B_2$ *be* $(n-1, k-1, \mu)$*-braid.*

*Let* $D_j^k = D_{1j}^k \cup D_{2j}^k$ *and* $D^n = D_1^n \cup D_2^n$. *Then* $(D^n, \bigcup_{j=1}^{\mu} D_j^k)$ *is an* $(n, k, \mu)$*-braid.*

PROOF. Let $U$ be a triangulation of $D^n$ such that pairs of subcomplexes $U_i$ of $U$ cover $D_i^n$, $i=1, 2, 3$. Let $x$ be a point of $\bigcup_{j=1}^{\mu} D_j^k$. If $x \varepsilon D_j^k - D_{3j}^{k-1}$, the pair $(St(x, D^n), St(x, D_j^k))$ is flat. If $x \varepsilon D_{3j}^{k-1}$, then $(St(x, D^n), St(x, D_j^k)) = (St(x, D_1^n), St(x, D_{1j}^k)) \cup (St(x, D_2^n), St(x, D_{2j}^k))$, where $(St(x, D_1^n), St(x, D_{1j}^k)) \cap (St(x, D_2^n), St(x, D_{2j}^k)) = (St(x, D_3^n), St(x, D_{3j}^k))$, which is flat because $B_3$ is locally flat.

Then by Lemma 2 $(St(x, D^n), St(x, D_j^k)) = (St(x, D_1^n \cup D_2^n), St(x, D_{1j}^k \cup D_{2j}^k))$ is flat. Hence $B_1 \cup B_2$ is an $(n, k, \mu)$*-braid.*

Similary the following is proved.

LEMMA 9. *Let* $B_i = (D_i^n, \bigcup_{j=1}^{\mu} D_{ij}^k)$ *be* $(n, k, \mu)$*-braids such that* $D_{1j}^k \cap D_{2j}^k = \partial D_{1j}^k = -\partial D_{2j}^k$, $i=1, 2, \cdots, \mu$, $D_1^n \cap D_2^n = \partial D_1^n = -\partial D_2^n$.

*Let* $S_j^k = D_{1j}^k \cup D_{2j}^k$, $j=1, 2, \cdots, \mu$, $S^n = D_1^n \cup D_2^n$ *be k-, n-spheres. Then* $(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ *is an* $(n, k, \mu)$*-link.*

LEMMA 10. *Let* $B=(D^n, \bigcup_{j=1}^{\mu} D_j^k)$ *be an* $(n, k, \mu)$*-braid and* $L_i = (S^n, \bigcup_{j=1}^{\mu} S_{ij}^k)$, $i=1, 2$ *be* $(n, k, \mu)$*-links such that* $L_1 \cap L_2 = B$.

*Let* $S_j^k = (S_{1j}^k \cup S_{2j}^k)$-Int $D_j^k$, $j=1, 2, \cdots, \mu$. *Then* $(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ *is an* $(n, k, \mu)$*-link.*

PROOF. Let $x \varepsilon S_j^k$. If $x \varepsilon S_{ij}^k - D_j^k$, $i=1, 2$, then $(St(x, S^n), St(x, S_j^k)) = (St(x, S^n), St(x, S_{ij}^k))$ which is flat. Suppose that $x \varepsilon \partial D_j^k$. Since $L_i$ and $B$ are locally flat pairs, the pairs $(St(x, S^n), St(x, S_{ij}^k))$ and $(St(x, D^n), St(x, D_j^k))$ are flat. Since $(St(x, S^n), St(x, S_{ij}^k)) = (St(x, S^n - \text{Int } D^n), St(x, S_{ij}^k - \text{Int } D_j^k)) \cup (St(x, D^n), St(x, D_j^k))$ and the intersection of the summand is $(St(x, \partial D^n), St(x, \partial D_j^k))$ which is flat, $(St x, S^n - \text{Int } D^n), St(x, S_{ij}^k - \text{Int } D_j^k))$ is flat by Lemma 2, $i=1, 2$. Since the pair $(St(x, S^n), St(x, S_j^k)) = (St(x, S_1^n - \text{Int } D^n), St(x, S_{1j}^k - \text{Int } D_j^k)) \cup (St(x, S_2^n - \text{Int } D^n), St(x, S_{2j}^k - \text{Int } D_j^k))$ and since the intersection of the summands is $(St(x, \partial D^n), St(x, \partial D_j^k))$, $(St(x, S^n), St(x, S_j^k))$ is flat by Lemma 1. Hence $L$ is an $(n, k, \mu)$*-link.*

LEMMA 11. *Let* $B=(D^n, \bigcup_{j=1}^{\mu} D_j^k)$ *be a trivial* $(n, k, \mu)$*-braid and* $H$ *is an orientation preserving automorphism of* $D^n$ *such that* $H(D_j^k) = D_j^k$, $j = 1, 2, \cdots, \mu$. *Then* $H$ *is isotopic to the identity.*

PROOF. Let us call the lemma *proposition* $B(n, k, \mu)$. By *proposition*

$L(n, k, \mu)$ we shall denote the similar proposition concerning with a trivial $(n, k, \mu)$-*link* $L$. The proposition $B(\mathrm{n}-k, 0, \mu)$ is trivial. In the first place let us prove that the proposition $L(n-1, k-1, \mu)$ implies the proposition $B(n, k, \mu)$. Let $H$ be an automorphism of $D^n$ such that $H(D_j^k)=D_j^k$, $j=1, 2, \cdots, \mu$.

Then $\partial H$ is also an automorphism of $\partial D^n$ such that $\partial H(\partial D_j^k)=\partial D_j^k$, $j=1, 2, \cdots, \mu$. Then by $L(n-1, k-1, \mu)$ $\partial H$ is isotopic to the identity, that is, ther is a level preserving homeomorphism $G: \partial D^n \times I_0 \longrightarrow \partial D^n \times I_0$ such that $G(x, 0)=(\partial H(x), 0)$, $G(\partial D_j^k, t)=(\partial D_j^k, t)$, $j=1, 2, \cdots, \mu$ and $G(x, 1)=(x, 1)$. Let us define a homeomorphism $a: \partial(D^n \times I_0) \longrightarrow \partial(D^n \times I_0)$ be taking $a|D^n \times 0 = H \times 0$, $a|\partial D^n \times t = G|\partial D^n \times t$, $t \varepsilon I_0$ and $a|D^n \times 1 = identity$.

Since $B$ is a trivial braid, $B \times I_0 = (D^n \times I_0, \bigcup_{j=1}^{\mu} D_j^k \times I_0)$ is also trivial. Hence there is a commutative diagram

$$\begin{array}{ccccc}
\bigcup_{j=1}^{\mu} D_j^k \times I_0 & \xrightarrow{\overline{H}} & \bigcup_{j=1}^{\mu} \widetilde{D}_j^k \times I_0 & \xrightarrow{\beta \times 1} & (\Delta^k \times I) \times I_0 \\
\Big\downarrow{\scriptstyle C \times 1} & & \Big\downarrow{\scriptstyle C \times 1} & & \Big\downarrow{\scriptstyle \times 0 \times 1} \\
D^n \times I_0 & \xrightarrow{\overline{H}} & \widetilde{D}^n \times I_0 & \xleftarrow{s \times 1} & (\Delta^k \times I) \times I^{n-k-1} \times I_0
\end{array}$$

where $1: I_0 \longrightarrow I_0$ is the identity map. Let $D_j^k = y_j * \partial D_j^k$, $j=1, 2, \cdots, \mu$ where $y_j$ is an inner point of $D_j^k$. Then we may assume that $\left(y_j, \frac{1}{2}\right)=\rho\left(\left(\Delta^k, -1 + \frac{2j}{\mu+1}\right), \frac{1}{2}\right)$ where $\rho = \overline{H}^{-1}(s \times 1)(\times 0 \times 1)$ and $\hat{\Delta}^k$ is the barycenter of $\Delta^k$. Let $p: (\Delta^k \times I) \times I^{n-k-1} \times I_0 \longrightarrow (\Delta^k \times I) \times I^{n-k-1} \times I_0$ be an automorphism defined by $p\left((\hat{\Delta}^k, t_1), t_2, \frac{1}{2}\right)=\left((\hat{\Delta}^k, t_1), t_2, \frac{1}{2}\right)$ for any $t_1 \varepsilon I$, $t_2 \varepsilon I^{n-k-1}$, $\frac{1}{2} \varepsilon I_0$, $p((x, t_1), t_2, t_3)=(s \times 1)^{-1} \overline{H} a \overline{H}^{-1}(s \times 1)((x, t_1), t_2, t_3)$ for any $((x, \mathrm{t}_1), t_2, t_3) \varepsilon\ \partial((\Delta^k \times I) \times I^{n-k-1} \times I_0)$ and $p((y, t_1), t_2, t_3) = \lambda\left((\hat{\Delta}^k, t_1), t_2, \frac{1}{2}\right) + (1-\lambda)((s \times 1)^{-1} \overline{H} a \overline{H}^{-1}(s \times 1))((x, t_1), t_2, t_4)$ provided $((y, t_1), t_2, t_3) = \lambda\left((\hat{\Delta}^k, t_1), t_2, \frac{1}{2}\right) + (1-\lambda)((x, t_1), t_2, t_4)$. We define a homeomorphism $K: D^n \times I_0 \longrightarrow D^n \times I_0$ by $K = H^{-1}(s \times 1) p (s \times 1)^{-1} H$.

Then $K$ is a required isotopy. Next, let us prove that the proposition $B(n, k, \mu)$ implies the proposition $L(n, k, \mu)$.

Let $H$ be a homeomorphism of $S^n$ such that $H(S_j^k)=S_j^k$, $j=1, 2, \cdots, \mu$.

By virtue of Lemma 5 and 6, it is assumed without loss of generality that there is a trivial $(n, k, \mu)$-*braid* $B(D^n, \bigcup_{j=1}^{\mu} D_j^k) \subset L$ such that $B$ and $H(B)$ are disjoint. Then again by Lemma 5 and 6, $H$ is isotopic to an automorphism $G$ of $S^n$ such that $G|D^n = identity$. Since $L$ and $B$ are trivial, $L-$

$\operatorname{Int} B$ is a trivial $(n, k, \mu)$-*braid* by lemma 0.

Then by $B(n, k, \mu)$ we may construct an isotopy $K$ between $G|S^n-\operatorname{Int} D^n$ and identity on $S^n-\operatorname{Int} D^n$ which keep $\partial(S^n-\operatorname{Int} D^n)$ fixed. Then $G$ is isotopic to the identity on $S^n$ and so $H$.

DEFINITION 7. Let $(S^n, S^{n-1})$ be a standard sphere pair, (i. e, an unknotted sphere pair) and let $S^n-S^{n-1}=D^n_+ \cup D^n_-$. Choose thin cylinders $D^1_j \times D^k_j$ embedded in $S^n, j=1, 2, \cdots, \mu$ satisfying

1) $(D^1_j \times D^k_j) \cap (\bigcup_{j=1}^{\mu} (S^k_{1j} \cup S^k_{2j})) \subset S^k_{1j} \cup S^k_{2j},\ j=1, 2, \cdots, \mu$

2) $(D^1_j \times D^k_j) \cap S^k_{1j}=0 \times D^k_j,\ (D^1_j \times D^k_j) \cap S^k_{2j}=1 \times D^k_j$ where $D^1_j \cong [0, 1]$ and

3) $S^k_{1j}$ induces an orientation on $0 \times D^k_j$ which is opposite to one induced from $D^1_j \times D^k_j$ and $S^k_{2j}$ induces an orientation on $1 \times D^k_j$ which is opposite to one induced from $D^1_j \times D^k_j$.

Since $S^k_{1j} \cup \partial(D^1_j \times D^k_j) \cup S^k_{2j}-0 \times \operatorname{Int} D^k_j \cup I \times \operatorname{Int} D^k_j$ is again $k$-sphere $S^k_j$, $(n, k)$-sphere pair $L=(S^n, \bigcup_{j=1}^{\mu} S^k_j)$ is also an $(n, k, \mu)$-*link*.

Let $l$ be strong $(n, k, \mu)$-*ltnk* type containing $L$. We define the *sum* to be $l=l_1+l_2$.

LEMMA 12. *The sum defined as above is well defined under $n-k \geq 3$, so that the set of strong $(n, k, \mu)$-link types $SI(n, k, \mu)$ from a commutative semigroup with 0 (the trivial type) provided $n-k \geq 3$.*

PROOF. Since $n-k \geqq 3$, $\partial(D^1_j \times D^k_j)$, $j=1, 2, \cdots, \mu$ are embedded locally flatly in $S^n$. Then by Lemma 10 $L=(S^n, \bigcup_{j=1}^{\mu} S^k_j)$ is an $(n, k, \mu)$-*link*. We must prove the sum is independent of a choice of (1) the embedding of $D^1_j \times D^k_j$; (2) representatives $L_i$.

(1) Let $\widetilde{D}^1_j \times \widetilde{D}^k_j$ be another cylinders is $S^n$, $j=1, 2, \cdots, \mu$ satisfying

1) $(\widetilde{D}^1_j \times \widetilde{D}^k_j) \cap (\bigcup_{j=1}^{\mu} (S^k_{1j} \cup S^k_{2j})) \subset S^k_{1j} \cup S^k_{2j},\ j=1, 2, \cdots, \mu$

2) $(\widetilde{D}^1_j \times \widetilde{D}^k_j) \cap S^k_{1j}=0 \times \widetilde{D}^k_j$ and

3) $(\widetilde{D}^1_j \times \widetilde{D}^k_j) \cap S^k_{2j}=1 \times \widetilde{D}^k_j$ where $\widetilde{D}^1_j=[0, 1]$.

Since there is an automorphism $f_{ij}$ of $S^k_{ij}$ $i=1, 2$, $j=1, 2, \cdots, \mu$ such that $f_{ij}(\widetilde{D}^k_j)=D^k_j$ and since $f_{ij}$ is isotopic to the identiiy by [2 (I)], we may suppose that 2)′ $(\widetilde{D}^k_j \times\ {}^k_j) \cap S^k_{1j}=0 \times D^k_j$

3)′ $(\widetilde{D}^1_j \times \widetilde{D}^k_j) \cap S_{2j}=1 \times D^k_j$

by an isotopy extended on the small neighborhood of $\bigcup_{j=1}^{\mu} (S^k_{1j} \cup S^k_{2j})$.

Next since $\widetilde{D}^1_j \times \widetilde{D}^k_j$ is a regular neighborhood of the arc $\widetilde{D}^1_j$ mod $\partial\widetilde{D}^1_j$, there is an ambient isotopy $G$ of $S^n$ which keeps $\bigcup_{j=1}^{\mu} (S^k_{1j} \cup S^k_{2j})$ and such that

$G(\tilde{D}_j^1\times\tilde{D}_j^k)=D_j^1\times D_j^k$ provided $n-k\geqq 3$. Hence $\tilde{L}=(S^n,\bigcup_{j=1}^{\mu}S_j^k)$ is ambient isotopic to the link $L=(S^n,\bigcup_{j=1}^{\mu}S_j^k)$ in the strong sense where $\tilde{S}_j^k=S_{1j}^k\cup\partial(\tilde{D}_j^1\times\tilde{D}_j^k)\cup S_{2j}^k-0\times\operatorname{Int}\tilde{D}_j^k\cup 1\times\operatorname{Int}\tilde{D}_j^k$.

(2) Let $\overset{*}{L}_i=(S^n,\bigcup_{j=1}^{\mu}\overset{*}{S}{}_{ij}^k)$, $i=1,2$ be other representatives of $l_i$ satisfying $\bigcup_{j=1}^{\mu}\overset{*}{S}{}_{1j}^k\subset\operatorname{Int}D_+^n$ and $\bigcup_{j=1}^{\mu}\overset{*}{S}{}_{2j}^k\subset\operatorname{Int}D_-^n$. And let $\overset{*}{D}{}_j^1\times\overset{*}{D}{}_j^k$, $j=1,2,\cdots,\mu$ be cylinders in $S^n$ satisfying 1), 2), 3) same as Definition 7.

Then $\overset{*}{L}=(S^n,\bigcup_{j=1}^{\mu}\overset{*}{S}{}_j^k)$ is an $(n,k,\mu)$-*link* by Lemma 10 where $\overset{*}{S}{}_j^k=\overset{*}{S}{}_{1j}^k\cup\partial(\overset{*}{D}{}_j^1\times\overset{*}{D}{}_j^k)\cup\overset{*}{S}{}_{2j}^k-0\times\operatorname{Int}\overset{*}{D}{}_j^k\cup 1\times\operatorname{Int}\overset{*}{D}{}_j^k$. Since $L_i$ and $\overset{*}{L}_i$, $i=1,2$ are strong isotopic, there is an ambient isotopy $F$ of $S^n$ such that $F(\overset{*}{S}{}_{ij}^k)=S_{ij}^k$, $i=1,2$; $j=1,2,\cdots,\mu$. Let $F(\overset{*}{D}{}_j^1\times\overset{*}{D}{}_j^k)=\tilde{D}_j^1\times\tilde{D}_j^k$, then $\tilde{L}=(S^n,\bigcup_{j=1}^{\mu}F(\overset{*}{S}{}_j^k))=(S^n,\bigcup_{j=1}^{\mu}\tilde{S}_j^k)$ is also an $(n,k,\mu)$-*link* strong isotopic to $\overset{*}{L}$ where $\hat{S}_j^k=S_{1j}^k\cup\partial(\tilde{D}_j^1\times\tilde{D}_j^k)\cup S_{2j}^k-0\times\operatorname{Int}\tilde{D}_j^k\cup 1\times\operatorname{Int}\tilde{D}_j^k$.

Then $\tilde{L}$ is ambient isotopic to $L$ by [12] provided $n-k\geqq 3$. Hence $\overset{*}{L}$ and $L$ belong to the same strong isotopy class $l$.

Consequently, the sum $l_1+l_2$ is well defined. The commutative and associative laws follow from the definition of sum and Lemma 7.

And the trivial type is cleary 0.

DEFINITION 8. Let $l$ be an $(n,k,\mu)$-*link* type with a representative $(n,k,\mu)$-*link* $L=(S^n,\bigcup_{j=1}^{\mu}S_j^k)$. We say $l$ is *link cobordant to zero*, written $l\sim 0$, if there exists an $(n+1,k+1,\mu)$-*braid* $B=(D^{n+1},\bigcup_{j=1}^{\mu}D_j^{k+1})$ such that $\partial B=L$, that is $\partial D_j^{k+1}=S_j^k$, $j=1,2,\cdots,\mu$ and $\partial D^{n+1}=S^n$.

Clearly the definition is independent of the reprsentative $L$.

We define the link type $-l$ to be represented by $-L=(-S^n,\bigcup_{j=1}^{\mu}(-S_j^k))$ where $L=(S^n,\bigcup_{j=1}^{\mu}S_{ij}^k)$. Let $l_i$ be strong link types, $i=1,2$. We say $l_1$ is *link cobordic* to $l_2$ written $l_1\sim l_2$ if $l_1-l_2\sim 0$. The link cobordant is a symmetric relation provided $n-k\geq 3$.

LEMMA 13. *Let $n-k\geq 3$ and let $l_1,l_2$ be strong $(n,k,\mu)$-link types such that $l_1\sim 0$. Then $l_1+l_2\sim 0$ if and only $l_2\sim 0$.*

PROOF. Let $L_i=(S^n,\bigcup_{j=1}^{\mu}S_{ij}^k)$, $i=1,2$ be representatives of $l_i$.

Choose thin cylinders $D_j^1\times D_j^k$ embedded in $S^n$ same as Definition 7, then $L=(S^n,\bigcup_{j=1}^{\mu}S_j^k)$ is a representative of $l_1+l_2$ by Lemma 12 where $S_j^k=S_{1j}^k$

$\cup \partial(D^1_j \times D^k_j) \cup S^k_{2j} - 0 \times \mathrm{Int}\, D^k_j \cup 1 \times \mathrm{Int}\, D^k_j$.

In the first place, suppose $l_2 \sim 0$. Same as Definition 7 we may suppose $\bigcup_{j=1}^{\mu} S^k_{1j} \subset D^n_+$ and $\bigcup_{j=1}^{\mu} S^k_{2j} \subset D^n_-$. Since $l_1 \sim 0$ and $l_2 \sim 0$ there are $(n+1, k+1, \mu)$-*braids* $\tilde{B}_i = (D^{n+1}, \bigcup_{j=1}^{\mu} D^{k+1}_{ij})$ such that $\partial \tilde{B}_i = L_i$.

And we may suppose $\bigcup_{j=1}^{\mu} D^{k+1}_{1j} \subset D^{n+1}_+$ and $\bigcup_{j=1}^{\mu} D^{k+1}_{2j} \subset D^{n+1}_-$ where $D^{n+1}_+ \cap S^n = D^n_+$ and $D^{n+1}_- \cap S^n = D^n_-$. We push $D^1_j \times \mathrm{Int}\, D^k_j$, $j = 1, 2, \cdots, \mu$ in $\mathrm{Int}\, D^{n+1}$ slightly. Then $\tilde{B} = (D^{n+1}, \bigcup_{j=1}^{\mu} D^{k+1}_j)$ is an $(n+1, k+1, \mu)$-*braid* using Lemma 8 such that $\partial \tilde{B} = L$ where $D^{k+1}_j = D^{k+1}_{1j} \cup D^1_j \times D^k_j \cup D^{k+1}_{2j}$, and hence $l_1 + l_2 \sim 0$.

Next, suppose that $l_1 + l_2 \sim 0$ and $l_1 \sim 0$. Then there are $(n+1, k+1, \mu)$-*braids* $\tilde{B} = (D^{n+1}, \bigcup_{j=1}^{\mu} D^{k+1}_j)$ and $\tilde{B} = (D^{n+1}_1, \bigcup_{j=1}^{\mu} D^{k+1}_{1j})$ such that $\partial B = L = (S_n, \bigcup_{j=1}^{\mu} S^k_j)$ and $\partial \tilde{B}_1 = L_1 = (S_n, \bigcup_{j=1}^{\mu} S^k_{1j})$. We suppose $D^{n+1} \cap D^{n+1}_1 = S^n$ and $D^{k+1}_j \cap D^{k+1}_{1j} = S^k_{1j}$, $j = 1, \cdots, \mu$. Since $\bigcup_{j=1}^{\mu} S^k_{1j} \subset D^n_+$ we may suppose $\bigcup_{j=1}^{\mu} D^{k+1}_{1j} \subset D^{n+1}_{1+}$ where $D^{n+1}_{1+} = D^{n+1}_1 \cap D^n_+$. Then $\tilde{L} = (\partial(D^{n+1} \cup D^{n+1}_{1+}), \bigcup_{j=1}^{\mu} (S^k_{2j} \cup (\partial(D^1_j \times D^k_j) - 1 \times \mathrm{Int}\, D^k_j)))$ is an $(n, k, \mu)$-*link* strong isotopic to $L_2 = (S_n, \bigcup_{j=1}^{\mu} S^k_{2j})$. Clearly $\partial (D^{n+1} \cup D^{n+1}_1, \bigcup_{j=1}^{\mu} (D^{k+1}_j \cup D^{k+1}_{1j})) = \tilde{l}$.

Hence $l_2 \sim 0$.

LEMMA 14. *For any strong $(n, k, \mu)$-link type $l$, $l - l \sim 0$ provided $n - k \geqq 3$.*

PROOF. Let an $(n, k, \mu)$-*link* $L = (S_n, \bigcup_{j=1}^{\mu} S^k_j)$ be a representative of $l$ and $B$ a trivial $(n, k, \mu)$-*braid* (see the first part of the proof of Lemma 5), $(L - \mathrm{Int}\, B) \times I$ is an $(n+1, k+1, \mu)$-*braid*.

Then $\partial(L - \mathrm{Int}\, B) \times I) = (L - \mathrm{Int}\, B) \cup (\partial B \times I) \cup (-L - \mathrm{Int}\,(-B))$ where $(L - \mathrm{Int}\, B) \cap (\partial B \times I) = \partial B$, $(-L - \mathrm{Int}\,(-B)) \cap (\partial B \times I) = -\partial B$. Since $B \cup (\partial B \times I) \cup (-B) = \partial(B \times I)$ is a trivial $(n, k, \mu)$-*link*, $\partial((L - \mathrm{Int}\, B) \times I)$ is a representative of $l + 0 - l$ by Lemma 12. Since $(L - \mathrm{Int}\, B) \times I$ is an $(n+1, k+1, \mu)$-*braid*, we have $l - l \sim 0$.

THEOREM 1. *Link cobordism is an equivalence relation between strong $(n, k, \mu)$-link types under $n - k \geq 3$. Link cobordism classes form an abelian group $L(n, k, \mu)$ under the sum operation. The identity element* 0 *of $L(n, k, \mu)$ is the class of the strong link type* 0. *For a link cobordism class $L$ containing a strong link type $l$, the inverse $-L$ is the class containing $-l$.*

PROOF. By Lemma 14 link cobordism is reflexive. It is clearly symmetric. To prove transitivity suppose that $l_1-l_2+l_2-l_3=(l_2-l_2)+l_1-l_3\sim 0$. By Lemma 14, $l_2-l_2\sim 0$ hence Lemma 13 implies $l_1-l_3\sim 0$, proving transitivity and the link cobordism is an equivalence relation. Next suppose that $l_1\sim\tilde{l}_1$ and $l_2\sim\tilde{l}_2$. Since $(l_1+l_2)-(\tilde{l}_1+\tilde{l}_2)=(l_1-\tilde{l}_1)+(l_2-\tilde{l}_2)\sim 0$, $l_1+l_2\sim\tilde{l}_1+\tilde{l}_2$ by Lemma 13. Hence the sum is well defined. Since the sum operation of link types is associative and commutative and since the inverse exists by Lemma 14, this complete the proof.

COROLLARY. *When $n-k\geqq 3$, the commutative semigroup of strong $(n, k, \mu)$-link types become a commutative group under sum operation.*

*(We also denote this group $SI(n, k, \mu)$). For a strong link type $l$ with a representative $L$, the inverse $-1$ is the type containing $-L$.*

PROOF. It is sufficiently to show that $l-l$ becomes the class containing a trivial link. By Lemma 14 $l-l\sim 0$ and using [10. Th. 7] $l-l$ is the class containing a trivial link provided $n-k\geqq 3$.

Complete the proof.

## § 3. The link homotopy group.

DEFINITION 9. Let $l$ be an $(n, k, \mu)$-*link* type with a representative $(n, k, \mu)$-*link* $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ and 0 be a trivial $(n, k, \mu)$-*link* type with a representative $L_0=(S^n, \bigcup_{j=1}^{\mu} \tilde{S}_j^k)$. We say $l$ *link homotopic to zero*, wtitten $l\overset{h}{\sim}0$, if there is a map

$G: \bigcup_{j=1}^{\mu} S_{0j}^k\times I_0\longrightarrow S^n$ satisfying

1 ) $G(S_{0j}^k, 0)=S_j^k$, $j=1, 2, \cdots, \mu$

2 ) $G(S_{0j}^k, 1)=\tilde{S}_j^k$, $j=1, 2, \cdots, \mu$ and

3 ) $G(S_{0i}^k, t)\cap G(S_{0j}^k, t)=\phi$ for any $t\in I_0$ and $i\neq j$.

We say that $l_1$ is *link homotopic to* $l_2$, written $l_1\overset{h}{\sim}l_2$, if $l_1-l_2\overset{h}{\sim}0$. Whenever $n-k\geqq 3$, the link homotopy is obviously well defined and an equivalence relation between strong $(n, k, \mu)$-*link* types and the set of strong $(n, k, \mu)$-*link* types are classified by this equivalence relation into classes which are called $(n, k, \mu)$-*link homotopy class.*

By the sum operation given by Definition 7, the set of link homotopy classes form a commutative semi-group with 0 provided $n-k\geqq 3$ (link homotopy class containing the trivial type). We now prove the existence of inverses.

LEMMA 15. *Let $l$ be an strong $(n, k, \mu)$-link type. If $l$ is link cobordant* *to zero, $l$ is link homotopie to zero.*

PROOF. Let $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ be a representative of $l$. Then by the assumption there exists an $(n+1, k+1, \mu)$-*braid* $B=(D^{n+1}, \bigcup_{j=1}^{\mu} D_j^k)$ such that $\partial B=L$. We take a trivial $(n+1, k+1, \mu)$-*braid* $\tilde{B}=(\tilde{D}^{n+1}, \bigcup_{j=1}^{\mu} D_j^{k+1})$ such that $\tilde{B}\subset B$. Then $D_j^{k+1}-\mathrm{Int}\,\tilde{D}_j^{k+1}\cong\partial D_j^{k+1}\times I_0=S_j^k\times I_0$ and $D^{n+1}-\mathrm{Int}\,\tilde{D}^{n+1}\cong\partial D^{n+1}\times I_0=S^n\times I_0=\partial\tilde{D}^{n+1}\times I_0=S^n\times I_0$ where $\tilde{S}^n=\partial\tilde{D}^{n+1}$

Although $\bigcup_{j=1}^{\mu} S_j^k\times I_0\subset\tilde{S}^n\times I_0$ is not a level preserving, it is a proper embedding $g$ such that $g(S_j^k, 0)=\partial\tilde{D}_j^{k+1}\subset(\tilde{S}^n, 0)$ and $g(S_j^k, 1)=S_j^k\subset(\tilde{S}^n, 1)$. And we may further suppose that $g(S_1^k, t), g(S_2^k, t), \cdots, g(S_\mu^k, t)$ are contained in the same set $(\tilde{S}^n, \tilde{t})$ for some $\tilde{t}\varepsilon I_0$. Let $p:\tilde{S}^n\times I_0\longrightarrow\tilde{S}^n$ be a projection onto the first factor. We define a map $G:\bigcup_{j=1}^{\mu} S_j^k\times I_0\longrightarrow\tilde{S}^n$ by $G=pg$. Then $G$ is the required homotopy between $l$ and $0$.

Hence we define the inverse of the link homotopy class containing $l$ by the class containing $-l$. Therefore we obtain the following.

THEOREM 2. *The set of $(n, k, \mu)$-link homotopy classes forms an abelian group by the sum operation provided $n-k\geq 3$. We call this group $(n, k, \mu)$-link homotopy group and denotes $H(n, k, \mu)$.*

## § 4. The structure of $L(n, k, \mu)$, $H(n, k, \mu)$ and the related results.

It is obviously that $L(n, k, 1)$ is equivalent to knot cobordism group defined by [11].

Hence the following is obvions.

THEOREM 3.

$$L(n, k, 1)\quad \begin{cases} \cong \{0\} \quad \text{if} \quad n-k\geq 3 & [13] \\ \cong \{0\} \quad \text{if} \quad n-k=2, \quad n=\text{even} & [8] \\ = \text{infitely generated if } n-k=2,\ n=3 & [1] \\ \qquad\qquad\qquad\qquad\qquad n\geq 5,\ \text{odd} & [8]. \end{cases}$$

THEOREM 4. *The group $SI(n, k, \mu)$ is isomorphic to the group $L(n, k, \mu)$ provided $n-k\geq 3$.*

PROOF. If a strong $(n, k, \mu)$-*link* type $l$ is equivalent to the trivial type, it is obvious that $l$ is link cobordic to zero.

Conversely $l$ is link cobordic to zero, it is equivalent to the trivial type provided $n-k\geqq 3$ [10. Th. 7]. And there is a natural homomorphism of $SI(n, k, \mu)$ onto $L(n, k, \mu)$. Complete the proof.

Denote by $\vee_{j=1}^{\mu} S_j^{n-k-1}$ the wedge of spheres $S_1^{n-k-1} \vee \cdots \vee S_{\mu}^{n-k-1}$.

Given an $(n, k, \mu)$-*link* $L=(S^n, \overset{\mu}{\underset{j=1}{\cup}} S_j^k)$, there exists a map $g$ of $\vee_{j=1}^{\mu} S_j^{n-k-1}$ in the closure of the complement $X$ of $\overset{\mu}{\underset{j=1}{\cup}} S_j^k$ in $S^n$ (i.e. $X=\overline{S^n - \overset{\mu}{\underset{j=1}{\cup}} S_j^k}$) whose homotopy class is well defined by the condition; $g(S_j^{n-k-1})$ is homotopic in $X$ to an $(n-k-1)$-sphere $\partial D_j^{n-k}$ which is the boundary of the fiber of a tubular neighborhood of $S_j^k$ and with linking number $+1$ with $S_j^k$.

LEMMA 16. *If $n-k \geq 3$ we obtain the isomorphism $\pi_i(\vee_{j=1}^{\mu} S_j^{n-k-1}) \cong \pi_i(X)$ for $i \leq n-2$ induced by $g$.*

PROOF. Using Alexander's duylity theorem (see [9]), we obtain

$$\widetilde{H}_i(X; Z) = \widetilde{H}^{n-i-1}(\overset{\mu}{\underset{j=1}{\cup}} S_j^k; Z) = \begin{cases} \overbrace{Z+\cdots+Z}^{\mu} & \text{if } i=n-k-1 \\ 0 & \text{otherwise.} \end{cases}$$

And $H_0(X) \cong Z$ since $X$ is connected and

$H_{n-1}(X) \cong G$ where $G$ is some abelian group, for example $G \cong Z$ if $\mu=2$. On the other hand

$$H_i(\vee_{j=1}^{\mu} S_j^{n-k-1}) = \begin{cases} \overbrace{Z+\cdots+Z}^{\mu} & i=n-k-1 \\ Z & i=0 \\ 0 & \text{othersise.} \end{cases}$$

Hence $H_i(X)=H_i(\vee_{j=1}^{\mu} S_j^{n-k-1})$ for $i \leqq n-2$. Since both $X$ and $\vee_{j=1}^{\mu} S_j^{n-k-1}$ are simply connected provided $n-k \geq 3$, $g^*: \pi_i(\vee_{j-1}^{\mu} S_j^{n-k-1}) \longrightarrow \pi_i(X)$ is an isomorphism for $i \leqq n-2$ by the theorem of J.H.C. Whitehead (see [9]).

LEMMA 17. *Let $L=(S^n, \overset{\mu}{\underset{j=1}{\cup}} S_j^k)$ be a homotopically trivial link. Then $L$ belongs to a trivial link type provided $2n \geqq 3k+4$.*

PROOF. By Lemma 16, $X$ is $(n-k-2)$-*connected.* Since $L$ is a homotopically trivial link, $S_j^k$ is homotopic to a point in $\overline{S^n - \underset{i \neq j}{\cup} S_i^k}$ for all j. Since $(n-k-2)-(2k-n+2) = 2n-3k-4 \geq 0$ and since $2n \geqq 3k+4$ implies $n-k \geqq 3$ for all positive integers $n$, $k$, using *Zeeman's Unknotting Theorem* [12, Chap. 8], the link $L$ is ambient isotopic to the trivial link for $2n \geqq 3k+4$.

THEOREM 5. *There exists an isomorphism between $\mathscr{L}(n, k, \mu)$ and $\mathscr{H}(n, k, \mu)$ provided $2n \geq 3k+4$.*

PROOF. There is a natural mapping $\omega$ of $\mathscr{L}(n, k, \mu)$ onto $\mathscr{H}(n, k, \mu)$ such that it maps a link cobordism class containing a strong link type $l$ to a link homotopy class containing $l$. The mapping $\omega$ is a homomorphism because $\omega([l_1]+[l_2])=\omega([l_1+l_2])=\{l_1+l_2\}=\{l_1\}+\{l_2\}$ where [ ] and { } mean cobordism class and homotopy class respectively. Next we will show $\omega$ is an monomorphism. Since $l_1\sim l_2$ implies $l_1\overset{h}{\sim}l_2$ by Lemma 15, it is sufficient to show that $l_1\overset{h}{\sim}l_2$ implies $l_1\sim l_2$. Since $l_1-l_2\overset{h}{\sim}0$, the strong link type $l=l_1-l_2$ is homotopically trivial. Hence $l$ is a trivial link type by Lemma 17 under the condition $2n\geqq 3k+4$. Hence $l_1\sim l_2$. Therefore $\omega$ is an isomorphism.

DEFINITION 10. Let $LM(n, k, \mu)$ be set of $(\mu\times\mu)$-matrixes $A$ of the form $A=\begin{bmatrix} e & \lambda_2^1 & \lambda_3^1 & \cdots\cdots & \lambda_\mu^1 \\ \lambda_1^2 & e & \lambda_3^2 & \cdots\cdots & \lambda_\mu^2 \\ \cdots & \cdots & \ddots & & \cdots \\ \lambda_1^\mu & \lambda_2^\mu & \cdots\cdots & \lambda_{\mu-1}^\mu & e \end{bmatrix}$ satisfying that

1) $\lambda_j^i$ is element of $\pi_k(S^{n-k-1})$
2) $e$ is the identity element of $\pi_k(S^{n-k-1})$
3) $\lambda_j^i=(\lambda_i^j)^\varepsilon$ where $\varepsilon=(-1)^{n-k}$

We call this matrix $(n, k, \mu)$-*link* matrix. Then $LM(n, k, \mu)$ becomes a group under the operation $*$ such that

$$\begin{bmatrix} e & \lambda_2^1 & \lambda_3^1 & \cdots & \lambda_\mu^1 \\ \lambda_1^2 & e & \lambda_3^2 & \cdots & \lambda_\mu^2 \\ & & \ddots & & \\ \lambda_1^\mu & \lambda_2^\mu & \cdots & \lambda_{\mu-1}^\mu & e \end{bmatrix} * \begin{bmatrix} e & K_2^1 & K_3^1 & \cdots & K_\mu^1 \\ K_1^2 & e & K_3^2 & \cdots & K_\mu^2 \\ & & \ddots & & \\ K_1^\mu & K_2^\mu & \cdots & K_{\mu-1}^\mu & e \end{bmatrix} = \begin{bmatrix} e & \lambda_2^1\cdot K_2^1 & \cdots & \lambda_\mu^1\cdot K_\mu^1 \\ \lambda_1^2\cdot\lambda_1^2 & e & \cdots & \lambda_\mu^2\cdot K_\mu^2 \\ & & \ddots & \\ \lambda_1^\mu\cdot K_1^\mu & \cdots & & e \end{bmatrix}$$

where $\cdot$ is the operation on $\pi_k(S^{n-k-1})$. Hence $LM(n, k, \mu)$ becomes an additive group under the usual sum operation of matrixes because $\pi_k(S^{n-k-1})$ is the abelian group under $n\geqq k+3$. We call it *the group of the $(n, k, \mu)$-link mairix* and denote also $LM(n, k, \mu)$.

THEOREM 6. *There is an isomorphism between $\mathscr{L}(n, k, \mu)$ and $LM(n, k, \mu)$ provided $2n\geq 3k+4$ and there is an epimorphism of $\mathscr{L}(n, k, \mu)$ onto $LM(n, k, \mu)$ provided $2n\geq 3k+3$ and $n-k\geqq 3$.*

PROOF. First we will show that there is a map of $\mathscr{L}(n, k, \mu)$ onto $LM(n, k, \mu)$. Let $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ be a representative of a link homotopy class $\{l\}$ containing a strong $(n, k, \mu)$-*link* type $l$, then $S_i^k\subset S^n$ defines an element $\lambda_j^i$ of $\pi_k(S^{n-k-1})\cong\pi_k(S^n-S_j^k)\,(i\neq j)$ under the condition $n-k\geqq 3$ and Lemma 16. Then $S_j^k\subset S^n$ defines an element $\lambda_i^j$ of $\pi_k(S^{n-k-1})\cong\pi_k(S^n-S_i^k)$ under the same conditions and $\lambda_i^j=(-1)^{n-k}\lambda_j^i$ by [12. Chap. 8].

And we define $e$ for $i=j$ formally. Let $\tilde{L}=(S^n, \bigcup_{j=1}^{\mu}\tilde{S}_j^k)$ be another representative of $\{l\}$. Then $L \overset{h}{\sim} \tilde{L}$ and $\tilde{S}_i^k$ defines the same element of $\pi_k(S^{n-k-1})$ as $\lambda_j^i$ corresponding to $S_i^k$. Hence $L$ and $\tilde{L}$ have same $(n, k, \mu)$-*link* matrix and this map is independent of the representative of $\{l\}$. Converely let $A=\begin{bmatrix} e & \lambda_2^1 & \lambda^1 & \cdots\cdots & \lambda_\mu^1 \\ \lambda_1^2 & e & \lambda^2 & \cdots\cdots & \lambda_\mu^2 \\ & & \cdots\cdots & & \\ \lambda_1^\mu & \lambda_2^\mu & \cdots\cdots & \lambda_{\mu-1}^\mu & e \end{bmatrix}$ be a given element of $LM(n, k, \mu)$. We shall construct a $(n, k, \mu$-)*link* $L$ having a given $(n, k, \mu)$-*link* matrix. Let $L_i=(S_i^n, \bigcup_{j\neq i} S_{ij}^k)$, $i=1, 2, \cdots, \mu$ be trivial $(n, k, \mu-1)$-*links* and $g_i: S_i^k \longrightarrow (S_i^n-\bigcup_{j\neq i} S_{ij}^k)$, $i=1, 2, \cdots, \mu$ be continuous maps such that $g_i(S_{ii}^k)$ represents the element $\lambda_j^i$ of $\pi_k(S_i^n-S_{ij}^k)$ for $i>j$ and the identity $e$ for $i<j$.

Since $\pi_p(S_i^n-\bigcup_{j\neq i} S_{ij}^k)\cong\pi_p(\bigvee_{j\neq i} S_j^{n-k-1})$ for $p\leqq n-2$ provided $n-k\geqq 3$ by Lemma 16, there exist embeddings $f_i: S_{ii}^k \longrightarrow (S_i^n-\bigcup_{j\neq i} S_{ij}^k)$ homotopic to $g_i$, $i=1, 2, \cdots, \mu$ provided $2n\geqq 3k+3$ and $n-k\geqq 3$ by *Irwin's Embedding Theorem* ([7], [12]). Hence $L_i=(S_i^n, \bigcup_{j\neq i} S_{ij}^k \cup f_i(S_{ii}^k))$ is an $(n, k, \mu)$-*link* such that $f_i(S_{ii}^k)$ represents the element $\lambda_j^i$ of $\pi_k(S^{n-k-1})=\pi_k(S_i^n-\bigcup_{j\neq i} S_{ij}^k)$ for $i>j$ and $e$ for $i<j$. Let $l_i$ be a strong $(n, k, \mu)$-*link* type containing $L_i$ constructed above. We construct a connected sum $S^n=\#_i S_i^n$, $i=1, 2, \cdots, \mu$ satisfying $S_i^n\cap S_{i+1}^n=(S_i^n-\bigcup_{j\neq i} S_{ij}^k\cup f_i(S_{ii}^k))\cap(S_{i+1}^n-\bigcup_{j\neq i+1} S_{i+1j}^k\cup f_{i+1}(S_{i+1,i+1}^k))=D_i^n$, $1,2,\cdots,\mu-1$ and $S_i^n\cap S_j^n=\phi$ if $j\neq i+1$ or $i-1$.

Let $l=l_1+l_2+\cdots+l_\mu$. Then it is obviously that $l$ is a strong $(n, k, \mu)$-*link* type having a given $(n, k, \mu)$-*link* matrix $A$. Hence there is a map $\mathscr{L}(n, k, \mu)$ onto $LM(n, k, \mu)$ provided $n-k\geqq 3$ and $2n\geqq 3k+3$.

Furthermore it is easy to show that this map is a homomorphism.

Next we will show that the epimorphism defined above is an isomorphism provided $2n\geqq 3k+4$. Let $E=\begin{bmatrix} e & e & \cdots\cdots & e \\ e & e & \cdots\cdots & e \\ \cdots & \cdots & \cdots & \cdots \\ e & e & \cdots\cdots & e \end{bmatrix}$ be the identity of $LM(n, k, \mu)$ and $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ be a representaitve of $E$ constructed as above under the conditions $n-k\geqq 3$ and $2n\geqq 3k+3$. Since $S_i^k \varepsilon e \varepsilon \pi_k(S_j^{n-k-1})\cong\pi_k(S^n-S_j^k)$ for any $j\neq i$ provided $n-k\geqq 3$ and since $\sum_{j\neq i}\pi_k(S_j^{n-k-1})=\pi_k(\bigvee_{j\neq i} S_j^{n-k-1})=\pi_k(S^n-\bigcup_{j\neq i} S_j^k)$ using Lemma 16 and [4] under the condition $2n\geqq 3k+4$, $S_i^k \varepsilon e \varepsilon \pi_k(S^n-\bigcup_{j\neq i} S_j^k)$. Hence for any $i$, $S_i^k$ is homotopically trivial in $S^n-\bigcup_{j\neq i} S_j^k$ and hence $L$ is homotopically trivial link. Therefore $\mathscr{L}(n, k, \mu)\cong LM(n, k, \mu)$

provided $2n \geqq 3k+4$.

From Theopem 4, 5 and 6 we obtain the main result.

THEOREM 7. *$SI(n, k, \mu)$ is isomorphic to $LM(n, k, \mu)$ provided* $2n \geq 3k+4$.

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## References

[1] R. FOX and J. MILNOR: Singularities of 2-spheres in 4-space and cobordism of knots, Osaka J. Math, 3 (1966) 257-67.

[2] V. K. A. M. GUGENHEIM: Piecewise liniar isotopy and embedding of elements and spheres (I), Proc. Lond. Math. Soc. (3) (1953), 29-53: and …, (II), ibid 3 (1953) 129-152.

[3] A. HAEFLIGER: Differentiable Links, Topology Vol. 1, 1962, 241-244.

[4] P. J. HILTON: On the homotopy groups of the union of spheres, J. Lond. Math. Soc. 30 (1955) 154-171.

[5] J. F. P. HUDSON and E. C. ZEEMAN: On regular neighborhoods, Proc. Lond. Math. Soc. (3) 14 (1964) 719-745.

[6] L. S. HUSCH: On relative regular neighborhoods, Proc. Lond. Math. Soc. (3) 19 (1969) 577-585.

[7] M. C. IRWIN: Embeddings of Polyhedral Manifold, Ann. of Math., Vol. 82, (1965) 1-14.

[8] M. KERVAIRE: Les noeuds de dimensions superieures, Bull. Soc. Math. France, 93 (1965) 225-271.

[9] A. KOMATSU, M. NAKAOKA and M. SUGAWARA; 位相幾何学 (岩波書店).

[10] W. B. R. LICKORISH: The piecewise linear unknotting of cones, Topology Vol. 4, (1965) 67-91.

[11] H. NOGUCHI: Obstuction to locally flat embeddings of combinatorial manifolds, Topology Vol. 5, (1966) 203-213.

[12] E. C. ZEEMAN: Seminar on combinatorial topology, I. H. E. S. (1963) (mimeographed notes), Paris.

[13] E. C. ZEEMAN: Unknotting combinatorial balls, Ann. of Math. Vol. 78 (1963) 501-526.

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