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On Riemannian manifolds satisfying a certain condition on the curvature tensors

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Kazunari YAMAUCHI

§ 0. Introduction

If a Riemannian manifold M is locally symmetric, then its curvature tensor R satisfies

$$(*) \quad R(X, Y)R = 0 \quad \text{for any tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R as a derivation of tensor algebra at each point of M . As a converse problem, there is a following conjecture by K. Momizu ([1]).

CONJECTURE. *Let M be a complete, irreducible Riemannian manifold with $\dim M \geq 3$. If M satisfies the condition $(*)$, then M is a locally symmetric space.*

For this conjecture, K. Nomizu gave an affirmative answer in case that M is a complete hypersurface in a Euclidean space ([2]). P. J. Ryan gave an affirmative answer in case that M is a complete hypersurface in a space of constant curvature ([3]). Furthermore there are some results in this direction.

In this paper we shall prove the following theorems

THEOREM 1. *Let $\bar{M}$ be an $n+1$ -dimensional Riemannian manifold satisfying the condition $\bar{R}(X, Y)\bar{R} = 0$, and M ($\dim M \geq 5$) be a hypersurface of $\bar{M}$ whose curvature tensor satisfies the condition $(*)$. If M satisfies $\lambda_i = \lambda (= \text{const} \neq 0)$, $1 \leq i \leq n$, then M is a space of constant curvature, where $\bar{R}$ and R are curvature tensors of $\bar{M}$ and M respectively, and λ_i is a eigenvalue of A , A is a field of symmetric endomorphism which corresponds to the second fundamental form h , that is, $h(X, Y) = g(AX, Y)$ for tangent vectors X and Y .*

THEOREM 2. *Let M be 3-dimensional Riemannian manifold whose curvature tensor satisfies the condition $(*)$. If the scalar curvature K of M is constant, then M is a locally symmetric space.*

I should like to express my hearty thanks to Prof. Y. Katsurada for her kind suggestions and many valuable criticisms.

§ 1. Proof of Theorem 1.

The following is a purely local argument. Let U be a neighborhood of a point $x \in M$ on which we choose a unit vector field ξ normal to M . For any vector field X and Y tangent to M , we have the formulas of Gauss and Weingarten

$$\begin{aligned}\bar{\nabla}_x Y &= \nabla_x Y + h(X, Y)\xi, \\ \bar{\nabla}_x \xi &= -AX,\end{aligned}$$

where $\bar{\nabla}$ and ∇ denote covariant differentiation for $\bar{M}$ and M respectively. Using these formulas we have the following equation

$$\bar{R}(X, Y)Z = R(X, Y)Z - (AX \wedge AY)Z + \{(\nabla_x h)(Y, Z) - (\nabla_Y h)(X, Z)\}\xi,$$

where, in general, $X \wedge Y$ denotes the endomorphism which maps Z upon $g(Z, Y)X - g(Z, X)Y$, g being the Riemannian metric of M .

At a point $x \in M$, let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_x(M)$ such that $Ae_i = \lambda_i e_i$, $1 \leq i \leq n$.

LEMMA: If $\lambda_i = \lambda = \text{const}$, then $(\nabla_x h)(Y, Z) = 0$.

$$\begin{aligned}\text{PROOF. } (\nabla_x h)(Y, Z) &= X \cdot h(Y, Z) - h(\nabla_x Y, Z) - h(Y, \nabla_x Z) \\ &= X \cdot g(AY, Z) - g(A\nabla_x Y, Z) - g(AY, \nabla_x Z) \\ &= X \cdot g(AY, Z) - g(\nabla_x(AY) - (\nabla_x A)Y, Z) - g(AY, \nabla_x Z) \\ &= g((\nabla_x A)Y, Z).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(\nabla_x A)Y &= \nabla_x(AY) - A(\nabla_x Y) \\ &= \nabla_x(\lambda Y) - \lambda \nabla_x Y \\ &= X\lambda \cdot Y + \lambda \nabla_x Y - \lambda \nabla_x Y \\ &= 0.\end{aligned}$$

Hence it follows that

$$(\nabla_x h)(Y, Z) = 0.$$

In the following computation, assuming that any two indices in i, j, k, l , have not the same value, we have

$$\begin{aligned}(\bar{R}(e_i, e_j)\bar{R})(e_i, e_l)e_j &= [\bar{R}(e_i, e_j), \bar{R}(e_i, e_l)]e_j - \bar{R}(\bar{R}(e_i, e_j)e_i, e_l)e_j \\ &\quad - \bar{R}(e_i, \bar{R}(e_i, e_j)e_l)e_j \\ &= \bar{R}(e_i, e_j)(R(e_i, e_l)e_j - \lambda^2(\delta_{il}e_i - \delta_{ij}e_l))\end{aligned}$$

$$\begin{aligned}
 & -\bar{R}(e_i, e_l) \left(R(e_i, e_j) e_j - \lambda^2 (\delta_{jj} e_i - \delta_{ij} e_j) \right) \\
 & -\bar{R} \left(R(e_i, e_j) e_i - \lambda^2 (\delta_{ij} e_i - \delta_{ii} e_j), e_l \right) e_j \\
 & -\bar{R} \left(e_i, R(e_i, e_j) e_l - \lambda^2 (\delta_{lj} e_i - \delta_{il} e_j) \right) e_j \\
 & = R(e_i, e_j) \left(R(e_i, e_l) e_j \right) - \lambda^2 R_{jil}^r (\delta_{rj} e_i - \delta_{ir} e_j) \\
 & - R(e_i, e_l) \left(R(e_i, e_j) e_j \right) + \lambda^2 R_{jil}^r (\delta_{ri} e_i - \delta_{ir} e_l) \\
 & + \lambda^2 \left(R(e_i, e_l) e_i - \lambda^2 (\delta_{il} e_i - \delta_{ii} e_l) \right) \\
 & - R \left(R(e_i, e_j) e_i, e_l \right) e_j - \lambda^2 R_{iil}^r (\delta_{rj} e_i - \delta_{ij} e_r) \\
 & - \lambda^2 \left(R(e_j, e_l) e_j - \lambda^2 (\delta_{lj} e_j - \delta_{jj} e_l) \right) \\
 & - R \left(e_i, R(e_i, e_j) e_l \right) e_j + \lambda^2 R_{iil}^r (\delta_{rj} e_i - \delta_{ij} e_r) \\
 & = \left(R(e_i, e_j) R \right) (e_i, e_l) e_j + \lambda^2 R_{ijil} e_j + \lambda^2 (R_{iil}^r - R_{jil}^r) e_r.
 \end{aligned}$$

Hence we obtain

$$\begin{aligned}
 \lambda^2 R_{ijil} &= 0, \\
 \lambda^2 (R_{iil}^r - R_{jil}^r) &= 0, \\
 \lambda^2 (R_{r iil} - R_{r jil}) &= 0 \quad (r \neq i, j, l).
 \end{aligned}$$

Under same computation, we have

$$\left(\bar{R}(e_i, e_j) \bar{R} \right) (e_k, e_l) e_i = \left(R(e_i, e_j) R \right) (e_k, e_l) e_i - \lambda^2 R_{jlik} e_i - \lambda^2 R_{jkl}^r e_r.$$

Hence we obtain

$$\begin{aligned}
 \lambda^2 R_{kjkl} &= 0, \\
 \lambda^2 R_{ljkl} &= 0, \\
 \lambda^2 R_{rjkl} &= 0 \quad (r \neq i, j, k, l).
 \end{aligned}$$

From the above results we have

$$\begin{aligned}
 R_{ijkl} &= 0, \\
 R_{ijjl} &= 0, \\
 R_{ijij} &= R_{iil}^{\text{put}} (= \mu).
 \end{aligned}$$

The sectional curvature $\rho(X, Y)$ for the plane spanned by X and Y is given by

$$\rho(X, Y) = \frac{g(R(X, Y) Y, X)}{g(X, X) g(Y, Y) - (g(X, Y))^2}.$$

Let $X = a^i e_i$, $Y = b^j e_j$, then using $R_{ijkl} = 0$, $R_{ijjl} = 0$, we have

$$\begin{aligned}\sum_{i=1}^n (a^i)^2 \cdot \sum_{j=1}^n (b^j)^2 \cdot \rho(X, Y) &= g(R(a^i e_i, p^j e_j) b^k e_k, a^l e_l) \\ &= \sum_{i=1}^n (a^i)^2 \cdot \sum_{j=1}^n (b^j)^2 \cdot \mu.\end{aligned}$$

Hence it follows that

$$\rho(X, Y) = \mu$$

Therefore M is a space of constant curvature since the sectional curvature $\rho(X, Y)$ does not depend on the direction.

§ 2. Proof of Theorem 2.

Now we can see that there is an orthonormal basis $\{e_1, e_2, e_3\}$ at each tangent space of M such that

$$(2.1) \quad R_{1212} = a, \quad R_{1313} = b, \quad R_{2323} = c,$$

otherwise zero. From (2.1) we have

$$R(e_i, e_j) = R_{ijij} e_i \wedge e_j$$

From the following relation ;

$$\begin{aligned}(R(e_i, e_j)R)(e_k, e_l) &= [R(e_i, e_j), R(e_k, e_l)] - R(R(e_i, e_j)e_k, e_l) \\ &\quad - R(e_k, R(e_i, e_j)e_l)\end{aligned}$$

we find that it is zero except in case that $k=i$, and $l \neq i, j (i \neq j)$. For this case we have

$$(R(e_i, e_j)R)(e_i, e_l) = R_{ijij}(R_{jlij} - R_{lilj})e_j \wedge e_l.$$

Thus we see that condition (*) is equivalent to

$$(2.2) \quad R_{ijij}(R_{jlij} - R_{lilj}) = 0$$

From (2.2) we have

$$\begin{aligned}a(c-b) &= 0, \\ b(c-a) &= 0, \\ c(b-a) &= 0.\end{aligned}$$

We consider the following two cases

$$\begin{aligned}\text{I.} \quad & a = b = c \\ \text{II.} \quad & a = b = 0, \quad c \neq 0\end{aligned}$$

In case of I, clearly M is a space of constant curvature.

In case of II, we obtain

$$\begin{aligned} R_{12} &= R_{13} = R_{23} = R_{11} = 0, \\ R_{22} &= R_{33} = R_{2323}. \end{aligned}$$

From the Bianchi's identity, we have

$$\nabla_1 R_{1k} + \nabla_2 R_{2k} + \nabla_3 R_{3k} = \frac{1}{2} \nabla_k K.$$

Let us put $k=3$. Then we have $\nabla_3 R_{33} = \nabla_3 R_{22} = 0$.

Putting $k=2$, we have $\nabla_2 R_{22} = \nabla_2 R_{33} = 0$.

From the Bianchi's identity we have

$$\nabla_1 R_{1kji} + \nabla_2 R_{2kji} + \nabla_3 R_{3kji} = \nabla_i R_{jk} - \nabla_j R_{ik}.$$

Put $j=k=2$ and $i=1$, then $\nabla_1 R_{22} = \nabla_1 R_{33} = 0$.

Hence it follows that

$$\nabla_j R_{ik} = 0$$

Therefore M is a locally symmetric space.

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