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On necessary and sufficient conditions for L²-well-posedness of mixed problems for hyperbolic equations II

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

We consider hyperbolic mixed problems (P, B_j) in a quadrant $\mathbf{R}_+ \times \mathbf{R}_+^n$:

$$\begin{aligned} Pu &= f && \text{in } t > 0, x_n > 0, x' \in \mathbf{R}^{n-1}, \\ B_j u &= 0 \quad (j=1, \dots, l) && \text{in } t > 0, x_n = 0, x' \in \mathbf{R}^{n-1}, \\ \partial_t^k u &= u_k \quad (k=0, 1, \dots, m-1) && \text{in } t = 0, x_n > 0, x' \in \mathbf{R}^{n-1} \end{aligned}$$

where $\mathbf{R}_+^n = \{x = (x', x_n) \in \mathbf{R}^n; x_n > 0\}$, $\partial_t = \frac{\partial}{\partial t}$, $D_{x_j} = (-i)\frac{\partial}{\partial x_j}$, $P = P(t, x; \partial_t, D_x)$ is a strictly t -hyperbolic operator of order m and $B_j = B_j(t, x'; \partial_t, D_x)$ are boundary operators of order $m_j < m$. Furthermore P is assumed non-characteristic with respect to the hyperplane $x_n = 0$.

This paper consists of two parts. The first part (§ 2 and § 3) is concerned with constant coefficient problems for homogeneous operators P and B_j . In previous paper [2] we assume that $\{B_j\}$ is normal; that is, $m_j \neq m_k$ if $j \neq k$ and the hyperplane $x_n = 0$ is non-characteristic for B_j . However, when $\{B_j\}$ is not always normal, it will be suitable for our purpose to use the following

DEFINITION. *The mixed problem (P, B_j) with homogeneous initial condition is L²-well-posed with decreasing order ν ($\nu \geq 0$, an integer) if and only if there exist positive constants a and C such that for any f with $e^{-at} f \in H_0^{\nu+1}(\mathbf{R}_+ \times \mathbf{R}_+^n)$ the problem has a unique solution u with $e^{-at} u \in H^m(\mathbf{R}_+ \times \mathbf{R}_+^n)$ satisfying*

$$\begin{aligned} (1.1) \quad & \sum_{j+|\alpha| \leq m-1} \int_0^\infty \int_{\mathbf{R}_+^n} e^{-2at} |(\partial_t^j D_x^\alpha u)(t, x)|^2 dt dx \\ & \leq C \sum_{j+|\alpha| \leq \nu} \int_0^\infty \int_{\mathbf{R}_+^n} e^{-2at} |(\partial_t^j D_x^\alpha f)(t, x)|^2 dt dx. \end{aligned}$$

In § 2 a necessary and sufficient condition for L²-well-posedness with decreasing order ν is given by the terms of compensating function (cf.

Theorem 4.1 in [2]). In §3 we show that under the L^2 -well-posedness with decreasing order ν Lopatinskiĭ's determinant $R(\tau_0, \sigma_0) \neq 0$ for $\operatorname{Re} \tau_0 > 0$ and $\sigma_0 \in \mathbf{R}^{n-1}$ is equivalent to the fact that $B_j(\tau_0, \sigma_0, \lambda)$ are linearly independent as polynomials in λ .

The second part is concerned with variable coefficient problems. In this part we always assume that $\{B_j\}$ is normal. For initial data $U(t_0) = (u_0, u_1, \dots, u_{m-1})$ we set up a convenient function space $\mathcal{U}(t_0)$; that is, $U(t_0) \in \mathcal{U}(t_0)$ if and only if $u_k \in H^{m-k}(\mathbf{R}_+^n)$ and satisfy the compatibility condition

$$\sum_{k=0}^{m_j} b_{j,k}(t, x'; D_x) u_k = 0 \quad \text{on } t = t_0, x_n = 0$$

where

$$B_j(t, x'; \partial_t, D_x) = \sum_{k=0}^{m_j} b_{j,k}(t, x'; D_x) \partial_t^k.$$

DEFINITION. *The mixed problem (P, B_j) is strongly L^2 -well-posed if and only if there exist positive constants T and C such that for an arbitrarily fixed time $t_0 \in [0, T)$ the problem (P, B_j) with $f \in H_0^1((t_0, T) \times \mathbf{R}_+^n)$ and $U(t_0) \in \mathcal{U}(t_0)$ has a unique solution $u \in \mathcal{E}^0((t_0, T), H^m(\mathbf{R}_+^n)) \cap \dots \cap \mathcal{E}^m((t_0, T), H^0(\mathbf{R}_+^n))$ which satisfies energy inequalities*

$$(1.2) \quad \|u(t, \cdot)\|_{m-1}^2 \leq C(\|U(t_0)\|_{m-1}^2 + \int_{t_0}^t \|f(s, \cdot)\|_0^2 ds),$$

$$(1.3) \quad \|u(t, \cdot)\|_m^2 \leq C(\|U(t_0)\|_m^2 + \int_{t_0}^t \|f(s, \cdot)\|_1^2 ds)$$

for any $t \in [t_0, T]$, where

$$\begin{aligned} \|u(t, \cdot)\|_k^2 &= \sum_{j=0}^k \|(\partial_t^j u)(t, \cdot)\|_{k-j}^2, \\ \|U(t_0)\|_k^2 &= \sum_{j=0}^k \|u_j(\cdot)\|_{k-j}^2, \\ \|u(\cdot)\|_k^2 &= \sum_{|\alpha| \leq k} \int_{\mathbf{R}_+^n} |D_x^\alpha u(x)|^2 dx. \end{aligned}$$

In §4 we show the following: If a variable coefficient problem (P, B_j) is strongly L^2 -well-posed, then each constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point is L^2 -well-posed (with decreasing order $\nu=0$), provided that the corresponding Lopatinskiĭ's determinant $R(t, x'; 1, 0) \neq 0$ on the boundary. Combining this and results in [1] we obtain a certain characterization of strongly L^2 -well-posed problems with real boundary condition for the case of second order.

This note is the supplement of our previous papers [1] and [2].

§ 2. Necessary and sufficient condition

In this section and the following we consider constant coefficient problems (P, B_j) with homogeneous initial condition. Here P and B_j are homogeneous operators.

We take Laplace transform in t and Fourier transform in x and $\hat{u}(\tau, \sigma, \lambda)$ and $\hat{u}(\tau, \sigma, x_n)$ denote the Fourier-Laplace image of $u(t, x', x_n)$ with respect to (t, x', x_n) and (t, x') respectively. By the assumption on P the number $l(m-l)$ of roots $\lambda_j^+(\tau, \sigma)$ ($\lambda_k^-(\tau, \sigma)$) of $P(\tau, \sigma, \lambda)=0$ in λ , which have positive (negative) imaginary part, is independent of $(\tau, \sigma) \in C_+ \times \mathbf{R}^{n-1}$, where $C_+ = \{\tau \in C; \operatorname{Re} \tau > 0\}$.

Taking now Fourier-Laplace transform the problem (P, B_j) becomes formally to the boundary value problem of ordinary differential equations depending on parameters $(\tau, \sigma) \in C_+ \times \mathbf{R}^{n-1}$:

$$(2.1) \quad \begin{aligned} P(\tau, \sigma, D_{x_n}) \hat{u}(\tau, \sigma, x_n) &= \hat{f}(\tau, \sigma, x_n) && \text{in } x_n > 0, \\ B_j(\tau, \sigma, D_{x_n}) \hat{u}(\tau, \sigma, x_n) &= 0 \quad (j=1, \dots, l) && \text{on } x_n = 0. \end{aligned}$$

Let $R(\tau, \sigma)$ be Lopatinskii's determinant; that is,

$$R(\tau, \sigma) = \det(B_j(\tau, \sigma, \lambda_k^+(\tau, \sigma))) / \prod_{j>k} (\lambda_j^+(\tau, \sigma) - \lambda_k^+(\tau, \sigma))$$

and $R_j(\tau, \sigma, x_n)$ be the determinant replacing j -column in $R(\tau, \sigma)$ by the transposed vector $(\exp(ix_n \lambda_1^+(\tau, \sigma)), \dots, \exp(ix_n \lambda_l^+(\tau, \sigma)))$. Then $R(\tau, \sigma)$ and $R_j(\tau, \sigma, x_n)$ ($j=1, \dots, l$) are analytic in $(\tau, \sigma) \in C_+ \times \mathbf{R}^{n-1}$. If $R(\tau, \sigma) \neq 0$ for some $(\tau, \sigma) \in C_+ \times \mathbf{R}^{n-1}$, then it is well known that for any $f \in C_0^\infty(\mathbf{R}_+)$ the problem (2.1) has a unique bounded solution $u \in C^\infty(\mathbf{R}_+)$ which is written by the form

$$(2.2) \quad \hat{u}(\tau, \sigma, x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda x_n} \hat{f}(\lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \int_0^\infty G(x_n, s, \tau, \sigma) f(s) ds$$

where

$$G(x_n, s, \tau, \sigma) = - \sum_{j=1}^l \frac{R_j(\tau, \sigma, x_n)}{R(\tau, \sigma)} \int_\Gamma \frac{B_j(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} e^{-i s \lambda} d\lambda$$

and $\Gamma = \Gamma(\tau, \sigma)$ denotes a simple closed curve in the lower half λ -plane enclosing all the roots $\lambda_j^-(\tau, \sigma)$.

Let Σ_+ be the set $\{(\tau', \sigma') \in C_+ \times \mathbf{R}^{n-1}; |\tau'|^2 + |\sigma'|^2 = 1\}$ and $\bar{\Sigma}_+$ be its closure. Furthermore let V be the zeros of $R(\tau, \sigma)$ in $C_+ \times \mathbf{R}^{n-1}$ and V' be $V \cap \Sigma_+$. V^c and V'^c denote the complement of V' and V in Σ_+ and $C_+ \times \mathbf{R}^{n-1}$ respectively. Then we obtain the following

THEOREM 2.1. Suppose that $R(\tau, \sigma)$ is identically not zero. Then the mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν if and only if the following condition is satisfied:

For every $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$ there exist a constant $C(\tau'_0, \sigma'_0)$ and a neighborhood $U(\tau'_0, \sigma'_0)$ such that

$$(2.3) \quad \|(D_{x_n}^k G)(x_n, s, \tau', \sigma')\|_{\mathcal{L}(H_0^\nu(s>0), L^2(x_n>0))} \leq C(\tau'_0, \sigma'_0) (Re \tau')^{-\nu-1}$$

for any $(\tau', \sigma') \in U(\tau'_0, \sigma'_0) \cap \Sigma_+ \cap V'^c$ and $k=0, 1, \dots, m-1$, where

$\|\cdot\|_{\mathcal{L}(H_0^\nu(s>0), L^2(x_n>0))}$ denotes the operator norm from $H_0^\nu(s>0)$ to $L^2(x_n>0)$.

Since the proof of the theorem is accomplished by the almost same considerations as those in Theorem 4.1 [2], we show only different points.

(I) SUFFICIENCY (EXISTENCE OF SOLUTIONS). Let S be the set $\{\sigma \in \mathbf{R}^{n-1}; \{R(\tau, \sigma) \text{ is identically zero in } \tau\}\}$. Then S is a null set with respect to Lebesgue measure in $\mathbf{R}^{n-1}$ because V is so in $\mathbf{C}_+ \times \mathbf{R}^{n-1}$. In what follows we assume $\sigma \notin S$ and for $(\tau, \sigma) \in \mathbf{C}_+ \times \mathbf{R}^{n-1}$ (τ', σ') denotes $(\rho^{-1}\tau, \rho^{-1}\sigma)$ where $\rho = (|\tau|^2 + |\sigma|^2)^{\frac{1}{2}}$.

LEMMA A. There exists an analytic extension $\tilde{G}(\tau, \sigma)$ in $\tau \in \mathbf{C}_+$ of $G(x_n, s, \tau, \sigma)$ as an operator from $H_0^\nu(s>0)$ to $L^2(x_n>0)$ such that $\tilde{G}(\tau, \sigma)f \in H^{m-1}(x_n>0)$ for $f \in H_0^\nu(s>0)$ and

$$(2.4) \quad \begin{aligned} & \|D_{x_n}^k(\tilde{G}(\tau, \sigma)f)\|_{L^2(x_n>0)} \\ & \leq C(\tau'_0, \sigma'_0) (Re \tau)^{-\nu-1} \rho^{k-m+1} \left(\sum_{\mu=0}^{\nu} \rho^{2(\nu-\mu)} \int_0^\infty |(D_s^\mu f)(s)|^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

for any (τ_0, σ_0) with $R(\tau_0, \sigma_0)=0$, (τ, σ) with $(\tau', \sigma') \in U(\tau'_0, \sigma'_0) \cap (\mathbf{C}_+ \times \mathbf{R}^{n-1})$ and $k=0, 1, \dots, m-1$. Here $C(\tau'_0, \sigma'_0)$ and $U(\tau'_0, \sigma'_0)$ are the same ones in Theorem 2.1.

PROOF. If $\sigma_0 \notin S$ and $R(\tau_0, \sigma_0)=0$, then τ_0 is an isolated point in $\mathbf{C}_+$. In virtue of (2.3) and the relations $(D_{x_n}^k G)(x_n, s, \tau, \sigma) = \rho^{k-m+1} (D_{x_n}^k G)(\rho x_n, \rho s, \tau', \sigma')$ ($k=0, 1, \dots, m-1$), we obtain for $f \in H_0^\nu(s>0)$ and $g \in L^2(x_n>0)$

$$(2.5) \quad \begin{aligned} & \left| \left(\int_0^\infty (D_{x_n}^k G)(x_n, s, \tau, \sigma) f(s) ds, g(x_n) \right)_{L^2(x_n>0)} \right| \\ & = \left| \int_0^\infty \overline{g(x_n)} dx_n \int_0^\infty \rho^{k-m+1} (D_{x_n}^k G)(\rho x_n, \rho s, \tau', \sigma') f(s) ds \right| \\ & = \left| \rho^{k-m-1} \int_0^\infty \overline{g(\rho^{-1} x_n)} dx_n \int_0^\infty (D_{x_n}^k G)(x_n, s, \tau', \sigma') f(\rho^{-1} s) ds \right| \\ & \leq \rho^{k-m-1} \|g(\rho^{-1} x_n)\|_{L^2(x_n>0)} \left\| \int_0^\infty (D_{x_n}^k G)(x_n, s, \tau', \sigma') f(\rho^{-1} s) ds \right\|_{L^2(x_n>0)} \\ & \leq \rho^{k-m-\frac{1}{2}} C(\tau'_0, \sigma'_0) (Re \tau')^{-\nu-1} \|g\|_{L^2(x_n>0)} \left(\sum_{\mu=0}^{\nu} \rho^{-2\mu} \int_0^\infty |(D_s^\mu f)(\rho^{-1} s)|^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &= C(\tau'_0, \sigma'_0)(Re\tau)^{-\nu-1} \rho^{\nu-m+k+\frac{1}{2}} \|g\|_{L^2(x_n>0)} \left(\sum_{\mu=0}^{\nu} \rho^{-2\mu} \int_0^{\infty} |(D_s^{\mu} f)(\rho^{-1}s)|^2 ds \right)^{\frac{1}{2}} \\
 &= C(\tau'_0, \sigma'_0)(Re\tau)^{-\nu-1} \rho^{k-m+1} \|g\|_{L^2(x_n>0)} \left(\sum_{\mu=0}^{\nu} \rho^{2\nu-2\mu} \int_0^{\infty} |(D_s^{\mu} f)(s)|^2 ds \right)^{\frac{1}{2}}
 \end{aligned}$$

where (τ, σ) with $(\tau', \sigma') \in (\tau'_0, \sigma'_0) \cap V'^c \cap \Sigma_+$. In particular, it follows from above that, with some $C(\tau_0, \sigma_0)$,

$$\begin{aligned}
 (2.6) \quad & \left| \left(\int_0^{\infty} (D_{x_n}^k G)(x_n, s, \tau, \sigma_0) f(s) ds, g(x_n) \right)_{L^2(x_n>0)} \right| \\
 & \leq C(\tau_0, \sigma_0) \|f\|_{H_0^{\nu}(s>0)} \|g\|_{L^2(x_n>0)}
 \end{aligned}$$

in a small neighborhood of τ_0 . Hence $\left(\int_0^{\infty} (D_{x_n}^k G)(x_n, s, \tau, \sigma_0) f(s) ds, g(x_n) \right)_{L^2(x_n>0)}$ has an analytic extension in C_+ . By Riesz theorem and (2.6) there exist operators $\tilde{G}_k(\tau, \sigma_0)$ ($k=0, 1, \dots, m-1$) from $H_0^{\nu}(s>0)$ to $L^2(x_n>0)$ such that $\tilde{G}_k(\tau, \sigma_0)f, g)_{L^2(x_n>0)}$ is analytic in C_+ and

$$\left(\tilde{G}_k(\tau, \sigma_0)f, g \right)_{L^2(x_n>0)} = \left(\int_0^{\infty} (D_{x_n}^k G)(x_n, s, \tau, \sigma_0) f(s) ds, g(x_n) \right)_{L^2(x_n>0)}$$

for $\tau \neq \tau_0$. In virtue of Banach-Steinhaus theorem there exist uniform derivatives

$$\left(\frac{d}{d\tau} \tilde{G}_k \right)(\tau, \sigma_0) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\tilde{G}_k(\tau+h, \sigma_0) - \tilde{G}_k(\tau, \sigma_0) \right) \quad \text{in } \mathcal{L}(H_0^{\nu}(s>0), L^2(x_n>0))$$

such that

$$\frac{d}{d\tau} \left(\tilde{G}_k(\tau, \sigma_0)f, g \right)_{L^2(x_n>0)} = \left(\left(\frac{d}{d\tau} \tilde{G}_k \right)(\tau, \sigma_0)f, g \right)_{L^2(x_n>0)}.$$

This shows that $\tilde{G}_k(\tau, \sigma_0)$ is an analytic extension in C_+ of $(D_{x_n}^k G)(x_n, s, \tau, \sigma_0)$ as an operator from $H_0^{\nu}(s>0)$ to $L^2(x_n>0)$ and $\|\tilde{G}_k(\tau, \sigma_0)f\|_{L^2(x_n>0)}$ is the same bound as (2.4). Since

$$\begin{aligned}
 & (-1)^k \left(\tilde{G}(\tau_0, \sigma_0)f, D_{x_n}^k g \right)_{L^2(x_n>0)} \\
 &= \lim_{\tau \rightarrow \tau_0} (-1)^k \left(\int_0^{\infty} G(x_n, s, \tau, \sigma_0) f(s) ds, (D_{x_n}^k g)(x_n) \right)_{L^2(x_n>0)} \\
 &= \lim_{\tau \rightarrow \tau_0} \left(\int_0^{\infty} (D_{x_n}^k G)(x_n, s, \tau, \sigma_0) f(s) ds, g(x_n) \right)_{L^2(x_n>0)} \\
 &= \left(\tilde{G}_k(\tau_0, \sigma_0)f, g \right)_{L^2(x_n>0)}
 \end{aligned}$$

where $\tilde{G}(\tau, \sigma_0) = \tilde{G}_0(\tau, \sigma_0)$ and $g \in C_0^{\infty}(\mathbf{R}_+)$. $D_{x_n}^k(\tilde{G}(\tau_0, \sigma_0)f)$ is in $L^2(x_n>0)$ and equal to $\tilde{G}_k(\tau_0, \sigma_0)f$. This finishes the proof.

Let us set

$$(2.7) \quad \hat{u}(\tau, \sigma, x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_n \lambda} \hat{f}(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \tilde{G}(\tau, \sigma) \hat{f}(\tau, \sigma, s)$$

where $f \in C_0^\infty(\mathbf{R}_+ \times \mathbf{R}_+^n)$. Then by Lemma A and the method of the proof of Theorem 4.1 in [2] we have the following

LEMMA B. $\hat{u}(\tau, \sigma, x_n) (\sigma \in S)$ is a solution in $H^m(x_n > 0)$ of (2.1) and satisfies

$$(2.8) \quad \begin{aligned} \|\hat{u}(\tau, \cdot, \cdot)\|_{m-1}^2 &\leq C(\operatorname{Re} \tau)^{-2(\nu+1)} \|f(\tau, \cdot, \cdot)\|_{\nu}^2, \\ \|\hat{u}(\tau, \cdot, \cdot)\|_m^2 &\leq C(\operatorname{Re} \tau)^{-2(\nu+1)} \|f(\tau, \cdot, \cdot)\|_{\nu+1}^2 \end{aligned} \quad (\operatorname{Re} \tau > 0)$$

where

$$\|\hat{u}(\tau, \cdot, \cdot)\|_k^2 = \sum_{j=0}^k \int_{\mathbf{R}^{n-1}} \rho^{2(k-j)} d\sigma \int_0^\infty |(D_{x_n}^j \hat{u})(\tau, \sigma, x_n)|^2 dx_n.$$

Let us set

$$u(t, x) = \frac{1}{(2\pi)^n} \int_{a-i\infty}^{a+i\infty} \int_{\mathbf{R}^{n-1}} \hat{u}(\tau, \sigma, x_n) d\sigma d\tau \quad (a > 0)$$

where $f \in C_0^\infty(\mathbf{R}_+ \times \mathbf{R}_+^n)$ and $\hat{u}(\tau, \sigma, x_n)$ is defined by (2.7). Then by Lemma B $u(t, x)$ is a solution of the problem (P, B_j) which satisfies $e^{-at} u \in H^m(\mathbf{R}_+ \times \mathbf{R}_+^n)$, (1.1) and

$$(2.9) \quad \begin{aligned} &a^{2(\nu+1)} \sum_{j+|\alpha| \leq m} \int_{\mathbf{R}_+ \times \mathbf{R}_+^n} e^{-2at} |(D_x^\alpha \partial_t^j u)(t, x)|^2 dt dx \\ &\leq C \sum_{j+|\alpha| \leq \nu+1} \int_{\mathbf{R}_+ \times \mathbf{R}_+^n} e^{-2at} |(\partial_t^j D_x^\alpha f)(t, x)|^2 dt dx. \end{aligned}$$

$C_0^\infty(\mathbf{R}_+ \times \mathbf{R}_+^n)$ is dense in $H_0^{\nu+1}(\mathbf{R}_+ \times \mathbf{R}_+^n)$. By (1.1), (2.9) and the limit process we obtain a solution $u(t, x)$ for $f \in H_0^{\nu+1}(\mathbf{R}_+ \times \mathbf{R}_+^n)$.

(II) NECESSITY AND UNIQUENESS OF SOLUTIONS.

LEMMA C. Suppose that for fixed $\tau_0 \in C_+$ $\{B_j(\tau_0, \sigma, D_{x_n})\}$ is normal in a bounded open set $D \subset \mathbf{R}^{n-1}$ and $R(\tau_0, \sigma) \neq 0$ for $\sigma \in D$. Let $\hat{u}(\sigma, x_n)$ be a function whose distribution derivatives in x_n up to m belong to $L^2(D \times \mathbf{R}_+)$. Then the problem (2.1) has the following uniqueness property; that is, if for any $\hat{\phi}(\sigma, x_n) \in C_0^\infty(D \times \mathbf{R}_+)$ whose support in σ is contained in D

$$\begin{aligned} (P(\tau_0, \sigma, D_{x_n}) \hat{u}, \hat{\phi})_{L^2(D \times \mathbf{R}_+)} &= 0, \\ (B_j(\tau_0, \sigma, D_{x_n}) \hat{u} \Big|_{x_n=0}, \hat{\phi} \Big|_{x_n=0})_{L^2(D)} &= 0 \quad (j=1, \dots, l), \end{aligned}$$

then $\hat{u} = 0$ in $L^2(D \times \mathbf{R}_+)$.

To prove Lemma C we may use the dual problem.

LEMMA D. Suppose that $\{B_j(\tau_0, \sigma_0, D_{x_n})\}$ is not normal and $R(\tau_0, \sigma_0) \neq 0$. Then there exist a point $(\tilde{\tau}_0, \tilde{\sigma}_0)$ sufficiently closed to (τ_0, σ_0) and a normal set $\{B'_j(\tilde{\tau}_0, \sigma, D_{x_n})\}$ in a sufficiently small neighborhood $D(\tilde{\sigma}_0)$ such that Lopatinskii's determinant for $(P(\tilde{\tau}_0, \sigma, D_{x_n}), B'_j(\tilde{\tau}_0, \sigma, D_{x_n}))$ does not vanish in $D(\tilde{\sigma}_0)$ and $(P(\tilde{\tau}_0, \sigma, D_{x_n}), B_j(\tilde{\tau}_0, \sigma, D_{x_n}))$ and $(P(\tilde{\tau}_0, \sigma, D_{x_n}), B'_j(\tilde{\tau}_0, \sigma, D_{x_n}))$ ($\sigma \in D(\tilde{\sigma}_0)$) have the same solutions.

PROOF. Let us set

$$B_j(\tau, \sigma, D_{x_n}) = \sum_{k=0}^{m'_j} b_{j,k}(\tau, \sigma) D_{x_n}^k.$$

Here we may assume that $m'_j \leq m_j$ and $b_{j,m'_j}(\tau, \sigma)$ is identically not zero in a neighborhood of (τ_0, σ_0) . Furthermore we may assume that $R(\tau, \sigma) \neq 0$ in a neighborhood of (τ_0, σ_0) . We carry out the following two processes in this neighborhood: First if $b_{j,m'_j}(\tau_0, \sigma_0) = 0$ then there is a point $(\tilde{\tau}_0, \tilde{\sigma}_0)$ closed to (τ_0, σ_0) such that $b_{j,m'_j}(\tilde{\tau}_0, \sigma) \neq 0$ in a neighborhood of $\tilde{\sigma}_0$. Thus we replace $B_j(\tau_0, \sigma_0, D_{x_n})$ by $b_{j,m'_j}(\tilde{\tau}_0, \sigma)^{-1} B_j(\tilde{\tau}_0, \sigma, D_{x_n})$. Second, after the first process, if $m'_j = m'_k$ then we replace B_j or B_k by $B_j - B_k$. Remark that in these processes it is invariant that Lopatinskii's determinant does not vanish. Hence, after each process, it does not occur the case that B_j and B_k ($j \neq k$) are monomials in D_{x_n} of same degree. Therefore Lemma D is obtained by carrying out successively these processes from B_j of highest order in D_{x_n} .

To prove the uniqueness let $\hat{u}(\tau, \sigma, x_n)$ ($\operatorname{Re} \tau \geq a$) be the Fourier-Laplace transform of a solution u of (P, B_j) ($f=0, e^{-at} u \in H^m(\mathbf{R}_+ \times \mathbf{R}_+^n)$). Then for an arbitrarily fixed point τ_0 with $\operatorname{Re} \tau_0 \geq a$ the two equations in Lemma C are valid for any $\hat{\varphi}(\sigma, x_n) \in C_0^\infty(\mathbf{R}^{n-1} \times \mathbf{R}_+)$. From the proof of Lemma D we see that the set Q of all the points $(\tilde{\tau}_0, \tilde{\sigma}_0)$ with $\operatorname{Re} \tilde{\tau}_0 \geq a$ satisfying the conclusion in Lemma D is almost everywhere equal to $\{\tau; \operatorname{Re} \tau \geq a\} \times \mathbf{R}^{n-1}$. Therefore it follows from Lemma C that $\hat{u} = 0$ in $L^2(Q \times \mathbf{R}_+)$, which implies that for some a' ($a' \geq a$) $\hat{u}(a' + i\eta, \sigma, x_n) = 0$ almost everywhere in (η, σ, x_n) .

Now we prove the necessity of Theorem 2.1. First, in the proof of theorem 4.1 in [2] pp. 142–144, a sequence $\{(\tau'_p, \sigma'_p)\}$ may be replaced by $\{(\tilde{\tau}'_p, \tilde{\sigma}'_p)\}$ where the conclusion of Lemma D is satisfied for each point $(\tilde{\tau}'_p, \tilde{\sigma}'_p)$. Here it may be assumed that if $p \rightarrow \infty$,

$$(\operatorname{Re} \tilde{\tau}'_p)^{p+1} \|(D_{x_n}^{k_0} G)(x_n, s, \tilde{\tau}'_p, \tilde{\sigma}'_p)\|_{\mathcal{L}(H_0^p(s>0), L^2(x_n>0))} \rightarrow \infty$$

because $\|(D_{x_n}^k G)(x_n, s, \tau, \sigma)\|_{\mathcal{L}(H_0^p(s>0), L^2(x_n>0))}$ is continuous in (τ, σ) .

Second we use Lemma C in order to show the inequality in lines 7–10 in [2] p. 143. Thus we may prove our assertion as we have done in [2].

§ 3. Lopatinskii's determinant

In this section we prove the following

THEOREM 3.1. *Suppose that a constant coefficient problem (P, B_j) is L^2 -well-posed with decreasing order ν and $R(\tau, \sigma)$ is identically not zero. Then $R(\tau_0, \sigma_0) \neq 0$ for $(\tau_0, \sigma_0) \in C_+ \times \mathbf{R}^{n-1}$ is equivalent to the fact that $B_j(\tau_0, \sigma_0, \lambda)$ ($j=1, \dots, l$) are linearly independent as polynomials in λ .*

From Theorem 3.1 we obtain immediately

COROLLARY 3.2. *Under the assumptions of Theorem 3.1. and the normality of $\{B_j\}$, $R(\tau, \sigma) \neq 0$ for any $(\tau, \sigma) \in C_+ \times \mathbf{R}^{n-1}$.*

PROOF OF THEOREM 3.1. $R(\tau_0, \sigma_0) \neq 0$ for $(\tau_0, \sigma_0) \in C_+ \times \mathbf{R}^{n-1}$ is equivalent to the fact that $B_j(\tau_0, \sigma_0, \lambda)$ are linearly independent modulo $\prod_{j=1}^l (\lambda - \lambda_j^+(\tau_0, \sigma_0))$ as polynomials in λ . Hence the necessity is obvious.

Let us set $B_j(\sigma, \tau, \lambda) = \sum_k b_{j,k}(\tau, \sigma) \lambda^k$. Since $B_j(\tau_0, \sigma_0, \lambda)$ are linearly independent, the matrix $(b_{j,k}(\tau, \sigma))$ has rank l ; that is, there exist $(k_1, \dots, k_l)$ and a neighborhood $U(\tau_0, \sigma_0)$ in $C_+ \times \mathbf{R}^{n-1}$ such that

$$(3.1) \quad \det(b_{j,k_h}(\tau, \sigma); \overset{j \downarrow}{h \rightarrow} 1, \dots, l) \neq 0 \text{ in } U(\tau_0, \sigma_0).$$

Using (3.1) we can construct a function v satisfying

$$(3.2) \quad B_j(\tau, \sigma, D_{x_n})v|_{x_n=0} = g_j \quad (j=1, \dots, l)$$

for any $g_j \in C$ and $(\tau, \sigma) \in U(\tau_0, \sigma_0)$. In fact, if $v(\tau, \sigma, x_n) = \sum_{h=1}^l v_{k_h}(\tau, \sigma) x_n^{k_h} (k_h!)^{-1}$ then (3.2) becomes to

$$\sum_{h=1}^l b_{j,k_h}(\tau, \sigma) v_{k_h}(\tau, \sigma) = g_j \quad (j=1, \dots, l).$$

In the rest of this section (τ, σ) are considered as parameters and belong to $U(\tau_0, \sigma_0) \cap V^c$, where $U(\tau_0, \sigma_0)$ is assumed, if necessary, sufficiently small.

Let u_1 be a solution of the Cauchy problem:

$$\begin{aligned} P(\tau, \sigma, D_{x_n})u_1 &= P(\tau, \sigma, D_{x_n})(\varphi v) & x_n > 0, \\ D_{x_n}^k u_1 &= 0 & (k=0, \dots, m-1) \quad x_n = 0 \end{aligned}$$

where $\varphi \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\varphi = 1$ ($0 \leq x_n \leq 2^{-1}$) and $\varphi = 0$ ($x_n \geq 1$). We consider the problem:

$$\begin{aligned} P(\tau, \sigma, D_{x_n})u &= P(\tau, \sigma, D_{x_n})(\varphi v - \varphi u_1) & x_n > 0, \\ B_j(\tau, \sigma, D_{x_n})u &= 0 & (j=1, \dots, l) \quad x_n = 0. \end{aligned}$$

Since $f \equiv P(\tau, \sigma, D_{x_n})(\varphi(v - u_1)) \in C_0^\infty(x_n > 0)$, the problem has a unique solution:

$$\begin{aligned} u(\tau, \sigma, x_n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_n \lambda} \hat{f}(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \int_0^{\infty} G(x_n, s, \tau, \sigma) f(\tau, \sigma, s) ds \\ &= \tilde{u}_1(\tau, \sigma, x_n) + \tilde{u}_2(\tau, \sigma, x_n). \end{aligned}$$

Since $|P(\tau, \sigma, \lambda)|^2 \geq C(Re\tau)^2 (|\tau|^2 + |\sigma|^2 + |\lambda|^2)^{m-1}$, we obtain for some constant $C(\tau_0, \sigma_0) > 0$

$$(3.3) \quad \|\tilde{u}_1(\tau, \sigma, x_n)\|_{H^{m-1}(x_n > 0)} \leq C(\tau_0, \sigma_0) \|f(\tau, \sigma, x_n)\|_{L^2(x_n > 0)}.$$

Furthermore it follows from the assumption and Theorem 2.1 that for some constant $C(\tau_0, \sigma_0) > 0$

$$(3.4) \quad \|\tilde{u}_2(\tau, \sigma, x_n)\|_{L^2(x_n > 0)} \leq C(\tau_0, \sigma_0) \|f(\tau, \sigma, x_n)\|_{H_0^s(x_n > 0)}.$$

From (3.3) and (3.4) we have

$$(3.5) \quad \|u(\tau, \sigma, x_n)\|_{L^2(x_n > 0)} \leq C(\tau_0, \sigma_0) \|f(\tau, \sigma, x_n)\|_{H_0^s(x_n > 0)}.$$

If we put $w = \varphi v - \varphi u_1 - u$, w is an L^2 -solution of

$$(3.6) \quad \begin{aligned} P(\tau, \sigma, D_{x_n})w &= 0 & x_n > 0, \\ B_j(\tau, \sigma, D_{x_n})w &= g_j & (j=1, \dots, l) \quad x_n = 0. \end{aligned}$$

Furthermore, by (3.5) and the construction of v and u_1 , we see that for some $C(\tau_0, \sigma_0) > 0$

$$(3.7) \quad \|w(\tau, \sigma, x_n)\|_{L^2(x_n > 0)}^2 \leq C(\tau_0, \sigma_0) \sum_{j=1}^l |g_j|^2.$$

On the other hand, if $R(\tau, \sigma) \neq 0$ then the problem (3.6) has a unique solution in $L^2(x_n > 0)$ which is written by the form

$$(3.8) \quad w(\tau, \sigma, x_n) = \sum_{j=1}^l \frac{R_j(\tau, \sigma, x_n)}{R(\tau, \sigma)} g_j.$$

Now we arrange the roots $\lambda_j^+(\tau, \sigma)$ into q -groups $\{\lambda_{k,h}^+(\tau, \sigma) \mid h=1, 2, \dots, k'\}$ ($k=1, \dots, q$) in a sufficiently small neighborhood $U(\tau_0, \sigma_0)$ such that $\lambda_{k,1}^+(\tau_0, \sigma_0) = \dots = \lambda_{k,k'}^+(\tau_0, \sigma_0)$. Let us set

$$\gamma_{k,1}(\tau, \sigma, x_n) = \exp(ix_n \lambda_{k,1}^+(\tau, \sigma)),$$

$$\gamma_{k,h}(\tau, \sigma, x_n) = (ix_n)^{h-1} \int_0^1 d\theta_1 \dots d\theta_{h-2} \int_0^1 \theta_1^{h-2} \dots \theta_{h-2} \exp(ix_n g_{k,h}(\tau, \sigma, \theta)) d\theta_{h-1},$$

$$B_j^{k,h}(\tau, \sigma) = \int_0^1 d\theta_1 \dots d\theta_{h-2} \int_0^1 \theta_1^{h-2} \dots \theta_{h-2} (\partial_{\lambda}^{h-1} B_j)(\tau, \sigma, g_{k,h}(\tau, \sigma, \theta)) d\theta_{h-1},$$

$$\begin{aligned} g_{k,h}(\tau, \sigma, \theta) &= \lambda_{k,1}^+(\tau, \sigma) + (\lambda_{k,2}^+(\tau, \sigma) - \lambda_{k,1}^+(\tau, \sigma)) \theta_1 + \\ &\dots + (\lambda_{k,h}^+(\tau, \sigma) - \lambda_{k,h-1}^+(\tau, \sigma)) \theta_1 \dots \theta_{h-1} \quad (h \geq 2). \end{aligned}$$

Then we have

$$\begin{aligned}
 R(\tau, \sigma) &= \begin{vmatrix} \cdot & \cdots & \cdot \\ B_1^{k,1}(\tau, \sigma) & \cdots & B_l^{k,1}(\tau, \sigma) \\ \cdot & \cdots & \cdot \\ B_1^{k,k'}(\tau, \sigma) & \cdots & B_l^{k,k'}(\tau, \sigma) \\ \cdot & \cdots & \cdot \end{vmatrix} / \Delta(\tau, \sigma) \\
 &= R'(\tau, \sigma) / \Delta(\tau, \sigma), \\
 R_j(\tau, \sigma, x_n) &= \begin{vmatrix} \cdot & \cdots & \cdot & \cdots & \cdot \\ B_1^{k,1}(\tau, \sigma) & \cdots & \gamma_{k,1}(\tau, \sigma, x_n) & \cdots & B_l^{k,1}(\tau, \sigma) \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ B_1^{k,k'}(\tau, \sigma) & \cdots & \gamma_{k,k'}(\tau, \sigma, x_n) & \cdots & B_l^{k,k'}(\tau, \sigma) \\ \cdot & \cdots & \cdot & \cdots & \cdot \end{vmatrix} / \Delta(\tau, \sigma) \\
 &= R'_j(\tau, \sigma, x_n) / \Delta(\tau, \sigma)
 \end{aligned}$$

where $\Delta(\tau, \sigma) \neq 0$ in $U(\tau_0, \sigma_0)$. It follows from (3.8) that

$$w(\tau, \sigma, x_n) = \sum_{k,h} \left(\sum_{j=1}^l \frac{A_j^{k,h}(\tau, \sigma)}{R'(\tau, \sigma)} g_j \right) \gamma_{k,h}(\tau, \sigma, x_n)$$

where $A_j^{k,h}$ is a cofactor of $R'(\tau, \sigma)$ with respect to $B_j^{k,h}(\tau, \sigma)$. Since $\gamma_{k,h}(\tau, \sigma, x_n)$ are linearly independent, it follows from (3.7) and the same method in [2] p. 146 that for some $C(\tau_0, \sigma_0) > 0$

$$\begin{aligned}
 (3.9) \quad & \left| \frac{A_j^{k,h}(\tau, \sigma)}{R'(\tau, \sigma)} \right| < C(\tau_0, \sigma_0) \\
 & (j=1, \dots, l, \quad k=1, \dots, q \quad \text{and} \quad h=1, \dots, k').
 \end{aligned}$$

By the definition of $A_j^{k,h}(\tau, \sigma)$ we have

$$R'(\tau, \sigma)^{-1} = \det \left(\frac{A_j^{k,h}(\tau, \sigma)}{R'(\tau, \sigma)} \right).$$

Hence it follows from this and (3.9) that for some $C(\tau_0, \sigma_0) > 0$

$$R'(\tau, \sigma) > C(\tau_0, \sigma_0).$$

In virtue of the continuity of $R(\tau, \sigma)$ we conclude that $R(\tau_0, \sigma_0) \neq 0$.

§ 4. Necessary condition for L^2 -well-posedness (The case of variable coefficients)

In this section we consider variable coefficient problems (P, B_j) . Here coefficients are smooth and constant except a compact set in $\mathbf{R}^{n+1}$.

Let $(P^0, B_j^0)_{(t, x')}$ be a constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point $(t, x', 0)$ and $R(t, x'; \tau, \sigma)$ be Lopatinskii's determinant for the problem $(P^0, B_j^0)_{(t, x')}$. Then we have the following

THEOREM 4.1. *Suppose that a variable coefficient problem (P, B_j) is strongly L^2 -well-posed and $R(t, x'; 1, 0) \neq 0$ for any boundary point $(t, x', 0)$. Then each constant coefficient problem $(P^0, B_j^0)_{(t, x')}$ is L^2 -well-posed (with $\nu=0$).*

PROOF. First we shall show that for an arbitrarily fixed boundary point $(t_0, x'_0, 0)$ ($0 \leq t_0 < T$) the problem $(P^0, B_j^0)_{(t_0, x'_0)}$ with $f=0$ and initial data $U(t_1) = (u_0, u_1, \dots, u_{m-1})$ ($0 \leq t_1 < T$) is strongly L^2 -well-posed. Here $u_k \in C_0^\infty(\mathbf{R}_+^n)$ for $k=0, 1, \dots, m-1$. By the assumption there exists a unique solution $v_\varepsilon \in \mathcal{E}^0((t_0, T), H^m(\mathbf{R}_+^n)) \cap \dots \cap \mathcal{E}^m((t_0, T), H^0(\mathbf{R}_+^n))$ of the problem (P, B_j) with $f=0$ and initial data $V_\varepsilon(t_0) = (v_{0,\varepsilon}, v_{1,\varepsilon}, \dots, v_{m-1,\varepsilon})$ which satisfy energy inequalities (1.2) and (1.3). Here $v_{k,\varepsilon}(x) = \varepsilon^{-k} u_k((x' - x'_0)\varepsilon^{-1}, x_n \varepsilon^{-1})$. Let us set $u_\varepsilon(s, y) = v_\varepsilon(t_0 + \varepsilon(s - t_1), x'_0 + \varepsilon y', \varepsilon y_n)$. Then $u_\varepsilon(s, y)$ becomes a solution of the equations:

$$\begin{aligned} & \text{for } s \in [t_1, t_1 + \varepsilon^{-1}(T - t_0)) \\ & P(t_0 + \varepsilon(s - t_1), x'_0 + \varepsilon y', \varepsilon y_n; \varepsilon^{-1} \partial_s, \varepsilon^{-1} D_y) u = 0 \quad \text{in } y \in \mathbf{R}_+^n, \\ (4.1) \quad & B_j(t_0 + \varepsilon(s - t_1), x'_0 + \varepsilon y'; \varepsilon^{-1} \partial_s, \varepsilon^{-1} D_y) u = 0 \quad (j=1, \dots, l) \\ & \text{in } y_n = 0, y' \in \mathbf{R}^{n-1}, \\ & \text{with initial data } (s=t_1) \end{aligned}$$

$$\begin{aligned} \partial_s^k u_\varepsilon &= \varepsilon^k v_{k,\varepsilon}(x'_0 + \varepsilon y', \varepsilon y_n) \\ &= u_k(y) \quad (k=0, 1, \dots, m-1) \quad \text{in } y \in \mathbf{R}_+^n. \end{aligned}$$

Furthermore it follows from (1.2) and (1.3) that u_ε satisfy

$$\begin{aligned} (4.2) \quad & \|u_\varepsilon(s, \cdot)\|_{m-1,\varepsilon} \leq C \|U(t_1)\|_{m-1,\varepsilon} \quad (t_1 \leq s \leq t_1 + \varepsilon^{-1}(T - t_0)) \\ & \|u_\varepsilon(s, \cdot)\|_{m,\varepsilon} \leq C \|U(t_1)\|_{m,\varepsilon} \end{aligned}$$

where

$$\|u(s, \cdot)\|_{k,\varepsilon} = |u(s, \cdot)|_k + \sum_{h=0}^{k-1} \varepsilon^{k-h} |u(s, \cdot)|_h$$

and $|u(s, \cdot)|_k$ denotes the norm obtained by replacing $|\alpha| \leq k$ in definition of $\|u(s, \cdot)\|_k$ by $|\alpha| = k$. Therefore, by (4.1) and (4.2), there exists a weak limit $u(s, \cdot)$ of a subsequence of $\{u_\varepsilon(s, \cdot)\}$ in $H^m(\mathbf{R}_+^n)$ as $\varepsilon \rightarrow 0$ such that $u(s, y)$ is a solution in $\mathcal{E}^0((t_1, T), H^m(\mathbf{R}_+^n)) \cap \dots \cap \mathcal{E}^m((t_1, T), H^0(\mathbf{R}_+^n))$ of the problem $(P^0, B_j^0)_{(t_0, x'_0)}$ and satisfies the energy inequalities

$$(4.3) \quad \begin{aligned} |u(s, \cdot)|_{m-1} &\leq C|U(t_1)|_{m-1}, \\ |u(s, \cdot)|_m &\leq C|U(t_1)|_m. \end{aligned}$$

Since $R(t_0, x'_0; 1, 0) \neq 0$ the problem $(P^0, B_j^0)_{(t_0, x'_0)}$ has a finite propagation speed (See [6]). Hence the problem $(P^0, B_j^0)_{(t_0, x'_0)}$ has a unique solution. Using Poincaré lemma and the finiteness of propagation speed it follows from (4.3) that for any s ($t_1 \leq s \leq T$)

$$(4.4) \quad \|u(s, \cdot)\|_{m-1} \leq C(s) \|U(t_1)\|_{m-1}$$

where $C(s)$ depends continuously on s and propagation speed. Thus we can define an operator $G(s, t_1)$ from initial data $U(t_1) \in C_0^\infty(\mathbf{R}_+^n)^m$ to the solution $u(s, y)$.

Next we shall show that the problem $(P, B_j)_{(t_0, x'_0)}$ with $f \in C_0^\infty((0, T) \times \mathbf{R}_+^n)$ and zero initial data is L^2 -well-posed ($\nu=0$). Let us set

$$u(t, x) = \int_0^t G(t, s) F(s) ds$$

where $F(s) = (0, \dots, 0, f(s, x))$. Then $u(t, x)$ becomes a solution of the problem $(P^0, B_j^0)_{(t_0, x'_0)}$ such that, by (4.4),

$$(4.5) \quad \|u(t, \cdot)\|_{m-1}^2 \leq C(t) \int_0^t \|f(t, \cdot)\|_0^2 dt \quad (0 \leq t \leq T)$$

By integrating (4.5) from 0 to T we obtain for some $C(T) > 0$

$$(4.6) \quad \int_0^T \|u(t, \cdot)\|_{m-1}^2 dt \leq C(T) \int_0^T \|f(t, \cdot)\|_0^2 dt.$$

Therefore the problem $(P^0, B_j^0)_{(t_0, x'_0)}$ with $f \in C_0^\infty((0, T) \times \mathbf{R}_+^n)$ and homogeneous initial-boundary conditions has a unique solution $u \in H^m((0, T) \times \mathbf{R}_+^n)$ satisfying (4.6). Only this fact is used in the proof of the necessity of Theorem 4.1 in [2]. Thus the proof is complete.

Finally we consider mixed problems of second order :

$$\begin{aligned} P &= \partial_t^2 - 2 \sum_{j=1}^n a_j(t, x) \partial_t \partial_{x_j} - \sum_{j,k=1}^n a_{jk}(t, x) \partial_{x_j} \partial_{x_k} + \text{first order term}, \\ B &= \partial_{x_n} - \sum_{j=1}^{n-1} b_j(t, x') \partial_{x_j} - c(t, x') \partial_t + h(t, x') \end{aligned}$$

where $\sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k > 0$ for any non zero vector $\xi \in \mathbf{R}^n$ and all the coefficients are real valued.

Combining Theorem 4.1 with results in [1] we obtain

THEOREM 4.2. *Suppose that $R(t, x'; 1, 0) \neq 0$ on the boundary. Then*

a variable coefficient problem (P, B) is strongly L^2 -well-posed if and only if each constant coefficient problem $(P^0, B^0)_{(t, x')}$ is L^2 -well-posed (with $\nu=0$).

REMARK. Let (P^0, B^0) denotes $(P^0, B^0)_{(t, x)}$ for a fixed point $(t, x', 0)$. Then the following statements (I), (II) and (III) are equivalent:

(I) (P^0, B^0) is L^2 -well-posed.

(II) Lopatinskiĭ's determinant for (P^0, B^0) does not vanish in $C_+ \times \mathbf{R}^{n-1}$ and (P^0, B^0) has no supersonic speeds.

(III) $a_{nn}c + a_n \geq 0$ and the quadratic form $H(\sigma) = (a_{nn}c + a_n)^2 (a_{nn}e - d^2) - 2(a_{nn}c + a_n)(a_{nn}a - a_n d)(a_{nn}b + d) - (a_{nn} + a_n^2)(a_{nn}b + d)^2$ is positive semi-definite, where

$$\begin{aligned} e &= \sum_{j,k=1}^{n-1} a_{jk} \sigma_j \sigma_k, & d &= \sum_{j=1}^{n-1} a_{nj} \sigma_j, \\ a &= \sum_{j=1}^{n-1} a_j \sigma_j, & b &= \sum_{j=1}^{n-1} b_j \sigma_j \quad (\sigma \in \mathbf{R}^{n-1}). \end{aligned}$$

The equivalence (I) and (III) has been proved in [1] and other equivalences are proved by using results in § 2 of [1] and [6].

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