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Vitali's theorem in vector lattices

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

Let Ω be a measure space with finite measure μ . Then, Vitali's theorem announces: *let $f_n (n=1, 2, \dots)$ be a sequence of summable functions on Ω with equi-absolutely continuous integrals and f_n converges to f in measure. Then,*

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu.$$

In this note, we shall generalize the Vitali's theorem to vector lattices.

§ 2. Convergences in vector lattices.

Let R be a σ -complete vector lattice i.e. $\bigcap_{n=1}^{\infty} a_n$ exists for every positive elements $0 \leq a_n \in R (n=1, 2, \dots)$. In the sequel, we assume that R is σ -complete. For $a_n \in R (n=1, 2, \dots)$, if $\bigcap_{m=1}^{\infty} (\bigcup_{n \geq m} a_n)$ and $\bigcup_{m=1}^{\infty} (\bigcap_{n \geq m} a_n)$ exist and equal to a , then we denote that

$$o\text{-}\lim_{n \rightarrow \infty} a_n = a.$$

In this case, we say that the sequence $\{a_n\}$ is *order-convergent* to a . It is easy to see that $o\text{-}\lim a_n = a$ iff there exist $b_n \downarrow 0$ (i.e. $b_1 \geq b_2 \geq \dots$ with $\bigcap_{n=1}^{\infty} b_n = 0$) such that

$$|a_n - a| \leq b_n$$

where $|x| = x \cup (-x) = x^+ + x^-$ for $x \in R$.

We shall define star-order-convergence as follows: a sequence $a_n (n=1, 2, \dots)$ is said to be *star-order-convergent* to a if for every subsequence of $\{a_n\}$, there exists its subsequence which is order-convergent to a . We shall denote

$$s\text{-}o\text{-}\lim a_n = a$$

if $a_n (n=1, 2, \dots)$ is star-order-convergent to a .

For a subset M of R , we denote $M^{\perp} = \{a : |a| \cap |b| = 0 \text{ for all } b \in M\}$.

If we can decompose $a \in R$ into as follows :

$$(A) \quad a = a_1 + a_2$$

with $a_1 \in M$ and $a_2 \in M^\perp$, then M is called normal.

If M is normal, then $M = M^{\perp\perp}$ and the decomposition (A) of a is uniquely determined. Namely,

$$a = a_1 + a_2, \quad a = a'_1 + a'_2, \quad a_1, a'_1 \in M, \quad a_2, a'_2 \in M^\perp$$

imply $a_1 = a'_1$ and $a_2 = a'_2$.

We see that the operator $[M]a = a_1$ is linear and lattice-homomorphic. $[M]$ is called a projection operator.

The normal subsets of R (or equivalently projection operators) constitutes a Boolean lattice by the usual order.

Let R be σ -complete. For $0 \leq p \in R$, the subset $\{p\}^{\perp\perp}$ is normal and

$$[\{p\}^{\perp\perp}]a = (\text{denoted by } [p]a) = \bigcup_{n=1}^{\infty} (np \cap a) \quad \text{for } a \geq 0.$$

In general, $[p]a = [p]a^+ - [p]a^-$.

$\{[N]; [p] \geq [N]\}$ is σ -complete as a Boolean lattice and for every $[N]$ with $[p] \geq [N]$ there exists $q \in R$ with $[N] = [q]$.

For arbitrary $p \in R$, $[p] = [|p|]$.

Let $[p_n]$ be a sequence of projection operators. $[P_n]$ is order-convergent to 0 if $\bigcap_{m=1}^{\infty} (\bigcup_{n \geq m} P_n) = 0$, i. e. there exists $[Q_n] \geq [P_n]$ with $[Q_1] \geq [Q_2] \geq \dots$ and $\bigcap_{n=1}^{\infty} [Q_n] = 0$.

We shall write $[P_n] \downarrow 0$ if $[P_1] \geq [P_2] \geq \dots$ and $\bigcap_{n=1}^{\infty} [P_n] = 0$.

We shall denote $[P_n] \downarrow\downarrow 0$ if for every subsequence of $[P_n]$ there exists its subsequence order-convergent to 0.

Now, we shall consider a special convergence in a σ -complete vector lattice.

We shall denote

$$(\otimes) - \lim_{n \rightarrow \infty} a_n = a$$

if there exists $[P_m] \downarrow\downarrow 0$ such that $(I - [P_m])a_n$ is star-order-convergent to $(I - [P_m])a$ ($m = 1, 2, \dots$). In the case of L_1 -space (the totality of summable functions) $f_n \rightarrow f$ (in measure) implies $f_n \rightarrow f$ (in above sense).

It is easy to see that if $(\otimes) - \lim_{n \rightarrow \infty} a_n = a$, then there exists $[p_m] \downarrow\downarrow 0$ such that $p_m \in R$ ($m = 1, 2, \dots$) and $(I - [p_m])a_n$ is star-order-convergent to $(I - [p_m])a$ for all $m = 1, 2, \dots$.

We see easily $[p_n] \downarrow 0$ and $[q_n] \downarrow 0$ imply $[p_n] \cup [q_n] \downarrow 0$. Hence, we see

that if $\bigotimes\text{-}\lim_{n \rightarrow \infty} a_n = a$ and $\bigotimes\text{-}\lim_{n \rightarrow \infty} b_n = b$, then

$$\begin{aligned}\bigotimes\text{-}\lim_{n \rightarrow \infty} (a_n + b_n) &= \bigotimes\text{-}\lim_{n \rightarrow \infty} a_n + \bigotimes\text{-}\lim_{n \rightarrow \infty} b_n, \\ \bigotimes\text{-}\lim_{n \rightarrow \infty} (a_n \cup b) &= a \cup b, \quad \bigotimes\text{-}\lim_{n \rightarrow \infty} (a_n \cap b) = a \cap b, \\ \bigotimes\text{-}\lim_{n \rightarrow \infty} |a_n - a| &= 0.\end{aligned}$$

§ 3. Equi-continuous subsets

Let $\bar{a}$ be a linear functional on R . $\bar{a}$ is said to be *o-continuous linear functional* if $o\text{-}\lim a_n = a$ implies $\bar{a}(a_n) \rightarrow \bar{a}(a)$.

It is easy to see that if $\bar{a}$ is an *o-continuous linear functional* on R , then

$$\sup_{0 \leq b \leq |a|} |\bar{a}(b)| < +\infty \quad \text{for all } a \in R.$$

The totality of *o-continuous linear functionals* is denoted by $\bar{R}$ and is called an *order-conjugate space* of R . Since R is reduced to 0 in some occasion, we assume that R is not trivial in the sense that for every $a \neq 0$, there exists $\bar{a} \in \bar{R}$ with $\bar{a}(a) \neq 0$.

$\bar{a} \geq \bar{b}$ means that $\bar{a}(a) \geq \bar{b}(a)$ for all $a \geq 0$.

By this order, $\bar{R}$ is a complete vector lattice. In the sequel, we assume that R is not trivial. A subset Γ of R is *equi-continuous* if for $a_n \downarrow 0$, $a_n \in \Gamma$ and $\varepsilon > 0$, there exists a natural number n_0 with

$$\bar{a}_{n_0}(|a|) \leq \varepsilon \quad \text{for all } a \in \Gamma.$$

By definition, if Γ is an equi-continuous subset of R , then $N[\Gamma] = \{b; |b| \leq |a| \text{ for some } a \in \Gamma\}$ is also an equi-continuous subset. It is known that Γ is equi-continuous iff Γ is relative compact by the weak topology induced by $\bar{R}$.

For $[P_n] \downarrow 0$ and $\bar{a} \in \bar{R}$, $\bar{a}[P_n](a) = \bar{a}([P_n]a)$ is also *o-continuous linear functional* for all n and $\bar{a}[P_n] \downarrow 0$.

§ 4. Vitali's theorem

Vitali's theorem for summable functions can be formulated as follows in the case of vector lattices.

THEOREM 1. *Let $a_n \in R$ ($n=1, 2, \dots$) be an equi-continuous sequences and $\bigotimes\text{-}\lim_{n \rightarrow \infty} a_n = a$. Then, a_n is weakly convergent to a (i.e. $\bar{a}(a_n) \rightarrow \bar{a}(a)$ for all $\bar{a} \in \bar{R}$).*

PROOF. Let $[p_n] \downarrow \downarrow 0$ and $0 \leq \bar{a} \in \bar{R}$. We shall prove that we can find

a natural number n_0 such that

$$\bar{a}([p_n]|a_m|) \leq \varepsilon \quad \text{for } n \geq n_0 \quad \text{and } m=1, 2, \dots.$$

If not, we find $\varepsilon > 0$ and $m_\nu \leq m_{\nu+1} \leq \dots$ and $n_\nu \leq n_{\nu+1} \leq \dots$ such that

$$\bar{a}([p_{n_\nu}]|a_{m_\nu}|) \geq \varepsilon \quad \nu = 1, 2, \dots.$$

By assumption, there exists a subsequence $[q_\nu]$ of $[p_{n_\nu}]$ ($\nu=1, 2, \dots$) order-convergent to 0 such that

$$\bar{a}([q_\nu]|a_{m_\nu}|) \geq \varepsilon.$$

This contradicts to the equi-continuity of $\Gamma = \{a_n\}$.

We shall prove that $s\text{-}0\text{-}\lim a_n = a$ implies $\bar{a}(a_n) \rightarrow \bar{a}(a)$.

If not, there exists n_ν ($\nu=1, 2, \dots$) such that

$$|\bar{a}(a_{n_\nu} - a)| \geq \varepsilon \quad \text{for some } \varepsilon > 0.$$

By assumption, we find a subsequence of $\{a_{n_\nu}\}$ order-convergent to a . This is a contradiction, since $\bar{a}$ is continuous by order-convergence.

Let $\bigotimes\text{-}\lim_{n \rightarrow \infty} a_n = a$. There exists $[a] \geq [p_m] \downarrow \downarrow 0$ such that

$$s\text{-}o\text{-}\lim_{n \rightarrow \infty} (I - [p_m]) a_n = (I - [p_m]) a.$$

Hence, choosing m such that $|\bar{a}([p_m](a_n - a))| < \varepsilon$, we have

$$|\bar{a}(a_n) - \bar{a}(a)| \leq |\bar{a}([p_m](a_n - a))| + |\bar{a}((I - [p_m])(a_n - a))| \leq 2\varepsilon$$

for sufficient large n .

This proves Theorem 1.

COROLLARY. Let $\bigotimes\text{-}\lim_{n \rightarrow \infty} a_n = a$. $\{a_n\}$ is weakly convergent to a if and only if $\{a_n\}$ is equi-continuous.

§ 5. $|w|$ -convergence.

By definition,

$|w|\text{-}\lim_{n \rightarrow \infty} a_n = a$ iff $\lim_{n \rightarrow \infty} \bar{a}(|a_n - a|) = 0$ for all $\bar{a} \in \bar{R}$. $|w|\text{-}\lim_{n \rightarrow \infty} a_n = a$ implies that a_n is weakly convergent to a . But, in general the converse is not true.

THEOREM 2. Under the assumption of Theorem 1, we have

$$|w|\text{-}\lim_{n \rightarrow \infty} a_n = a.$$

PROOF. If $\{a_n\}$ is equi-continuous, then $\{x_n\}$ is equi-continuous where $x_n = |a_n - a|$. We see easily

$$\bigotimes\text{-}\lim_{n \rightarrow \infty} a_n = a \quad \text{iff} \quad \bigotimes\text{-}\lim_{n \rightarrow \infty} x_n = \bigotimes\text{-}\lim_{n \rightarrow \infty} |a_n - a| = 0.$$

By Theorem 1, $\{x_n\}$ is weakly convergent to 0. But this means that

$$|\omega| - \lim_{n \rightarrow \infty} a_n = a.$$

REMARK. If R is a space of summable functions defined on a finite measure space, under the assumption of Theorem 1, we have $\lim_{n \rightarrow \infty} \|a_n - a\| = 0$ by Theorem 2.

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References

- [1] H. NAKANO: Modulated semi-ordered linear spaces, Maruzen, Tokyo, 1950.
- [2] I. NATANSON: Theory of functions of a real variable, (English translation) Ungar, New York, 1955.

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