Title	第4回関数空間セミナー報告集
Author(s)	大久保，和義；中路，貴彦
Description	1995年12月25日（月）～12月27日（水）
Citation	Hokkaido University technical report series in mathematics, 41, 1
Issue Date	1995-01-01
DOI	https://doi.org/10.14943/5167
Doc URL	https://hdl.handle.net/2115/5481
Type	departmental bulletin paper
File Information	41.pdf

第4回 関数空間セミナー 報告集

1995年12月25日（月）〜12月27日（水）

代表者　大久保 和義・中路 貴彦

Series #41. February, 1996

HOKKAIDO UNIVERSITY

TECHNICAL REPORT SERIES IN MATHEMATICS

♯ 7: S. Izumiya,G. Ishikawa (Eds.), "特異点と微分幾何" 研究集会報告集,1988.

♯ 8: K. Kubota (Ed.), 第13回偏微分方程式論札幌シンポジウム予稿集, 76 pages. 1988.

♯ 9: Y. Okabe (Ed.), ランジュヴァン方程式とその応用予稿集, 64 pages. 1988.

♯ 10: I. Nakamura (Ed.), Superstring 理論と K3 曲面, 91 pages. 1988.

♯ 11: Y. Kamishima (Ed.), 1988年度談話会アブストラクト集 Colloquium Lectures, 73 pages. 1989.

♯ 12: G. Ishikawa, S. Izumiya and T. Suwa (Eds.), "特異点論とその応用" 研究集会報告集 Proceedings of the Symposium "Singularity Theory and its Applications," 317 pages. 1989.

♯ 13: M. Suzuki, "駆け足で有限群を見てみよう"1987年7月北大での集中講義の記録, 38 pages. 1989.

♯ 14: J. Zajac, Boundary values of quasiconformal mappings, 15 pages. 1989

♯ 15: R. Agemi (Ed.), 第14回偏微分方程式論札幌シンポジウム予稿集, 55 pages. 1989.

♯ 16: K. Konno, M.-H. Saito and S. Usui (Eds.), Proceedings of the Meeting and the workshop "Algebraic Geometry and Hodge Theory" Vol. I, 258 pages. 1990.

♯ 17: K. Konno, M.-H. Saito and S. Usui (Eds.), Proceedings of the Meeting and the workshop "Algebraic Geometry and Hodge Theory" Vol. II, 235 pages. 1990.

♯ 18: A. Arai (Ed.), 1989年度談話会アブストラクト集 Colloquium Lectures, 72 pages. 1990.

♯ 19: H. Suzuki (Ed.), 複素多様体のトポロジー Topology of Complex Manifolds, 133 pages. 1990.

♯ 20: R. Agemi (Ed.), 第15回偏微分方程式論札幌シンポジウム予稿集, 65 pages. 1991.

♯ 21: Y. Giga, Y. Watatani (Eds.), 1990年度談話会アブストラクト集 Colloquium Lectures, 105 pages. 1991.

♯ 22: R. Agemi (Ed.), 第16回偏微分方程式論札幌シンポジウム予稿集, 50 pages. 1991.

♯ 23: Y. Giga, Y. Watatani (Eds.), 1991年度談話会・特別講演アブストラクト集 Colloquium Lectures, 89 pages. 1992.

♯ 24: K. Kubota (Ed.), 第17回偏微分方程式論札幌シンポジウム予稿集, 29 pages. 1992.

♯ 25: K. Takasaki, "非線型可積分系の数理" 1992.9.28～10.2 北海道大学での集中講義　講義録, 52 pages. 1993.

♯ 26: T. Nakazi (Ed.), 第1回関数空間セミナー報告集, 93 pages. 1993.

♯ 27: K. Kubota (Ed.), 第18回偏微分方程式論札幌シンポジウム予稿集, 40 pages. 1993.

♯ 28: T. Hibi (Ed.), 1992年度談話会・特別講演アブストラクト集 Colloquium Lectures, 108 pages. 1993.

♯ 29: I. Sawashima, T. Nakazi (Eds.), 第2回関数空間セミナー報告集, 79 pages. 1994.

♯ 30: Y. Giga, Y.-G. Chen, 動く曲面を追いかけて, 講義録, 62 pages. 1994.

♯ 31: K. Kubota (Ed.), 第19回偏微分方程式論札幌シンポジウム予稿集, 33 pages. 1994.

♯ 32: T. Ozawa (Ed.), 1993年度談話会・特別講演アブストラクト集 Colloquium Lectures, 113 pages. 1994.

♯ 33: Y. Okabe (Ed.), The First Sapporo Symposium on Complex Systems, 24 pages. 1994.

♯ 34: A. Arai, Infinite Dimensional Analysis on an Exterior Bundle and Supersymmetric Quantum Field Theory, 10 pages. 1994.

♯ 35: S. Miyajima, T. Nakazi (Eds.), 第3回関数空間セミナー報告集, 104 pages. 1995.

♯ 36: N. Kawazumi (Ed.), リーマン面に関連する位相幾何学, 63 pages. 1995.

♯ 37: I. Tsuda (Ed.), The Second & Third Sapporo Symposium on Complex Systems, 190 pages. 1995.

♯ 38: M. Saito (Ed.), 1994年度談話会・特別講演アブストラクト集 Colloquim Lectures, 100 pages. 1995.

♯ 39: S. Izumiya (Ed.), 接触幾何学と関連分野研究集会報告集, 186 pages. 1995.

♯ 40: H. Komatsu, A. Kishimoto (Eds.), 作用素論・作用素環論研究集会予稿集, 61 pages. 1995.

第4回 関数空間セミナー　報告集

１９９５年１２月２５日（月）～１２月２７日（水）

代表者　大久保　和義
　　　　中路　貴彦

目　次

A Note on a Problem of an Extermal Problem in H^1

J. Inoue

January 26, 1996

Abstract : It is shown that, for each $p(0 < p < 1)$, there exist an exposed point of the uint ball of the classical Hardy space $H^1(D)$, which has the property that f^{-1} dose not belong to H^p.

In the study of the classical Hardy space $H^1 = H^1(D)$ of the unit disc D, it is an interesing and also an important problem to characterize the exposed points (of the uint ball of H^1).

Characterization of the exposed points are studied by several authors. such as [1], [3], [4], [5], [6], [7], [8]. and etc.

On the other hand, some means to constract various concrete examples of the exposed points are known:

(a). If $f \in H^1, \|f\|_1 = 1$, and $\Re f \geq 0$ on $T = \partial D$,
then f is an exposed point [8].

(b). If $f \in H^1, \|f\|_1 = 1$, and f^{-1} belongs to H^1, then f is an exposed point [8]

(c). If $f \in H^1, \|f\|_1 = 1$, and if there exists some non zero $k \in H^\infty$ such that $\Re[kf] \geq 0$, then f is an exposed point [5].

With use of (a), (b) and (c), we can construct various concrete examples of exposded points easily.

For example, if q is an inner function, then $(1+q)/\|(1+q)\|_1$ is an exposed point by (a).

Also, by use of (b), we can know easily that if f is an analytic polynomial with $\|f\| = 1$, then f is an exposed point if and only if f has no zero on D, and has at most simple zeros on T.

On the other hand, we can easily see that if f satisfy (a), (b) or (c) , then f^{-1} must belongs to H^p for each $0 < p < 1$.

In this short note, we show that, for each $0 < p < 1$, we can construct an exposed point sucht that f^{-1} dose not belong to H^p.

In the following, we need next definition and theorem [2].

Defintion 1. If f is a function in H^1 with norm 1, we put
$S_f = \{g \in H^1 : \|g\|_1 = 1, \ \ g(e^{it})/f(e^{it}) \geq 0.a.e.t\}$

S_f is the solution set of the well knouwn extermal probelm in H^1. f is an exposed point (of the unit ball of H^1) if and only if $S_f = \{f\}$.

Theorem 1[1]. Suppose f is a function in H^1 with $\|f\|_1 = 1$. If f^{-1} is locally belongs to H^1 on the unit circle except a finite set A of T, and $f^{-1} \in H^p$ for some $p > 0$, then the linear span of S_f has finite dimension.

We now prove the following theorem.

Theorem 2. For each $p(0 < p < 1)$, there exists an exposed point of the unit ball of H^1 sucn that f^{-1} dose not belong to H^p.

Proof. Let u be a positive measurable function on $(0, 2\pi)$ such that
(i) u is locally integrable $(0, 2\pi)$,
(ii) If we put $g = e^{iu(t)}$, we have $g^{-1}(e^{it}) \notin L^p(T)$, but $g^{-1}(e^{it}) \in L^{p/2}(T)$,
(iii) $g(e^{it})/(1 - e^{it})^2 \notin L^1(T)$.
Let $f(z)$ be an outer function defined by

$$f(z) = \lambda \exp \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log g(e^{it}) dt/2\pi \quad (z \in D)$$

where λ is a positive constant to ensure $\|f\|_1$ =1.

Then by the definition of f, f^{-1} locally belongs to H^1 except point in $A = \{1\}$, $f^{-1}(e^{it}) \notin H^p$, but $f^{-1} \in H^{p/2}$. Therefore by theorem 1, we have that the linear span of S_f is of finite dimensional. Since f is outer, these imply that if f is not exposed, f must be devided in H^1 by a function of the form $(z - \alpha)^2$ for some $\alpha \in T$ [5]. But it is impossible by the condition (i) and (iii), and we can conclude that f is an exposed point of the unit ball of H^1. Q.E.D.

References

[1] K. deLeeuw and W. Rudin, Exteme points and extremal problems in H^1 , Pacific J. Math. 8(1958), 467-485.

[2] J. Inoue and T. Nakazi, Finite dimensional solution sets of extemarproblems in H^1, Operator Theory. Advances and Applications, Vol. 62(1993), 115-124.

[3] E. Hayashi, Solution sets of extramal problems in H^1, Proc. Amer. Math. soc. 93(1985), 690-696.

[4] H. Helson, Large anslytic functions II , in Analysis and Partial differntial Equations(ed. Cora Sadoky, Marcvel Dekker, 1991), 217-220.

[5] T. Nakazi, Exposed points and extremal problems in H^1, J.Functi. Anal. 53(1983)., 224-230.

[6] T. Nakazi, Sum of two inner functions and exposed points in H^1, Proc. Edinburgh Math. Soc. 35(1992), 349-357.

[7] D. Sarason, Exposed points in H^1, I, Oper. Theory Adv. Appl. 41(1989), 485-496.

[8] K. Yabuta, Some uniqeness theorem for $H^p(U^n)$ functions, Tohoku Math. J.24(1972), 353-357.

Jyunji Inoue

Department of Mathematics

Hokkaido University

Sapporo 060 Japan

Scaling Functions associated with Unimodular Wavelets for L^2 *

Atsushi Nobuta †

Abstract

We construct Scaling Functions associated with Unimodular Wavelets for L^2. We also find a necessary and sufficient condition for a class of Unimodular Wavelet for L^2 constructed in reference [1] to be associated with Multiresolution Analysis.

1 Introduction

ここでは、Y.-H. Ha, H. Kang, J. Lee, & J. K. Seo [1] で論じられた Unimodular Wavelets に対応する Scaling Function の構成及びその特定をその目的とする。Unimodular Wavelets とは、Meyer's Equations を基に Unimodular 条件 (i.e. $|\widehat{\psi}(\xi)| = 1$ for all $\zeta \in \mathrm{supp}\widehat{\psi}$) を課して構成された Orthonormal Wavelets のことである。$\psi \in L^2(\mathbf{R})$ が $L^2(\mathbf{R})$ に対する Orthonormal Wavelet であるとは、$\psi \in L^2(\mathbf{R})$, $j, k \in \mathbf{Z}$ に対して、記号 $\psi_{j,k}$ を

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$$

としたとき、族 $\{\psi_{j,k}\}$ が $L^2(\mathbf{R})$ に対して Orhtonormal basis となることである。この Orthonormal Wavelets を構成するには、基本的に 2 つの方法がある。1 つは、Meyer's Equations に基づく方法で、もう 1 つは、Multiresolution Analysis (MRA) に基づく方法である。[1] において構成された Unimodular Wavelets は前者に基づいている。この Meyer's Equations は Orthonormal Wavelets に対する必要十分条件で Y. Meyer によって与えられ、[4] において論じられている。証明は、[1] を参照されたい。

Theorem 1.1 (Meyer's Equations : [1])
$\psi \in L^2(\mathbf{R})$ が、$L^2(\mathbf{R})$ に対して *Orthonormal Wavelet* となるための必要十分条件は、次の等式 (W.1) ~ (W.4) を満たすことである。

(W1) $\sum_{k=-\infty}^{\infty} |\widehat{\psi}(\xi + 2k\pi)|^2 = 1$

(W2) $\sum_{k=-\infty}^{\infty} \widehat{\psi}(\xi + 2k\pi)\overline{\widehat{\psi}(2^j(\xi + 2k\pi))} = 0$, for $j \geq 1$

(W3) $\sum_{j=-\infty}^{\infty} |\widehat{\psi}(2^{-j}\xi)|^2 = 1$

(W4) $\sum_{l=0}^{\infty} \widehat{\psi}(2^l(\xi + 2p_0\pi))\overline{\widehat{\psi}(2^l\xi)} = 0$, for $p_0 \in 2\mathbf{Z} + 1$

これらの等式は、ほとんど至るところの意味で、成り立つことを意味する。

---

*The author wishes to thank Professor J.Inoue for helpful and good guidance regarding the subject of this paper

†Department of Mathematics, Faculty of Science, Hokkaido University

この Meyer's Equations に Unimodular 条件を課すことにより、Theorem 1.1 の (W1)⇒(W4) , (W3)⇒(W2) が示せ、Unimodular Wavelets の必要十分条件は、(W.1)(W.3) を満たすこととなる ([1])。さらに、$\mathbf{K} = \mathrm{supp}\widehat{\psi}$ とすると Unimodular Wavelets の必要十分条件 (W.1)(W.3) は、この $\mathbf{K}$ に対する必要十分条件として、次のように特徴付けられる。

Theorem 1. 2 (Unimodular Wavelets for $L^2(\mathbf{R})$: [1])
$\psi \in L^2(\mathbf{R})$ を *Unimodular Function* とし、

$$\mathbf{K} = \mathrm{supp}\widehat{\psi},\ \mathbf{K}^+ = \mathbf{K} \cap (\,0,\infty),\ \mathbf{K}^- = \mathbf{K} \cap (\,-\infty,0)$$

とすると、ψ が $L^2(\mathbf{R})$ に対して *Orthonormal Wavelet* となるための必要十分条件は、次の 2 つの条件を満たすことである。

(1) 適当な $a \in \mathbf{R}$ に対して、τ_aは、null-set を除いて、$\mathbf{K}$上 one-to-one で、かつ

$$m(\ [\,a, a+2\pi) - \tau_a(\mathbf{K})\ ) = 0$$

(2) 適当な $a > 0, b < 0$ に対して、δ_a, δ_b は、null-set を除いて、各々 $\mathbf{K}^+, \mathbf{K}^-$上 one-to-one で、かつ

$$m(\ [\,a, 2a) - \delta_a(\mathbf{K}^+)\ ) = 0\ ,\ m(\ (2b, b\,] - \delta_b(\mathbf{K}^-)\ ) = 0$$

ここで、m は $\mathbf{R}$ 上の *Lebesgue measure* であり、関数τ_a, δ_aは

$$\tau_a : \mathbf{R} \longrightarrow [\,a, a+2\pi)\ ,\ a \le \tau_a(x) := x + 2^{\exists 1}k(x)\pi \le a + 2\pi\ ,\ \text{for } a \in \mathbf{R}$$
$$\delta_a : [\,0,\infty) \longrightarrow [\,a, 2a)\ ,\ a \le \delta_a(x) := 2^{\exists 1 j(x)}x \le 2a\ ,\ \text{for } a > 0$$

でまた、$a < 0$ ならば、$x < 0$ に対して、$\delta_a(x) = -\delta_{-a}(-x)$ とする。

これを用いて次に、[1] で構成された Unimodular Wavelets の 3 つクラスをあげる。

Theorem 1. 3 ([1])

$$\widehat{\psi}(\xi) = e^{i\frac{\xi}{2}}\chi_{\mathbf{K}}(\xi) \tag{1.1}$$

$$\mathbf{K} = [\,2a - 4\pi,\ a - 2\pi] \cup [\,a,\ 2a]\ ,\ 0 < a < 2\pi$$

とすると、ψ は $L^2(\mathbf{R})$ に対して *Orthonormal Wavelet* となる。逆に、$\mathbf{K}^+, \mathbf{K}^-$ が各々 1 つの区間からなる *Unimodular Wavelets* はこの形式のみである。

Theorem 1. 4 ([1])

$$\mathbf{K}^+ = \left[\frac{2^j}{2^{j+1}-1}\pi,\ \pi\right] \cup \left[2^j\pi,\ 2^j\pi + \frac{2^j}{2^{j+1}-1}\pi\right]\ ,\ \mathbf{K}^- = -\mathbf{K}^+\ ,\ j > 0$$

とすると、式 (1.1) の ψ は、$L^2(\mathbf{R})$ に対して *Orthonormal Wavelet* となる。逆に、$\mathbf{K}^+$ が互いに素な 2 つの区間からなり、$\mathbf{K}^-$ が原点に関して $\mathbf{K}^+$ と対称な区間からなる *Unimodular Wavelets* はこの形式のみである。

Theorem 1. 5 ([1])
$j \ge 2,\ 1 \le p \le 2^j - 2$ に対して

$$\mathbf{K}^- = \left[-2\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi,\ -\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi\right]$$

$$\mathbf{K}^+ = \left[\frac{2(p+1)}{2^{j+1}-1}\pi,\ \frac{2(2p+1)}{2^{j+1}-1}\pi\right] \cup \left[\frac{2^{j+1}(2p+1)}{2^{j+1}-1}\pi,\ \frac{2^{j+2}(p+1)}{2^{j+1}-1}\pi\right]$$

とすると、式 (1.1) の ψ は $L^2(\mathbf{R})$ に対して *Orthonormal Wavelet* となる。逆に、$\mathbf{K}^-$ が 1 つの区間で、$\mathbf{K}^+$ が互いに素な 2 つの区間からなる *Unimodular Wavelets* はこの形式のみである。

次の節では、この論文の目的である Unimodular Wavelets に対応する Scaling Functions の必要十分条件を与え、上の 3 つの例が MRA に対応するか否かを考察する。

2 Statement of results

この節では、[1] においては Meyer's Equations を基に Unimodular Wavelets を構成したのに対し、MRA に基づいた Unimodular Wavelets の構成を試みた結果として、著者が得られた結果を述べる。そのために、まず MRA の定義を与えておく。

Definition 2. 1 (Multiresolution Analysis : [2],[3])

$$\{\mathrm{V}_j\}_{j\in\mathbf{Z}} \ : \ \text{sequence of closed subspace of } L^2(\mathbf{R})$$

としたとき、$\{\mathrm{V}_j\}_{j\in\mathbf{Z}}$ が次の条件 (R1)~(R4) を満たすとき、$L^2(\mathbf{R})$ の MRA という。

(R1) $\mathrm{V}_j \subset \mathrm{V}_{j+1}$

(R2) $\overline{\bigcup_{j=-\infty}^{\infty} \mathrm{V}_j} = L^2(\mathbf{R})$, $\bigcap_{j=-\infty}^{\infty} \mathrm{V}_j = \{0\}$

(R3) $f(x) \in \mathrm{V}_j \iff f(2x) \in \mathrm{V}_{j+1}$

(R4) $\exists\, \phi \in \mathrm{V}_0$ s.t. $\{\phi(x+k)\}_{k\in\mathbf{Z}}$; *orthonormal basis* for V_0

この ϕ を $\{\mathrm{V}_j\}$ を生成する *Scaling Function* という。

ここで、$\{\mathrm{V}_j\}_{j\in\mathbf{Z}}$ を MRA 、ϕ を Scaling Function とし、また $\mathbf{W}_j$ を V_j の直交補空間 ($\mathrm{V}_j \oplus \mathbf{W}_j = \mathrm{V}_{j+1}$) とする。このとき、$\{\psi(\cdot - k) : k \in \mathbf{Z}\}$ が $\mathbf{W}_0$ の Orthonormal basis になるような関数 ψ をみつければ、この ψ が Orthonormal Wavelet となる。一般に、Orthonormal Wavelet ψ が Scaling Function ϕ に対応していると

$$\widehat{\psi}(2\zeta) = e^{i\zeta}\overline{\mathrm{m}_0(\zeta+\pi)}\widehat{\phi}(\zeta) \tag{2.1}$$

を満たす。ここで、m_0 は

$$\widehat{\phi}(2\zeta) = \mathrm{m}_0(\zeta)\widehat{\phi}(\zeta)$$

を満たす 2π 周期で、$L^2([\ 0,\ 2\pi\ ])$ に属す関数である。

これらを用いて $L^2(\mathbf{R})$ に対しての Unimodular Wavelets に対応する Scaling Functions もまた Unimodular Function となるので、対応する Scaling Functions の必要十分条件は次のように特徴付けられる。

Theorem 2. 1 (Scaling Functions associated with Unimodular Wavelets for $L^2(\mathbf{R})$: [5])

$$\widehat{\phi} = \chi_{\mathrm{M}} \ , \ \mathrm{M} \subset \mathbf{R} \ : \ \text{measurable}$$

とすると、ϕが $L^2(\mathbf{R})$ に対して *Scaling Function* となるための必要十分条件は、次の 3 つの条件を満たすことである。

(1) $\lim_{n\to\infty} m([\ -\pi,\ \pi\ ]\backslash 2^n\mathrm{M}) = 0$

(2) $\mathrm{M}\subset 2\mathrm{M}$

(3) 適当な $a \in \mathbf{R}$ に対して、τ_aは、null-set を除いて、M 上 one-to-one で、かつ

$$m(\ [\ a, a+2\pi) - \tau_a(\mathrm{M})\ ) = 0$$

ここで、m は $\mathbf{R}$ 上の *Lebesgue measure* である。

ここで m_0 を $\mathrm{m}_0(\zeta)=\sum_{l\in\mathbf{Z}}\widehat{\phi}(2(\zeta+2l\pi))$ とおくと、上の Scaling Function ϕ に対して

$$\widehat{\phi}(2\zeta)=\mathrm{m}_0(\zeta)\widehat{\phi}(\zeta)$$

を満たす 2π 周期で、$L^2([\ 0,\ 2\pi\ ])$ に属す関数となる。従って、(2.1) より

$$\begin{aligned}\widehat{\psi}(\zeta) &= e^{i\frac{\zeta}{2}}\overline{\mathrm{m}_0\left(\frac{\zeta}{2}+\pi\right)}\widehat{\phi}\left(\frac{\zeta}{2}\right)\\ &= e^{i\frac{\zeta}{2}}\sum_{l\in\mathbf{Z}}\widehat{\phi}(\zeta+2\pi(2l+1))\widehat{\phi}\left(\frac{\zeta}{2}\right)\end{aligned}$$

を得る。よって、Theorem 2.1 の Scaling Function ϕ は、

$$\widehat{\psi}(\zeta)=e^{i\frac{\zeta}{2}}\left(\sum_{l\in\mathbf{Z}}\widehat{\phi}(\zeta+2\pi(2l+1))\right)\widehat{\phi}\left(\frac{\zeta}{2}\right) \tag{2.2}$$

によって Unimodular Wavelet ψ に対応している。

これを用いて、Theorem 1.3 ~ Theorem 1.5 の Unimodular Wavelets が、どのような Scaling Functions に対応しているかについて考察する。但し、Th.1.3 と Th.1.4 については、[1] の結果によりわかっている。実際、Th.1.3 に対しては Scaling Functions として

$$\widehat{\phi}(\xi)=\chi_{[\ a-2\pi,\ a\ ]}(\xi)\ ,\ 0<a<2\pi$$

が対応しており、Th.1.4 では、$j=1$ のときのみ MRA に対応し、Unimodular Wavelet ψ と Scaling Function ϕ は、

$$\widehat{\psi}(\xi)=e^{i\frac{\xi}{2}}\ \chi_{\mathbf{K}}(\xi)$$

$$\mathbf{K}=\left[\ -\frac{8}{3}\pi,\ -2\pi\ \right]\cup\left[\ -\pi,\ -\frac{2}{3}\pi\ \right]\cup\left[\ \frac{2}{3}\pi,\ \pi\ \right]\cup\left[\ 2\pi,\ \frac{8}{3}\pi\ \right]$$

$$\widehat{\phi}(\xi)=\chi_{\mathrm{M}}(\xi)$$

$$\mathrm{M}=\left[\ -\frac{4}{3}\pi,\ -\pi\ \right]\cup\left[\ -\frac{2}{3}\pi,\ \frac{2}{3}\pi\ \right]\cup\left[\ \pi,\ \frac{4}{3}\pi\ \right]$$

である。従って、後は Th.1.5 について考察するわけだが、[1] により p が奇数のときは MRA に対応しないことが得られているので、p として偶数のみを扱う。すると、MRA に対応するための必要十分条件が次のように得られる。

Theorem 2. 2 (Scaling Functions associated with Unimodular Wavelets of Theorem 1.5 : [5])
ψ を Th.1.5 において構成された *Unimodular Wavelet* で、p を偶数とする。このとき ψ が、*MRA* に対応するための必要十分条件は、

$$\mathbf{X}:=\mathbf{X}(j,p,k)=\frac{(2^j-2^k)p+2^j-2^{k-1}}{2^{j+1}-1}$$

が整数となる $k\ (1\leq k\leq j-1)$ が存在することである。

また、そのとき対応する *Scaling Function* ϕ は

$$\widehat{\phi}(\xi)=\chi_{\mathrm{M}}(\xi)$$

$$\mathrm{M}=\left[\ -\left(1-\frac{2p+1}{2^{j+1}-1}\right)\pi,\ \frac{2(p+1)}{2^{j+1}-1}\pi\ \right]\cup\left(\bigcup_{l=1}^{j}\left[\frac{2^l(2p+1)}{2^{j+1}-1}\pi,\ \frac{2^{l+1}(p+1)}{2^{j+1}-1}\pi\ \right]\right)$$

である。

この Theorem 2.2 は、Theorem 2.1 の条件 (3) における区間 $[\,a,\ a+2\pi)$ を複素平面の単位円 $\mathbf{T}$ に置き換えることにより得られた。また、MRA に対応する必要十分条件が、$\mathbf{X}$ が整数となる k が存在するときのみであることが大変興味深いものであると思われる。以下に、$\mathbf{X}$ が整数となる k $(1 \leq k \leq j)$ が存在するときの j,p を j=2∼20 まで列記しておく。

j=2 ,p=2
j=3 ,p=6
j=4 ,p=4,10,14
j=5 ,p=30
j=6 ,p=8,20,42,54,62
j=7 ,p=36,90,126
j=8 ,p=16,84,170,238,254
j=9 ,p=72,438,510
j=10,p=32,136,292,340,682,730,886,990,1022
j=11,p=660,1386,2046
j=12,p=64,272,584,1188,1364,2730,2906,3510,3822,4030,4094
j=13,p=528,2340,5850,7662,8190
j=14,p=128,2184,5460,10922,14198,16254,16382
j=15,p=1056,4680,10836,21930,28086,31710,32766
j=16,p=256,2080,4368,9288,18724,21140,21844,43690,44394,46810
56246,61166,63454,65278,65534
j=17,p=17544,38052,93018,113526,131070
j=18,p=512,4160,16912,34952,37448,75044,84628,87380,174762,177514
187098,224694,227190,245230,257982,261630,262142
j=19,p=8256,149796,174420,349866,374490,516030,524286
j=20,p=1024,33824,69904,304292,349524,699050,744282,978670
1014750,1047550,1048574

最後に、Theorem 2.1 と Theorem 2.2 の証明に関しては参考文献の [5] を参照されたい。

References

[1] Young-Hwa Ha, Hyeonbae Kang, Jungseob Lee, and Jin Keun Seo. *Unimodular Wavelets for* L^2 *and the Hardy Space* H^2. Michigan Math. J. 41(1994). 345-361.

[2] I.Daubechies. Ten Lectures on Wavelets. SIAM-NSF Regional Conference Series. 61. SIAM. Philadelphia.1992.

[3] S.Mallat. *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbf{R})$. Trans. Amer. Math. Soc. 315(1989). 69-88.

[4] P.G.Lemarié. *Analyse multi-échelles et ondelette á support compact.* Les Ondelettes en 1989. (P.G.Lemarié,ed.). Lecture Notes in Math, 1438. 26-38. Springer, Berlin, 1990.

[5] 信田 篤. *Scaling Functions associsted with Unimodular Wavelets for* L^2 *and the Hardy Space* H^2. 北海道大学大学院理学研究科数学専攻 修士論文 1996.

Martingale transforms in a Banach space

Toshitada SHINTANI

1. Notations. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{A}_1, \mathcal{A}_2, \ldots$ a nondecreasing sequence of sub-σ-fields of $\mathcal{A}$. Let X be a Banach space with norm $|\cdot|$ and the Radon-Nikodým property. Let $f=(f_1,f_2,\ldots)$ be an X-valued martingale with norm $\|f\|_1 = \sup_n E|f_n| < \infty$ relative to $\mathcal{A}_1, \mathcal{A}_2, \ldots$. Here E denotes expectation : integration over Ω with respect to P. Let $v=(v_1,v_2,\ldots)$ be a real-valued predictable sequence, that is, $v_k:\Omega \to R$ is $\mathcal{A}_k$-measurable, $k \geqslant 1$. Then $g=(g_1,g_2,\ldots)$, defined by $g_n = \sum_{k=1}^{n} v_k(f_{k+1} - f_k)$ with $|v| \leqslant 1$ in absolute value, is the transform of the martingale f by v.

2. Real-valued case. Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$. If Z is a random variable with finite mean, by the Radon-Nikodým theorem, for Z there is a $\mathcal{B}$-measurable function φ which is satisfying

$$\int_A Z(\omega)\, dP = \int_A \varphi(\omega)\, dP \quad \text{for every } A \in \mathcal{B}$$

and which decides the correspondence $Z \longrightarrow \varphi$, i. e., $F: Z(\omega) \longmapsto \varphi(\omega)$. This function φ is unique up to a set of P-measure zero, and any such function, denoted by $E(Z/\mathcal{B})(\omega)$, is called the conditional expectation of $Z(\omega)$ relative to $\mathcal{B}$. Therefore, the above correspondence F is written by

$$E(Z/\mathcal{B})(\omega) = \varphi(\omega) = F(Z(\omega)/\mathcal{B}) \text{ for almost all } \omega \in \Omega.$$

If $f=(f_1,f_2,\ldots)$ is a martingale then, for almost all ω ,

$$F(f_{n+1}(\omega) / \mathcal{A}_n) = f_n(\omega) \quad (n=1,2,\ldots)$$

and F is linear: for all $\lambda, \mu \in R$

$$F(\lambda \cdot Y(\omega) + \mu \cdot Z(\omega) / \mathcal{B}) = F((\lambda \cdot Y + \mu \cdot Z)(\omega)/\mathcal{B})$$
$$= E(\lambda \cdot Y + \mu \cdot Z / \mathcal{B})(\omega) = \lambda \cdot F(Y(\omega)/\mathcal{B}) + \mu \cdot F(Z(\omega)/\mathcal{B}).$$

Let X= R, that is, let $f=(f_1,f_2,\ldots)$ be an L^1-bounded and real-valued martingale. Then $|\cdot|$ denotes the absolute value.

Theorem 1. If $\|f\|_1 < \infty$ then $\sum_{n=1}^{\infty} | f_{n+1} - f_n | < \infty$ a. e., that is, f is of bounded variation.

Proof. Suppose that there exists a subset M of Ω such that $P(M) \neq 0$ and $\sum_{n=1}^{\infty} | f_{n+1}(\omega) - f_n(\omega) | = \infty$ for all $\omega \in M$.

Then, for any $G = G(\omega) > 0$ there is a number $N = N(G, \omega) > 0$ such that $\sum_{k=1}^{n} | f_{k+1}(\omega) - f_k(\omega) | > G$ on M $(\forall n \geqslant N)$.

So there are a number $k = k(\omega) \leqslant n$ and a positive real number $G' = G'(\omega)$ such that $| f_{k+1}(\omega) - f_k(\omega) | = G' > 0$ for each $\omega \in M$. Here, set $G' = G'(\omega') = | f_{k(\omega)+1}(\omega') - f_{k(\omega)}(\omega') |$ for each $\omega \in M$ $(\omega' \in \Omega,\ M \subset \Omega)$.

G' is well-defined on Ω and $G' > 0$ when $\omega' = \omega$, i. e., $G' > 0$ on M.

Now, when $\omega' = \omega$ $| f_{k+1}(\omega) - f_k(\omega) |$ is defined on M.

By the definition of the absolute value

$$| f_{k+1}(\omega) - f_k(\omega) |$$
$$= \begin{cases} f_{k+1}(\omega) - f_k(\omega) & \text{on } A \overset{\text{def.}}{=\!=} \{\omega \,;\, f_{k+1}(\omega) \geqq f_k(\omega)\} \ (\subset M) \\ -(f_{k+1}(\omega) - f_k(\omega)) & \text{on } M \setminus A. \end{cases}$$

Since $k(\omega) = k < \infty$, $\{k(\omega) ; \omega \in M\} \subset \{1, 2, \dots, n, \dots\}$. Thus,

$$E\, | f_{k(\omega)}(\omega') | \leqslant \sup_{\lambda \in \{k(\omega);\, \omega \in M\}} E |f_\lambda| \leqslant \sup_{\lambda \in \{1,2,\dots,n,\dots\}} E |f_\lambda| = \sup_n E |f_n| < \infty .$$

So $| f_{k+1} - f_k | \in L^1$. For almost all $\omega \in A$

$$E(| f_{k+1} - f_k | / \mathcal{A}_k)(\omega) = F(| f_{k+1} - f_k | (\omega) / \mathcal{A}_k)$$
$$= F(| f_{k+1}(\omega) - f_k(\omega) | / \mathcal{A}_k)$$
$$= F(f_{k+1}(\omega) - f_k(\omega) / \mathcal{A}_k)$$
$$= f_k(\omega) - f_k(\omega) = 0.$$

For almost all $\omega \in M \setminus A$

$$E(| f_{k+1} - f_k | / \mathcal{A}_k)(\omega) = F(| f_{k+1}(\omega) - f_k(\omega) | / \mathcal{A}_k)$$
$$= F(f_k(\omega) - f_{k+1}(\omega) / \mathcal{A}_k)$$
$$= f_k(\omega) - f_k(\omega) = 0.$$

Therefore $E(| f_{k+1} - f_k | / \mathcal{A}_k)(\omega) = 0$ for almost all $\omega \in M$.

On the other hand, if $E(G' / \mathcal{A}_{k(\omega)})(\omega') = 0$ $(k = k(\omega))$ for almost all $\omega' \in \Omega$ then $E(G'(\omega')) = E(E(G' / \mathcal{A}_{k(\omega)})(\omega')) = 0.$

Thus, $G' = 0$ a. e. for each $\omega \in M$. This contradicts to $G' > 0$ on M.

So $E(G'/\mathcal{A}_k)(\omega') \neq 0$ for some $\omega' = \omega \in M$.

Since $P(M) \neq 0$, $M \neq \phi$. Then

$$\begin{aligned} 0 &= F(\,|f_{k+1}(\omega) - f_k(\omega)|\,/\mathcal{A}_k\,) \\ &= F(\,G'(\omega)\,/\mathcal{A}_k\,) \\ &= E(\,G'/\mathcal{A}_k\,)(\omega) \\ &\neq 0 \quad \text{for some } \omega \in M. \end{aligned}$$

This is a contradiction. Thus there is not such M.

Therefore $\sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| < \infty$ for almost all $\omega \in \Omega$.

3. Vector-valued case. Let $Z(\omega)$ be a Bochner-integrable function on a probability space $(\Omega, \mathcal{A}, P)$ taking values in X.

Let $\mathcal{B}$ be a sub-σ-field contained in $\mathcal{A}$. Then the conditional expectation $E(Z/\mathcal{B})(\omega)$ of $Z(\omega)$ relative to $\mathcal{B}$ is defined as a Bochner-integrable function on $(\Omega, \mathcal{A}, P)$ such that $E(Z/\mathcal{B})$ is $\mathcal{B}$-measurable and that $\int_A Z(\omega)dP = \int_A E(Z/\mathcal{B})(\omega)dP$, $\forall A \in \mathcal{B}$, where the integrals are Bochner-integrals.

Therefore, by above correspondence $F: Z(\omega) \longmapsto E(Z/\mathcal{B})(\omega)$, similarly in the real-valued case

$$E(Z/\mathcal{B})(\omega) \text{ is written by } F(Z(\omega)/\mathcal{B})$$

for almost all $\omega \in \Omega$. Let f be an X-valued and L^1_X-bounded martingale. Then

$F(f_{n+1}(\omega)/\mathcal{A}_n) = f_n(\omega)$ (n=1,2,...) and F is linear.

Theorem 2. If $\|f\|_1 < \infty$ then $\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty$ a. e..

Proof. Suppose that there exists a subset M of Ω such that $P(M) \neq 0$ and $\sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| = \infty$ for all $\omega \in M$.

Then, for any $G = G(\omega) > 0$ there is a number $N = N(G, \omega) > 0$ such that $\sum_{k=1}^{n} |f_{k+1}(\omega) - f_k(\omega)| > G$ on M ($\forall\, n \geqslant N$).

So there are a number $k = k(\omega) \leqslant n$ and a positive real number $G' = G'(\omega)$ such that $|f_{k+1}(\omega) - f_k(\omega)| = G' > 0$ for each $\omega \in M$.

Then, $\vec{g}(\omega') \overset{\text{def.}}{=\!=} f_{k(\omega)+1}(\omega') - f_{k(\omega)}(\omega')$ for each $\omega \in M$ ($\omega' \in \Omega$, $M \subset \Omega$) such that $|\vec{g}(\omega)| = G'(\omega) > 0$ when $\omega' = \omega$, i. e., $\vec{g} = \vec{g}(\omega) \neq \vec{0}$ on M. Since f is a martingale, for almost all $\omega' \in \Omega$

$$E(f_{k(\omega)+1} - f_{k(\omega)} / \mathcal{A}_{k(\omega)})(\omega') = \vec{0}.$$

Thus, $E(\vec{g}(\omega')) = \int_{\Omega} E(\vec{g} / \mathcal{A}_{k(\omega)})(\omega')\, dP(\omega')$

$= \vec{0}$ (Here E denotes the Bochner integral)

$\overset{\text{def.}}{\Longleftrightarrow} E|\vec{g}| = 0$ (E is the Lebesgue integral)

$\Longleftrightarrow |\vec{g}| = 0$ a. e.

$\Longleftrightarrow \vec{g}(\omega') = \vec{0}$ for almost all $\omega' \in \Omega$ and for each $\omega \in M$.

So $\vec{g}(\omega) = \vec{0}$ on M $(\subset \Omega)$.

This is a contradiction on M. Thus, there is not such M.

Therefore $\sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| < \infty$ for almost all $\omega \in \Omega$.

4. Martingale transforms.

Theorem 3. If $\|f\|_1 < \infty$ then the martingale transform g by v converges a. e. in X without the assumption that v is predictable.

In fact,

$$|g_\infty(\omega)| \leq \sum_{n=1}^{\infty} |v_n(\omega)| |f_{n+1}(\omega) - f_n(\omega)| \leqslant \sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| < \infty$$

for almost all $\omega \in \Omega$.

Theorem 4. Let X be Banach space. Then we have the followings:

$$\begin{tikzcd}
\begin{array}{l}\text{" X has RN-property" and}\\ \text{" all X-valued martingale f}\\ \quad\text{is } L^1\text{-bounded "}\end{array} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \begin{array}{l}\text{" X has RN-property" and}\\ \text{" all X-valued martingale f}\\ \quad\text{is } L^p\text{-bounded" } (1<p<\infty)\end{array} \\
\sum_{n=0}^{\infty}\|f_n - f_\infty\|_p < \infty \ (1<p<\infty) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \sum_{n=0}^{\infty}|f_n - f_\infty| < \infty \text{ a. e.} \arrow[d, Rightarrow] \\
\sum_{n\geqslant 0}\|f_{n+1} - f_n\|_p \leqq {}^{\exists}C_p \cdot \sup_n \|f_n\|_p \arrow[d, Rightarrow, "(1<p<\infty)"] & \sum_{n=0}^{\infty}|f_{n+1} - f_n| < \infty \text{ a. e.} \\
\text{" X is super-reflexive } (\Rightarrow \text{ reflexive)"} &
\end{tikzcd}$$

i.e., " X has RN-property and $\|f\|_1 < \infty \Rightarrow$ X is reflexive "

(Thus, $X \in$ UMD)

References

[1] T. Shintani, L^p-convergence of an extended stochastic integral, II (in Japanese), Hokkaido Univ. Tech. Report series in Math., No. 35 (1995), 56-61.

[2] ——————, Martingale transforms in a Banach space (1995) (preprint).

Tomakomai National College of Technology

Interpolating Sequences and Embedding Theorems in Weighted Bergman Spaces

広島大・理　山田雅博
(Masahiro Yamada)

§1. 序

D を複素平面における開単位円板、H を D 上の解析関数全体とする。また、$0 < p, q < \infty$、ν, μ を D 上の有限な Borel 正測度とする。ここでは次の問題を考える。ある定数 $0 < C < \infty$ が存在して、全ての $f \in H$ に対して

$$\left(\int_D |\, f \,|^q \, d\nu\right)^{\frac{1}{q}} \leq C\left(\int_D |\, f \,|^p \, d\mu\right)^{\frac{1}{p}}$$

となるとき、ν と μ は (q,p) に関する (ν,μ)-Carleson 不等式を満たすと呼ぶことにする。ここで、$L_a^q(\nu), L_a^p(\mu)$ を各々 $L^q(\nu) \cap H, L^p(\mu) \cap H$ とするとき、上の条件は $L_a^p(\mu)$ が $L_a^q(\nu)$ に連続に埋め込まれる事と同値である。

<u>問題.</u> ν と μ が (q,p) に関する (ν,μ)-Carleson 不等式を満たすための必要十分条件を見つけよ。

$a, z \in D$ に対して、 ϕ_a を原点 0 を a に写す D 上の一次変換、

$$\beta(a,z) = 1/2 \log(1+ |\, \phi_a(z) \,|)/(1- |\, \phi_a(z) \,|)$$

とする。このとき、 $0 < r < \infty$ なる r に対して、

$$D_r(a) = \{z \in D \mid \beta(a,z) < r\}$$

とする。さらに、

$$\hat{\mu}_r(a) = \frac{1}{m(D_r(a))} \int_{D_r(a)} d\mu$$

と定義する。ここで、m は正規化された 2 次元 Lebesgue 測度を表す。

<u>定理 1 .</u> $0 < p \leq q$ とする。さらに、$d\mu = w dm_\alpha, w$ は (A_s) -条件を満足し、ある $0 < R < \infty$ が存在して $\epsilon_R(\mu, \alpha) < 1$ とする。このとき、(q,p) に

関して (ν,μ)-Carleson 不等式が満たされる事と、ある $0<r<\infty$ が存在して $(1-\mid a\mid^2)^{2(1-q/p)}\hat{\nu}_r(a)/\hat{\mu}_r(a)^{q/p}\le C \quad (a\in D)$ なることは同値である。

定理２．$0<q<p$ かつ、$p\le 1$ とする。さらに、$d\mu = wdm_\alpha, w$ は (A_s) -条件を満足し、$\epsilon_R(\mu,\alpha)\to 0(R\to\infty)$ とする。このとき、(q,p) に関して (ν,μ)-Carleson 不等式が満たされる事と、ある $0<r<\infty$ が存在して $\hat{\nu}_r/\hat{\mu}_r\in L^t(\mu)$ なることは同値である。ここで、$1/t+1/(p/q)=1$ とする。

定理３．$0<q<p$ かつ、$1<p$ とする。さらに、$d\mu = wdm_\alpha, w$ は $(A_p(\alpha))_\partial$ -条件を満足し、$(1-p)(1+\alpha)<\theta<0$ なる θ が存在して、$\delta_R(\mu,\alpha,\theta)\to 0(R\to\infty)$ とする。このとき、(q,p) に関して (ν,μ)-Carleson 不等式が満たされる事と、ある $0<r<\infty$ が存在して$\hat{\nu}_r/\hat{\mu}_r\in L^t(\mu)$ なることは同値である。ここで、$1/t+1/(p/q)=1$ とする。

以下では定理の証明の概略を述べる。また、C は定数を表すものとする。

§2. $p\le q$ の場合について

任意の $a\in D$ に対して、$K_a(z)=(1-\bar{a}z)^{-2}, k_a(z)=K_a(z)/K_a(a)^{1/2}$ とおく。$\alpha>-1$ に対して、

$$\tilde{\mu}_\alpha(a)=\int \mid k_a\mid^{2+\alpha}d\mu$$

と書き、これを Berezin 変換と呼ぶ。$w\ge 0$ を D 上の可積分関数とする。$1<p<\infty$ に対して、w が (A_p)-条件を満足するとは、ある $0<r,C<\infty$ が存在して、すべての $a\in D$ について

$$\hat{w}_r(a)(w^{-1/(p-1)})^\wedge_r(a)^{p-1}\le C$$

なるときを言う。また、$0<R<\infty$ に対して、

$$\epsilon_R(\mu,\alpha)=\sup_{a\in D}\left(\int_{D\setminus D_r(a)}\mid k_a\mid^{2+\alpha}d\mu\right)\tilde{\mu}_\alpha(a)^{-1}$$

と定義する。さらに、$\alpha>-1$ に対して、$dm_\alpha=(1-\mid z\mid^2)^\alpha dm$ と書く事にする。以上を準備して、定理１の証明の概略を述べる。

定理 1 の証明. ある $0 < r < \infty$ が存在して $(1-\mid a\mid^2)^{2(1-q/p)}\hat{\nu}_r(a)/\hat{\mu}_r(a)^{q/p} \leq C \quad (a \in D)$ であると仮定する。任意の $f \in H$ と $a \in D$ に対して、$\mid f \mid$ の劣調和性と Hőlder の不等式より、

$$\mid f(a)\mid^q \leq C\Big(\int_{D_r(a)} \mid f\mid^p \, d\mu\Big)^{q/p}(1-\mid a\mid^2)^{-2q/p}\hat{\mu}_r(a)^{-q/p}$$

が成立する。ここで、両辺を ν で積分し、右辺の D を分割することを考える。補題4.3.6[6;p62] より、ある自然数 N と $\{\lambda_n\} \subset D$ が存在して $D = \cup D_r(\lambda_n)$ かつ、全ての $z \in D$ は高々 N 個の $D_{2r}(\lambda_n)$ にしか属さないように D を分割できる。よって、

$$\begin{aligned}
&\int_D \mid f(a)\mid^q \, d\nu \\
&\leq \ C\sum \int_{D_r(\lambda_n)}\Big(\int_{D_r(a)} \mid f\mid^p \, d\mu(1-\mid a\mid^2)^{-2}\hat{\mu}_r(a)^{-1}\Big)^{q/p} d\nu(a) \\
&\leq \ C\sum\Big(\int_{D_{2r}(\lambda_n)} \mid f\mid^p \, d\mu(1-\mid \lambda_n\mid^2)^{-2}\hat{\mu}_r(\lambda_n)^{-1}\Big)^{q/p} \nu(D_r(\lambda_n))
\end{aligned}$$

となり、D の分割の仕方と $q/p \geq 1$ であることから、Carleson 不等式が得られる。逆に Carleson 不等式が成立していたと仮定する。このとき、$f = k_a^{(2+\alpha)/p}$ を不等式に代入すると、$\epsilon_R(\mu, \alpha) < 1$ なる仮定から直ちに定理の条件が導かれる。

§3. $p > q$ の場合について

まず、$(A_p(\alpha))_\partial$ -条件について述べる。可積分関数 w が $(A_p(\alpha))_\partial$ -条件を満足するとは、ある $0 < C < \infty$ が存在して、すべての $a \in D$ について

$$\tilde{w}_\alpha(a)(w^{-1/(p-1)})^{\sim}_\alpha(a)^{p-1} \leq C$$

なるときを言う。また、$0 < R < \infty$ と実数 θ に対して、

$$\delta_R(\mu, \alpha, \theta) = \sup_{a\in D}\Big(\int_{D\setminus D_r(a)} \mid K_a\mid^{1+\alpha/2} (1-\mid z\mid^2)^\theta d\mu\Big)\Big((1-\mid a\mid^2)^\theta\tilde{\mu}_\alpha(a)\Big)^{-1}$$

と定義する。

定理１では、(ν,μ)-Carleson 不等式がみたされるための必要条件を得るために f として特殊な関数を不等式に代入した。定理２と定理３の証明においても、それは同様であるが、定理１で用いられたような具体的な関数ではない。この場合に用いられる関数 f はある補間点列問題の解であり、その存在に関する議論が必要となる。まず、それについて述べる。

$A=\{a_j\}$ を D の点列とする。また、

$$R_A = 1/2 \inf_{i\neq j} \beta(a_i, a_j)$$

とおく。さらに、

$$s(\mu,p,a)=s(a)=\inf\left\{\int \mid f\mid^p d\mu; f(a)=1, f\in H\right\}$$

と定義し、これを Riesz's 関数と呼ぶ。このとき、$L^p_a(\mu)$ から l^p への線型写像 T_A を

$$T_A f = \{s(a_j)^{1/p} f(a_j)\}$$

によって定義する。点列 A が $L^p_a(\mu)$ における補間点列であるとは、T_A が全射であるとき、すなわち任意の $\{c_j\}\in l^p$ に対して $c_j = s(a_j)^{1/p} f(a_j)$ なる f が存在するときをいう。この問題について、次の結果が得られる。

定理４. $d\mu = w dm_\alpha$ とする。

（１）$0<p\leq 1$ とする。w は (A_s) -条件を満足し、$\epsilon_R(\mu,\alpha)\to 0 (R\to\infty)$ とする。このとき、ある $0<R_0<\infty$ が存在して、$R_0\leq R_A$ ならば A は補間点列となる。

（２）$1<p$ とする。w は $(A_p(\alpha))_\theta$ -条件を満足し、$(1-p)(1+\alpha)<\theta<0$ なる θ が存在して、$\delta_R(\mu,\alpha,\theta)\to 0(R\to\infty)$ とする。このとき、ある $0<R_0<\infty$ が存在して、$R_0\leq R_A$ ならば A は補間点列となる。

定理４についての証明は省略する。しかし、これを用いて定理２、定理３の事実が得られる事を注意しておく。定理２、定理３において (ν,μ)- Carleson 不等式が成立するための十分条件を得る証明は、定理１の証明と比較する意味でも興味深い。その概略をこれから述べる。

定理２、定理３における十分性の証明. 定理１の証明と同様にして、任意の $f\in H$ と $a\in D$ に対して、$\mid f\mid$ の劣調和性と Hölder の不等式より、

$$\mid f(a)\mid^q \leq C\left(\int_{D_r(a)} \mid f\mid^q d\mu\right)(1-\mid a\mid^2)^{-2}\hat{\mu}_r(a)^{-1}$$

が得られる。ここで、両辺を ν で積分し、Fubini の定理を用いて、

$$\begin{aligned}
&\int_D \mid f(a) \mid^q d\nu(a) \\
&\leq \ C\int_D \mid f(z) \mid^q \int_{D_r(z)} (1- \mid a \mid^2)^{-2} \hat{\mu}_r(a)^{-1} d\nu(a) d\mu(z) \\
&\leq \ C\int_D \mid f(z) \mid^q (\hat{\nu}_r(z)/\hat{\mu}_r(z)) d\mu(z)
\end{aligned}$$

となる。ここで、もう一度 Hőlder の不等式を用いることによって、Carleson 不等式が得られる。

上記の証明は、定理 1 のそれよりも比較的簡単であることが解る。これは、$\nu = \mu$ の場合について考えると、$q < p$ のとき、$L_a^p(\mu)$ が $L_a^q(\mu)$ に常に含まれている事からも予想されるであろうと思われる。また、これらの結果は特殊な測度 μ についてしか得られていないが、μ に条件をつけない場合には、たとえ $\nu = m$ のときであっても問題は非常に難しいように思われる。しかし、その場合について問題を解決することが重要であろうと思っている。

References

1. D.Luecking, Multipliers of Bergman spaces into Lebesgue spaces, Proc. Edinburgh Math. Soc. 29(1986), 125-131.
2. T.Nakazi and M.Yamada, (A_2)-conditions and Carleson inequalities in Bergman spaces, to appear.
3. V.L.Oleinik and B.S.Pavlov, Embedding theorems for weighted class of harmonic and analytic functions, J. Soviet Math. 2(1974), 135-142.
4. R.Rochberg, Interpolation by functions in the Bergman spaces, Mich. Math. J. 29(1982), 229-236.
5. M.Yamada, Interpolating sequences and embedding theorems in weighted Bergman spaces, to appear.
6. K.Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York,1990.

A GEOMETRICAL STRUCTURE IN THE FURUTA INEQUALITY

EIZABURO KAMEI

1. Introduction The Furuta inequality [8] is one of recent developments in operator theory, which is a beautiful extension of the Löwner-Heinz inequality [14,18] : If $A \geq B \geq 0$, then

$$(1) \qquad (A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

holds for $r \geq 0$, $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$, in which the case $q = \frac{p+2r}{1+2r}$ is essential, i.e.,

$$(2) \qquad (A^r A^p A^r)^{\frac{1+2r}{p+2r}} \geq (A^r B^p A^r)^{\frac{1+2r}{p+2r}}$$

for $p \geq 1$ and $r \geq 0$. We note that the case $r = 0$ in (1) is the Löwner-Heinz inequality. An elementary proof of it is given in [9] and some nice applications are discussed in [11,12].

Now, as in previous papers [2,3,4,5,6,10,15,16], the theory of operator means established by Kubo and Ando [17] is quite useful for the investigation of the Furuta inequality. Actually it induces the power mean $\sharp_\alpha$ $(0 \leq \alpha \leq 1)$ corresponding to the operator monotone function $t \to t^\alpha$ $(t \geq 0)$, i.e.,

$$A \sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$$

for positive invertible operators A and B. Then it is easily checked that the Furuta inequality (2) is equivalent to the following inequality in terms of operator means ;

$$(3) \qquad A^{-2r} \sharp_{\frac{1+2r}{p+2r}} A^p \geq A^{-2r} \sharp_{\frac{1+2r}{p+2r}} B^p$$

for $p \geq 1$ and $r \geq 0$. Moreover, as an alternative formula of the Furuta inequality, we obtained in [16 ; Theorem] and also [3 ; Cor. 2] that if $A \geq B \geq 0$ and they are invertible, then

$$(4) \qquad B^{-2r} \sharp_{\frac{1+2r}{p+2r}} A^p \geq A \geq B \geq A^{-2r} \sharp_{\frac{1+2r}{p+2r}} B^p$$

for $p \geq 1$ and $r \geq 0$. Very recently, we [6] attempt a mean theoretic approach to the grand Furuta inequality [13] which is a parametric formula interpolating the Furuta inequality and the Ando-Hiai inequality [1] and is proved by using the original Furuta inequality (1) : If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0,1]$,

$$F_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^{p}A^{-t/2})^{s}A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}A^{-r/2}$$

is a decreasing function of both r and s for all $r \geq t$, $p \geq 1$ and $s \geq 1$. In particular, the inequality

$$A^{1-t} = F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s)$$

holds for $r \geq t$, $p \geq 1$ and $s \geq 1$.

Recalling our attempts on the Furuta inequality, we recognize that the following special case of the Furuta inequality (2) plays an important role : If $A \geq B \geq 0$, then

$$A \geq (A^{r}B^{p}A^{r})^{\frac{1}{p+2r}} \tag{5}$$

for $p \geq 1$ and $r \geq 0$, see [6; Lemma 2], [3; Lemma] and [16; Cor. 2].

In this note, we pay our attension to the Furuta inequality (5) again. Precisely we use it in Furuta's repeating method established in his inequality and obtain the following inequality: If $A \geq B \geq 0$ and A is invertible, then for each $\alpha \in [0,1]$,

$$A^{-s} \natural_{\frac{(p+s-n)\alpha+n}{p+s}} B^{p} \leq B^{(p+s-n)\alpha-s+n} \tag{6}$$

holds for $p \geq 1$ and $n+1 > s \geq n$ for some nonnegative integer n. We here note that if we take $\alpha = \frac{1+s-n}{p+s-n}$ in the above (6), then we have (4) by replacing $s = 2r$; in other words, (4) is just the case $\alpha = \frac{1+s-n}{p+s-n}$ in (6). Furthermore the inequality (6) is discussed in a general setting, so that we will have some geometrical view for the Furuta inequality; we will recognize the figure like a gingko leaf in it.

2. The Furuta inequality.

At the beginning, we start with a simple application of (5), which is the heart of this paper.

Lemma 1. *If $A \geq B \geq 0$ and A is invertible, then for each $\alpha \in [0,1]$,*

$$A^{-s} \natural_{\alpha} B^{p} \geq A^{-s-1} \natural_{\alpha+\frac{1-\alpha}{p+s+1}} B^{p} \tag{7}$$

holds for $p \geq 1$ and $s \geq 0$.

Next we prepare an elementary fact as a base for repeating the use of Lemma 1.

Lemma 2. *Let $p \geq 1$ and $n+1 > s \geq n$ for some nonnegative integer n. For a given $\alpha \in [0,1]$, a sequence $\{\alpha_k\}$ is defined by $\alpha_0 = \alpha$ and*

$$\alpha_{k+1} = \alpha_k + \frac{1-\alpha_k}{p+s-n+k+1} \quad (k = 0,1,2,\cdots,n-1). \tag{8}$$

Then each α_k is expressed as

$$\alpha_k = \frac{\alpha(p+s-n)+k}{p+s-n+k} \tag{9}$$

for $k = 1,2,\cdots,n$ and in particular

$$\alpha_n = \frac{\alpha(p+s-n)+n}{p+s}. \tag{10}$$

If $\alpha = \frac{1+s-n}{p+s-n}$, then $\alpha_n = \frac{1+s}{p+s}$.

Theorem 3. *If $A \geq B \geq 0$ and A is invertible, then for each $\alpha \in [0,1]$,*

$$(6) \qquad A^{-s} \natural_{\frac{(p+s-n)\alpha+n}{p+s}} B^p \leq B^{(p+s-n)\alpha-s+n}$$

holds for $p \geq 1$ and $n+1 > s \geq n$ for some nonnegative integer n.

Remark. If we take $\alpha = \frac{1+s-n}{p+s-n}$ in (6) for given $p \geq 1$ and $n+1 > s \geq n$, then Lemma 2 and Theorem 3 imply (4) by replacing $s = 2r$. It also means that (5) is fundamental among the Furuta inequalities.

3. Gingko leaf in the Furuta inequality.

First of all, we consider the continuous analogue of the preceding discussion, which is seen on both variables s and p. To do this, Lemma 1 must be reformed as follows:

Lemma 4. *If $A \geq B \geq 0$ and A is invertible, then for each $\alpha \in [0,1]$ and $\epsilon \in (0,1]$,*

$$A^{-s} \natural_{\alpha} B^p \geq A^{-s-\epsilon} \natural_{\alpha+\frac{(1-\alpha)\epsilon}{p+s+\epsilon}} B^p$$

holds for $p \geq 1$ and $s \geq 0$.

Next we have the following inequality on p:

Lemma 5. *If $A \geq B \geq 0$ and A is invertible, then for each $\alpha \in [0,1]$ and $\delta \in (0,1]$,*

$$A^{-s} \natural_{\alpha} B^p \geq A^{-s} \natural_{\frac{(p+s)\alpha}{p+s+\delta}} B^{p+\delta}$$

holds for $p \geq 1$ and $s \geq 0$.

Combining with Lemmas 4 and 5, the following theorem is obtained:

Theorem 6. *If $A \geq B \geq 0$ and A is invertible, then for each $\alpha, \epsilon, \delta \in [0,1]$,*

$$(12) \qquad A^{-s} \natural_{\alpha} B^p \geq A^{-s-\epsilon} \natural_{\frac{(p+s)\alpha+\epsilon}{p+s+\epsilon+\delta}} B^{p+\delta}$$

and

$$(13) \qquad B^{-s} \natural_{\alpha} A^p \leq B^{-s-\epsilon} \natural_{\frac{(p+s)\alpha+\epsilon}{p+s+\epsilon+\delta}} A^{p+\delta}$$

hold for $p \geq 1$ and $s \geq 0$.

We here note that Theorem 3 is continuously interpolated by Theorem 6. Finally we propose a geometrical structure in the Furuta inequality. For simplicity, we consider the scalar case, i.e., positive operators A and B are positive numbers a and b respectively. Also we assume that $0 < b < 1 < a$.

Now it follows from Theorem 6 that

$$(14) \qquad B \leq A = B^{-2} \natural_1 A \leq B^{-2} \natural_{\frac{3}{4}} A^2 \leq B^{-2} \natural_{\frac{3}{5}} A^3 \leq B^{-2} \natural_{\frac{1}{2}} A^4 \leq \cdots,$$

$$(15) \qquad 1 \leq B^{-2} \natural_{\frac{2}{3}} A \leq B^{-2} \natural_{\frac{1}{2}} A^2 \leq B^{-2} \natural_{\frac{2}{5}} A^3 \leq B^{-2} \natural_{\frac{1}{3}} A^4 \leq \cdots$$

and

$$(16) \qquad B^{-1} \leq B^{-2} \sharp_{\frac{1}{3}} A \leq B^{-2} \sharp_{\frac{1}{4}} A^2 \leq B^{-2} \sharp_{\frac{1}{5}} A^3 \leq B^{-2} \sharp_{\frac{1}{6}} A^4 \leq \cdots .$$

In addition, since $C = C \sharp_0 A$ for $A, C \geq 0$, we have

$$B^{-2} = B^{-2} \sharp_0 A = B^{-2} \sharp_0 A^2 = B^{-2} \sharp_0 A^3 = B^{-2} \sharp_0 A^4 = \cdots .$$

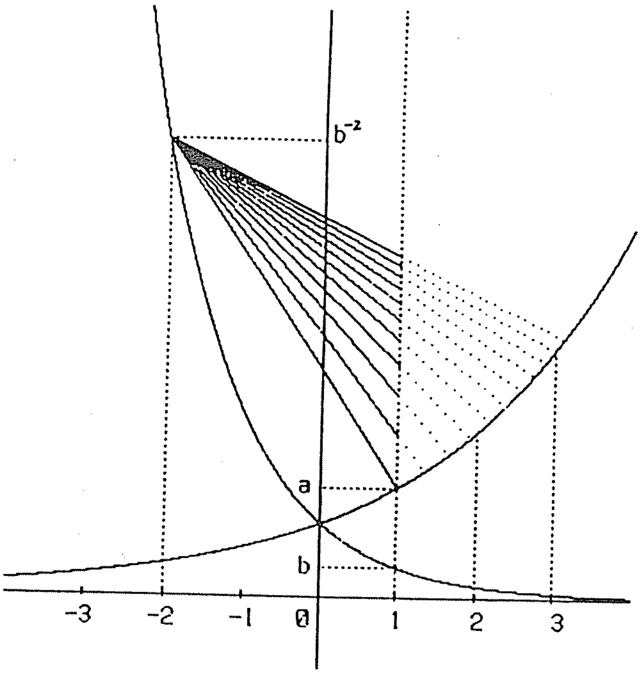
We now draw the graphs of fuctions $y = a^x$ and $y = b^x$, and fix the point $Q = (-2,\ b^{-2})$. Then the segment spanned by Q and $P_n = (n,\ a^n)$ intersects the axis $x = 1$ at the point $M_n = (1,\ b^{-2} \sharp_{\frac{3}{n+2}} a^n)$ exactly and $\frac{|QM_n|}{|QP_n|} = \frac{3}{n+2}$. Moreover the segment QM_n consists of the path $\{B^{-2} \sharp_\alpha A^n\ ;\ 0 \leq \alpha \leq \frac{3}{n+2}\}$; the sequence (14) of inequalities appears on the axis $x = 1$. Similarly, (15) and (16) do on the axis $x = 0$ and $x = -1$ respectively. Incidentally, the following sequence of inequalities appears on the axis $x = \beta$ $(-2 \leq \beta \leq 1)$;

$$B^\beta \leq B^{-2} \sharp_{\frac{\beta+2}{3}} A \leq B^{-2} \sharp_{\frac{\beta+2}{4}} A^2 \leq B^{-2} \sharp_{\frac{\beta+2}{5}} A^3 \leq B^{-2} \sharp_{\frac{\beta+2}{6}} A^4 \leq \cdots .$$

Combinig the above discussion with Theorem 6, for each $s \geq 0$ the set

$$\{B^{-s} \sharp_\alpha A^p\ ;\ p \geq 1, 0 \leq \alpha \leq 1\}$$

looks like a gingko leaf as the following figure.



REFERENCES

1. T.Ando and F.Hiai, *Log-majorization and complementary Golden-Thompson type inequalities*, Linear Alg. and its Appl. **197/198** (1994), 113-131.
2. M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator theory **23** (1990), 67-72.
3. M.Fujii,T.Furuta and E.Kamei, *Furuta's inequality and its application to Ando's Theorem*, Linear Alg. and its Appl. **149** (1991), 91-96.
4. M.Fujii,T.Furuta and E.Kamei, *Operator functions associated with Furuta's inequality*, Linear Alg. and its Appl. **179** (1993), 161-169.
5. M.Fujii and E.Kamei, *Furuta's inequality and a generaalization of Ando's Theorem*, Proc. Amer. Math. Soc., **115** (1992), 409-413.
6. M.Fujii and E.Kamei, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc., to appear.
7. M.Fujii and E.Kamei, *A geometrical structure in the Furuta inequality*, Math. Japon., to appear.
8. T.Furuta, $A \geq B \geq 0$ *assures* $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ *for* $r \geq 0, p \geq 0, q \geq 1$ *with* $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc. **101** (1987), 85-88.
9. T.Furuta, *Elementary proof of an order preserving inequality*, Proc. Japan Acad. **65** (1989), 126.
10. T.Furuta, *A proof via operator means of an order preserving inequality*, Linear Alg. and its Appl. **113** (1989), 12-130.
11. T.Furuta, *Two operator functions with monotone property*, Proc. Amer. Math. Soc. **111** (1991), 511-516.
12. T.Furuta, *Application of an order preserving inequalities*, Operator theory : Advances and Applications, Birkhouser Verlag Basel **59** (1992), 180-190.
13. T.Furuta, *Extension of the Furuta inequality and log-majorization by Ando-Hiai*, to appear in Linear Alg. and its Appl. (1994).
14. E.Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415-438.
15. E.Kamei, *Furuta's inequality via operator means*, Math. Japon. **33** (1988), 737-739.
16. E.Kamei, *A satellite to Furuta's inequality*, Math. Japon. **33** (1988), 883-886.
17. F.Kubo and T.Ando, *Means of positive linear operators*, Math. Ann. **246** (1980), 205-224.
18. K.Löwner, *Über monotone Matrixfunktionen*, Math. Z. **38** (1934), 177-216.

MOMODANI SENIOR HIGHSCHOOL, IKUNO, OSAKA 544, JAPAN

Symmetric Riemmanian space of type AI and generalized Numerical Ranges

Hiroshi Nakazato

Department of Mathematics, Faculty of Science Hirosaki University

Hirosaki, 036 JAPAN

Abstract. *In this talk I treat with certain generalized numerical numerical ranges of 3 by 3 complex matrices.*

1. Reduction of the parameters

Question. *Suppose that A, B are 3×3 complex diagonal matrices: $A = diag\{a_1, a_2, a_3\}$, $B = diag\{b_1, b_2, b_3\}$. Set*

$$W(A,B) = \{tr(A\ U\ \ B\ U^*) : U \text{ is a 3 by 3 unitary matrix}\}.$$

Describe the boundary of the compact subset $W(A,B)$ of the complex plane C via the eigenvalues $a_1, a_2, a_3, b_1, b_2, b_3$.

By a theorem of Cheung and Tsing (cf.[2]), the range $W(A,B)$ is star- shaped with respect to the point $(1/3)(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$. Therefore the range $W(A,B)$ is determined by its boundary. In the paper [1], Au-Yeung and Poon characterize the compact set

$$\Omega = \{U \circ \overline{U} : U \text{ is a 3 by 3 unitary matrix}\},$$

where $\overline{U}$ is the complex conjugate of the matrix U and o denotes the Hadamard (Shur, entry-wise) product of matrices. By using their characterization of Ω and the equation

$$2(x^2\ y^2 + x^2\ z^2 + y^2\ z^2) - (x^4 + y^4 + z^4) = (x+y-z)(x+z-y)(y+z-x)(x+y+z)$$

for real numbers x, y, z, and the Euler angles of rotation in $\mathbf{R}^3$, we can characterize the boundary of Ω in the 4-dimensional affine space of all real matrices whose column sums and row sums are 1.

Lemma 1.

$$\partial\Omega = \{g \circ g : g \in SO(3)\}.$$

This lemma leads us the following theorem.

Theorem1. *Suppose that A and B are 3 by 3 complex diagonal matrices. Then the equation*

$$W(A,B) = \{tr(A\ g\ B\ g^t) : g \in SO(3)\}$$

holds, where g^t denotes the transpose of the matrix g.

2. Critical Points

Now we reduce the problem to a special case. If the eigenvalues a_1, a_1, a_3 of A lie on a straight line on the complex plane, then the range $W(A,B)$ is convex and coincides with the convex hull of the 6 points

$$\{a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + a_3 b_{\sigma(3)} : \sigma \in S_3\}.$$

Therefore we may assume that $a_i \neq a_j$ for $1 \leq i < j \leq 3$ and the three points a_1, a_2, a_3 lie on a circle with radius $r \in (0,\infty)$ on the complex plane. Since $W(A,B) = W(B,A)$, we may assume that the eigenvalues b_1, b_2, b_3 of B also lie on a circle. By using rotations and translations, we may assume that $A = diag\{a_1, a_2, a_3\}$ and $B = diag\{b_1, b_2, b_3\}$ are elements of the group $SU(3)$ satisfying $a_i \neq a_j, b_i \neq b_j$ for $1 \leq i < j \leq 3$. We take a square root $C = diag\{c_1, c_2, c_3\} \in SU(3)$ of the matrix A,i.e., $c_i^2 = a_i$ $(1 \leq i \leq 3)$ and $c_1 c_2 c_3 = 1$. Then we have the relations $(c_i + c_j)(c_i - c_j) \neq 0$ for $1 \leq i < j \leq 3$. We obtain a fundamental equation

$$tr(A\ g\ B\ g^t) = tr(C\ g\ B\ g^t\ C)$$

for every $g \in SO(3)$. We consider the real analytic map ψ of the 3-dimensinal Lie group $SO(3)$ into the plane $\mathbf{C} \simeq \mathbf{R}^2$:

$$g \mapsto tr(C\ g\ B\ g^t\ C).$$

We remark that for every $g \in SO(3)$ the element $C\ g\ B\ g^t\ C$ belongs to the 5-dimensinal compact symmetric Riemannian space

$$M = \{X : X \text{ is a 3 by 3 unitary matrix}, \det(\mathrm{X}) = 1, \mathrm{X}^{\mathrm{t}} = \mathrm{X}\}.$$

We research the rank of the Jacobian matrix of the map ψ at every $g \in SO(3)$. For almost every $g \in SO(3)$, the rank is equal to 2. We say that g is a *critical point* if the rank at g is less than 2. We remark that if $\psi(g)$ is a boundary point of $W(A, B)$, then the point g is necessarily critical. We obtain the following theorem.

Theorem2. *Suppose that* $C = diag\{c_1, c_2, c_3\}$ *and* $B = diag\{b_1, b_2, b_3\}$ *are elements of* $SU(3)$ *satisfying the relations* $(c_i + c_j)(c_i - c_j) \neq 0$, $b_i \neq b_j$ *for* $1 \leq i < j \leq 3$. *Set* $X = X(g) = C\ g\ B\ g^t\ C$, $X = \{x_{ij} = x_{ij}(g) : 1 \leq i, j \leq 3\}$ *for every* $g \in SO(3)$. *Then an element* $g \in SO(3)$ *is a critical point of the map* ψ, *if and only if the three complex numbers* x_{12}, x_{13}, x_{23} *lie on a straight line passing through the origin* 0 *on the complex plane* $\mathbf{C}$. Moreover for the points x_{12}, x_{13}, x_{23} to enjoy this condition , it is necessary and sufficient that one of the following conditions holds: 1) The matrix $g = \{g_{pq} : 1 \leq p, q \leq 3\}$ has an entry g_{ij} for which $g_{ij} = 1$ or $g_{ij} = -1$; 2) Some eigenvalue of the unitary matrix X has multiplicity ≥ 2.

3. Boundary of $W(A, B)$

We define a simple closed curve Γ on the plane $\mathbf{C}$ by the equation

$$\Gamma = \{2\ \exp(it) + \exp(-2\ i\ t) : 0 \leq t \leq 2\pi\}$$

$$= \{z = x + iy : (x, y) \in \mathbf{R}^2, (\mathrm{x}^2 + \mathrm{y}^2)^2 + 24\mathrm{xy}^2 - 8\mathrm{x}^3 + 18(\mathrm{x}^2 + \mathrm{y}^2) - 27 = 0\}.$$

The curve Γ is called a *deltoid*. We denote by D the closed domain surrounded by Γ:

$$D = \{2\ r\ \exp(it) + r\ \exp(-2\ i\ t) : 0 \leq t \leq 2\pi, 0 \leq r \leq 1\}.$$

Then we have the equation

$$D = \{\exp(is) + \exp(it) + \exp(iu) : (s, t, u) \in \mathbf{R}^3, \mathrm{s} + \mathrm{t} + \mathrm{u} \equiv 0 \text{ mod.} 2\pi\}.$$

For the point $z = \exp(is) + \exp(it) + \exp(iu)$ with $(s, t, u) \in \mathbf{R}^3, \mathrm{s} + \mathrm{t} + \mathrm{u} \equiv 0 \text{ mod.} 2\pi$

to belong the boundary Γ, it is necessary and sufficient that the condition

$$(\exp(it)-\exp(is))(\exp(it)-\exp(iu))(\exp(is)-\exp(iu))=0$$

holds. By using this condition and Theorem 2, we can obtain the following theorem.

Theorem3. *Suppose that* $A=diag\{a_1,a_2,a_3\}, B=diag\{b_1,b_2,b_3\}$ *are elements of the group* $SU(3)$ *with* $a_i \neq a_j, b_i \neq b_j$ *for* $1 \leq i < j \leq 3$. *Set*

$$V_+ = \{a_1b_1+a_2b_2+a_3b_3, a_1b_2+a_2b_3+a_3b_1, a_1b_3+a_2b_1+a_3b_2\},$$

$$V_- = \{a_1b_1+a_2b_3+a_3b_2, a_1b_3+a_2b_2+a_3b_1, a_1b_2+a_2b_1+a_3b_3\}.$$

Then the boundary $\partial W(A,B)$ *of the range* $W(A,B)$ *in the plane* $\mathbf{C}$ *satisfies the inclusion*

$$\partial W(A,B) \subset \Gamma \cup \{tz_1+(1-t)z_2 : 0 \leq t \leq 1, z_1 \in V_+, z_2 \in V_-\}.$$

Remark. *We assume the assumtions of Theorem 3 hold. Then, for every* $z_1 \in V_+, z_2 \in V_-$ *the straight line* $L(z_1,z_2)$ *passing through* z_1, z_2, *i.e.,*

$$L(z_1,z_2)=\{tz_1+(1-t)z_2 : t \in \mathbf{R}\}$$

is a tangent line of the deltoid Γ *at some non-singular point of* Γ *or at one of 3 cusps of* Γ.

References

[1] Y.H.Au-Yeung, Y.T.Poon :3 x 3 orthostochastic matrices and the numerical ranges, Linear and Multilinear Algebra, 27(1979) pp.69-79

[2] W.S.Cheung, N.K.Tsing: The C-numerical range of matrices is star-shaped, preprint, 1995

[3]H.Nakazato:Set of 3 x 3 orthostochastic matrices, preprint, 1995.

Ho-Kalman Realization by Computer Algebra

by
Fumio Kubo
Dept. Math., Toyama Univ.

Abstract. The Ho-Kalman realization of a minimal system is practiced for several numerical examples using *Mathematica* 2.2.0.

1. Systems Realization

A linear system in time domain is described by

$$
\begin{aligned}
&x' = Ax + Bu &&\text{system equation}\\
&y = Cx &&\text{output equation}
\end{aligned}
$$

where

A : n by n matrix
B : n by r , C : m by n

and

x : n dimensional vector valued function, called state variable,
u : r dimensional , input
y : m dimensional , output

is transformed into an algebraic equation in frequency domain

$$Y(s)= C(sI - A)^{-1}B\ U(s)+ C(sI - A)^{-1}x(0)$$

The main portion $G(s)=C(sI - A)^{-1}B$ of this equation is called the transfer function, which is a matrix whose elements are rational functions. In particular, the degree of the numerators less than that of the denominator for each entries. The matrix function of this type is called proper.

For the analysis, the transfer function is given from the matrices (A,B,C), while in engineering synthesis, the input-output relation is given first as a proper matrix function. And there occurs an inverse problem. Given proper matrix function, construct the triplet (A,B,C) whose transfer function is the given matrix function. This is called a realization.

Analysis : (A,B,C) ===> G(s)
Synthesis : G(s) ===>(A,B,C)

The properness of the function is a condition for the existence of the realizations. The triplet is determined not uniquely. And it is natural to ask for the smallest realization. It is known that the minimality or irreducibility condition for the system is stated in several ways.

Ho-Kalman's theorem goes as follows :

Theorem. Let

$$\gamma(s) := \text{LCM of the denominators of the entries of } G(s),$$

$$\nu := \deg \gamma(s),$$

and

$$G(s) := \Sigma_n s^{-n} J_n.$$

If H, K be Hankel block-matrices with first row $(J_0, J_1, \ldots, J_{\nu-1})$ and $(J_1, J_2, \ldots, J_\nu)$ respectively, and S, T be matrices with SHT be Row-reduced form of H. Then A=SKT, B=SH, C=HT (with suitable restriction) gives a minimal system .

2. Computation using *Mathematica* 2.2.0
The source file of the *Mathematica* Notebook will be given.

(Default Kernel) In[65]:=

```
(* Give a PROPER rational matrix function and its
matrix form. *)
MatrixForm[G[s_]={{1/(s*(s+1)),1/(s+1)},
{2/(s+1),1/(s+2)}}]

(*The rectangular size of the transfer matrix,
where m is vertical, r is horizontal. *)
m=Dimensions[G[s]][[1]];
r=Dimensions[G[s]][[2]];

(* To obtain the LCM polynomial of the denominators,
the list of denominators
is put on a temporaly file named temp1. While, by
the following simple Table command gives a matrix
of denominators, one need to flatten them. *)
Flatten[Table[Denominator[G[s][[i,j]]],{i,1,m},
{j,1,r}]]>>temp1;

(* Thus the LCM is given by cut-and-pasting from
temp1. *)
gam[s_]=PolynomialLCM[s*(1 + s), 1 + s,
1 + s, 2 + s];

(* The degree of the LCM polynomial will be
called nu. *)
nu=Exponent[gam[s], s];

(* To obtain the Laurent expansion,
the command "Series" is available, *)
MatrixForm[g[z_]=Normal[Series[G[s]/.s->1/z,
{z,0,2*nu}]]];

(* Thus the matricial coefficients of g[z] gives
an  m by r  matrices called J[p].
They are given as the  (p+1)th derivative at zero
divided by (p+1)!.  *)
J[p_]=Expand[Derivative[p+1][g][0]/(p+1)!];

H=Partition[Flatten[Table[Table[Flatten[
Table[J[j+i-2][[k]],{j,1,nu}]],
{k,1,r}],{i,1,nu}]],nu*m];
MatrixForm[H];

(*  We need also shifted Hankel matrix. *)
SH=Partition[Flatten[Table[Table[Flatten[
Table[J[j+i-1][[k]],{j,1,nu}]],
{k,1,r}],{i,1,nu}]],nu*m];
MatrixForm[SH];

(* Now we get the McMillan degree. *)
McMillan=nu*m-Dimensions[NullSpace[H]][[1]];

(* To show that McMillan is really the McMillan
degree. *)
MatrixForm[Minors[H,McMillan+1]];
MatrixForm[Minors[H,McMillan]];
```

```
(* Now, the next step is to obtain the (m*nu)
square invertible matrix ss and the r*nu square
invertible matrix ttsuch that ss.H.tt is a
row-reduced form.*)

r1={{1,0,0,0,0,0},{0,1,0,0,0,0},{1,0,1,0,1,0},
{0,0,0,1,0,0},{0,0,0,0,1,0},{0,0,0,0,0,1}};
r2={{1,0,0,0,0,0},{0,1,0,0,0,0},{0,0,1,0,0,0},
{0,0,0,1,0,0},{0,0,0,0,0,1},{0,0,0,0,1,0}};
r3={{1,-1,-1,1,-1,0},{0,1,0,0,0,0},{0,0,1,0,0,0},
{0,0,0,1,0,0},{0,0,0,0,1,0},{0,0,0,0,0,1}};
l1={{1,0,0,0,0,0},{0,1,0,0,0,0},{0,0,1,0,0,0},
{0,0,0,1,0,0},{0,0,1,0,1,0},{0,0,0,0,0,1}};
l2={{1,0,0,0,0,0},{0,1,0,0,0,0},{0,0,1,0,0,0},
{0,0,0,1,0,0},{0,0,0,0,0,1},{0,0,0,0,1,0}};
r4={{1,0,0,0,0,0},{0,1,2,2,-4,0},{0,0,1,0,0,0},
{0,0,0,1,0,0},{0,0,0,0,1,0},{0,0,0,0,0,1}};
r5={{1,0,0,0,0,0},{0,1,0,0,0,0},{0,0,1,0,0,0},
{0,0,0,1,3,0},{0,0,0,0,1,0},{0,0,0,0,0,1}};
l3={{1,0,0,0,0,0},{0,1,0,0,0,0},{0,1,1,0,0,0},
{0,2,0,1,0,0},{0,-4,0,0,1,0},{0,0,0,0,0,1}};
l4={{1,0,0,0,0,0},{0,1,0,0,0,0},{0,0,1,0,0,0},
{0,0,0,1,0,0},{0,0,0,3,1,0},{0,0,0,0,0,1}};
l5={{1,0,0,0,0,0},{0,1,0,0,0,0},{0,0,1,0,0,0},
{0,0,-2/3,1,0,0},{0,0,0,0,1,0},{0,0,0,0,0,1}};
l6={{1,0,0,0,0,0},{0,1,0,0,0,0},{0,0,1,3/2,0,0},
{0,0,0,1,0,0},{0,0,0,0,1,0},{0,0,0,0,0,1}};
l7={{1,0,0,0,0,0},{0,1,0,0,0,0},{0,0,-1/3,0,0,0},
{0,0,0,3/2,0,0},{0,0,0,0,1,0},{0,0,0,0,0,1}};
MatrixForm[l7.l6.l5.l4.l3.l2.l1.H.r1.r2.r3.r4.r5];

(*Å@Thus we get the final reduced form, namely
a (non-square) zero one matrices whose entries
are 1 if i=j and 1<=i<=McMillan
and 0 otherwise.
And the product of all left multipliers gives the
invertible matrix ss, and the product of right
multipliers gives  tt. *)
ss=l7.l6.l5.l4.l3.l2.l1;
tt=r1.r2.r3.r4.r5;
MatrixForm[ss];
MatrixForm[tt];

(* Several zero one matrices whose entries are 1
if i=j and 1<=i<=McMillan, r, m and 0 otherwise
are defined. *)
MatrixForm[III=Table[If[i==j,1,0],{i,1,McMillan},
{j,1,m*nu}]];
MatrixForm[JJJ=Table[If[i==j,1,0],{i,1,McMillan},
{j,1,r*nu}]];
MatrixForm[KKK=Table[If[i==j,1,0],{i,1,r},
{j,1,r*nu}]];
MatrixForm[LLL=Table[If[i==j,1,0],{i,1,m},
{j,1,m*nu}]];

(* Now the Ho-Kalman realization  (AA,BB,CC)
is given. *)
```

```
MatrixForm[AA=III.ss.SH.tt.Conjugate[
Transpose[JJJ]]]
MatrixForm[BB=III.ss.H.Conjugate[
Transpose[KKK]]]
MatrixForm[CC=LLL.H.tt.Conjugate[
Transpose[JJJ]]]

(* Is it really a realization ? *)
MatrixForm[Together[Cancel[CC.Inverse[
s*IdentityMatrix[McMillan]-AA].BB]]];

(* Is this a irreducible(=minimal) ? *)
MatrixForm[PP=Table[Flatten[Table[
(MatrixPower[AA,k].BB)[[j]]
,{k,0,McMillan-1}]],{j,1,McMillan}]];

MatrixForm[QQ=Table[Flatten[Table[
(MatrixPower[Conjugate[Transpose[AA]],k].
Conjugate[Transpose[CC]])[[j]],
{k,0,McMillan-1}]],{j,1,McMillan}]];

(* Check the rank of PP, which is expected
to be equat to McMillan*)
NullSpace[PP];
Dimensions[NullSpace[PP]][[1]];

(* Check the rank of QQ, which is expected
to be equat to McMillan*)
NullSpace[QQ];
Dimensions[NullSpace[QQ]][[1]];
```

```
(Default  Kernel)  Out[66]//MatrixForm=
      1              1
  ---------        -----
  s (1 + s)        1 + s
     2               1
   -----           -----
   1 + s           2 + s

(Default  Kernel)  Out[115]//MatrixForm=
  0     -1    -3    -1
  0     -2    -2    0
  0     0     -1    0
  0     1     2     -1

(Default  Kernel)  Out[116]//MatrixForm=
  0     1
  2     1
  -1    0
  0     0

(Default  Kernel)  Out[117]//MatrixForm=
  1    0    0    0
  0    1    0    0
```

COMMUTATIVE BANACH ALGEBRAS AND BSE-NORM

Sin-Ei Takahasi
Department of Basic Technology, Applied Mathematics and Physics
Yamagata University, Yonezawa 992, Japan

Yasuji Takahashi
Department of System Engineering, Okayama Prefectural University
Soja, Okayama 719-11, Japan

and

Osamu Hatori
Department of Mathematical Science, Graduate School of Science and Technology
Niigata University, Ikarashi, Niigata 951, Japan

Abstract. We consider a class of commutative Banach algebras A which satisfy the condition: $\| x^\wedge \|_{\mathrm{BSE}} = \| x \|$ for every $x \in A$. It is clear that all function algebras on a locally compact Hausdorff space with supremum norm belong to such a class. Here we show that group algebras, commutative H^*-algebras, $L^p(G)$ $(1 < p < \infty$ for compact $G)$, $\ell^1(S)$ and some semigroup algberas also belong to this class.

1. Let A be a commutative Banach algebra with Gelfand space Φ_A. Denote by span(Φ_A) the linear span of Φ_A in the dual space A^* of A. An arbitrary linear functional p in span(Φ_A) has the unique expansion

$$p = \sum_{\varphi \in \Phi_A} p^\wedge(\varphi)\varphi ,$$

where $p^\wedge$ is a complex-valued function on Φ_A with finite support. We denote by $C_{\mathrm{BSE}}(\Phi_A)$ the set of all continuous complex-valued functions σ on Φ_A which satisfy the following: there exists a positive real number β such that

$$\left| \sum_{\varphi \in \Phi_A} p^\wedge(\varphi)\sigma(\varphi) \right| \leq \beta \| p \|_{A^*}$$

for every $p \in \mathrm{span}(\Phi_A)$ and denote by $\| \sigma \|_{\mathrm{BSE}}$ the infimum of such β. In this case, $C_{\mathrm{BSE}}(\Phi_A)$ becomes a semisimple commutative Banach algebra with norm $\| \sigma \|_{\mathrm{BSE}}$ (see [6, Lemma 1]). We call $C_{\mathrm{BSE}}(\Phi_A)$ the induced algbera of A. For an element x in A, let $x^\wedge$ be the Gelfand transform of x and set $A^\wedge = \{x^\wedge : x \in A\}$. Then it is easy to see that

$$\| x^\wedge \|_{\mathrm{BSE}} = \sup \{| p(x)| : p \in \mathrm{span}(\Phi_A), \|p \|_{A^*} \leq 1\} \leq \| x \|$$

for every $x \in A$ and so $A^\wedge \subseteq C_{\mathrm{BSE}}(\Phi_A)$. Here we are interested in a class of commutative Banach algebras A which satisfy the condition: $\| x^\wedge \|_{\mathrm{BSE}} = \| x \|$ for every $x \in A$. Of course, all function algebras on a locally compact Hausdorff space with supremum norm belong to such a class. We further show that group algebras, commutative H^*-algebras, $L^p(G)$ $(1 < p < \infty$ for compact $G)$, $\ell^1(S)$ and some semigroup algberas also belong to this class.

2. We need the following lemma to prove the facts stated in the preceding section. It seems that the lemma is a known result, but we give a proof for the sake of completeness.

Lemma. Let X be a Banach space and S a subspace of the dual space X^* of X. Then the following are equivalent:

(i) $\| x \| = \sup\{| f(x)| : f \in S, \| f \| \leq 1\}$ for every $x \in X$.

(ii) $\{ f \in S : \| f \| \leq 1\}$ is weak*-dense in the unit ball $\{ f \in X^* : \| f \| \leq 1\}$ of X^*.

Proof. (i) $\Rightarrow$ (ii). Suppose (i). Let K be the weak*-closure of $\{ f \in S : \| f \| \leq 1\}$. Then K is a weak*-closed, convex, balanced set of X^* and hence K equals the bipolar K^{00} of K with respect to the dual pair $\{(X^*, \sigma(X^*, X)), (X, \| \ \|_X)\}$. However K^{00} contains the unit ball $\{ f \in X^* : \| f \| \leq 1\}$. In fact, let $f \in X^*$ with $\| f \| \leq 1$. Consider an arbitrary element x in the polar K^0 of K. Then $|k(x)| \leq 1$ for every $k \in K$. Also we have from (i) that $\| x \| = \sup\{| f(x)| : f \in K\}$ and hence $| f(x)| \leq \| f \| \| x \| \leq 1$. In other words, $f \in K^{00}$. Cosequently, we have

$$K \subseteq \{ f \in X^* : \| f \| \leq 1\} \subseteq K^{00} = K.$$

and obtain (ii).

(ii) $\Rightarrow$ (i). Suppose (ii). Let $x \in X$ and $\varepsilon > 0$. Take a functional $f \in X^*$ such that $\| f \| \leq 1$ and $\| x \| \leq | f(x)| + \varepsilon / 2$. By (ii), we can find a linear functional $g \in \{ f \in S : \| f \| \leq 1\}$ such that $| f(x) - g(x)| < \varepsilon / 2$. Then

$$\| x \| \leq |g(x)| + \varepsilon \leq \sup\{| f(x)| : f \in S, \| f \| \leq 1\} + \varepsilon$$

and so $\| x \| \leq \sup\{| f(x)| : f \in S, \| f \| \leq 1\}$ since ε is arbitrary. The converse inequality is trivial, hence we obtain (i). Q. E. D.

3. Let $L^1(G)$ be the group algebra on a locally compact Abelian group G. G. I. Gaudry [1] precisely showed that the norm of a bounded regular Borel measure μ on G is given by the following equation:

$$\| \mu \| = \sup_p \left| \int_G p(t^{-1}) d\mu(t) \right| ,$$

where p is any trigonometric polynomial for which $\| p \|_\infty \leq 1$. In particular, if μ belongs to $L^1(G)$, then the above equation can be rewritten as follows:

$$\| f \|_1 = \| f^\wedge \|_{\mathrm{BSE}} \quad (f \in L^1(G)).$$

Therefore $L^1(G)$ belongs to our class. Here we will give a short proof of this result as follows: Let $g \in L^\infty(G)$ with $\| g \|_\infty \leq 1$. For any compact subset K of G and $\varepsilon > 0$, take a continuous function $g_{K, \varepsilon}$ on G with compact support such that

$\|g_{K,\varepsilon}\|_\infty \leq 1$ and $g = g_{K,\varepsilon}$ on $K \cap (K_\varepsilon)^c$ for some K_ε with $\int_{K_\varepsilon} dx < \varepsilon$. Moreover, take a trigonometric polynomial $p_{K,\varepsilon}$ such that $\left\| p_{K,\varepsilon} \right\|_\infty \leq 1$ and $\left\| p_{K,\varepsilon} - g_{K,\varepsilon} \right\|_\infty \leq \varepsilon$. In this case, it is easy to see that

$$w^*\text{–}\lim_{K,\varepsilon} p_{K,\varepsilon} = g.$$

In other words, the set of trigonometric polynomials p on G with $\|p\|_\infty \leq 1$ is weak*-dense in the unit ball of $L^\infty(G)$. Therefore $\|f\|_1 = \|f^\wedge\|_{\mathrm{BSE}}$ for every $f \in L^1(G)$ by Lemma 2.

4. Let A be a commutative H^*-algebra. Let $\{e_\alpha\}$ be the complete orthogonal family of irreducible self-adjoint idempotents in A. Then there exists a one-to-one correpondence between points of Φ_A and the family $\{e_\alpha\}$. For each e_α, let φ_α be the corresponding element of Φ_A and $e'_\alpha = e_\alpha / \|e_\alpha\|$. Note that for each α, we have $\varphi_\alpha(x) = \langle x, e'_\alpha \rangle / \|e_\alpha\|$ $(x \in A)$. An arbitrary element p of span(Φ_A) has a unique expansion $p = \sum_\alpha p^\wedge(\alpha)\varphi_\alpha$. Then span($\Phi_A$) is norm-dense in A^* and hence the set of $p \in$ span(Φ_A) with $\|p\|_{A^*} \leq 1$ is weak*-dense in the unit ball of A^*. Therefore we see from Lemma 2 that A belongs to our class .

5. Let G be a compact Abelian group and $1 < p < \infty$. Then $L^p(G)$ is a commutative Banach algebra under convolution and the linear functionals in span(Φ_A) correspond precisely to the trigonometric polynomials on G. However since the set of trigonometric polynomials on G is L^q norm-dense in $L^q(G)$, it follows that span(Φ_A) is norm-dense in $(L^p(G))^*$, where q is the conjugate number of p. Therefore we see from Lemma 2 that $L^p(G)$ belongs to our class.

6. Let S be any set and $\ell^1(S)$ the set of all complex-valued functions x on S such that $\|x\|_1 = \sum_{s \in S} |x(s)| < \infty$. Then $\ell^1(S)$ is a commutative Banach algebra under the pointwise operations and norm $\|\ \|_1$. For each $s \in S$, define $\varphi_s(x) = x(s)$ $(x \in \ell^1(S))$. Then $\Phi_{\ell^1(S)} = \{\varphi_s : s \in S\}$. Let $\xi \in \ell^\infty(S)$ with $\|\xi\|_\infty \leq 1$ and let Λ be the family of finite subsets of S. For each $\lambda \in \Lambda$, define $f_\lambda = \sum_{s \in \lambda} \xi(s)\varphi_s$. Then $f_\lambda \in \Phi_{\ell^1(S)}$ and $\|f_\lambda\| \leq 1$ $(\lambda \in \Lambda)$. In this case, we can easily see that $\{f_\lambda\}$ converges *-weakly to the linear functional on $\ell^1(S)$ corresponding to ξ. Therefore we see from Lemma 2 that $\ell^1(S)$ belongs to our class.

7. Let k be a non-negative integer and $N_k = \{k, k+1, k+2, ...\}$. Then N_k is an Abelian semigroup under addition. Let $L^1(N_k)$ be the semigroup algebra on N_k. In this

case, we see that $L^1(N_k)$ belongs to our class. Suppose $k \geq 1$. For each $z \in \Delta_0 = \{z \in \boldsymbol{C} : 0 < |z| \leq 1\}$, we define

$$\varphi_z(a) = \sum_{n=k}^{\infty} a_n z^n$$

for each $a = (a_n)_{n \in N_k} \in L^1(N_k)$. Then $\Phi_{L^1(N_k)} = \{\varphi_z : z \in \Delta_0\}$. Let $b = (b_n)_{n \in N_k} \in L^\infty(N_k)$ with $\| b \|_\infty \leq 1$. For any $m \geq k$, set

$$z_1 = 1, z_2 = e^{2\pi i/(m+1)}, \dots, z_{m+1} = e^{2m\pi i/(m+1)} .$$

Moreover, set $b_0 = 1, b_1 = 1, \dots, b_{k-1} = 1$ and then there are the unique complex numbers $c_1, c_2, \dots, c_{m+1}$ which satisfy the following

$$\begin{cases} c_1 + c_2 + \dots + c_{m+1} = b_0 \\ c_1 z_1 + c_2 z_2 + \dots + c_{m+1} z_{m+1} = b_1 \\ \quad \vdots \\ c_1 z_1{}^m + c_2 z_2{}^m + \dots + c_{m+1} z_{m+1}{}^m = b_m . \end{cases}$$

In this case, it is obvious that

$$|c_1 z_1{}^n + c_2 z_2{}^n + \dots + c_{m+1} z_{m+1}{}^n| \leq 1$$

for every $n \geq k$. We now define

$$p_m = \sum_{n=1}^{m+1} c_n \varphi_{z_n} \quad (m = k, k+1, \dots\dots).$$

Then $p_m \in \Phi_{L^1(N_k)}$ with $\| p_m \| \leq 1$ for each $m \geq k$. Also, we can easily see that $\{ p_m \}_{m \geq k}$ converges *-weakly to the linear functional on $L^1(N_k)$ corresponding to b.

Similarly, the above argument holds for the case of $k = 0$. Therefore we see from Lemma 2 that $L^1(N_k)$ belongs to our class.

8. If A is one of a unital commutative C^*-algebra, the disk algebra, the classical Hardy algebra, a group algera on a discrete Abelian group, a commutative H^*-algebra, $L^p(G)$ $(1 < p < \infty$ for compact $G)$, $\ell^1(S)$ or $L^1(N_k)$, then $A^\wedge = C_{\mathrm{BSE}}(\Phi_A)$ (see [6, Theorems 3 and 7] and [7, Theorems, 3, 4, 5 and 6]). Therefore these Banach algebras are isometrically isomorphic to the induced algebras.

9. For any commutative Banach algebra A, we define

$$J_A = \sup \{ \left| \sum_{\varphi \in \Phi_A} p^\wedge(\varphi) \right| : p \in \mathrm{span}(\Phi_A), \ \| p \|_{A^*} \leq 1 \} .$$

Then we have $1 \leq J_A \leq \infty$. If $J_A < \infty$, then A posesses a weak approximate identity in the sense of Jones-Lahr (cf. [3]) bounded by J_A from Helly's theorem. Therefore

$$\| T^\wedge \|_{\mathrm{BSE}} \leq J_A \| T \|$$

for every $T \in M(A)$ such that $T^\wedge \in C_{\mathrm{BSE}}(\Phi_A)$. Here $M(A)$ denotes the multiplier algebra of A an $T^\wedge$ denotes the Helgason-Wang transform of T (cf. [2, 4, 9]). In this case, we can observe that if $J_A = 1$ and $\| x^\wedge \|_{\mathrm{BSE}} = \| x \|$ for every $x \in A$, then

$$\| T^\wedge \|_{\mathrm{BSE}} = \| T \|$$

for every $T \in M(A)$ such that $T^\wedge \in C_{\mathrm{BSE}}(\Phi_A)$. In particular since $J_{L^1(G)} = 1$, Guadry's eaquation follows from the fact that $L^1(G)$ belongs to our class.

It is not true that $J_A = 1$ for any commutative Banach algebra A. In fact, we see that

$$J_{\ell^1(S)} = \begin{cases} n, & \text{if } \#S = n < \infty \\ \infty, & \text{if } S \text{ is an infinite set}. \end{cases}$$

Acknowledgment. The authors wish to thank Mr. A. Uchiyama, Dr. K. Tanahashi and Dr. S. Yamagami for helpful suggestions about the fact that $L^1(N_k)$ belongs our class.

References

1. G. I. Gaudry, Topics in harmonic analysis. Lecture notes, Department of Mathematics, Yale University, New Haven, Ct. 1969.
2. S. Helgason, Multipliers of Banach algebras, Ann. of Math., 64(1956), 240-254.3 8.
3. C. A. Jones and C. D. Lahr, Weak and norm approximate identities are different, Pacific J. Math. 72(1977), 99-104.
4. R. Larsen, *An Introduction to the Theory of Multipliers* (Springer-Verlag, New York- Heiderberg, 1971).
5. W. Rudin, *Fourier Analysis on Groups,* Interscience Publishers, Inc., New York, 1962.
6. S.-E. Takahasi and O. Hatori, Commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein-type theorem, Proc. Amer. Math. Soc., 110(1990), 149-158.
7. S.-E. Takahasi and O. Hatori, Commutative Banach algebras and BSE-inequalities, Math. Japonica 37(1992), 607-614.
8. 高橋眞映, 第３４回実函数論・函数解析合同シンポジウム講演集録, 1995, 7月。
9. J.-K. Wang, Multipliers of commutative Banach algebras, Pacific J. Math., 11 (1961), 1131-1149.
10. J. G. Wendel, On isometric isomorphism of group algebras, Pacific J. Math. 1(1951), 305-311.
11. J. G. Wendel, Left centralizers and isomorphisms of group algebras, Pacific J. Math. 2(1951), 251-261.

可換Banach環の最大正則部分環とApostol環について
(The greatest regular subalgebras and the Apostol algebras of commutative Banach algebras)

新潟大学大学院　自然科学研究科　　羽鳥　理
(Osamu Hatori)

Abstract. We study the spectra of Fourier multipliers on R^n. We show that the Apostol algebra and the greatest regular closed subalgebra of the algebra of L^p-multipliers on R^n whose Fourier transforms are continuous and vanish at infinity coincide with each other and they are *not* ideals of the algebra if $1 < p < 2$.

可換Banach環Aにおいて$a \in A$に対してA上の有界線形作用素T_aを$T_a b = ab$により定める。$Dec\, A$によりT_aがA上のdecomposable作用素であるような$a \in A$全体を表し、$Reg\, A$によりAの閉部分環でそれ自身が正則なBanach環の中でで最大のものを表す。Neumann [8]はAが半単純の時は、Gelfnad変換がGelfand空間上hull-kernel位相に関して連続なもの全体と$Dec\, A$とが一致することを示し、$Reg\, A \subset Dec\, A$も示した。Neumann [8]やLaursen-Neumann [7], [9]はこのことを自然なスペクトルを持つ測度を決定する問題に適用して、興味深い結果を得た。同様の方法でコンパクトabel群上のL^p-multiplierの場合を扱うことができる。一方R^nをはじめとする非コンパクト局所コンパクトabel群ではL^pが環にならないのでLaursen-Neumannの方法は適用できないが、その上のL^p-multiplierについてはどのようであるか興味あるところでありこの小論のテーマでもある。

定義1　XをBanach空間とし、TをX上の有界線形作用素とする。Tは次の条件を満たすときdecomposableであると言われる：複素平面Cの開被覆U, Vに対して$Y + Z = X$なる不変部分空間Y, Zが存在して$T|Y$（resp. $T|Z$）のスペクトル$\sigma(T|Y)$（resp. $\sigma(T|Z)$）がU（resp. V）に含まれる。

定義2　Aを可換Banach環とし$a \in A$とする。A上の積作用素T_aを$T_a b = ab,\ b \in A$と定めるとT_aはA上の有界作用素になる。

$$Dec\, A = \{a \in A : T_a \text{は decomposable}\}$$

と定める。

ApostolはAが単位元を持つとき$Dec\, A$はAの閉部分環をなす事を示した。このことから$Dec\, A$はApostol環と呼ばれることがある。Neumann [8]はAが半単純であれば単位的であることを仮定しなくとも$DecA$は閉部分環であることを示した。

定理N　Aを半単純可換Banach環とする。この時$a \in Dec\, A$であることとaのGelfnad変換がGelfnad空間上hull-kernel位相に関して連続であることは同値である。特に、$Dec\, A$

はAの閉部分環である。また$Reg\,A \subset Dec\,A$である。

単位的半単純可換Banach環は最大の閉正則部分環を持つことを証明したのはAlbrecht [1]である。その後、Inoue-Takahasi [2]とNeumann [9]により任意の可換Banach環に対する存在定理が独立に示された。

問題　半単純可換Banach環Aに対して$Reg\,A = Dec\,A$となるか。

特に、局所コンパクトabel群G上の測度環$M(G)$のとき最大正則閉部分環とApostol環が一致するかどうか興味あるところである。これに関してLaursen-Neumann [7], [9]は可換Banach環とその乗作用素環を調べることにより、Fourier-Stieltjes変換が無限遠点で0になるような測度全体（$M(G)$の閉部分環）$M_0(G)$に対して以下で述べるような結果を得た。$M_{00}(G)$はGelfnad変換がGの相対群$\widehat{G}$の外側で0になるような測度全体からなる$M_0(G)$の閉イデアルとする。また、

$$NS(M_0(G)) = \{\mu \in M_0(G) : \overline{\check{\mu}(\Phi_{M_0(G)})} = \overline{\hat{\mu}(\widehat{G})}\}$$

と定める。ここで、$\Phi_{M_0(G)}$は$M_0(G)$の極大イデアル空間、$\mu \in M_0(G)$に対して$\check{\mu}$はμの$M_0(G)$におけるGelfand変換、$\hat{\mu}$はFourier-Stieltjes変換を表す。よって、$NS(M_0(G))$はスペクトルがFourier-Stieltjes変換により与えられる測度全体である。Gが非離散のときは$M_0(G) \neq NS((M_0(G))$であることがVaropoulos [11], [12]により知られている。

定理LN

$$RegM_0(G) = DecM_0(G) = M_{00}(G) \subset NS(M_0(G))$$

である。特に、Gがコンパクトの時には$\subset$は$=$で置き換えられ、さらに、スペクトルが可算集合であるような（$M_0(G)$に属する）測度全体とも一致する。

この定理はZafran [13]の定理とShimizu-Izuchi [5](cf.[3], [4])の定理を含んでいる。

定義3　$1 \leq p < \infty$とする。$L^p(G)$上の有界線形作用素Tで平行移動不変なものをL^p-multiplierといい、その全体を$M_p(G)$で表す。TのFourier変換を$\widehat{T}$で表す。

$M_p(G)$は作用素ノルムで可換Banach環になる。$M_1(G)$と$M(G)$は等距離同型であることがよく知られている。$T \in M_1(G)$を測度と同一視したとき$\widehat{T}$はFourier-Sstieltjes変換に他ならない。$1 < p < 2$のとき$M_1(G) \subset M_p(G) \subset M_2(G)$であり、Planchrelの等式から$\widehat{M_2(G)} = L^\infty(\widehat{G})$なので$T \in M_p(G)$のFourier変換は$\widehat{G}$上の本質的有界関数であるが、連続関数とは限らない。$M_p(G)$と$M_{p'}(G)$とは同一視できるので、以下では$1 < p < 2$と仮定する。

定義4

$$C_0M_p(G) = \{T \in M_p(G) : \widehat{T} \in C_0(\widehat{G})\},$$

$$NS(C_0M_p(G)) = \{T \in C_0M_p(G) : \overline{\check{T}(\Phi_{C_0M_p(G)})} = \overline{\widehat{T}(\widehat{G})}\},$$

$$C_{00}M_p(G) = \{T \in C_0M_p(G) : \check{T}\text{は}\widehat{G}\text{の外側で}0\}$$

とする。ここで$C_0(\widehat{G})$は$\widehat{G}$上の複素数値連続関数で無限遠点で0になるもの全体である。

$C_0M_1(G) = M_0(G)$, $NS(C_0M_1(G)) = NS(M_0(G))$であるから非離散な$G$に対しては$C_0M_1(G) \neq NS(C_0M_1(G))$である。Zafran [14], [15]により$G = R^n, Z^n, T^n$の場合に$C_0M_p(G) \neq NS(C_0M_p(G))$であることが知られている。$G$がコンパクトの場合は$\widehat{G}$が離散空間で、$M_p(G)$が可換Banach環$L^p(G)$の（可換Banach環としての）乗作用素全体になるのでLausen-Neumannの方法によって次が分かる。

定理1　Gはコンパクトabel群とする。この時、

$$\begin{aligned} RegC_0M_p(G) &= DecC_0M_p(G) = C_{00}M_p(G) \\ &= NS(C_0M_p(G)) = \{T \in C_0M_p(G) : T\text{のスペクトルは可算集合}\} \end{aligned}$$

である。

Gがコンパクトでない場合は$L^p(G)$が可換Banach環ではないのでLausen-Neumannの方法は適用できない。一方Laursen-Neumannと同様な結果が期待できそうにも見える。しかし、$G = Z^n$の場合$\widehat{G}$がコンパクトだから$1 \in C_0M_p(G)$となり、$DecC_0M_p(G)$は$C_0M_p(G)$のイデアルとはならない。よって$DecC_0M_p(G) \neq C_{00}M_p(G)$である。一方Gelfnad空間を$L^1\widehat{(G)}$のレベル集合に分解する方法を用いて次が示された。

定理2　Gは任意の局所コンパクトabel群とする。このとき

$$RegC_0M_p(G) = DecC_0M_p(G) \subset NS(C_0M_p(G))$$

が成立する。さらに、上式の左辺は、$NS(C_0M_p(G))$に含まれ和について閉じた集合のうちで極大である。

上で述べたように$RegC_0M_p(Z^n) = DecC_0M_p(Z^n)$は$C_0M_p(Z^n)$のイデアルではない。$G = R^n$の場合が興味もたれる。次のような結果が得られた。

定理3　$RegC_0M_p(R^n) = DecC_0M_p(R^n)$は$C_0M_p(R^n)$のイデアルではない。また、$C_{00}M_p(R^n) = \{0\}$である。

証明は以下の補題4を用いて$n = 1$の場合を証明し、Saeki [10]の定理を用いて任意のnの場合に拡張する。

補題4

$$C_0\widehat{M_p}(R) \bigcap C_c(R) \setminus Dec\widehat{C_0M}_p(R) \neq \emptyset$$

ここで$C_c(R)$はR上の複素数値連続関数でコンパクト台を持つもの全体を表す。

補題4の証明ではJodeit [6]のmultiplierのextension theoremを用いる。

参考文献

[1] E. Albrecht, *Decomposable systems of operators in harmonic analysis*, In Toeplitz Centennial (I. Gohberg, ed.), Birkhäuser, Basel 1982, pp. 19–35

[2] J. Inoue and S.-E. Takahasi, *A note on the largest regular subalgebra of a Banach algebra*, Proc. Amer. Math. Soc. 116(1992), 961–962

[3] K. Izuchi, *On measures whose spectra are countable sets*, Sci. Rep. Res. Inst. Engrg. Kanagawa Univ. No.2(1979), 73–80

[4] K. Izuchi, *An L-subspace generated by a certain measure with countable spectrum*, Colloq. Math. 44(1981), 327–332

[5] K. Izuchi and C. Shimizu, *On measures with countable spectra*, Approximation theory in functional analysis (Proc. Sympos., Re. Inst. Math. Sci., Kyoto Univ., Kyoto,1975) Sûrikaisekikenkyûsho Kôkyûroku No.265(1976), 1–9 (Japanese) MR48# 4653

[6] M. A. Jodeit,Jr *Restriction and extensions of Fourier multipliers*, Studia Math. 34(1970), 215–226

[7] K. B. Laursen and M. M. Neumann, *Decomposable multipliers and applications to harmonic analysis*, Studia Math. 101(1992), 193–214

[8] M. M. Neumann, *Banach algebras, decomposable convolution operators, and a spectral mapping property*, In Function Spaces (K. Jarosz, ed.), Marcel Dekker, New York 1992, pp. 307–323

[9] M. M. Neumann, *Commutative Banach algebras and decomposable operators*, Mh. Math. 113(1992), 227–243

[10] S. Saeki, *Translation invariant operators on groups*, Tôhoku Math. J 22(1970), 409–419

[11] N. Varopoulos, *Sur les measures de Radon d'un groupe localement compact abélian*, C. R. Acad. Sci. Paris 258(1964), 3805–3808

[12] N. Varopoulos, *Sur les measures de Radon d'un groupe localement compact*, C. R. Acad. Sci. Paris 258(1964), 4896–4899

[13] M. Zafran, *On the spectra of multipliers*, Pacific J. Math. 47(1973), 609–626

[14] M. Zafran, *The spectra of multiplier transformations on the L_p spaces*, Ann. Math. 103(1976), 355–374

[15] M. Zafran, *The function operating on multiplier algebras*, J. Funct. Anal. 26(1977), 289–314

Current Address: *Department of Mathematical Science, Graduate school of Science and Technology, Niigata University, Niigata 950-21 Japan*

Nicely placed sets の積集合についての注意

城西大学理学部　山口 博

Abstract. G. Godefroy introduced the concept of a nicely placed set on a discrete abelian group. Let G and H be a compact abelian group and its closed subgroup respectively, and let $\tau : G^{\wedge} \to G^{\wedge}/H^{\perp}$ be the natural homomorphism. Let $E^{\sim}$ be a nicely placed set in $G^{\wedge}/H^{\perp}$. We show that $\tau^{-1}(E^{\sim})$ is a nicely placed set in $G^{\wedge}$.

Godefroy（[3]）により、discrete abelian group にたいして、nicely placed set と Shapiro set という概念が導入されている。そして、可算 discrete abelian groups において、２つの nicely placed sets（Shapiro sets）の積集合が nicely placed set（Shapiro set）になることが示されている。ここでは、この結果についての注意を述べる。

G を compact abelian group とし、$G^{\wedge}$ を G の dual group とする。m_G を G の Haar measure とする。また、$M(G)$、$L^1(G)$ は通常の測度環、群環とする。$E \subset G^{\wedge}$ にたいして、$M_E(G) = \{\mu \in M(G) : \mu^{\wedge} = 0 \text{ on } E^c\}$、$L^1{}_E(G) = M_E(G) \cap L^1(G)$ とおく。

<u>定義</u> 1.　$0 < p < 1$ とする.

(i)　$E \subset G^{\wedge}$: nicely placed set

$\Leftrightarrow$　$L^1{}_E(G)$ の unit ball は $L^p(G)$ で closed.

(ii)　$E \subset G^{\wedge}$: Shapiro set

$\Leftrightarrow$　$E' \subset E$; E' : nicely placed set.

(iii)　$E \subset G^{\wedge}$: (＊) を満たす

$\Leftrightarrow$　$\mu \in M_E(G) \Rightarrow \mu_a, \mu_s \in M_E(G)$.

但し、$\mu = \mu_a + \mu_s$ は μ の m_G に関する Lebesgue 分解.

(iv)　$E \subset G^{\wedge}$: Riesz set $\Leftrightarrow$ $M_E(G) \subset L^1(G)$.

<u>注意</u> 1.（i）　$E \subset G^{\wedge}$: nicely placed set $\Leftrightarrow$ $L^1{}_E(G)$ の unit ball は測度収束の位相で closed.

(ii)　Riesz set, Shapiro set の部分集合はそれぞれRiesz set, Shapiro set である. しかし、(＊) を満たす集合の部分集合が (＊) を満たす集合とは限らない.　又、nicely placed set の部分集合が nicely placed set とはかぎらない.

(iii)　$G = \mathbb{T}$ (i.e, $G^{\wedge} \cong \mathbb{Z}$) のとき、[3, 3.8 Example, p.322] で $\mathbb{Z}$ の Riesz set で nicely placed set でないものが構成されている.

(iv)　(cf. [3, Lemma 1.1]).

$E \subset G^{\wedge}$: Riesz set ⇔ $\forall E' \subset E$; E' : （＊）を満たす集合.

これらの集合の間には、次の関係がある。

$$\begin{array}{ccc} \{\text{ nicely placed set }\} & \subset & \{\text{ (＊)を満たす集合 }\} \\ \cup & & \cup \\ \{\text{ Shapiro set }\} & \subset & \{\text{ Riesz set }\} \end{array}$$

Godefroy は次の定理を得た。

定理 1 ([３、Theorem 2.7]).

G_1, G_2 : metrizable compact abelian groups.

(i) $E_i \subset G_i^{\wedge}$: nicely placed set $(i = 1, 2)$

⇒ $E_1 \times E_2$: nicely placed set in $G_1^{\wedge} \oplus G_2^{\wedge}$.

(ii) $E_i \subset G_i^{\wedge}$: Shapiro set $(i = 1, 2)$

⇒ $E_1 \times E_2$: Shapiro set in $G_1^{\wedge} \oplus G_2^{\wedge}$.

次に、(G, X) を compact abelian group G が locally compact Hausdorff space X に作用する位相変換群とする。$\sigma \in M^{1}(X)$ を quasi-invariant measure とし、$\mu \in M(X)$ に対して、$sp(\mu)$ を μ の spectrum とする。

定義 2. $0 < p < 1$.

$E \subset G^{\wedge}$: σ-nicely placed set

⇔ $L^1_E(\sigma)$ の unit ball が $L^p(\sigma)$ で closed.

但し、$L^1_E(\sigma) = \{f \in L^1(\sigma) : sp(f) \subset E\}$.

Finet and Tardivel-Nachef は次の定理を得た。

定理 2 ([２、Theorem 2.1]).

G が metrizable compact abelian group とする。

$E \subset G^{\wedge}$: nicely placed set ⇒ E : σ-nicely placed set.

ところで、次の補題が成り立つ。

補題 1. G : compact abelian group. $E \subset G^{\wedge}$ に対して、次は同値.

（i）　E : nicely placed set in $G^\wedge$.

（ii）　$E \cap \Gamma$: nicely placed set in Γ for any countable subgroup Γ of $G^\wedge$.

定理2と補題1により、次の結果が得られる。

定理 3.　G : compact abelian group.　H : closed subgroup of G.
$E^\sim \subset G^\wedge / H^\perp$: nicely placed set.　τ : $G^\wedge \to G^\wedge / H^\perp$: natural map.
$\Rightarrow$　$E = \tau^{-1}(E^\sim)$: nicely placed set in $G^\wedge$.

注意 2.　定理3は Shapiro set に対しては成り立たない。

注意 3.　G : compact abelian group.　E, E' : nicely placed sets in $G^\wedge$.
$\Rightarrow$　$E \cap E'$: nicely placed set in $G^\wedge$.

定理3と注意3より、次の系が得られる。

系 1.　G_1, G_2 : compact abelian groups.
$E_i \subset G_i^\wedge$: nicely placed set　$(i = 1, 2)$.
$\Rightarrow$　$E_1 \times E_2$: nicely placed set in $G_1^\wedge \oplus G_2^\wedge$.

補題 2.　G : compact abelian group.　$E \subset G^\wedge$.　すると、次は同値.

（i）　E : Shapiro set in $G^\wedge$.

（ii）　$E \cap \Gamma$: Shapiro set in Γ for any countable subgroup Γ of $G^\wedge$.

定理1（ii）と補題2により、次の系が得られる。

系 2.　G_1, G_2 : compact abelian groups.
$E_i \subset G_i^\wedge$: Shapiro set　$(i = 1, 2)$.
$\Rightarrow$　$E_1 \times E_2$: Shapiro set in $G_1^\wedge \oplus G_2^\wedge$.

参考文献

[1] C. Finet, Lacunary sets for groups and hypergroups, J. Austral. Math. Soc. 54 (1993), 39-60.

[2] C. Finet and V. Tardivel-Nachef, Lacunary sets on transformation groups, Hokkaido Math. J. 23 (1994), 1-19.

[3] G. Godefroy, On Riesz subsets of abelian discrete groups, Israel J.Math. 61 (1988), 301-331.

[4] P. Harmand, D. Werner and W. Werner, M-ideals in Banach spaces and Banach Algebras, Lecture Notes in Mathematics, Vol. 1547, Springer, Berlin-Heidelberg, 1993.

[5] J.H. Shapiro, Subspaces of $L^p(G)$ spanned by characters, $0 < p < 1$, Israel. J. Math. 29 (1978), 248-264.

[6] H. Yamaguchi, On the product of a Riesz set and a small p set, Proc. Amer. Math. Soc. 81 (1981), 273-278.

An application of orthoisomorphisms to non-commutative L^p-isometries

新潟大学理学部・渡辺恵一

0. Introduction

考えている問題は，前回の第3回関数空間セミナーで報告したものと同じである．今回の報告の主旨は，Bunce-Wright の直交保存同型に関する定理を用いると，我々の定理もやや良くなるという事である．

$1 < p < \infty,\ p \neq 2$ とする．Yeadon [Y1] は，半有限 von Neumann 環 に付随した非可換 L^p-空間の間の線型等距離作用素について詳細な構造定理を得た (1981)．これは，古典的著書 [B] の中で Banach が ℓ^p 及び $L^p(0,1)$ 上の線型等距離全射の構造を述べたことに起源を持つと考えられる．
半有限とは限らない von Neumann 環に付随した非可換 L^p-空間へ上の定理を拡張するというのが，我々の問題である。その際の困難は，半有限の場合には central になるような非可換性の障害物，例えば Radon-Nikodym derivative が現れるため，土台の von Neumann 環と非可換 L^p-空間のあいだに共通の領域が得られにくい，ということから来る様に思われる．

我々は，Haagerup の非可換 L^p-空間を用いる．その元は有界でもなく，土台の von Neumann 環に付随してもいないのだが，とにかく作用素であって，その非負部分は土台の von Neumann 環の前双対空間の情報を持っている．

1.

前回の話とかなり重複するので，Haagerup の，半有限とは限らない von Neumann 環に付随した非可換 L^p-空間の基本的事項及び記号については第3回関数空間セミナー報告集を参照して頂くことにする．

L^p-isometry から Jordan $*$-同型を誘導する際に重要なのは Clarkson の不等式の等号成立条件である．

Theorem (Kosaki, 1984).
$a, b \in L^p(\mathcal{M})$ に対して，
$$||a+b||_p{}^p + ||a-b||_p{}^p = 2(||a||_p{}^p + ||b||_p{}^p)$$
$$\iff ab^* = a^*b = 0.$$

[W1] では次が示されている．

Theorem 1.

$\mathcal{M}_i, i = 1, 2,$ *σ-finite von Neumann algebras*

φ_0 *(resp. ψ_0), faithful normal state on $\mathcal{M}_1$ (resp. $\mathcal{M}_2$)*

$T : L^p(\mathcal{M}_1; \varphi_0) \longrightarrow L^p(\mathcal{M}_2; \psi_0)$, *onto, $*$-preserving, linear isometry*

$\Longrightarrow {}^{\exists}J : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, *onto Jordan $*$-isomorphism,* $J(s(a)) = s(T(a)),\quad a \in L^p(\mathcal{M}_1; \varphi_0)_{sa}$.

ここで, $s(a)$ は *support projection* を表す.

Clarkson の不等式の等号成立条件により, $s(a) \mapsto s(T(a))$ が写像になることを示し, Dye の projection orthoisomorphism の定理により, それが Jordan $*$-isomorphism であることを見るのが証明の方針である. このため, von Neumann 環の σ-有限性, 及び T の $*$-保存性, 全射を仮定しなければならなかった.

Theorem (Dye, 1955).

$J : (\mathcal{M}_1)_{proj.} \longrightarrow (\mathcal{M}_2)_{proj.}$, *onto, 1 to 1,* $ef = 0 \Leftrightarrow J(e)J(f) = 0$,

$\mathcal{M}_1$ は I_2-直和成分を持たない

$\Longrightarrow J$ は *onto Jordan $*$-isomorphism* (の制限).

Theorem 1 改.

$\mathcal{M}_i, i = 1, 2,$ *von Neumann algebras*

φ_0 *(resp. ψ_0), faithful normal semifinite weight on $\mathcal{M}_1$ (resp. $\mathcal{M}_2$)*

$T : L^p(\mathcal{M}_1; \varphi_0) \longrightarrow L^p(\mathcal{M}_2; \psi_0)$, *linear isometry*

$\Longrightarrow {}^{\exists}J : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, *Jordan $*$-isomorphism,*

$$|T(h_\varphi^{1/p})| = h_{\varphi \circ J^{-1}}^{1/p}, \quad \varphi \in (\mathcal{M}_1)_{*,+}.$$

前回は自己共役作用素の support を見て projection の対応を引き起こしたが, 今回は前双対空間の非負部分の対応を考える;

$$\beta : (\mathcal{M}_1)_{*,+} \longrightarrow (\mathcal{M}_2)_{*,+}, \text{ by } h_{\beta(\varphi)}^{1/p} = |T(h_\varphi^{1/p})|, \quad \varphi \in (\mathcal{M}_1)_{*,+}.$$

β は, Clarkson の不等式の等号成立条件から直交保存と分かり, 次の定理により, $(\mathcal{M}_1)_*$ 上に線型に拡張される. (次の定理は, Christensen, Yeadon らの, von Neumann 環の射影束上の有限加法的確率測度に関する定理から導かれる.)

Theorem 2.

$\rho : \mathcal{M}_{*,+} \longrightarrow [0, \infty)$,

(1) $\rho(\alpha\varphi) = \alpha\rho(\varphi), \alpha \geq 0$

(2) $\varphi_n \perp \varphi_m (n \neq m), {}^{\exists}\sum \varphi_n \Rightarrow \rho(\sum \varphi_n) = \sum \rho(\varphi_n)$

(3) $\rho(\varphi) \leq ||\varphi||$

(4) $||\varphi_n - \varphi|| \to 0 \Rightarrow \rho(\varphi_n) \to \rho(\varphi)$

$\Longrightarrow {}^{\exists 1}x \in \mathcal{M}_+ \ ; \ \rho(\varphi) = \varphi(x), \ \varphi \in \mathcal{M}_{*,+}$.

そこで, Bunce-Wright の定理が使え, 少し計算すれば Theorem 1 改を得る. 以下は Bunce-Wright の定理を injective の場合に述べたものである.

Theorem (Bunce-Wright, 1993).

$\beta : (\mathcal{M}_1)_* \longrightarrow (\mathcal{M}_2)_*$, *1 to 1, positive, continuous linear,*

$\varphi \perp \psi \Rightarrow \beta(\varphi) \perp \beta(\psi)$

$\Longrightarrow {}^{\exists}J : \mathcal{M}_1 \longrightarrow (\mathcal{M}_2)_\beta$, *onto weak*-continuous Jordan *-isomorphism,*

$\beta^*(J(x)) = \beta^*(1)x, \ x \in \mathcal{M}_1$.

ここで, $(\mathcal{M}_2)_\beta$ は $\{s(\beta(\varphi)); \varphi \in (\mathcal{M}_1)_{*,+}\}$ が生成する *σ-weak closed *-*部分環である.

2.

Theorem 1 改により, 非可換 L^p-空間の間に線型等距離写像があれば, 土台の von Neumann 環の間に Jordan *-同型がある, という部分は, なんら特別な仮定無しに証明された事になる. 残る部分は, 誘導された Jordan *-同型を使って, もとの L^p-等距離写像の構造を書き表す事である.

<u>以下 T は全射と仮定する.</u> $\mathcal{M}_2$ 上には, ψ_0 と $\varphi_0 \circ J^{-1}$ の 2 つの faithful normal semifinite weight がある.

κ を, ψ_0 から $\varphi_0 \circ J^{-1}$ への weight の変更に関連した, $\mathcal{M}_2 \rtimes_{\sigma^{\psi_0}} \mathbb{R}$ から $\mathcal{M}_2 \rtimes_{\sigma^{\varphi_0 \circ J^{-1}}} \mathbb{R}$ の上への自然な *-isomorphism とする. κ は可測作用素環の間の *-同型 $\tilde{\kappa}$ に拡張され, $\tilde{\kappa}$ の制限は $L^p(\mathcal{M}_2; \psi_0)$ から $L^p(\mathcal{M}_2; \varphi_0 \circ J^{-1})$ の上への positive linear isometry である (cf. [W1; Lemma 2.1, Lemma 2.2]).

一方, モジュラー自己同型群の一意性から,

$$\sigma^{\varphi_0 \circ J^{-1}} = J \circ \sigma^{\varphi_0} \circ J^{-1}.$$

(注. J が $*$-反同型ならば$\sigma_t^{\varphi_0\circ J^{-1}} = J\circ\sigma_{-t}^{\varphi_0}\circ J^{-1}$ となるので, 上の表式は不正確なのだが, Jordan $*$-同型が $*$-同型と $*$-反同型の直和であることを用いると, 接合積の段階では結果的に同じことになる.)

J の拡張であるような $\mathcal{M}_1 \rtimes_{\sigma^{\varphi_0}} \mathbb{R}$ から $\mathcal{M}_2 \rtimes_{\sigma^{\varphi_0\circ J^{-1}}} \mathbb{R}$ の上への Jordan $*$-isomorphism $\tilde{J}$ が一意に存在する. さらに, $\tilde{J}$ は 可測作用素環の間の Jordan $*$-isomorphism に拡張され, $\tilde{J}$ の $L^p(\mathcal{M}_1;\varphi_0)$ への制限は $L^p(\mathcal{M}_1;\varphi_0)$ から $L^p(\mathcal{M}_2;\varphi_0\circ J^{-1})$ の上への positive linear isometry である (cf. [W1; Section 4]).

このように, $L^p(\mathcal{M}_1;\varphi_0)$ から $L^p(\mathcal{M}_2;\psi_0)$ の上への標準的な positive linear isometry $\tilde{\kappa}^{-1}\circ\tilde{J}$ が得られる. [W2] では, von Neumann 環が σ-有限で T が positive であるとき, $T=\tilde{\kappa}^{-1}\circ\tilde{J}$ を示した;

Theorem 3.

$\mathcal{M}_i, i=1,2,$ *$\underline{\sigma\text{-finite}}$ von Neumann algebras*

$T: L^p(\mathcal{M}_1;\varphi_0) \longrightarrow L^p(\mathcal{M}_2;\psi_0)$, *onto, positive, linear isometry,*

$J: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, *Thm. 1* の *Jordan $*$-isomor.,*

$\tilde{\kappa}: L^p(\mathcal{M}_2;\psi_0) \longrightarrow L^p(\mathcal{M}_2;\varphi_0\circ J^{-1})$, *state* 取りかえの *$*$-isomor.*(の制限)

$\Longrightarrow T=\tilde{\kappa}^{-1}\circ\tilde{J}$ の $L^p(\mathcal{M}_1;\varphi_0)$ への制限.

ただし, $\tilde{J}$ は J の自然な拡張.

今回は von Neumann 環の σ-有限性を仮定すること無く, $*$-保存性を持つ T の構造定理を得た;

Theorem 3 改.

$T: L^p(\mathcal{M}_1;\varphi_0) \longrightarrow L^p(\mathcal{M}_2;\psi_0)$, *onto, $*$-preserving, linear isometry,*

$J: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, *Thm. 1* 改の *Jordan $*$-isomor.,*

$\tilde{\kappa}: L^p(\mathcal{M}_2;\psi_0) \longrightarrow L^p(\mathcal{M}_2;\varphi_0\circ J^{-1})$, *weight* 取りかえの *$*$-isomor.*

$\Longrightarrow {}^{\exists 1}z \in \mathcal{M}_2$ *: central symmetry,*

$T = z\cdot\tilde{\kappa}^{-1}\circ\tilde{J}.$

REFERENCES

[A] Araki, H., *An application of Dye's theorem on projection lattices to orthogonally decomposable isomorphisms*, Pacific J. Math. **137** (1989), 1–13.

[B] BANACH, S., *Théorie des Operations Linéaries*, Warsaw, 1932.

[BW] Bunce, L. J. and Wright, J. D. M., *On orthomorphisms between von Neumann preduals and a problem of Araki*, Pacific J. Math. **158** (1993), 265–272.

[C] Christensen, E., *Measures on projections and physical states*, Comm. Math. Phys. **86** (1982), 529–538.

[D] Dye, H. A., *On the geometry of projections in certain operator algebras*, Ann. Math. **66** (1955), 73–88.

[H] Haagerup, U., *L^p-spaces associated with an arbitrary von Neumann algebra*, Colloq. Internat. CNRS **274** (1979), 175–184.

[Ka] Kadison, R. V., *Isometries of operator algebras*, Ann. Math. **54** (1951), 325–338.

[Ko] Kosaki, H., *Applications of uniform convexity of non-commutative L^p-spaces*, Trans. Amer. Math. Soc. **283** (1984), 265–282.

[S] STRĂTILĂ, Ş, *Modular theory in Operator Algebras*, Abacus Press, Tunbridge Wells, England, 1981.

[T] Terp, M., *L^p-spaces associated with arbitrary von Neumann algebras*, Notes, Copenhagen University, 1981.

[W1] Watanabe, K., *On isometries between non-commutative L^p-spaces associated with arbitrary von Neumann algebras*, J. Operator Theory **28** (1992), 267–279.

[W2] ——, *Finite measures on preduals and non-commutative L^p-isomrtries*, J. Operator Theory **33** (1995), 371–379.

[Y1] Yeadon, F. J., *Isometries of non-commutative L^p-spaces*, Math. Proc. Camb. Phil. Soc. **90** (1981), 41–50.

[Y2] ——, *Finitely additive measures on projections in finite W^*-algebras*, Bull. London Math. Soc. **16** (1984), 145–150.

Topological property of an invariant set with respect to a family of functions

Fukiko Takeo
Ochanomizu University

Abstract

The invariant set with respect to a family $\{f_1, \ldots, f_m\}$ of functions has much relation with a quotient space of a sequence space. By considering the relation between the number of end points and the equivalence class, we shall give the characterization of the topological property of the quotient space.

§1. 序

$\mathbf{R^d}$ 上の関数族 $\{f_1, \ldots, f_m\}(m \geq 2)$ に対し、f_j が縮小写像であるとき、この関数族に対する $\mathbf{R^d}$のコンパクトな不変部分集合 K が存在し、

$$K = f_1(K) \cup \ldots \cup f_m(K)$$

が成り立つ [2]。この不変集合の位相的性質について、畑 [1] は研究し、端点の数等に関する結果を与えている。$E = \{1, 2, \ldots, m\}$ の元を成分に持つ無限列の集合 $E^{(\omega)}$を考えると、$x = x_1x_2 \ldots \in E^{(\omega)}$ に対し，$\lim_{n\to\infty} f_{x_1}f_{x_2} \cdots f_{x_n}(K)$ は Kの点となり、以下のような上への写像ψが存在する [2]。

$$\psi : E^{(\omega)} \to K$$

$E^{(\omega)}$ に $U_n(x) = \{y \in E^{(\omega)} \mid x_1x_2 \ldots x_n = y_1y_2 \ldots y_n\}$ を基本近傍系とする位相を入れると、$E^{(\omega)}$は 全不連結な完全集合となる。ψが一対一だと、Kは全不連結な完全集合となるが、一対一でないときは、いろんな位相の集合となる。ψによる像が同じものに対し、同値類$\sim$を入れることにより、$E^{(\omega)}$の商空間 $E^{(\omega)}/_\sim$が考えられる。亀山 [3] は、この商空間の位相的構造を研究し、連結性等に関する結果を得ている。本研究では、商空間の商位相に基づく基本近傍系を具体的に求め、それを用いて端点を定義し、端点の数を用いて位相的構造の特徴付けを行い、畑 [1] の結果の別証も含むいくつかの結果を得た。

§2. 定義

$E = \{1, 2, \ldots, m\}$ に対し，

- $E^{(\omega)}$: E の元を成分に持つ無限列の集合
- $E^{(n)}$: E の元を成分に持つ長さ n の列の集合
- $E^{(*)}$: E の元を成分に持つ有限列の集合, i.e. $E^{(*)} = \cup E^{(n)}$
- $j \in E$ と $x = x_1x_2 \ldots \in E^{(\omega)}$ に対し，

  $jx = jx_1x_2 \ldots$ とする，

- $n \in \mathbf{N} \cup \{0\}$ に対し，写像 $P_n : E^{(\omega)} \to E^{(*)} \cup \{\phi\}$ は最初の n 成分への射影，即ち

$$P_n(x_1 x_2 \ldots) = x_1 x_2 \ldots x_n$$

- 写像 $\sigma : E^{(\omega)} \to E^{(\omega)}$ は shift 作用素，即ち

$$\sigma(x_1 x_2 \ldots) = x_2 x_3 \ldots$$

- 同値関係 $\sim$ は次の (1),(2) を満たすとき不変であるという
  (1) $x \sim y$ ならば $jx \sim jy$　　　$(\forall j \in E)$
  (2) $jx \sim jy$ ならば $x \sim y$　　　$(\forall j \in E)$
- $x \in E^{(\omega)}$ に対して Qx を x の同値類, 即ち,
  $Qx = \{y \in E^{(\omega)} | x \sim y\}$　とする.
- $A := \{x \in E^{(\omega)} | \ \exists y \in Qx \text{ s.t. } P_1 x \neq P_1 y\}$
- $A_j := \{x \in A \mid P_1 x = j\}$, $E_j := \{x \in E^{(\omega)} \mid P_1 x = j\}$
- F_n は A_jの要素の数が n 個である j の集合，即ち

$$F_n = \{j \in E \mid \sharp(A_j) = n\}$$

- $U_n(x) = \{y \in E^{(\omega)} \mid P_n y = P_n x\}$
- $V_n(x) = \{y \in E^{(\omega)} \mid P_n Qy \subset P_n Qx\}$
- $q : E^{(\omega)} \to E^{(\omega)}/_\sim$ を natural quotient map
- $\tilde{U}_n(q(x)) = \{q(y) \in E^{(\omega)}/_\sim \mid P_n Qy \subset P_n Qx\}$

§3. 結果

以下，$\sim$ は不変な同値関係であり，A の要素の数が有限，即ち $\sharp A < \infty$ であるとする.

補題 1 $\sharp A < \infty$ のとき，$x \in A$ は非周期的である.

補題 2 次が成り立つ.

1. $\cup\{U_n(x') \mid x' \in Qx\} \backslash V_n(x) \subset \{y \mid \ l(y) \leq n-1\})$

2. $\tilde{U}_n(q(x)) = q(V_n(x))$

3. $V_n(x)$ は開集合.

命題 1 $E^{(\omega)}/_\sim$ において $\{\tilde{U}_n(q(x)) \mid n \in \mathbf{N},\ q(x) \in E^{(\omega)}/_\sim\}$ は商位相の基底をなす.

補題 3 $\tilde{U}_n(q(x))$ の境界 $\partial\tilde{U}_n(q(x))$ は集合

$$\{q(y) \mid P_n y = P_n x' \text{ なる } x' \in Qx \text{ が存在し, かつ } P_n Q y \not\subset P_n Q x\}$$

$$\subset \{q(y) \mid l(y) \leq n-1\}$$

である.

Remark. $E^{(\omega)}/_\sim$ が連結なら, $P_1 A = E$ である.

以下, $E^{(\omega)}/_\sim$ が連結の場合を考える.

定義 $q(x) \in E^{(\omega)}/_\sim$ に対し, ある $N \in \mathbf{N}$ が存在して, $n \geq N$ ならば$\partial\tilde{U}_n(q(x))$ が一点のみからなるとき, $q(x)$ は $E^{(\omega)}/_\sim$ の端点であるという.

定理 1 *1.* 次の *(a)(b)* は同値である.

(a) $q(x) \in E^{(\omega)}/_\sim$ が, $E^{(\omega)}/_\sim$ の端点である

(b) *i.* $Qx = \{x\}$ かつ

ii. ある $N \in \mathbf{N}$ が存在し, $n \geq N$ に対して

$$x_n x_{n+1} \notin P_2 A, \qquad x_n \in F_1$$

が成り立つ.

2. $q(x)$ が端点なら, $q(\sigma x)$ も端点である.

3. $q(x)$ が端点なら, 任意の $s \in E$に対し $sx \in A$ または $q(sx)$ も端点である.

定理 2 次は同値である.

1. 端点が存在する.

2. $F_1 \neq \phi$ かつ

$$i_j i_{j+1} \notin P_2 A (j = 1, 2, \ldots, n-1), \ \ i_n i_1 \notin P_2 A$$

を満たす $\{i_1, i_2, \ldots, i_n\} \subset F_1$ $(n \geq 1)$ が存在する.

命題 2 $\sharp(F_1) \geq 3$ ならば、$E^{(\omega)}/_\sim$の端点が無限に多く存在する。

命題 3 $\sharp(F_1) = 2$ ならば、$E^{(\omega)}/_\sim$の端点が数は 2 または無限個である。

補題 4 $P_1 x,\ P_1 y \in F_1$ $(P_1 x \neq P_1 y)$ なる $x, y \in A^c$ に対して

$$\sigma x,\ \sigma y,\ \sigma z \in \{x, y\} \ \ (\forall z \in A)$$

とする. このとき

1. $a, b \in \{P_1 x, P_1 y\}$ に対し, 次の *(a)* と *(b)* は同値である.

(a) $ab \in P_2 A$

  (b) $ab \notin \{P_2x, P_2y\}$

2. 次の (a) と (b) は同値である.

  (a) $z_nz_{n+1} \notin P_2A$ $(\forall n)$ かつ $z_j \in \{x_1, y_1\}(\forall j)$

  (b) $z \in \{x, y\}$

Remark. σx, $\sigma y \in \{x, y\}$ のとき, $s = P_1x$, $t = P_1y$ とすると x, y のとりうる pair は

$$\begin{cases} x = \bar{s} \\ y = \bar{t} \end{cases} \quad \begin{cases} x = \bar{s} \\ y = t\bar{s} \end{cases} \quad \begin{cases} x = s\bar{t} \\ y = \bar{t} \end{cases} \quad \begin{cases} x = \bar{s}t \\ y = \bar{t}s \end{cases}$$

の 4 通りだけである。

定理 3 次は同値である.

1. $E^{(\omega)}/_\sim$ は区間 $[0,1]$ と同相である

2. (a) $\sharp(F_1) = 2$, $\sharp(F_2) = m - 2$

  (b) 任意の $a \in A$ に対し,$\sharp(Qa) = 2$ である。

  (c) $\cup_{j=1}^{m}\{jx, jy\} = A \cup \{x, y\}$ を満たす $x, y \in A^c$ が存在する.

命題 4 $\sharp(F_1) = 1$ ならば、Eの要素の数は 3 以上である。

命題 5 次は同値である。

1. 端点の数は 1

2. (a) $\sharp(F_1) = 1$, *i.e.* $F_1 = \{s\}$, かつ $ss \notin P_2A$,

  (b) 任意の $r \in E\backslash\{s\}$ に対し $r\bar{s} \in A$ である。

命題 6 次は同値である。

1. 端点の数 n は 3 以上の有限個

2. (a) $\sharp(F_1) = 1$, *i.e.* $F_1 = \{s\}$, かつ $ss \notin P_2A$,

  (b) 次の *i. ii. iii,* を満たす $r \in E\backslash\{s\}$ と $n(\geq 3)$ がある。

    i. $r\bar{s} \notin A$ であり、 $t \in E\backslash\{r, s\}$ に対しては $t\bar{s} \in A$ である。

    ii. どんな $t \in E\backslash\{s\}$ と $l(0 \leq l \leq n-2)$ に対しても $t\underbrace{s\ldots s}_{l}r\bar{s} \in A$ である。

    iii. $\underbrace{s\ldots s}_{n-1}r\bar{s} \in A$ かつ $t \in E\backslash\{r, s\}$ に対し $t\bar{s} \in A$ である.

参考文献

[1] M.Hata. *On the structure of Self-Similar Sets.* Japan J. Appl. Math. 2(1985), 381-414.

[2] J.E.Huchinson. *Fractals and self-similarity.* Indiana Univ. Math. J. 30(1981) 713-747.

[3] A. Kameyama. *Self-Similar Sets from the Topological point of View.* Japan J. Indust. Appl. Math. 10(1993) 85-95.

Simultaneous Contractibility

T. Ando(北星学園大学　安藤　毅)

Abstract　A matrix with spectral raidus < 1 is called S-stable. S-stability of a matrix A is characterized by existence of positive definite H such that $H > A^*HA$; H is called a contractizer of A. The problem is when two S-stable matrices A, B admit a common contractizer. We treat also characaterization of the set of matices that have one and the same contractizer.

1. Norm

$\mathbb{M}_n$ で $n \times n$ の行列のなす線形空間とする。$\mathbb{M}_n^+$ で positive definite 行列の作る cone とする。各 $A \in \mathbb{M}_n$ を $\mathbb{C}^n \longmapsto \mathbb{C}^n$ の線形写像と考える。

$\|\cdot\|$ を $\mathbb{C}^n$ の通常の内積による norm とし，$\|A\|$ を行列 A の operator norm とする。$\mathbb{C}^n$ の一般の norm $|||\cdot|||$ での operator norm を $|||\cdot|||$ による **matrix norm** という。

[1]　$\mathbb{C}^n$ の上の norm $|||\cdot|||$ に対して，
内積から作られる norm である $\Longleftrightarrow$ Jordan-Neumann の平行四辺の恒等式が成り立つ $\Longleftrightarrow$ ある正則行列 S で

$$|||x||| = \|Sx\| \; (x \in \mathbb{C}^n).$$

内積による norm に関する matrix norm を **Hilbertian** matrix norm という。

[2]　$|||\cdot|||$ が Hilbertian $\Longleftrightarrow$ ある正則行列　S で

$$|||A||| = \|SAS^{-1}\| \quad (A \in \mathbb{M}_n).$$

複素平面上の単位円板上で analytic で値がまた単位円板に入る複素関数 $f(z)$ の全体を **Schur class** といい $\mathcal{S}$ で表す。

Theorem (von Neumann). *Hilbertian norm* $|||\cdot|||$ に関して

$$|||A||| < 1, f \in \mathcal{S} \Longrightarrow |||f(A)||| < 1.$$

Theorem (Foias). $|||\cdot|||$ が *matrix norm* で，

$$|||A||| < 1, f \in \mathcal{S} \Longrightarrow |||f(A)||| < 1$$

の性質を持てば，$|||\cdot|||$ は *Hilbertian* である。

行列 A が $\|\cdot\|$ の closed unit ball の端点であることと unitary 行列であることは同じであるから次が云える。

[3] Hilbertian matrix norm の closed unit ball の端点の集合は乗法群をなす。

2. S-stable matrix

$$r(A) \equiv \textbf{ spectral radius } = \max\{|\lambda_i(A)|\ ;\ i = 1, 2, \dots, n\}.$$

[4]　どの matrix norm $|||\cdot|||$ に関しても次が成り立つ。

(1) $r(A) \le |||A|||$,

(2) **(Gelfand formula)**　$r(A) = \lim_{k\to\infty} |||A^k|||^{1/k}$.

$r(A) < 1$ のとき，A を **S-stable** (= Schur stable) という。これは（離散型）発展方程式

$$x_{k+1} = Ax_k \quad (k = 0, 1, \dots)$$

の解 x_k $(k = 1, 2, \dots)$ がどんな初期値 x_0 に関しても安定すなわち $x_k \to 0 (k \to \infty)$ となることと同値である。実際，解は

$$x_k = A^k x_0 \quad (k = 0, 1, \dots).$$

3. Contractible matrix

S-stable 行列 A がある matrix norm $|||\cdot|||$ に関して (strict) contraction となるとき A は **w-contractible** であるという。この matrix norm として Hilbertian norm がとれるとき **s-contractible** という。

[5]　A が s-contractible $\iff$ ある $H \in \mathbb{M}_n^+$ があり $A^*HA < H$.

このような H を A の **contractizer** という。

A に関する **Stein map** $\Phi_A : \mathbb{M}_n \longmapsto \mathbb{M}_n$ を

$$\Phi_A(X) \equiv X - A^*XA \quad (X \in \mathbb{M}_n)$$

で定義する。

Theorem (Rota construction).　*A が S-stable $\iff$ Φ_A が invertible.*

このとき逆写像 Φ_A^{-1} は

$$\Phi_A^{-1}(X) = \sum_{k=0}^{\infty} A^{*k} X A^k$$

で与えられ *(completely) positive linear map* となる。

Corollary. *H が A の contractizer $\iff$ $H \in \Phi_A^{-1}(\mathbb{M}_+^n)$.*

Corollary. *A S-stable $\iff$ A w-contractible $\iff$ A s-contractible.*

4. Simultaneous contractibility

S-stable 行列の集合 $\mathcal{M}$ が **simultaneously w-contactible** とはある matrix norm $|||\cdot|||$ で
$$|||A||| < 1 \quad (A \in \mathcal{M}).$$
この norm として Hilbertian norm がとれるとき **simultaneously s-contractible** という。

[6] $\mathcal{M}$ が simultaneously w-contractible $\Longrightarrow$ $\mathcal{M}$ は有界。

[7] $\mathcal{M}$ が simultaneously s-contractible $\Longleftrightarrow$ ある $H > 0$ があり $A^*HA < H \quad (A \in \mathcal{M})$.

[8] $\mathcal{M}$ が simultaneously s-contractible $\Longleftrightarrow$ $\bigcap_{A\in\mathcal{M}} \Phi_A^{-1}(\mathrm{M}_n^+) \neq \emptyset$.

[9] $\mathcal{M}$ が simultaneously w-contractible $\Longrightarrow$ $\mathcal{M}$ を含む最小の convex, multiplicative semi-group は有界で，その元はすべて S-stable である。

[10] $\mathcal{M}$ が simultaneously s-contractible $\Longrightarrow$ $\mathcal{M}$ を含む最小の convex, Schur-class functional calculus で閉じた multiplicative semi-group の有界で，その元はすべて S-stable である。

Lemma. A, B が可換 $\Longrightarrow$ Φ_A, Φ_B は可換。

Corollary. 可換な *S-stable* 行列の有限集合 $\mathcal{M} = \{A_1, \dots, A_m\}$ は *simultaneously s-contractible.* 実際
$$(\Phi_{A_1}^{-1} \circ \Phi_{A_2}^{-1} \circ \dots \Phi_{A_m}^{-1})(\mathrm{M}_n^+) \subset \bigcap_{A\in\mathcal{M}} \Phi_A^{-1}(\mathrm{M}_n^+).$$

Theorem. $\mathcal{M}$ が有界な可換族で
$$\sup\{r(A); A \in \mathcal{M}\} < 1$$
ならば *simultaneously s-contractible* である。

$\mathcal{M}$ に対して
$$\mathcal{M}^k \equiv \{A_1A_2\dots A_k \ ; \ A_i \in \mathcal{M}\} \quad (k = 1, 2, \dots)$$
の表示を使おう。

Lemma (Daubechies - Lagarias). $\mathcal{M}$ を有界な乗法半群とする。
(generalized Gelfand formula)
$$\lim_{k\to\infty} \sup_{Z\in\mathcal{M}^k} |||Z|||^{1/k} = \lim_{k\to\infty} \sup_{Z\in\mathcal{M}^k} r(Z)^{1/k}.$$

Theorem (Brayton-Tong). 有界な乗法半群 $\mathcal{M}$ が
$$\lim_{k\to\infty} \sup_{Z\in\mathcal{M}^k} \|Z\|^{1/k} < 1$$
ならば $\mathcal{M}$ は *simultaneously w-contractible* である。

5. 2個の場合

A, B を S-stable 行列とする。

Proposition. 次は互いに同値である：
(1) A, B は *simultaneously s-contractible,*
(2) $\exists X > 0 \quad \Phi_B \circ \Phi_A^{-1}(X) > 0,$ (3) $\exists Y > 0 \quad \Phi_A \circ \Phi_B^{-1}(Y) > 0.$
次は互いに同値である：
(i) A, B は *not simultaneously s-contractible* ,
(ii) $\exists S > 0 \quad \Phi_A^{-1} \circ \Phi_B(S) < 0,$ (iii) $\exists T > 0 \quad \Phi_B^{-1} \circ \Phi_A(T) < 0.$

Proposition.

$$\lim_{k\to\infty} \sup_{Z\in\{A,B\}^k} \|Z\|^{1/k} < 1/\sqrt{2} \implies A, B \text{ は } \textit{simultaneously s-contractible.}$$

Proposition. A, B が共に *nilpotent* ならば *simultaneously s-contractible.*

Lemma. A, B が *rank one* で $A = a \otimes b^*, B = c \otimes d^*$ のとき

$$\lim_{k\to\infty} \sup_{Z\in\{A,B\}^k} \|Z\|^{1/k} = \max\{r(A), r(B), r(AB)\}$$

Counter example.

$$\lim_{k\to\infty} \sup_{Z\in\{A,B\}^k} \|Z\|^{1/k} < 1$$

であるが *simultaneously s-contractible* ではない *rank one* A, B がある。

6. Contraction domain

$H > 0$ に対して，これを contractizer とする S-stable 行列の全体を H の **contraction domain** と呼び $\mathcal{C}_H$ で表す：

$$\mathcal{C}_H \equiv \{A \,;\, H > A^*HA\} = \{A \,;\, H \in \Phi_A^{-1}(\mathbb{M}_n^+)\}.$$

$\mathcal{C}_H$ は有界 (absolutely) convex, multiplicative 開集合である。

Lemma (Lyubic). $\mathbb{M}$ が有界 *(absolutely) convex, multiplicative* 開集合が *matrix norm* になる必要十分条件はこの性質をもったものの中で *maximal* なことである。

Theorem. 有界 *(absolutely) convex, multiplicative* 開集合 $\mathcal{M}$ について次は同値である。
(1) $\mathcal{M} = \mathcal{C}_H$ *for some* $H > 0,$
(2) $\mathcal{M}$ は *maximal* でかつ *Schur-class* の *functional calculus* で閉じている,
(3) $\mathcal{M}$ の *closure* の端点の集合は乗法群をなす。

7. Reference

1. R.K. Brayton and C.H. Tong,
   Constructive stability and asymptotic stability of dynamical systems,
   IEEE Trans. Circuits and Systems, C-27(1980), 1121-1130

2. I. Daubechies and J.C. Lagarias,
   Set of matrices all infinite products of which converge,
   Linear Alg. Appl. 161(1992), 227-263.

3. C. Foias,
   Sur certains theoremes de J. von Neumann concernant les ensembles spectraux,
   Acta Sci. Math. (Szeged), 18(1957), 15-20.

4. J. v. Neumann,
   Eine Spektraltheorie fur allgemeine Operatoren eines unitaren Raumes,
   Math. Nachrichten 4(1951), 258-281.

5. G.-C. Rota,
   On models for linear operators,
   Comm. Pure Appl. Math. 8(1960), 469-472.

CHAOS IN ITERATED CUBIC MAPS

Kiyoko NISHIZAWA
DEP. MATH. FAC.SCI. JOSAI UNIVERSITY
1-1, Keyakidai, Sakado, Saitama 350-02, JAPAN
e-mail: kiyoko@euclides.josai.ac.jp

1995.12.26

Keywords: chaotic dynamical systems - topological entropy - bifurcation.
AMS(MOS)subject classification: 14Q05, 58F03, 58F20

1 Introduction.

System of iterated maps of the interval, viewed as dynamical systems, is considered as an important model for the chaotic behavior in certain physical, chemical and biological systems. Since there are many notions of chaos, in this paper we consider **topological chaos**, meaning positivity of the **topological entropy**.

For a parametrized family of functions, we have a vague general question; "how does the **complexity** of a dynamical system vary with parameters ?" One measure of complexity would be the numbers of periodic point of various periods. And the topological entropy of a map is also one of the particular useful indicator of the complexity of the system. In [9], Milnor and Thurston considered the topological entropy $h(f)$ and growth number $s(f) = \exp h(f)$ of continuous maps f.

Definition. For a piecewise monotone map f considered as a map from the compact interval $[-\infty, \infty]$ to itself, the topological entropy $h(f)$ of f is defined as follows:

$$h(f) = \lim_{k \to \infty} \frac{1}{k} \log(l(f^{\circ k}))$$

where $l(f^{\circ k})$ is the number of **laps** of k-th iterate of f.

Except in very special cases, the entropy cannot be computed using these definitions. Thus the problem of finding an **algorithm** to compute topological entropy to any accuracy is discussed in many papers ([1],[2]).

The quadratic(logistic) family of maps is important as a population growth model in theoretical population dynamics and an example of a family of simple maps with extremely complicated dynamics. Milnor and Thurston([9]), based on the Douady-Hubbard-Sullivan argument(unpublished), proved that this family has only orbit-creation values and no orbit-annihilation values and that the topological entropy $h(f_\lambda)$ is monotone increasing as a function of λ.

The bulk of the present paper is devoted to the study of the transition to chaos for the cubic polynomials. Some conjectures concerning the entropy in case of cubic maps were enunciated by Milnor in [7]. In section 3 ,we shall discuss these problems. Our main result is a classification of the routes to topological chaos along an algebraic curve defined in the moduli space of the real cubic polynomials.

2 Algebraic curves defined in Moduli space of the cubic polynomials.

We consider the family of cubic maps $x \mapsto g(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ $(c_3 \neq 0, c_i \in \mathbf{C})$. For such a cubic map g , we have two normal forms ; $x^3 - 3Ax \pm \sqrt{B}$, $A, B \in \mathbf{C}$. Therefore, the complex affine conjugacy class of g can be represented by (A, B). The **moduli space**, denoted by $\mathcal{M}$, consisting of all affine conjugacy classes of cubic maps, can be identified with the coordinate space $\mathbf{C}^2 = \{(A, B)\}$ ([7]).

Moduli space of the real cubic polynomials. By a suitable real affine transformation, any **real** cubic map $g(x)$ is transformed to a unique map $f(x) = \sigma x^3 - 3Ax + \sqrt{|B|}$, where $\sigma := sgn(g''')$, $A, B \in \mathbf{R}$. The real affine conjugacy class of g or f can be represented by $(A, B) \in \mathbf{R}^2$ if $B \neq 0$. But if $B = 0$, σ should be added as an essential class invariant, as $x \mapsto x^3 - 3Ax$ and $x \mapsto -x^3 - 3Ax$ belong to different classes. Thus ,due to J.Milnor([7]), the real Moduli space, or real cut of the moduli space $\mathcal{M}$, of real affine conjugacy classes of real cubic maps can be described as the disjoint union of the upper half-plane $\mathbf{H}^+ = \{(A, B)|B \geq 0\}$ and the lower half-plane $\mathbf{H}^- = \{(A, B)|B \leq 0\}$ We denote this space by $\mathcal{M}_R$.

A complex cubic map f, or the corresponding point $(A, B) \in \mathcal{M}$, belongs to the **connectedness locus** if the orbits of both critical points p_i such that $f'(p_i) = 0$, $i = 1, 2$, are bounded. And f is **hyperbolic** if both of these critical orbits converge towards attracting periodic orbits. The set of all hyperbolic points in the moduli space $\mathcal{M}$ forms an open set. Each connected component of this open set is called a **hyperbolic component**. By M.Rees([15]), each hyperbolic component contains a unique post-critically finite complex cubic map. This map is called a **center map** or **Thurston map** and the coordinates (A, B) of f called a **center** in the moduli space. The centers are roughly classified into four diffeent types, as follows. In the following t, p, q denote integers. A center is of the type $\mathcal{A}_p$ if two critical points p_1, p_2. of the center map coinside and has the period p : $f^p(p_1) = p_1$. In fact, only possible values for p in this case are $1, 2$. A center is of the type $\mathcal{B}_{p+q}$ if $f^p(p_1) = p_2$ and $f^q(p_2) = p_1$; of the type $\mathcal{C}_{(t)q}$ if $f^t(p_1) = p_2$ and $f^q(p_2) = p_2$; of the type $\mathcal{D}_{p,q}$ if $f^p(p_1) = p_1$ and $f^q(p_2) = p_2$. These exhaust all types of centers.

Center curves in the moduli space.
The **center curves** CDp, BCp, which are algebraic curves, can be defined according to the above four renormalization-type. We show how the equations of these curves are obtained by induction on p ([10], [13] and [14]).

Theorem 2.1 : Defining equation of a center curve *For a given p, there exist an algebraic curve CDp containing all centers of the type $\mathcal{C}_{(k)p}$ and $\mathcal{D}_{k,p}$, and another algebraic curve BCp containing all centers of the type $\mathcal{B}_{p+k}$ and $\mathcal{C}_{(p)k}$.*

For example, we obtain precisely the following curves;

$$\begin{aligned}
\mathit{CD1} &: B = 4A(A+\tfrac{1}{2})^2, \\
\mathit{BC1} &: B = 4A(A-\tfrac{1}{2})^2, \\
\mathit{CD2} &: B^2 - 8A^3B + 4A^2B - 5AB + 2B + 16A^6 - 16A^5 \\
&\qquad\qquad -12A^4 + 16A^3 - 4A + 1 = 0, \\
\mathit{BC2} &: B^3 - 12A^3B^2 - 6AB^2 + 2B^2 + 48A^6B + 24A^3B + 21A^2B \\
&\qquad -6AB + B - 64A^9 + 96A^7 - 20A^5 - 12A^3 - A = 0,
\end{aligned}$$

We can embed $\mathbf{C}^2$ canonically in $\mathbf{P}^2(\mathbf{C}) : (A, B) \to (1 : A : B)$. Then an affine algebraic curve $V_0 = \{(A, B) \in \mathbf{C}^2 : h(A, B) = 0\}$ uniquely determines a projective algebraic curve $V = \{(C : A : B) \in \mathbf{P}^2(\mathbf{C}) : H(C : A : B) = 0\}$ in $\mathbf{P}^2(\mathbf{C})$ such that $h(A, B) = H(1 : A : B)$ and $V \cap \mathbf{C}^2 = V_0$.

Definition. For a center curve V_0, the corresponding projective algebraic curve V is called the **projective center curve**. We denote by PBCp and PCDp, these curves corresponding to BCp and CDp respectively.

We give some algebraic-geometric properties of these curves.

Theorem 2.2 :([10], [14])
- The interseciton with the line at infinity: *Each projective center curve and the line at infinity, $L_\infty : C = 0$, intersect at the point $P_\infty = (0 : 0 : 1)$ only. P_∞ is singular and its multiplicity can be calculated explicitly.*
- Irreducibility and Singurarity: *For projective center curves PCDi and PBCi for only $i = 1$ and 2,*

PCD1 and PBC1 are irreducible curves of degree 3. PCD2 is an irreducible curve of degree 6. P_∞ is 4-fold and $(0.25, -0.4375)$ is an ordinary double point.
PBC2 is an irreducible curve of degree 9. P_∞ is 6-fold and four ordinary double points are as follows :

$$\begin{aligned}
&(-0.1341351918179714, -1.37344484910264), \\
&(-0.5531033117555605, -0.6288238268413773), \\
&(0.3041906503790061 * i + 0.3436192517867655, \\
&\qquad\qquad 0.6886343379400248 - 0.04267412324347224 * i), \\
&(0.3436192517867655 - 0.3041906503790061 * i, \\
&\qquad\qquad 0.04267412329900053 * i + 0.6886343379735695),
\end{aligned}$$

- Principal part of the center curves and Genus : *The principal part at P_∞ of PCD1 and of PBC1 is $(C^2 - 4A^3)^1$, of PCD2 is $(C^2 - 4A^3)^2$, and of PBC2 is $(C^2 - 4A^3)^3$. The curves PCD1 and PBC1 are rational. Hence the genus is 0. The genus of PCD2 is* 1. *The genus of PBC2 is* 3.

The irreducibility of each projective center curve is determined based on Kaltofen's algorithms on *risa-asir* (computer algebra system by FUJITSU CO.LTD.) ([16] , [6]). To calculate genus g of each projective center curve Γ, we determine the principal part at P_∞ of the curves by using Newton Polygons and apply the Plücker's formula. I am grateful to Y. Komori([4]) for helpful suggestions on the genus.

We would like to state the following conjectures for the projective center curves:

Conjectures
- All projective center curves are irreducible.
- All singular points except P_∞ are ordinary double points.
- Especially, for real graph of center curves, the sigular point exists only in $\mathcal{R}_1$.
- The pricipale part at P_∞ of every projective center curve has a form $(C^2 - 4A^3)^k$.

3 Monotonicity of topological entropy along center curve.

The conjecture that the toplogical entropy $h(f)$ of a real cubic map f depends **monotonely** on its parameters was enunciated by Milnor in [7] and [8]. Namely, each locus of constant entropy in parameter space is connected. Another conjecture due to Milnor([7], [8]) is a **maximum and minimum principle** for entropy: the maximum and minimum values for the entropy function on any closed region in the moduli space must occur the boundary. Real graphs of BC1 and CD2-2 are shown in **Figure 1** and of CD1, BC1, and CD2-2 in **Figure 2** together with the equi-growth number lines. A glance at these figures suggests that the topological entropy vary monotonously along a part of real graph of center curve. We proposed in [10],[14] this entropy-monotonicity conjecture alog a center curve. Recentely we can prove that our monotonicity conjecture is true on CD1, and on any center curve([11],[12]). I am grateful to Y. Komori([5]) for helpful suggestions on this conjecture. He can prove that our monotonicity conjecture is true for any center curves and his idea can be applied to suitable family of polynomials with higher degree.

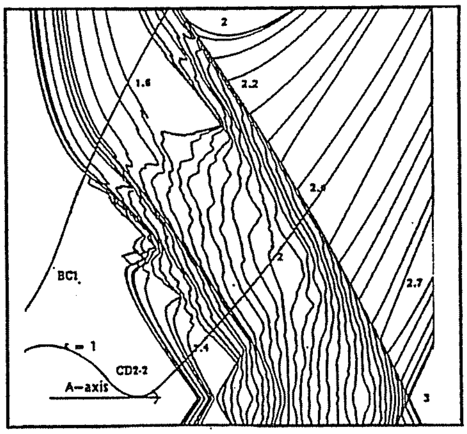


Figure 1: Real graphs of BC1 , CD2-2 with the equi-growth number lines. The region is [.57, 1.03] × [0, .43] in $(A, b)-$ plane

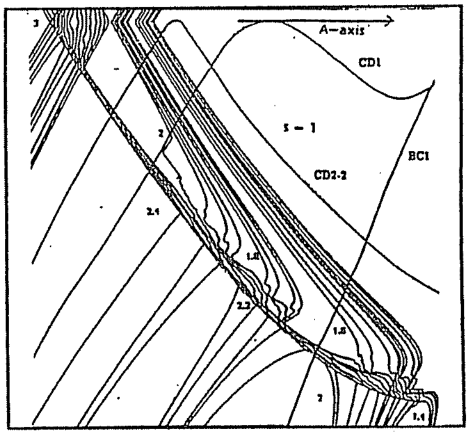


Figure 2: Real graphs of CD1, BC1, CD2-2 with the equi-growth number lines. The region is $[-1.05, -.09] \times [0, -1.35]$ in $(A, b')-$)plane

Theorem 3.1 : entropy-monotonicity *The topological entropy of a cubic mapf is monotonely continuous if f varies along a part(called a bone)of center curve CD1.*

References

[1] L. Block and J. Keesling, Computing the topological entropy of maps of the interval with three monotone pieces. J. Statist. Phys.,**66**(1992)755-774.

[2] L. Block, J. Keesling, Shihai Li, K. Peterson, An Improved Algorithm for Computing Topological Entropy. J. Statist. Phys.,**55**(1989) 929-939.

[3] Ittai Kan, H. Koçak, J. Yorke, Antimonotonicity:Concurrent creation and annihilation of periodic orbits, Ann. Math., **130**(1978) 219-252.

[4] Y. Komori, Principal part and blowing-up, private communication , 1994.3.22.

[5] Y. Komori, Monotonicity of topological entropy in the cubic maps, private communication, 1994.10.

[6] S. Landau, Factoring polynomials over algebraic number fields, SIAM J. Comput., **14**(1985) 184–195.

[7] J. Milnor, Remarks on iterated cubic maps. Preprint # 1990/6, SUNY StonyBrook.

[8] J. Milnor, Entropy for Bimodal map, resume of international conference of dynamical systems, Denmark, June, 1993.

[9] J. Milnor & W. Thurston, On iterated maps of the interval. L.N.Math. 1342, *Springer-Verlag*, (1988) 465–563 .

[10] K. Nishizawa, COMPLEX DYNAMICAL SYSTEMS:ALGEBRAIC CURVES IN THE MODULI SPACE OF THE CUBIC MAPS, Proceedings of 5-th International Colloquium on Differential Equations, SCT Pub.(1995) 130–139.

[11] K. Nishizawa, TOPLOGICAL CHAOS IN ITERATED CUBIC MAPS, To appear in Proceedings of 6-th International Colloquium on Differential Equations,

[12] K. Nishizawa, CHAOS IN ITERATED CUBIC MAPS: Topological Entropy and Bifurcation. Proceedings of ATCM 1: Innovatiev Use of Technology for Teaching and Research in mathematics, Singapore 1995 697–706.

[13] K. Nishizawa & A. Nojiri, Center curves in the moduli space of the real cubic maps, Proc. Japan Acad. Ser.A, **69**(1993) 179–184.

[14] K. Nishizawa & A. Nojiri, Algebraic geomerty of center curves in the moduli space of the cubic maps, Proc. Japan Acad. Ser.A, **70**(1994) 99–103.

[15] M. Rees, Components of degree two hyperbolic rational maps, Invent. Math., **100**(1990) 357–382.

[16] K. Yokoyama, M. Noro, & T. Takeshima, On factoring multi-variate polynomials over algebraically closed fields, RISC-Linz Series, no.90-26.0 (1990) 1-8.

EQUIVALENCE OF THE MCSHANE AND BOCHNER INTEGRALS FOR FUNCTIONS WITH VALUES IN HILBERTIAN (UCs-N) SPACES WHICH ARE NUCLEAR

北海道教育大学 札幌校　櫻田 邦範

1．はじめに

the generalized Riemann integral は，その名が示すように 通常の Riemann 積分の定義と類似して定義されるが，Riemann 和 に用いられる partitions の族の違いにより，実数値関数の場合には２つの一般化が研究されている。一方は Henstock 積分であり，他方は McShane 積分である。実数値関数の場合には，Henstock 積分は 狭義 Denjoy 積分と同値であり，McShane 積分 は Lebesgue 積分と同値であることが知られている。

1990 年，R.A. Gordon は，実数値関数に対する McShane 積分の定義を Banach 空間値関数の場合に拡張し，その積分の性質を検討するとともに，可測な Pettis 積分可能な関数は，すべて McShane 積分可能であることを示した [2]。

1992 年，S.S. Cao は，実数値関数に対する Henstock 積分の定義を Banach 空間値関数の場合に拡張し，実数値の場合に多くの結果の証明に重要な役割を演ずる Henstock's Lemma について検討し，それが有限次元の Banach 空間の場合には成立するが，無限次元の場合には，必ずしも成立しないことを示した [1]。

1994 年，S. Nakanishi は，Henstock 積分の定義を (UCs-N) 空間値関数の場合に拡張し，その性質を検討すると共に，この積分が，1984 年に同氏によって定義・研究された (UCs-N) 空間値 Bochner 積分を含むことを示した。しかも特に興味深いことには，L. Schwartz の超関数の理論に現れる S, S', D, D' などの空間をその代表的な例として含む，核型 (UCs-N) 空間に値を取る関数の Henstock 積分に対しては，Henstock's Lemma が成立することを示している [11]。

ここでは，核型 (UCs-N) 空間に値を取る関数に対する McShane 積分と Bochner 積分の同値性に関し，次節の結果をご報告したい。なお，これらの結果は文献 [14] をまとめたものである。この報告で用いられる the fundamental terminology and notations は，[5]，[11] および [14] に従う。(UCs-N) spaces および (UCs-N) space valued Bochner integrals の定義については，[7]，[8] および [11] を；Hilbertian (UCs-N) spaces which are nuclear 等の定義については [11] を参照されたい。

Definition 1. Let X be a (Cs-N) space with a sequence of semi-norms $\{p_n\}$: $(X, \{p_n\})$, which is r-separated and complete. An X-valued function f defined on $[a, b]$ is said to be *McShane integrable to a vector* $z \in X$ on $[a, b]$ if for every $n \in N$ there is a positive function $\delta_n(\xi)$ on $[a, b]$ such that for any division D of $[a, b]$ given by

$$a = t_0 < t_1 < \cdots < t_h = b \quad \text{and} \quad \{\xi_1, \xi_2, \cdots, \xi_h\} \subset [a, b]$$

satisfying $[t_{i-1}, t_i] \subset (\xi_i - \delta_n(\xi_i), \xi_i + \delta_n(\xi_i))$ for $i = 1, 2, \cdots, h$, we have

$$p_n\left(\sum_{i=1}^{h} f(\xi_i)(t_i - t_{i-1}) - z\right) < 1/2^n,$$

or alternatively,

$$p_n\left(\sum f(\xi)(v-u)-z\right)<1/2^n$$

where $[u,v]$ denotes a typical interval in D with $[u,v]\subset(\xi-\delta_n(\xi),\xi+\delta_n(\xi))$. Such a division D is called a δ_n-fine Lebesgue division, or simply a δ_n-fine L division, and sometimes it is denoted by $D=\{([t_{i-1},t_i],\xi_i):i=1,2,\cdots,h\}$ or $D=\{[u,v],\xi\}$. The vector z is uniquely determined. The *integral* of f on $[a,b]$ is given by the vector z, and it is written $\int_a^b f(t)dt$. The function f is McShane integrable on a measurable set $A\subset[a,b]$ if the function $f\chi_A$ is McShane integrable on $[a,b]$.

Definition 2. Let X be a (UCs-N) space with component spaces $(X_\alpha,\{p_n^\alpha\})$ $(\alpha\in\Sigma)$, which is r-separated and complete. An X-valued function f defined on $[a,b]$ is said to be *McShane integrable to a vector* $z\in X$ on $[a,b]$ if there is a component space X_α such that:
(1) The image of $[a,b]$ by f is contained in X_α;
(2) f is McShane integrable to z on $[a,b]$ as a $(X_\alpha,\{p_n^\alpha\})$-valued function.
If it is necessary to indicate such an X_α explicitly, for convenience we will say that f is McShane integrable (X_α) to z on $[a,b]$. The vector z is uniquely determined. The *integral* of f on $[a,b]$ is given by the vector z, and it is written $\int_a^b f(t)dt$. The function f is McShane integrable on a measurable set $A\subset[a,b]$ if the function $f\chi_A$ is McShane integrable on $[a,b]$.

2．主結果

Theorem 1. *Let* $(X,\{p_n\})$ *be a Hilbertian* (CN) *space which is nuclear, and let* f *be an* X*-valued measurable function on* $[a,b]$*. The function* f *is Bochner integrable on* $[a,b]$ *if and only if* f *is McShane integrable on* $[a,b]$*, and both integrals coincide.*

Theorem 2. *Let* X *be a* (UCN) *space such that each component space* $(X_\alpha,\{p_n^\alpha\})$ $(\alpha\in\Sigma)$ *is a Hilbertian* (CN) *space which is nuclear, and let* f *be an* X*-valued function defined on* $[a,b]$*. Suppose that* f *is* μ*-measurable* (X_α) *for some* $\alpha\in\Sigma$*. The function* f *is Bochner integrable on* $[a,b]$ *if and only if* f *is McShane integrable on* $[a,b]$*, and both integrals coincide.*

Theorem 3. *Let* X *be a Hilbertian* (UN) *space which is nuclear, with component spaces* (X_α,p_α) *endowed with* $\langle\ ,\ \rangle_\alpha$ $(\alpha\in\Sigma)$*, and let* f *be an* X*-valued function defined on* $[a,b]$*. Suppose that* f *is* μ*-measurable* (X_α) *for some* $\alpha\in\Sigma$*. The function* f *is Bochner integrable on* $[a,b]$ *if and only if* f *is McShane integrable on* $[a,b]$*, and both integrals coincide.*

Theorem 4. *Let* X *be a Hilbertian* (UCs-N) *space which is nuclear, with component spaces* $(X_\alpha,\{p_n^\alpha\})$ $(\alpha\in\Sigma)$*, and let* f *be an* X*-valued function defined on* $[a,b]$*. Suppose that* f *is* μ*-measurable* (X_α) *for some* $\alpha\in\Sigma$*. The function* f *is Bochner integrable on* $[a,b]$ *if and only if* f *is McShane integrable on* $[a,b]$*, and both integrals coincide.*

3．定理の証明の方針

定理１の証明は，次の様に４つの補題を準備する方法によってなされる。

Lemma 1. *Let X be a* (Cs-N) *space with a sequence of semi-norms $\{p_n\}$, which is r-separated and complete. If f is Bochner integrable on $[a,b]$, then it is McShane integrable on $[a,b]$, and both integrals coincide.*

Proof. (cf, [11, Proposition 15]) Since f is Bochner integrable on $[a,b]$, f is μ-measurable on $[a,b]$ and there is a sequence $\{f_i\}$ of X-valued simple functions on $[a,b]$ such that $\lim_{i\to\infty} \int_a^b p_n(f_i(t) - f(t))dt = 0$ for every $n \in N$. Moreover, since the Bochner integral of f on $[a,b]$: $\int_a^b f(t)dt$ is the vector r-$\lim \int_a^b f_i(t)dt$ in $(X, \{p_n\})$, we have $\lim_{i\to\infty} p_n(\int_a^b f_i(t)dt - \int_a^b f(t)dt) = 0$ for every $n \in N$. Let $n \in N$. Then, there is an i_0 such that for every $i \geqq i_0$ we have

$$\int_a^b p_n(f_i(t) - f(t))dt < 1/2^{n+2} \text{ and } p_n\left(\int_a^b f_i(t)dt - \int_a^b f(t)dt\right) < 1/2^{n+2}.$$

Fix an i with $i \geqq i_0$. By [14, Proposition 4 and 2.(3)], f_i is McShane integrable and Bochner integrable on $[a,b]$, and both integrals coincide. Hence, there is a positive function $\delta'_n(\xi)$ on $[a,b]$ such that for any δ'_n-fine L division $D = \{[u,v], \xi\}$ of $[a,b]$ we have

$$p_n\left(\sum f_i(\xi)(v-u) - \int_a^b f_i(t)dt\right) < 1/2^{n+2}.$$

Note that, for real valued functions, the McShane integral and the Lebesgue integral are equivalent, and both integrals coincide (see [4]). Since the real valued function $p_n(f_i(t) - f(t))$ is Lebesgue integrable, it is McShane integrable and both integrals coincide. Therefore, there is a positive function $\delta''_n(\xi)$ on $[a,b]$ such that for any δ''-fine L division $D = \{[u,v], \xi\}$ of $[a,b]$ we have

$$\left|\sum p_n(f_i(\xi) - f(\xi))(v-u) - \int_a^b p_n(f_i(t) - f(t))dt\right| < 1/2^{n+2}.$$

Define a positive function δ_n on $[a,b]$ by $\delta_n(\xi) = \min\{\delta'_n(\xi), \delta''_n(\xi)\}$. Then, for any δ_n-fine L division $D = \{[u,v], \xi\}$ of $[a,b]$ we have

$$\begin{aligned}
&p_n\left(\sum f(\xi)(v-u) - \int_a^b f(t)dt\right) \\
&\leqq p_n\left(\sum f(\xi)(v-u) - \int_a^b f_i(t)dt\right) + p_n\left(\int_a^b f_i(t)dt - \int_a^b f(t)dt\right) \\
&< p_n\left(\sum f(\xi)(v-u) - \sum f_i(\xi)(v-u)\right) + p_n\left(\sum f_i(\xi)(v-u) - \int_a^b f_i(t)dt\right) + 1/2^{n+2} \\
&< \sum p_n(f_i(\xi) - f(\xi))(v-u) + 1/2^{n+2} + 1/2^{n+2} \\
&\leqq \left|\sum p_n(f_i(\xi) - f(\xi))(v-u) - \int_a^b p_n(f_i(t) - f(t))dt\right| + \int_a^b p_n(f_i(t) - f(t))dt + 1/2^{n+1} \\
&< 1/2^{n+2} + 1/2^{n+2} + 1/2^{n+1} = 1/2^n.
\end{aligned}$$

Thus, f is McShane integrable to the Bochner integral of f on $[a,b]$.

Lemma 2. *Let $(X, \{p_n\})$ be a Hilbertian* (CN) *space which is nuclear, and let f be an X-valued McShane integrable function on $[a,b]$ with the primitive F. Then, for every $n \in N$, there exists a positive function $\delta_n(\xi)$ on $[a,b]$ such that for any δ_n-fine L division $D = \{[u,v], \xi\}$ of $[a,b]$ we have*

$$\sum p_n(f(\xi)(v-u) - F([u,v])) < 1/2^n.$$

Lemma 3. *Let* $(X, \{p_n\})$ *be a Hilbertian* (CN) *space which is nuclear, and let* f *be an* X*-valued function on* $[a, b]$. *The function* f *is McShane integrable on* $[a, b]$ *if and only if for every* $n \in N$ *there is a positive function* $\delta_n(\xi)$ *on* $[a, b]$ *such that for any two* δ_n*-fine L divisions* $D_1 = \{([t_{i-1}, t_i], \xi_i) : i = 1, 2, \cdots, h\}$ *and* $D_2 = \{([t_{i-1}, t_i], \eta_i) : i = 1, 2, \cdots, h\}$ *of* $[a, b]$ *we have*

$$\sum_{i=1}^{h} p_n(f(\xi_i) - f(\eta_i))(t_i - t_{i-1}) < 1/2^n.$$

Lemma 4. *Let* $(X, \{p_n\})$ *be a Hilbertian* (CN) *space which is nuclear, and let* f *be an* X*-valued function on* $[a, b]$. *If* f *is McShane integrable on* $[a, b]$, *then* $p_m(f(t))$ *is McShane integrable on* $[a, b]$ *for every* $m \in N$.

Proof of Theorem 1 By Lemma 1 the " only if " part holds. The " if " part is proved as follows (cf. [15, Theorem 1, p133]). Since f is an X-valued measurable function, there is a sequence $\{f_i\}$ of simple functions such that $\lim_{i \to \infty} p_m(f(t) - f_i(t)) = 0$ μ-a.e. on $[a, b]$ for every $m \in N$. Fix $n \in N$. For every $i \in N$, let

$$g_i^n(t) = \begin{cases} f_i(t), & \text{if} \quad p_n(f_i(t)) \leqq 2p_n(f(t)), \\ 0, & \text{if} \quad p_n(f_i(t)) > 2p_n(f(t)). \end{cases}$$

Note that, for real valued functions, the McShane integral and the Lebesgue integral are equivalent (see [4]). Hence, by [14, Proposition 4] and Lemma 4, $\{g_i^n\}_{i \in N}$ is a sequence of simple functions on $[a, b]$, and it satisfies $p_n(g_i^n(t)) \leqq 2p_n(f(t))$ and $\lim_{i \to \infty} p_n(f(t) - g_i^n(t)) = 0$ μ-a.e. on $[a, b]$. Since the real valued function $p_n(f(t))$ is McShane integrable on $[a, b]$ from Lemma 4, it is Lebesgue integrable on $[a, b]$. Moreover, the sequence of real valued measurable functions $\{p_n(f(t) - g_i^n(t))\}_{i \in N}$ satisfies that $p_n(f(t) - g_i^n(t)) \leqq 3p_n(f(t))$ for all $i \in N$. Hence, using the Lebesgue-Fatou Lemma, we obtain

$$\lim_{i \to \infty} \int_a^b p_n(f(t) - g_i^n(t))dt = 0.$$

Now, for each $n \in N$, since $\lim_{i \to \infty} \int_a^b p_n(f(t) - g_i^n(t))dt = 0$, take an $i(n) \in N$ such that $\int_a^b p_n(f(t) - g_{i(n)}^n(t))dt < 1/2^n$. Put $h_n(t) = g_{i(n)}^n(t)$ for every $n \in N$. Then, for each $m \in N$, if $n > m$, then $\int_a^b p_m(f(t) - h_n(t))dt \leqq \int_a^b p_n(f(t) - g_{i(n)}^n(t))dt < 1/2^n$. Thus we have a sequence $\{h_n\}$ of X-valued simple functions on $[a, b]$ such that

$$\lim_{n \to \infty} \int_a^b p_m(f(t) - h_n(t))dt = 0$$

for every $m \in N$. Therefore f is Bochner integrable on $[a, b]$.

定理 4 の証明は，定理 1 の証明とほぼ同じ様に，次の 4 つの補題の準備の後になされる。定理 2，3 の証明も同様な方針でなされる。(see [14])

Lemma 5. *Let* X *be a* (UCs-N) *space with component spaces* $(X_\alpha, \{p_n^\alpha\})$ $(\alpha \in \Sigma)$, *and let* f *be an* X*-valued function defined on* $[a, b]$. *If* f *is Bochner integrable on* $[a, b]$, *then it is McShane integrable on* $[a, b]$, *and both integrals coincide.*

Lemma 6. *Let X be a Hilbertian* (UCs-N) *space which is nuclear, with component spaces* $(X_\alpha, \{p_n^\alpha\})$ $(\alpha \in \Sigma)$. *If f is an X-valued McShane integrable function on $[a,b]$ with the primitive F, then there is an $\alpha \in \Sigma$ such that f is McShane integrable (X_α) on $[a,b]$, and that, for every $n \in N$, there is a positive function $\delta_n(\xi)$ on $[a,b]$ such that for any δ_n-fine L division $D = \{[u,v],\xi\}$ of $[a,b]$ we have*

$$\sum p_n^\alpha(f(\xi)(v-u) - F([u,v])) < 1/2^n.$$

Lemma 7. *Let X be a Hilbertian* (UCs-N) *space which is nuclear, with component spaces* $(X_\alpha, \{p_n^\alpha\})$ $(\alpha \in \Sigma)$, *and let f be an X-valued function defined on $[a,b]$. The function f is McShane integrable on $[a,b]$ if and only if there is an $\alpha \in \Sigma$ so that the image of $[a,b]$ by f is contained in X_α, and that, for every $n \in N$, there is a positive function $\delta_n(\xi)$ on $[a,b]$ such that for any two δ_n-fine L divisions $D_1 = \{([t_{i-1},t_i],\xi_i) : i = 1,2,\cdots,h\}$ and $D_2 = \{([t_{i-1},t_i],\eta_i) : i = 1,2,\cdots,h\}$ of $[a,b]$ we have*

$$\sum_{i=1}^{h} p_n^\alpha(f(\xi_i) - f(\eta_i))(t_i - t_{i-1}) < 1/2^n.$$

Lemma 8. *Let X be a Hilbertian* (UCs-N) *space which is nuclear, with component spaces* (X_α, p_α) $(\alpha \in \Sigma)$, *and let f be an X-valued function defined on $[a,b]$. If the function f is McShane integrable on $[a,b]$, then there is an $\alpha \in \Sigma$ so that, for every $\beta \in \Sigma$ with $\beta \geqq \alpha$, $p_m^\beta(f(t))$ is McShane integrable on $[a,b]$ for every $m \in N$.*

REFERENCES

1. S.S. Cao, *The Henstock integral for Banach-valued functions*, Southeast Asian Bull. Math. **16** (1992), 35–40.
2. R.A. Gordon, *The McShane integral of Banach-valued functions*, Illinois J. **34** (1990), 557–567.
3. P.Y. Lee and T.S. Chew, *A better convergence theorem for Henstock integrals*, Bull. London Math. Soc. **17** (1985), 557–564.
4. E.J. McShane, *A unified theory of integration*, Amer. Math. Monthly **80** (1973), 349–359.
5. S. Nakanishi, *The method of ranked spaces proposed by Professor Kinjiro Kunugi*, Math. Japon. **23** (1978), 291–323.
6. S. Nakanishi, *Main spaces in distribution theory treated as ranked spaces and Borel sets*, Math. Japon. **26** (1981), 179–201.
7. S. Nakanishi, *On ranked union spaces and dual spaces*, Math. Japon. **28** (1983), 353–370.
8. S. Nakanishi, *Integration of ranked vector space valued functions*, Math. Japon. **33** (1988), 105–128.
9. S. Nakanishi, *Ranked-vector-space valued measure and the Radon-Nikodym theorem*, Math. Japon. **34** (1989), 253–274.
10. S. Nakanishi, *Some ranked vector spaces*, Math. Japon. **34** (1989), 789–813.
11. S. Nakanishi, *The Henstock integral for functions with values in nuclear spaces*, Math. Japon. **39** (1994), 309–335.
12. W.F. Pfeffer, *The Riemann approach to integration: local geometric theory*, Cambridge University Press, New York, 1993.
13. K. Sakurada, Y. Abe, T. Ishida, I. Hashimoto, *A remark on dual spaces of* (CUCs-N) *spaces*, J. Hokkaido Univ. Educ.(Sec. II A) **37** (1986), 23–27.
14. K. Sakurada, *Equivalence of the McShane and Bochner integrals for functions with values in Hilbertian* (UCs-N) *spaces which are nuclear*, in preprint.
15. K. Yoshida, *Functional Analysis*, Springer-Verlag, New York, 1974.

MATHEMATICS LABORATORY, SAPPORO CAMPUS, HOKKAIDO UNIVERSITY OF EDUCATION, 5-3-1 AINOSATO, KITA-KU, SAPPORO 002, JAPAN

On Canonical Solutions to the Hamburger Moment Problem

AKIO ARIMOTO
Musashi Institute of Technology
Tamazutsumi 1-28-1, Setagaya-ku
Tokyo 158, Japan

Let μ be a positive Borel measure which has all power moments on R:

$$\int_{-\infty}^{\infty} x^k d\mu(x) < \infty \ , \quad k = 0,1,2,\cdots$$

and let

$$V_\mu = \left\{ v \middle| \int x^n dv = \int x^n d\mu, n = 0,1,2,3,\cdots \right\}.$$

We say that μ is determinate if V_μ is one point set and indeterminate if V_μ contains more than one point. If μ is indeterminate, V_μ is compact in the weak* topology and a convex set. Naimark(1947) proved that μ is an extreme point of V_μ if and only if $\overline{\wp} = L^1(\mu)$, where $\wp$ is a set of polynomials. In this case μ is called V-extremal. Also μ is called N-extremal if μ is indeterminate and $\overline{\wp} = L^2(\mu)$. It is known that an N-extremal measure is V-extremal (and that the converse is false). Let consider the Riesz's functon

$$R_\mu(z) = \sup\left\{ |p(z)| : p \in \wp, \|p\|_{L^2(\mu)} \leq 1, z \in C \right\}$$

and the Mergelian's function

$$M_\mu(z) = \sup\left\{ |p(z)| : p \in \wp, \|p\|_{L^2(\mu')} \leq 1, z \in C \right\},$$

where $d\mu' = (1+x^2)^{-1} d\mu$.

Riesz has shown that

Theorem.A ([1])
μ is indeterminate if and only if $R_\mu(z) < \infty, \ \Im z \neq 0$.

Furthermore we can prove that for $d\tilde{\mu} = (1+x^2) \ d\mu$,

$$\frac{1}{1+|z|} \frac{1}{R_\mu(z)} \leq \left\| \frac{1}{x-z} + \wp \right\|_{L^2(\tilde{\mu})} \leq \frac{1+|z|}{|\Im z|} \frac{1}{R_\mu(z)}, \quad \Im z \neq 0$$

and that

$$\frac{1}{1+|z|} \frac{1}{M_\mu(z)} \leq \left\| \frac{1}{x-z} + \wp \right\|_{L^2(\mu)} \leq \frac{1+|z|}{|\Im z|} \frac{1}{M_\mu(z)}, \quad \Im z \neq 0.$$

From the second inequality we can easily see that $M_\mu(z) = +\infty$ for some z, $(\Im z \neq 0)$ if and only if polynomials are dense in $L^2(\mu)$. Hence both inequalities enable us to see that an indeterminate measure μ is N-extremal if and only if $R_\mu(z) < \infty$, but

$M_\mu(z)=+\infty\quad(\Im z\neq 0)$. An entire function $f\colon \mathrm{C}\to\mathrm{C}$ is said to be of **minimal exponential type** if

$$\forall\varepsilon>0,\exists C_\varepsilon:\ |f(z)|\le C_\varepsilon e^{\varepsilon|z|}\qquad \text{for } z\in\mathrm{C}$$

The set of entire functions of minimal exponential type will be denoted by $\mathscr{E}_0$. We say that $f(z)$ belongs to the Hamburger class $\mathscr{H}$ if $f(z)$ is a real $\mathscr{E}_0$ function with simple real zeros $a_1,a_2,\cdots(0\le|a_1|\le|a_2|\le\cdots\le|a_n|,|a_n|\to\infty)$ and satisfies

$$\frac{a_n^{2m}}{|f'(a_n)|}\to 0,\ \ n\to\infty, m=0,1,2,\cdots$$

Theorem B.(Hamburger 1944 [1]) *If* μ *is N-extremal then it can be represented as* $\mu=\sum\alpha_n\delta_{a_n}$ *, when* $f(z)\in\mathscr{H}$, $\{a_1,a_2,\cdots\}=f^{-1}(0)$, $\alpha_n>0\ \ (n=1,2,3,\cdots)$,

(i) $\sum\alpha_n a_n^{2m}<\infty,\quad m=0,1,2,\cdots,$

(ii) $$\sum\frac{1}{\alpha_n(1+a_n^2)f'(a_n)^2}<\infty$$

and

(iii) $$\sum\frac{1}{\alpha_n f'(a_n)^2}=\infty.$$

Hamburger (and Akhiezer [1]) stated that the converse is also true.

(*) $$\sum\frac{1}{\alpha_n f'(a_n)^2(1+a_n{}^2)}<\infty$$

and

(**) $$\sum\frac{1}{\alpha_n f'(a_n)^2}=\infty$$

for some $f(z)\in\mathscr{H}$. $\{a_1,a_2,\cdots\}=f^{-1}(0)$ and $\alpha_n>0\ \ (n=1,2,3,\cdots)$,
imply that $\mu=\sum\alpha_n\delta_{a_n}$ is N-extremal.

But this is wrong. In fact, Koosis [5] constructed a counter example. Let $f(z)=S(z)T(z)$ where $S(z)$ and $T(z)$ are infinite products such as

$$S(z)=\prod\left(1-\frac{z}{a_n}\right),\quad T(z)=\prod\left(1-\frac{z}{b_n}\right),\quad a_n=2^n,\quad b_n=2^n\left(1+2^{\frac{-3}{4}n^2}\right)$$

He has shown that $\mu=\sum\frac{1}{f'(a_n)^2}\delta_{a_n}+\sum\frac{1}{f'(b_n)^2}\delta_{b_n}$ satisfies the conditions (*),(**) but is not N-extremal. To prove this he used the following theorem.

Theorem C ([5],[2]) *For* $\mu=\sum\frac{1}{f'(a_n)^2}\delta_{a_n}$, $f\in\mathscr{H}$.

$\overline{\wp} \neq L^2(\mu)$ *if and only if there exists a nonzero* $\varphi(z) \in \mathcal{E}_0$ *such that*

(i) $\sum |\varphi(a_n)|^2 < \infty$,

(ii) $\forall n \geq 0,: \lim_{|y| \to \infty} \dfrac{|y^n \varphi(iy)|}{|f(iy)|} = 0$

For necessary and sufficient condition, Ito([4]) has proved the following.

Theorem D. μ *is N-extremal if and only if* μ *can be represented as*

$\mu = \sum_{n=1}^{\infty} \alpha_n \delta_{a_n}, \alpha_n > 0, (n = 1,2,\cdots)$, *where there exist a pair* $T(z), S(z)$ *of real* $\mathcal{E}_0$ *functions satisfying*

a) S *has only real simple zeros* $a_n, 0 \leq |a_1| \leq |a_2| \leq \cdots \leq |a_n|, |a_n| \to \infty$

b) $$\alpha_n = \frac{1}{(1 + a_n{}^2) T(a_n) S'(a_n)} > 0, \ (n = 1,2,3,\cdots)$$

c) $$\sum \frac{a_n^{2m}}{T(a_n) S'(a_n)} < \infty, \ m = 0,1,2,\cdots$$

d) $$\sum \frac{T(a_n)}{S'(a_n)} < \infty,$$

e) $$\sum \frac{|\theta_n|^2}{T(a_n) S(a_n)} < \infty \ , \ \sum \frac{a_n^m \theta_n}{T(a_n) S'(a_n)} = 0, \ m = 0,1,2,\cdots \Rightarrow \ \theta_n = cT(a_n)$$

The condition e) is equivalent to the conditions

e') $\overline{\wp_i}$ is co-dimension 1 in $L^2(\mu)$, $\wp_i = (x - i)\wp$

or

e") $\varphi(z) \in \mathcal{E}_0$, such that $\left| y^k \dfrac{\varphi(iy)}{S(iy)} \right| \to 0, \ |y| \to \infty, k = 0,1,2,\cdots \ \sum \dfrac{T(a_n)}{S'(a_n)} |\varphi(a_n)|^2 < \infty$

$\Rightarrow \ \varphi(z) = const$

There is another necessary and sufficient condition by Sodin that has been kindly informed to me recently.

We say that an entire function g is a *Hamburger divisor* of $f \in \mathcal{H}$ if $g(z) \in \mathcal{H}$ and $g^{-1}(0) \subset f^{-1}(0)$.

TheoremE.([6]) μ *is N-extremal if and only if there exits an* $f \in \mathcal{H}$ *and* $\alpha_n > 0, (n = 1,2,\cdots)$ *such that*

(i) $\mu = \sum \alpha_n \delta_{a_n}$, $\{a_1, a_2, \cdots\} = f^{-1}(0)$.

(ii) $\sum \alpha_n a_n^{2m} < +\infty, m = 0,1,2,\cdots$

(iii) $\sum \dfrac{1}{\alpha_n f'(a_n)^2 (1+a_n{}^2)} < +\infty$

(iv) $\sum_{a_n \in g^{-1}(0)} \dfrac{1}{\alpha_n g'(a_n)^2} = +\infty$ *for each Hamburger divisor* g *of* f .

We can now give some examples for N-extremal measures.

Theorem F([3])

$\Lambda = \{a_n\}$ *be an increasing sequence of positive numbers such that*

(i) $n_\Lambda(r) = \max\{n | a_n \le r\} \approx r^\rho$ *for* $r \to \infty$*, where* $0 < \rho < \dfrac{1}{2}$

(ii) There exists a constant d>0 such that the discs $\{z| \ |z - a_n| \le d a_n^{1-\rho}\}$ *are disjoint.*

Then $f(z) = \prod_{n=1}^{\infty} \left(1 - \dfrac{z}{a_n}\right)$ *belongs to* $\mathscr{H}$ *and* $\mu = \sum_{n=1}^{\infty} \dfrac{1}{f'(a_n)^2} \delta_{a_n}$ *is an N-extremal measure.*

Example for Theorem D. Let

$$S(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^\alpha}\right), \quad \alpha > 2$$

$$T(z) = -\prod_{n=1}^{\infty} \left(1 - \frac{z}{n^\alpha (1+\varepsilon_n)}\right), \quad \varepsilon_n = \frac{1}{n^\beta}, \quad \alpha + 1 < \beta < 3\alpha$$

Then $\alpha_n = \dfrac{1}{(1+a_n{}^2) T(a_n) S'(a_n)} > 0, a_n = n^\alpha$, and $\mu = \sum_{n=1}^{\infty} \alpha_n \delta_{a_n}$ is N-extremal.

References

[1]Akhiezer, N.I., The classical moment problem, Oliver and Boyd, Edinburgh,1965

[2] Berg,Ch. Indeterminate moment problems and the theory of entire functions T.J.Stielties conference ,1994 at Delft Univ. of Tech.

[3]Fryntov,A. On a uniqueness theorem related to polynomial approximation on discrete sets,Mathematical Physics, Analysis and Geometry(Kharkov) 1,1994

[4]Ito,T. A characterization of N-extremal measures for the Hamburger moment problem, unpublished 1991

[5]Koosis,Measure Orthogonales extremales pour l'approximation ponderee par des polynomes,C.R.Acad.Sci.Paris,Ser I 311(1990),503-506

[6]Sodin,M;Borichev,A. Hamburger divisor and the canonical solution of Hamburger moment problem, private communication (1995/8/23)

e-mail: arimoto@ie.musashi-tech.ac.jp

REPRODUCING KERNELS AND THEIR APPLICATIONS

SABUROU SAITOH

Department of Mathematics, Faculty of Engineering
Gunma University, Kiryu 376, Japan
E-mail address: ssaitoh@eg.gunma-u.ac.jp

1. INTEGRAL TRANSFORMS IN HILBERT SPACES

We shall formulate an integral transform as follows:

$$f(p) = \int_T F(t)\overline{h(t,p)}dm(t), \quad p \in E. \tag{1.1}$$

Here, the input $F(t)$ (source) is a function on a set T, E is an arbitrary set, $dm(t)$ is a σ-finite positive measure on the dm measurable set T, and $h(t,p)$ is a complex-valued function on $T \times E$ which determines the transform of the system.

We shall assume that $F(t)$ is a member of the Hilbert space $L_2(T, dm)$ satisfying

$$\int_T |F(t)|^2 dm(t) < \infty. \tag{1.2}$$

The space $L_2(T, dm)$ whose norm gives an energy integral will be the most fundamental space as the input function space. In other spaces we shall modify them in order to meet to our situation, or as a prototype case we shall consider primarily or, as the first stage, the linear transform (1.1) in our situation.

As a natural result of our basic assumption (1.2), we assume that

$$\text{for any fixed} \quad p \in E, \quad h(t,p) \in L_2(T, dm) \tag{1.3}$$

for the existence of the integral in (1.1).

2. IDENTIFICATION OF THE IMAGES OF LINEAR TRANSFORMS AND THE INTERRELATIONSHIP BETWEEN THE INPUT AND OUTPUT FUNCTIONS

For its importance and simplicity, we shall formulate the integral transform (1.1) in the following general and abstract form:

Let $\mathcal{F}(E)$ be a linear space comprising of all complex-valued functions on an abstract set E. Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space equipped with inner product $(\cdot,\cdot)_{\mathcal{H}}$. Let

$$\mathbf{h} : E \longrightarrow \mathcal{H}$$

be a Hilbert space $\mathcal{H}$-valued function on E. Then, we shall consider the linear mapping L from $\mathcal{H}$ into $\mathcal{F}(E)$ defined by

$$f(p) = (L\mathbf{f})(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}. \tag{2.1}$$

The fundamental problems in the linear mapping (2.1) will be firstly the identification (characterization) of the images $f(p)$ and secondly will be the relationship between $\mathbf{f}$ and $f(p)$.

The key which solves these fundamental problems is to form the function $K(p,q)$ on $E \times E$ defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}}. \tag{2.2}$$

We let $\mathcal{R}(L)$ denote the range of L for $\mathcal{H}$ and we introduce the inner product in $\mathcal{R}(L)$ induced from the norm

$$\|f\|_{\mathcal{R}(L)} = \inf\{\|\mathbf{f}\|_{\mathcal{H}}; f = L\mathbf{f}\}. \tag{2.3}$$

Then, we obtain

Theorem 2.1. *For the function $K(p,q)$ defined by (2.2), the space $[\mathcal{R}(L), (\cdot,\cdot)_{\mathcal{R}(L)}]$ is a Hilbert (possibly finite-dimensional) space satisfying the properties that*

(i) for any fixed $q \in E$, $K(p,q)$ belongs to $\mathcal{R}(L)$ as a function in p,
and
(ii) for any $f \in \mathcal{R}(L)$ and for any $q \in E$,

$$f(q) = (f(p), K(p,q))_{\mathcal{R}(L)}.$$

Further, the function $K(p,q)$ satisfying (i) and (ii) is uniquely determined by $\mathcal{R}(L)$. Furthermore, the mapping L is an isometry from $\mathcal{H}$ onto $\mathcal{R}(L)$ if and only if

$$\{\mathbf{h}(p); p \in E\} \quad \textit{is complete in} \quad \mathcal{H}. \tag{2.4}$$

In Theorem 2.1, the properties (i) and (ii) of the function $K(p,q)$ will be called the "reproducing property" of $K(p,q)$ in (or for) the Hilbert space $\mathcal{R}(L)$, and then the kernel $K(p,q)$ is called "reproducing kernel". A Hilbert space admitting a reproducing kernel will be called a "reproducing kernel Hilbert space" — RKHS.

For Theorem 2.1 itself, see [57], [19], [33], [31] and, [38], [39] and [50], pages 84-85 for the detailed comments.

Theorem 2.1 itself will not become a fundamental theorem in linear transforms. In order to realize Theorem 2.1 as a fundamental theorem in linear transforms we will need the idea of reproducing kernel Hilbert spaces. As we shall see in the next section, since a reproducing kernel Hilbert space is uniquely determined by the reproducing kernel $K(p,q)$, conversely, we shall write it by H_K; that is,

$$\mathcal{R}(L) = H_K$$

in Theorem 2.1. When we consider Theorem 2.1 that the range $\mathcal{R}(L)$ of L for $\mathcal{H}$ forms precisely the Hilbert space H_K admitting the reproducing kernel $K(p,q)$ defined by (2.2) and the RKHS H_K admits an intuitively determined inner product $(\cdot,\cdot)_{H_K}$ for the members of H_K which are functions on E, apart from the linear transform (2.1) and, of course, apart from the space $\mathcal{H}$, we will be able to realize Theorem 2.1 as a fundamental theorem in linear transforms. Then, Theorem 2.1 will be stated in the form

Theorem 2.2. *The images $f(p)$ of the linear transform (2.1) for $\mathcal{H}$ form precisely the functional Hilbert space H_K admitting the reproducing kernel $K(p,q)$ defined by (2.2) which is uniquely determined by the reproducing kernel $K(p,q)$. Then, we have the inequality*

$$\|f\|_{H_K} \leqq \|\mathbf{f}\|_{\mathcal{H}}.$$

Furthermore, for any $f \in H_K$, there exists a uniquely determined member $\mathbf{f}^*$ of $\mathcal{H}$ such that

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \quad on \quad E$$

and

$$\|f\|_{H_K} = \|\mathbf{f}^*\|_{\mathcal{H}}.$$

3. ELEMENTARY PROPERTIES OF REPRODUCING KERNEL HILBERT SPACES

As we saw in Theorem 2.1, when we consider linear transforms in the framework of Hilbert spaces, we meet to naturally the idea of reproducing kernel Hilbert spaces. Therefore, we shall review the basic properties of reproducing kernel Hilbert spaces. Basic references are [6], [57] and [29].

4. INVERSION FORMULA FOR LINEAR TRANSFORMS

Our inversion formula will give a new viewpoint and a new method for integral equations of Fredholm of the first kind which are fundamental in the theory of integral equations. The characteristics of our inversion formula are as follows:

(i) Our inversion formula is given in terms of the reproducing kernel Hilbert space H_K which is intuitively determined as the image space of the integral transform (1.1).
(ii) Our inversion formula gives the visible component F^* of F with the minimum $L_2(T, dm)$ norm.
(iii) The inverse F^* is, in general, given in the sense of the strong convergence in $L_2(T, dm)$.

(iv) Our integral equation (1.1) is, in general, an ill-posed problem, but our solution F^* is given as a solution of a well-posed problem in the sense of Hadamard (1902, 1923).

At this moment, we can see why we meet ill-posed problems; that is, because we do not consider the problems in the natural image spaces H_K, but in some artificial spaces.

5. DETERMINATION OF THE LINEAR SYSTEM

In Theorem 2.2, conversely by using an isometrical mapping $\widetilde{L}$ from a Hilbert space H_K admitting a reproducing kernel $K(p,q)$ on E onto a Hilbert space $\mathcal{H}$ and by using the reproducing kernel $K(p,q)$, we can determine the linear system function $\mathbf{h}(p)$ which is a Hilbert $\mathcal{H}$-valued function on E as follows :

$$\mathrm{g}_{\widetilde{L}}(q) = \widetilde{L}K(\cdot,q), \tag{5.1}$$

which is called the "generating vector" of $\widetilde{L}$. See [50] and [9] for many concrete examples.

In the sequel, we shall examine the isometrical identities and inversion formulas in the following typical integral transforms.

6. THE LAPLACE TRANSFORM

7. THE FOURIER TRANSFORM

8. PALEY-WIENER'S THEOREM FOR ENTIRE FUNCTIONS

9. IN THE HEAT EQUATION

10. IN THE WAVE EQUATION

11. ANALYTIC AND HARMONIC FUNCTIONS OF CLASS L_2

12. IN THE MEYER WAVELETS

We shall refer to applications to the following topics.

13. SAMPLING THEOREM

14. BEST APPROXIMATIONS BY THE FUNCTIONS IN A RKHS

15. ANALYTIC EXTENSION FORMULAS

16. APPLICATIONS TO RANDOM FIELDS ESTIMATIONS

17. APPLICATIONS TO SCATTERING AND INVERSE PROBLEMS

18. NONHARMONIC INTEGRAL TRANSFORMS

19. NONLINEAR TRANSFORMS

20. REPRESENTATIONS OF INVERSE FUNCTIONS

21. OBSERVATION AND CONTROL

For a survey article for these topics, see S. Saitoh:

One approach to some general integral transforms and its applications, Integral Transforms and Special Functions **3** *(1995), 49-84.*

REFERENCES

1. H. Aikawa, N. Hayashi and S. Saitoh, *The Bergman space on a sector and the heat equation*, Complex Variables **15** (1990), 27-36.
2. H. Aikawa, N. Hayashi and S. Saitoh, *Isometrical identities for the Bergman and the Szegő spaces on a sector*, J. Math. Soc. Japan **43** (1991), 196-201.
3. H. Aikawa, N. Hayashi, I. Onda and S. Saitoh, *Analytical extensions of the members of the Bergman and Szegő spaces on some tube domains*, Arch. Math. **56** (1991), 362-369.
4. H. Aikawa, *Infinite order Sobolev spaces, analytic continuation and polynomial expansions*, Complex Variables **18** (1992), 253-266.

5. T. Ando and S. Saitoh, *Restrictions of reproducing kernel Hilbert spaces to subsets*, Preliminary reports, Suri Kaiseki Kenkyu Jo, Koukyu Roku **743** (1991), 164-187.
6. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337-404.
7. R. P. Boas and D. V. Widder, *An inverse formula for the Laplace integral*, Duke Math. J. **6** (1940), 1-26.
8. S. Bochner and W. T. Martin, "Several complex variables," Princeton Univ. Press, Princeton N. J., 1948.
9. J. Burbea, *Total positivity of certain reproducing kernels*, Pacific J. Math. **67** (1976), 101-130.
10. P. L. Butzer and R. J. Nessel, "Fourier Analysis and Approximations," Acad. Press, New York and London, 1971.
11. D. -W. Byun, *Integral transforms and approximation of functions*, Thesis, Gunma University (1993).
12. D. -W. Byun, *Isometrical mappings between the Szegö and the Bergman-Selberg spaces*, Complex Variables **20** (1992), 13-17.
13. D. -W. Byun and S. Saitoh, *A real inversion formula for the Laplace transform*, Zeitschrift für Analysis und ihre Anwendungen **12** (1993), 597-603.
14. D. -W. Byun and S. Saitoh, *Approximation by the solutions of the heat equation*, J. Approximation Theory **78** (1994), 226-238.
15. D. -W. Byun and S. Saitoh, *Best approximation in reproducing kernel Hilbert spaces*, Proc. of the 2th International Colloquium on Numerical Analysis, VSP-Holland (1994), 55-61.
16. C. K. Chui, "An Introduction to Wavelets," Academic Press, New York, 1992.
17. I. Daubechies, "Ten Lectures on Wavelets," Society for Industrial and Applied Mathematics, Philadelphia, 1992.
18. P. J. Davis, "Interpolation & Approximation," Dover Books on Advanced Mathematics, New York, 1973.
19. A. Devinatz, *Integral representations of positive definite functions*, Trans. Amer. Math. Soc. **74** (1953), 56-76.
20. B. A. Fuks, *Introduction to the theory of analytic functions of several complex variables*, Transl. Math. Mono. vol. 8, Amer. Math. Soc., Providence, R.I. (1963).
21. B. A. Fuks, *Special chapters in the theory of analytic functions of several complex variables*, Transl. Math. Mono. vol. 14, Amer. Math. Soc., Providence, R.I. (1965).
22. N. Hayashi and S. Saitoh, *Analyticity and smoothing effect for the Schrödinger equation*, Ann. Inst. Henri Poincaré **52** (1990), 163-173.
23. N. Hayashi and S. Saitoh, *Analyticity and global existence of small solutions to some nonlinear Schrödinger equation*, Commun Math. Phys. **139** (1990), 27-41.
24. N. Hayashi, *Global existence of small analytic solutions to nonlinear Schrödinger equations*, Duke Math. J. **60** (1990), 717-727.
25. N. Hayashi, *Solutions of the (generalized) Korteweg-de Vries equation in the Bergman and the Szegö spaces on a sector*, Duke Math. J. **62** (1991), 575-591.
26. V. Isakov, "Inverse Source Problems," Mathematical Surveys and Monographs Number 34, American Mathematical Society, 1990.
27. A. J. Jerri, *The Shannon sampling theorem-its various extensions and applications*, a tutorial review, Proceedings of the IEEE **65** (1977), 1565-1596.
28. D. Klusch, *The sampling theorem, Dirichlet series and Hankel transforms*, J. of Computational and Applied Math. **44** (1992), 261-273.
29. M. G. Kreĭn, *Hermitian positive kernels on homogeneous spaces*, I. Amer. Math. Soc. Transl.(2) **34** (1963), 69-108.
30. W. T. Martin, *Analytic functions and multiple Fourier integrals*, Amer. J. Math. **62** (1940), 673-679.
31. M. Z. Nashed and G. Wahba, *Convergence rates of approximate least squares solutions of linear integral and operator equations of the first kind*, Math. Computation **28** (1974), 69-80.
32. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex plane*, Amer. Math. Soc. Colloq. Publ. vol 19, Amer. Math. Soc. Providence R. I. (1934).
33. E. Parzen, *Statical inference on time series by RKHS methods*, Proc. of 12th Biennical seminar of the Canadian Mathematical Congress, Amer. Math. Soc. Providence, R.I. (1970).
34. M. Plancherel and G. Pólya, *Founctions entières et integrals de Fourier multiple*, Comment. Math. Helv. **9** (1936-37), 224-248;, **10** (1937-38), 110-163.
35. A. G. Ramm, "Random Fields Estimation Theory," Longman Scientific, Wiley, New York, 1990.
36. A. G. Ramm, "Multidimensional Inverse Scattering Problems," Longman Scientific & Technical, 1992.
37. L. I. Ronkin, *Introduction to the theory of entire functions of several variables*, Transl. Math. Mono. vol.44, Amer. Math. Soc., Providence R. I. (1974).
38. S. Saitoh, *Integral transforms in Hilbert spaces*, Proc. Japan Acad. **58** (1982), 361-364.
39. S. Saitoh, *Hilbert spaces induced by Hilbert space valued functions*, Proc. Amer. Math. Soc. **89** (1983), 74-78.

40. S. Saitoh, *The Weierstrass transform and an isometry in the heat equation*, Applicable Analysis **16** (1983), 1-6.
41. S. Saitoh, *Fourier transforms with weighted functions and the Green's functions*, Applicable Analysis **16** (1983), 123-130.
42. S. Saitoh, *Some fundamental interpolation problems for analytic and harmonic functions of class L_2*, Applicable Analysis **17** (1984), 87-106.
43. S. Saitoh, *Some isometrical identities in the wave equation*, International J. Math. and Math. Sci. **7** (1984), 117-130.
44. S. Saitoh, *Integral transforms by Green's function on R^n*, Applicable Analysis **17** (1984), 157-167.
45. S. Saitoh, *Integral transforms in linear equations of parabolic type with constant coefficients*, Applicable Analysis **18** (1984), 13-27.
46. S. Saitoh, *The Laplace transform of L^p functions with weights*, Applicable Analysis **22** (1986), 103-109.
47. S. Saitoh, *Generalizations of Paley-Wiener's theorem for entire functions of exponential type*, Proc. Amer. Math. Soc. **99** (1987), 465-471.
48. S. Saitoh, *Cauchy integrals for L_2 functions*, Arch. Math. **51** (1988), 451-454.
49. S. Saitoh, *Fourier-Laplace transforms and the Bergman spaces*, Proc. Amer. Math. Soc. **102 no.4** (1988), 985-992.
50. S.Saitoh, "Theory of reproducing kernels and its applications," Pitman Res. Notes in Math. Series 189, Longman Scientific & Technical, England, 1988.
51. S.Saitoh, *Isometrical identities and inverse formulas in the one-dimensional Schrödinger equation*, Complex Variables **15** (1990), 135-148.
52. S.Saitoh, *Isometrical identities and inverse formulas in the one-dimensional heat equation*, Applicable Analysis **40** (1991), 139-149.
53. S.Saitoh, *Inequalities for the solutions of the heat equation*, General Inequalities 6, Birkhäuser Verlag, Basel Boston (1992), 351-359.
54. S.Saitoh, *Representations of the norms in Bergman-Selberg spaces on strips and half planes*, Complex Variables **19** (1992), 231-241.
55. S.Saitoh, *The Hilbert spaces of Szegö type and Fourier-Laplace transforms on $\mathbb{R}^n$*, Generalized Functions and Their Applications (1993), 197-212, Plenum Publishing Coorporation, New York.
56. S.Saitoh, *Analyticity of the solutions of the heat equation on the half space $\mathbb{R}^n_+$*, Proc. of the 4th International Colloquium on Differential Equations, VSP-Holland (1994), 265-275.
57. L. Schwartz, *Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associès (noyaux reproduisants)*, J. Analyse Math. **13** (1964), 115-256.
58. E. M. Stein and G. Weiss, "Introduction to Fourier analysis on Euclidean spaces," Princeton University Press, 1971.
59. K. Yao, *Application of reproducing kernel Hilbert spaces - Bandlimited signal models*, Information and Control **11** (1967), 429-444.
60. D. V. Widder, "The Laplace transform," Princeton University Press, 1972.

Generalization of Gaussian estimates and interpolation of the spectrum in L^p

Shizuo Miyajima (宮島　静雄)
Science Univ. of Tokyo(東京理科大学理学部)

Abstract

Recently, it has been revealed that the semigroups satisfying Gaussian estimates inherit some of the nice properties enjoyed by the Gaussian semigroup itself. W. Arendt gave a result in this direction by proving the invariance of the spectrum of the generators of consistent C_0-semigroups with Gaussian estimates. We generalize this result to the semigroups "dominated" by the one generated by the fractional power $-(I-\Delta)^{\alpha}$ $(1/2 < \alpha \leq 1)$.

Let $\Omega \subset \boldsymbol{R}^N$ be an open set, and suppose that a C_0-semigroup $T_p = \{T_p(t)\}_{t \geq 0}$ on $L^p(\Omega)$ with generator A_p is given for each $1 \leq p < \infty$. The family $\{T_p\}_p$ of C_0-semigroups is called consistent if $T_p(t) = T_q(t)$ holds on $L^p(\Omega) \cap L^q(\Omega)$ for all $t \geq 0$ and $p, q \in [1, \infty)$. The most typical example of a consistent family of C_0-semigroups is the one generated by the Laplacian. Namely, let

$$K(z,t) := \frac{1}{(4\pi t)^{N/2}} \exp\{-|z|^2/4t\} \quad (z \in \boldsymbol{R}^N, t > 0)$$

be the Gaussian kernel and set

$$(G_p(t)f)(x) := \int_{\mathbf{R}^N} K(x-y,t) f(y)\, dy \quad (f \in L^p(\boldsymbol{R}^N))$$

and $G_p(0) := I$. Then each $G_p := \{G_p(t)\}_{t \geq 0}$ is a C_0-semigroup on $L^p(\boldsymbol{R}^N)$ which is called the Heat semigroup or Gaussian semigroup, and it forms a consistent family of C_0-semigroups. It is well-known that the spectrum of the generator of G_p is independent of p.

This example can be generalized to allow a perturbation of the Laplacian by some potentials. Let us consider rather a simple case of locally integrable positive potentials. Suppose $V \in L^1_{loc}(\boldsymbol{R}^N)$ and $V \geq 0$. Then $T_p(t) := \text{s-}\lim_{n \to \infty} e^{t(\Delta - V \wedge n)}$ exists in each $L^p(\boldsymbol{R}^N)$ and defines a consistent family of C_0-semigroups. For this fact, we refer the reader to Voigt [6], [7] and Hempel–Voigt [2]. The generator A_p of T_p is an extension of the operator sum $\Delta - V$ in $L^p(\boldsymbol{R}^N)$, and hence gives a realization of the Schrödinger operator with potential V. Hempel–Voigt [2] showed that the spectrum of A_p is independent of p. (Actually, they proved more general theorem which allows V to be negative.)

On the other hand, the semigroup T_p above has an important property. In fact, by applying the Trotter product formula to $e^{t(\Delta - V \wedge n)}$ and going to the limit as $n \to \infty$, we obtain the following (pointwise) inequality:

$$|T_2(t)f| \leq e^{t\Delta}|f| = G_2(t)|f| \quad (f \in L^2(\boldsymbol{R}^N)).$$

It is known that a similar estimate holds even if V has a non-zero negative part, provided it is "sufficiently small" (cf. Simon [4, p. 474]).

Generalizing this situation, the notion of (upper) Gaussian estimate has been introduced:

Definition. *A C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ on $L^2(\Omega)$ $(\Omega \subset \boldsymbol{R}^N)$ is said to admit a Gaussian estimate if there exist constants M, $b > 0$ and $w \in \boldsymbol{R}$ such that*

$$T(t)|f| \leq Me^{wt}G_2(bt)|f|$$

holds for all $t \geq 0$ and $f \in L^2(\Omega)$. Here G_2 denotes the Gaussian semigroup on $L^2(\boldsymbol{R}^N)$, and $|f|$ on the right-hand side of the inequality is naturally considered as an element in $L^2(\boldsymbol{R}^N)$.

Arendt [1] recently proved the following theorem, which generalizes in some direction the result of Hempel–Voigt.

Theorem.(Arendt[1]) *Assume that a C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ on $L^2(\Omega)$ $(\Omega \subset \boldsymbol{R}^N)$ admit a Gaussian estimate. Then T naturally induces consistent C_0-semigroups T_p on $L^p(\Omega)$ $(1 \leq p < \infty)$ such that $T_2 = T$, and $\rho_\infty(A_p)$ is independent of $p \in [1, \infty)$ where A_p is the generator of T_p, and $\rho_\infty(A_p)$ denotes the connected component of the resolvent set of A_p containing a right half-plane. If we further assume that T consists of self-adjoint operators, then the spectrum of A_p is independent of p.*

Roughly speaking, this result of Arendt says that a C_0-semigroup on L^2 "dominated" by the Gaussian semigroup also has similar spectral property as the Gaussian semigroup itself. So we are naturally led to the question whether a C_0-semigroup on L^2 also has such a property provided it is "dominated" by some well-behaved semigroup other than the Gaussian semigroup. In fact, we can obtain an affirmative answer to this question. To state the result, let us introduce the following notion by generalizing that of Gaussian estimate.

Definition. *A C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ on $L^2(\Omega)$ $(\Omega \subset \boldsymbol{R}^N)$ is said to be essentially dominated by a C_0-semigroup $S = \{S(t)\}_{t\geq 0}$ on $L^2(\boldsymbol{R}^N)$ if there exist constants M, $b > 0$ and $w \in \boldsymbol{R}$ such that*

$$T(t)|f| \leq Me^{wt}S(bt)|f| \qquad (\forall f \in L^2(\Omega),\ \forall t \geq 0)$$

holds. (On the right-hand side, $|f|$ is naturally considered as an element in $L^2(\boldsymbol{R}^N)$.)

Our task is to examine what happens if we take for S a semigroup other than the Gaussian semigroup. Note that S should be a positive (=order preserving) semigroup if there exists a semigroup essentially dominated by S. On the other hand recall that if $\{e^{tA}\}_{t\geq 0}$ is a bounded positive C_0-semigroup, then $\{e^{-t(-A)^\alpha}\}_{t\geq 0}$ $(0 \leq \alpha \leq 1)$ is also a positive C_0-semigroup. Here $(-A)^\alpha$ denotes the fractional power of $-A$ of order α. So the first case we should consider seems to be the case of $S(t) = e^{-t(-\Delta)^\alpha}$. But at present we are only able to prove the case of $S(t) = e^{-t(I-\Delta)^\alpha}$ (see the remarks following our Theorem).

Theorem. *Assume that a* C_0*-semigroup* T *on* $L^2(\Omega)$ $(\Omega \subset \boldsymbol{R}^N)$ *is essentially dominated by* $S_\alpha = \{e^{-t(I-\Delta)^\alpha}\}_{t\geq 0}$ *(on* $L^2(\boldsymbol{R}^N)$*) for some* $\alpha \in (\frac{1}{2}, 1]$. *Then there exist consistent* C_0*-semigroups* T_p *on* $L^p(\Omega)$ $(1 \leq p < \infty)$ *such that* $T_2 = T$ *and* $\rho_\infty(A_p)$ *is independent of* $p \in [1, \infty)$*, where* A_p *is the generator of* T_p.

Corollary. *Assume that the generator* A *of* T *is self-adjoint and that* T *is essentially dominated by* S_α *for some* $\alpha \in (\frac{1}{2}, 1]$. *Then there exist consistent* C_0*-semigroups* T_p *on* $L^p(\Omega)$ $(1 \leq p < \infty)$ *such that* $T = T_2$ *and* $\sigma(A_p)$ *is independent of* $p \in [1, \infty)$*, where* A_p *is the generator of* T_p.

Remarks. (1) In the case of $\alpha = 1$, the result above is equivalent to that of Arendt.

(2) By an inspection of the proof of our theorem, we can see that the essential domination by $e^{-t(\varepsilon-\Delta)^\alpha}$ for an $\varepsilon > 0$ and $1/2 < \alpha \leq 1$ implies the same conclusion as in our main theorem. So, it may be conjectured that the assumption of essential domination by $e^{-t(I-\Delta)^\alpha}$ can be relaxed to that by $e^{-t(-\Delta)^\alpha}$. Note that $0 \leq e^{-t(\varepsilon-\Delta)^\alpha} \leq e^{-t(-\Delta)^\alpha}$ holds for all $t \geq 0$. At present, we are not able to prove or disprove this conjecture. However, we would like to note that the Fourier multiplier theory yields the p-independence of the spectrum of $(-\Delta)^\alpha$ in $L^p(\boldsymbol{R}^N)$ with a suitable domain ([5, p. 96]).

Though the method of the proof of this theorem follows the line devised by Arendt [1], we must prepare some our own estimates.

Firstly, we note that $S_\alpha(t)$ is represented as an integral operator with a kernel $K_\alpha(x-y, t)$ $(x, y \in \boldsymbol{R}^N, t > 0)$: $S_\alpha(t)f = K_\alpha(\cdot, t) * f$ $(f \in L^2(\boldsymbol{R}^N)$ ($*$ denotes the convolution).

The following two estimates concerning $K_\alpha(z, t)$ are crucial to our proof.

Proposition 1 *Let* $0 < \alpha \leq 1$ *and* $0 < \delta < 1$. *Then there exists a constant* $C_{\alpha,\delta} > 0$ *such that*

$$0 \leq K_\alpha(t, x) \leq C_{\alpha,\delta} \frac{e^{-\delta|x|}}{t^{N/2\alpha}} \quad (0 < {}^\forall t < \infty). \tag{1}$$

Proposition 2 *Let* $\frac{1}{2} < \alpha \leq 1$ *and* $0 < \delta < 1$. *Then there exists a constant* $C_{\alpha,\delta} > 0$ *such that*

$$0 \leq K_\alpha(t, x) \leq C_{\alpha,\delta} \frac{e^{-\delta|x|}}{|x|^{N+1}} t^{1/2\alpha} \quad (0 < {}^\forall t \leq 1). \tag{2}$$

For details see Miyajima–Ishikawa [3].

References

[1] Arendt, W., *Gaussian estimates and interpolation of the spectrum in L^p*, Differential and Integral Equations **7**(no. 5)(1994), 1153–1168.

[2] Hempel, R. and J. Voigt, *The spectrum of a Schrödinger operator in $L^p(\boldsymbol{R}^\nu)$ is p-independent*, Comm. Math. Phys. **104**(1986), 243–250.

[3] Miyajima, S. and M. Ishikawa, *Generalization of Gaussian estimates and interpolation of the spectrum in L^p*, to appear in SUT Journal of Mathematics.

[4] Simon, B., *Schrödinger semigroups*, Bulletin(New Series) of the AMS, **7**(1982), 447–526.

[5] E. M. Stein, "Singular integrals and differentiability properties of functions," Princeton Univ. Press, Princeton, New Jersey, 1970.

[6] J. Voigt, *Absorption semigroups, their generator, and Schrödinger semigroups*, J. Funct. Anal. **67**(1986), 167–205.

[7] J. Voigt, *Absorption semigroups*, J. Operator Theory **20**(1988), 117–131.

Shizuo Miyajima
Dep. of Mathematics, Fac. of Science
Wakamiya-cho 26, Shinjyuku-ku
Tokyo, 162 Japan

A GENERALIZATION OF BOCHNER'S THEOREM ON BANACH SPACES

MICHIE MAEDA
お茶の水女子大学　理学部　数学

Abstract. We treat the Sazonov topology on a dual Banach space. Also, we show that there exists a new sufficient Sazonov topology on every real separable dual Banach space.

１．Bochner 問題

有限次元空間で知られているBochnerの定理は次のようにのべられる。

［Bochner の 定理］ R^n 上で定義された複素数値関数 f が次の条件を満たすときに f はRⁿ 上のBorel確率測度の特性関数になる。

（１） fは正型。

（２） f（０）＝１。

（３） fは原点で連続。

この定理の逆、すなわちBorel確率測度の特性関数は（１）（２）（３）の条件を満たすことは明らかである。
この定理を無限次元ベクトル空間上に拡張することを考える。まず実可分な無限次元ヒルベルト空間では（３）の条件をあるベクトル位相に関して連続と変えると成り立つことがSazonovによって示された（１９５８年）。

［Sazonov の定理］ Hを実可分ヒルベルト空間とする。τ_{HS}をH'上のHilbert－Schmidt operatorをすべて連続にするような最弱位相とする。このときH'上で定義された複素数値関数fが次の条

件をもつことはｆがＨ上のＢｏｒｅｌ確率測度の特性関数になるための必要十分条件である。

（１）　ｆは正型。

（２）　ｆ（０）＝１。

（３）　ｆは原点でτ_{HS}－位相に関して連続。

このような条件を満たす位相のことをＳａｚｏｎｏｖ－位相（Ｓ－位相）という。　次に実可分無限次元Ｂａｎａｃｈ空間について調べる。これについてはＭｏｕｃｈｔａｒｉの次の二つの結果がある（１９７３年、１９７５年）。

［Ｍｏｕｃｈｔａｒｉの定理］

Ⅰ．実可分Ｂａｎａｃｈ空間Ｅが、Ｓａｚｏｎｏｖ－位相をもてば、ＥはL_0（Ω，Ｐ）の部分空間と同型で、ｃｏｔｙｐｅ２となる。

Ⅱ．ＥがL_0の部分空間と同型で、ｍｅｔｒｉｃ　ａｐｐｒｏｘｉｍａｔｉｏｎ　ｐｒｏｐｅｒｔｙ　を持てば、Ｅ’上にＳ－位相が存在する。

ここでのⅠより、Ｂａｎａｃｈ空間に対しては、つねにＳ－位相が存在するということはないので、Ｓ－位相の定義を必要性と十分性の二つに分けて考える。

２．必要Ｓａｚｏｎｏｖ－位相と十分Ｓａｚｏｎｏｖ－位相

Ｅを実可分Ｂａｎａｃｈ空間とし、Ｅ’を、そのｔｏｐｏｌｏｇｉｃａｌ　ｄｕａｌ　とする。Ｅ上のすべてのＢｏｒｅｌ確率測度μの特性関数$\hat{\mu}$がＥ’上のベクトル位相τに関して連続であるならば、このτを必要Ｓａｚｏｎｏｖ－位相（ＮＳ－位相）という。

逆にＥ’上で定義された複素数値関数ｆが　［(1)ｆは正型、(2)ｆ（０）＝１、(3)ｆはＥ’上のベクトル位相τに関して連続］　の三つの条件を満たせば、ｆはＥ上のＢｏｒｅｌ確率測度μの特性関数になるということがいえるときに、この位相τを十分Ｓａｚｏｎｏｖ－位相（ＳＳ－位相）という。

τがＮＳ－位相でありかつＳＳ－位相であるならば、τはＳ－位相である。

出来れば、最弱なＮＳ－位相と最強なＳＳ－位相を求めたい。　最弱ＮＳ－位相は既に知られている。　これについて説明する前に、いくつかの概念とそれらの間の関係について述べる。

μをＥ上のｃｙｌｉｎｄｒｉｃａｌ　ｍｅａｓｕｒｅ、$\hat{\mu}$をμの特性関数

とする。またTをE'から$L_0(\Omega, P)$へのlinear random functionとする。a_1, a_2, ・・・, $a_n \in E'$に対してTa_1, Ta_2, ・・・, Ta_nによるdistributionが$\mu_{a_1 \cdots a_n}$となるようなcylindrical measure μが存在する。これをμ^Tと表す。逆にcylindrical measure μに対して$\mu = \mu^T$となるTが存在することも知られている。

Tがdecomposed operatorであるとは、あるE－値確率変数Φが存在して$Ta(\omega) = <\Phi(\omega), a>$　$(\forall a \in E')$が成り立つことである。

μがE上のcylindrical measureであるときに、$\hat{\mu}$がτ－連続であるということと$\mu = \mu^T$となるTがτ－連続であるということとは同値である。またμがBorel measureであるということとTがdecomposedであるということも同値である。

［定義］E'上の、すべてのdecomposed operatorを連続にする最弱位相をτ_0という。

このとき次の定理が成り立つ。

［定理］Eが実可分Banach空間であるとする。このときE'上の位相τ_0は最弱NS－位相である。

3．位相　M

ここで新しい位相を定義する。

Xをlocally convex Hausdorff空間とし、［P］を次のような条件を満たすX上の連続なseminormの族とする。

X_pをp∈［P］によってassociateされるBanach空間とする。

任意のp∈［P］に対して、連続なHilbert的seminorm qが存在して、次の条件を満たす。

① $p(x) \leqq q(x)$,　$(\forall x \in X)$

② (i, X_q, X_p)はabstract Wiener spaceになる。

［定義］　$\{x \in X ; p(x) \leqq 1, p \in [P]\}$を原点の基本近傍系とする

ようなベクトル位相を　位相Ｍ　という。

（注意）ｐがＨｉｌｂｅｒｔ的で、ｉがＨｉｌｂｅｒｔ－Ｓｃｈｍｉｄｔ ｏｐｅｒａｔｏｒである場合は位相Ｆという。　Ｅ’上に位相Ｆを考えるとＳＳ－位相になることは、既知のことである。

［定理］　Ｅが実可分Ｂａｎａｃｈ空間であるとき、位相ＭはＥ’上のＳＳ－位相である。

証明の概略：ϕをＥ’上で定義された正型で$\phi(0)=1$を満たすような複素数値関数とする。更にϕは位相Ｍに関して連続であると仮定する。このとき$\phi=\hat{\mu}$となるｃｙｌｉｎｄｒｉｃａｌ　ｍｅａｓｕｒｅ　μ　が存在する。このμはまた$\mu=\mu^{T}$となるｌｉｎｅａｒ　ｏｐｅｒａｔｏｒ　Ｔを決定する。Ｔは$E' \to L_0(\Omega, P)$　へ　Ｍ－連続である。Ｎｉｋｉｃｈｉｎの定理（［5］）により　Ｔは$E' \to L_{1/2}(\Omega, P)$へのＭ－連続なｏｐｅｒａｔｏｒと考えることができる。これよりｐ∈［Ｐ］が存在して次がいえる。

$\forall \varepsilon > 0$　に対して　$\exists \delta > 0$,　$p(x') < \delta$ならば

$$\int |Tx'(\omega) - T0(\omega)| \quad dP < \varepsilon$$

が成り立つ。

そこで［Ｐ］の作り方からａｂｓｔｒａｃｔ　Ｗｉｅｎｅｒ　ｓｐａｃｅ　(i, E'_q, E'_p)が得られる。　Ｔは、$j : E' \to E'_q$,　$i : E'_q \to E'_p$,　$\phi : E'_p \to L_{1/2}$, に分解出来る。　更に$\phi \circ i$はＰｉｅｔｓｃｈの定理（［9］）により次のように分解できる。

$U : E'_q \to H$,　$V : H \to L_{1/2}$, ここでＨはヒルベルト空間.

Ｕはヒルベルト空間からヒルベルト空間へのＨｉｌｂｅｒｔ－Ｓｃｈｍｉｄｔ ｏｐｅｒａｔｏｒになるので、そのｄｕａｌ　ｏｐｅｒａｔｏｒ　Ｕ’もＨｉｌｂｅｒｔ－Ｓｃｈｍｉｄｔ　ｏｐｅｒａｔｏｒ　になる。このことを用いてμが測度に拡張できることが示せる。

4．例

ヒルベルト空間上のＳ－位相は前述のτ_{HS}以外に、ｍｅａｓｕｒａｂｌｅ ｓｅｍｉｎｏｒｍをすべて連続にする最弱位相$\tau_{\mathfrak{m}}$がある。この$\tau_{\mathfrak{m}}$はτ_{HS}より強い位相である。そこでτ_{HS}より強く$\tau_{\mathfrak{m}}$より弱いベクトル位相はすべてＳ－位

相になる。τ_{HS}は位相Fでありτ_0でもある。$\tau_{\mathfrak{M}}$は位相Mである。τ_{HS}は最弱S－位相であるが、最強のS－位相または最強のSS－位相が存在するかどうかはこの場合でもわかってはいない。ただ$\tau_{\mathfrak{M}}$より強いS－位相は見つかっていない。

REFERENCES

1. L. Gross, *Measurable functions on Hilbert space*, Trans. Amer. Math. Soc. **105** (1962), 372-390.
2. L. Gross, *Harmonic analysis on Hilbert Space*, Mem. Amer. Math. Soc. **46** (1963).
3. H. H. Kuo, *Gaussian measures in Banach spaces*, Lecture Notes in Math. **463**, Springer-Verlag, New York, 1975.
4. W. Linde, *Probability in Banach spaces - Stable and Infinitely Divisible Distributions*, Wiley, New York, 1983.
5. B. Maurey, *Théoremes de Nikishin : Théoremes de factorisation pour les applications linéaires àvaleursdansunespace*$L^0(\Omega,\mu)$, Sém. Maurey-Schwartz 1972-1973, Exp. 10, 11 et 12, Ecole Polytechnique, (1972).
6. D. H. Mouchtari, *Certain general questions of the theory of probability measures in linear spaces*, Theor. Probab. Appl. **18** (1973), 64-75.
7. D. H. Mouchtari, *Sur l'existence d'une topologie du type de Sazonov sur un espace de Banach*, Sém. Maurey-Schwartz 1975-1976, Exp. 17, Ecole Polytechnique, (1975).
8. Y. Okazaki, *Bochner's theorem on measurable linear functionals of a Gaussian measure*, Ann. Probab. **9** (1981), 663-664.
9. A. Pietsch, *Absolutely p-summing mappings in normed spaces* (in German), Studia Math. **28** (1967), 333-353.
10. V. V. Sazonov, *On characteristic functionals*, Theor. Verojatnost. i Primenen **3**, (1958), 201-205.
11. O. G. Smolyanov and S. V. Fomin, *Measures on linear topological spaces*, Russian Math. Surveys **31** (1976), 1-53. (From Uspekhi Mat. Nauk **31** (1976), 3-56.)

A generalization of Hanner's inequality

by

Aoi Kigami, Yoshiaki Okazaki and Yasuji Takahashi

1. Introduction

In [2], we extended the Hanner's 2-element inequality in L^p to the n-element inequality. However the main result in [2] was restricted to the real valued functions in L^p and the general complex case was left open. In this talk, we show that the n-element version of the Hanner's inequality is also valid for the complex valued L^p-functions. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be the independent Rademacher sequence and $x_1, x_2, \cdots x_n \in L^p$. We prove

$$E \| \sum_{i=1}^{n} \varepsilon_i x_i \|^p \geqq E | \sum_{i=1}^{n} \varepsilon_i \| x_i \| |^p \quad \text{for} \quad 1 \leqq p \leqq 2,$$

$$E \| \sum_{i=1}^{n} \varepsilon_i x_i \|^p \leqq E | \sum_{i=1}^{n} \varepsilon_i \| x_i \| |^p \quad \text{for} \quad 2 \leqq p < \infty .$$

2. Hanner's inequality

Let $1 \leqq p < \infty$, (S, Σ, μ) be a probability space and $L^p = L^p(S, \Sigma, \mu)$. The norm of L^p is given by $\| x \| = (\int | x(t) |^p d \mu (t))^{1/p}$. Hanner [1] proved the following inequalities. Let ε_1 , ε_2 be the independent Rademacher random variables with the distribution $\varepsilon_i = \pm 1$ with probability 1/2. hen the Hanner's inequality is given by

$$E \| \sum_{i=1}^{2} \varepsilon_i x_i \|^p \geqq E | \sum_{i=1}^{2} \varepsilon_i \| x_i \| |^p \quad \text{for} \quad 1 < p \leqq 2,$$

$$E \| \sum_{i=1}^{2} \varepsilon_i x_i \|^p \leqq E | \sum_{i=1}^{2} \varepsilon_i \| x_i \| |^p \quad \text{for} \quad 2 \leqq p < \infty ,$$

where E means the mathematical expectation.

Lemma 1. Let g_1 and g_2 be the independent Gaussian random variables with mean 0 and variance 1 on a probability space (Ω, P). Let $\varphi : C \to L^p(\Omega, P ; \mathbb{R})$ be, for $z = u + iv \in C$, $\varphi(z)(\omega) = c_p(u\, g_1(\omega) + v\, g_2(\omega))$, where $L^p(\Omega, P ; \mathbb{R})$ is the real valued L^p space and $c_p = (\int |g_1(\omega)|^p dP(\omega))^{-1/p}$. Then it hold that

1. φ is real linear, and 2. φ is isometry, $\| \varphi(z) \| = |z|$.

Proof. 1. is clear. To show 2, we calculate the L^p-norm of $\varphi(z)$.

$$\| \varphi(z) \|^p = c_p^p \int |\varphi(ug_1(\omega) + vg_2(\omega))|^p \, dP(\omega)$$

$$= c_p^p (\sqrt{u^2 + v^2})^p \int \left| \frac{u}{\sqrt{u^2 + v^2}} g_1(\omega) + \frac{v}{\sqrt{u^2 + v^2}} g_2(\omega) \right|^p dP(\omega)$$

$$= (\sqrt{u^2 + v^2})^p .$$

We have used the fact that the distributions of $sg_1 + tg_2 (s^2 + t^2 = 1, s,t \in \mathbb{R})$ and g_1 are identical, hence the last integral is c_p^{-p}.

Lemma 2. It holds that for $1 \leqq p \leqq 2$

$$E \mid \sum_{i=1}^{n} \varepsilon_i z_i \mid^p \geqq E \mid \sum_{i=1}^{n} \varepsilon_i |z_i| \mid^p ,$$

and for $2 \leqq p < \infty$, the converse inequality is valid.

Proof. Let φ be the mapping given in Lemma 1. We have

$$E \mid \sum_{i=1}^{n} \varepsilon_i z_i \mid^p = E \| \varphi(\sum_{i=1}^{n} \varepsilon_i z_i) \|^p = E \| \sum_{i=1}^{n} \varepsilon_i \varphi(z_i) \|^p$$

$$\geqq E \mid \sum_{i=1}^{n} \varepsilon_i \| \varphi(z_i) \| \mid^p = E \mid \sum_{i=1}^{n} \varepsilon_i |z_i| \mid^p ,$$

where the above inequality is the Hanner's inequality for the real L^p-functions $\{\varphi(z_i)\}$ (see [2]) and the last equality follows from Lemma 1.

Lemma 3 (Hanner [1]). Let $\alpha \geqq 0$ and $u \geqq 0$. Let $f(u)$ be

$$f(u) = | u^{1/p} + \alpha |^p + | u^{1/p} - \alpha |^p .$$

If $1 \leqq p \leqq 2$, then $f(u)$ is a convex function, and if $2 \leqq p < \infty$, then $f(u)$ is a concave function.

Lemma 4. Let $u_1, u_2, \cdots, u_n \geqq 0$ and let $F(u_1, u_2, \cdots, u_n)$ be

$$F(u_1, u_2, \cdots, u_n) = E \mid \sum_{i=1}^{n} \varepsilon_i u_i^{1/p} \mid^p .$$

Then regarding F as a function of each u_i, F is convex for $1 \leqq p \leqq 2$ and F is concave for $2 \leqq p < \infty$.

Proof. The Lemma follows from Lemma 3. See also Kigami, Okazaki and Takahashi [2].

Theorem 1. Let n be a natural number, $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be independent Rademacher random variables and $x_1, x_2, \cdots, x_n$ be functions in L^p.

(1) If $1 \leqq p \leqq 2$, then it holds that

$$E\| \sum_{i=1}^{n} \varepsilon_i x_i \|^p \geqq E \left| \sum_{i=1}^{n} \varepsilon_i \|x_i\| \right|^p .$$

(2) If $2 \leqq p < \infty$, then it holds that

$$E\| \sum_{i=1}^{n} \varepsilon_i x_i \|^p \leqq E \left| \sum_{i=1}^{n} \varepsilon_i \|x_i\| \right|^p .$$

Proof. (1) Suppose that $1 \leqq p \leqq 2$. By Lemma 2, we have

$$\begin{aligned} E\| \sum_{i=1}^{n} \varepsilon_i x_i \|^p &= E\left(\int_S \left| \sum_{i=1}^{n} \varepsilon_i(\omega) x_i(t) \right|^p d\mu(t) \right) \\ &= \int_S E \left| \sum_{i=1}^{n} \varepsilon_i(\omega) x_i(t) \right|^p d\mu(t) \\ &\geqq \int_S E \left| \sum_{i=1}^{n} \varepsilon_i(\omega) \, | x_i(t) | \right|^p d\mu(t) \\ &= E\| \sum_{i=1}^{n} \varepsilon_i \, | x_i | \, \|^p , \end{aligned}$$

where $|x_i|(t) = |x_i(t)|$. So we can suppose that each x_i is a non-negative function, $x_i(t) \geqq 0$. By Lemma 3 and by the Jensen's inequality, we obtain that

$$\begin{aligned} &\int_S F(x_1(t)^p , x_2(t)^p , \cdots, x_n(t)^p) \, d\mu(t) \\ &\geqq F(\int_S x_1(t)^p d\mu(t), \int_S x_2(t)^p d\mu(t), \cdots , \int_S x_n(t)^p d\mu(t)) , \end{aligned}$$

where F is the function given in Lemma 4. This proves (1).

(2) The case where $2 \leqq p < \infty$ is obtained by the manner same to the case (1). In this case, F is concave and we obtain the converse inequality

$$\begin{aligned} &\int_S F(x_1(t)^p , x_2(t)^p , \cdots, x_n(t)^p) \, d\mu(t) \\ &\leqq F(\int_S x_1(t)^p d\mu(t), \int_S x_2(t)^p d\mu(t), \cdots, \int_S x_n(t)^p d\mu(t)) , \end{aligned}$$

by the Jensen's inequality. This completes the proof.

Remark. In the case where $p = 1$, Hanner's 2-element inequality

$$\| x_1+x_2 \| + \| x_1-x_2 \| \geqq \| x_1 \| + \| x_2 \| + | \| x_1 \| - \| x_2 \| |$$

is nothing but the triangular inequality. So this 2-element inequality is valid in all Banach spaces. But the n-element inequality

$$E \| \sum_{i=1}^{n} \varepsilon_i x_i \|^p \geqq E | \sum_{i=1}^{n} \varepsilon_i \| x_i \| |^p$$

is not necessarily valid in all Banach spaces. If this n-element inequality is valid for every n, then the Banach space is of cotype 2, see [2].

Remark. Hlawka obtained the following inequality. Let x, y, z be the elements in a Hilbert space. Then it holds that

$$||x+y+z|| + ||x|| + ||y|| + ||z|| \geqq ||x+y|| + ||y+z|| + ||z+x||.$$

This inequality is derived from our inequality as follows. Let u, v, w be the elements in L^1 (or in a Banach space imbedded isometrically in L^1) then we have

$$||u+v+w||+||u+v-w||+||u-v+w||+||-u+v+w||$$

$$\geqq ||u||+||v||+||w||+| ||u||+||v||-||w|| | \\ +| ||u||-||v||+||w|| |+| ||-||u||+||v||+||w|| |$$

$$\geqq ||u||+||v||+||w|| + ||u||+||v||+||w|| = 2(||u||+||v||+||w||),$$

where the first inequality is the Hanner's inequality for $p=1$, $n=3$, and the second inequality is the triangular inequality in R. If we put $x= u+v-w$, $y=u-v+w$, $z=-u+v+w$, then we get the Hlawka's inequality. Hlawka's inequality is rewritten as

$$E|| \sum_{j=1}^{3} \varepsilon_j x_j || \geqq 1/2 \sum_{j=1}^{3} ||x_j||$$

for $x_1, x_2, x_3 \in L^1$.

References

[1] O. Hanner, On the uniform convexity of L^p and ℓ^p, Arkiv for Mat. 3(1956), 239-244.

[2] A. Kigami, Y. Okazaki and Y. Takahashi, A generarization of the Hanner's inequality and the type 2(cotype 2) constant of a Banach space, Bulletin of the Kyushu Institute of Technology (Mathematics, Natural Science) No. 42 (March 1995), 29-34.

[3] A. Kigami, Y. Okazaki and Y. Takahashi, A generarization of the Hanner's inequality, Bulletin of the Kyushu Institute of Technology (Mathematics, Natural Science) No. 43 (March 1996), to appear.

Department of Control Engineering and Science
Kyushu Institute of Technology
Kawazu, Iizuka 820, Japan

Department of Control Engineering and Science
Kyushu Institute of Technology
Kawazu, Iizuka 820, Japan

and

Department of System Engineering
Okayama Prefectural University
Kuboki, Soja 719-11, Japan

von Neumann-Jordan constant for Banach spaces of cotype (q,p)

Yasuji Takahashi
Department of System Engineering, Okayama Prefectural University
Kuboki, Soja 719-11, Japan
Mikio Kato
Department of Mathematics, Kyushu Institute of Technology
Tobata, Kitakyushu 804, Japan
Yoshiaki Okazaki
Department of Control Engineering and Science, Kyushu Institute of Technology
Kawazu, Iizuka 820, Japan

Abstract. We denote by $C_{NJ}(X)$ the von Neumann-Jordan (NJ-) constant for a Banach space X and by $\widetilde{C}_{NJ}(x)$ the infimum of all NJ-constants for equivalent norms of X. It is well-known that $1 \leqq C_{NJ}(X) \leqq 2$ for all Banach spaces X; $C_{NJ}(X)=1$ if and only if X is a Hilbert space (Jordan and von Neumann [4]); and $C_{NJ}(L_p) = 2^{2/t-1}$, where $1 \leqq p \leqq \infty$, $1/p + 1/p' = 1$ and $t = \min\{p, p'\}$ (Clarkson[1]). We show that if $\widetilde{C}_{NJ}(X) = 2^{2/t-1}$ with $1 \leqq t \leqq 2$, then X and X′ (dual of X) are of type r for all $r < t$; and in general, the converse is not true. We also show that if X is of cotype (q,p) with $0 < q \leqq p < 2$ and $\widetilde{C}_{NJ}(X) \leqq 2^{2/p-1}$, then $t = \sup\{r ; X$ is of type $r\}$ implies that $\widetilde{C}_{NJ}(X) = 2^{2/t-1}$.

Definitions and notations

Let X be a Banach space and X′ be the (topological) dual of X.

von Neumann-Jordan constant : $C_{NJ}(X)$ is defined as the smallest constant C for which

$$1/C \leqq (\| x+y \|^2 + \| x-y \|^2)/ 2(\| x \|^2 + \| y \|^2) \leqq C$$

hold for all x, y $\in$ X with $(x, y) \neq (0, 0)$. ($C_{NJ}(X)$ is called NJ-constant of X). Denote by $\widetilde{C}_{NJ}(X)$ the infimum of all NJ-constants for equivalent norms of X.

Remark 1. Despite its fundamental nature very little is known about the NJ-constant. Recently, the first and second authors proved that if X is uniformly convex, then $C_{NJ}(X) < 2$; and if $C_{NJ}(X) < 2$, then X is super-reflexive. Thus, X is super-reflexive if and only if $\widetilde{C}_{NJ}(X) < 2$ (see Kato and Takahashi [5]).

Type and cotype inequalities : X is said to be of (Rademacher) type p, $1 \leqq p \leqq 2$, if there exists a constant $M > 0$ such that for all $x_1, x_2, \ldots, x_n$ in X

$$E\| \Sigma_j \ r_j x_j \| \leqq M \ (\Sigma_j \ \| x_j \|^p)^{1/p},$$

where E denotes the expectation, and $r_j = r_j(t)$ are the Rademacher functions ($r_j(t) = \mathrm{sgn}(\sin 2^j \pi t)$, $0 \leqq t \leqq 1$).

X is said to be of (Rademacher) cotype q, $2 \leqq q \leqq \infty$, if there exists a constant $M > 0$ such that for all $x_1, x_2, \ldots, x_n$ in X

$$E\| \Sigma_j \ r_j x_j \| \geqq M^{-1} (\ \Sigma_j \ \| x_j \|^q)^{1/q}.$$

Remark 2. The smallest constant M satisfying the above inequalities is said to be type p (resp. cotype q) constant, and denoted by $T_p(X)$ (resp. $C_q(X)$). It is well-known that if X is of type p, then X' is of cotype q ($1/p + 1/q = 1$), and the converse is also true if X is B-convex. It is also known that every Banach space is of type 1 and cotype ∞. For further informations on type and cotype for Banach spaces, see Maurey and Pisier [7].

Super-property : Let (P) be a property for Banach spaces. X is said to have super-(P) if any Banach space Y finitely representable in X has (P). It is well-known that if X is of type p (resp. cotype q) and Y is finitely representable in

X, then Y is of type p (resp. cotype q); namely type and cotype are the super-properties. On the other hand, reflexivity is not the super-property. X is said to be super-reflexive if any Banach space Y finitely representable in X is reflexive. In general, uniform convexity implies super-reflexivity, and the converse is not true. However, if X is super-reflexive, then it admits an equivalent uniformly convex norm (Enflo [2]).

Banach spaces of cotype (q, p) and the Λ_p-ideal property

Let X and Y be Banach spaces, and denote by L(X, Y) the set of all bounded linear operators from X into Y. An operator T in $L(X', L_p)$, $0 < p \leqq 2$, is said to be a Λ_p-operator ($T \in \Lambda_p(X', L_p)$) if $\exp(- \|Tx'\|^p)$, $x' \in X'$, is a characteristic function (ch.f.) of a Radon measure μ on X. (The measure μ is symmetric and p-stable.)

Λ_p-ideal property : X is said to have the Λ_p-ideal property if for all $T \in \Lambda_p(X', L_p)$ and all $S \in L(L_p, L_p)$ we have $ST \in \Lambda_p(X', L_p)$. It is known that every Banach space has the Λ_p-ideal property for p = 2 or $p \in (0, 1)$. It is also known that the Λ_p-ideal property is a super-property, that is, if X has the Λ_p-ideal property and Y is finitely representable in X, then Y has the Λ_p-ideal property (Linde [6]).

Banach spaces of cotype (q,p) : Let us denote by τ_p the weakest vector topology on X' making all operators $T \in \Lambda_p(X', L_p)$ continuous. X is said to be of M-cotype p, $0 < p \leqq 2$, if any τ_p-continuous cylindrical measure on X is Radon. X is said to be of cotype (q, p), $0 < p, q \leqq 2$, if any τ_p-continuous q-stable cylindrical measure on X is Radon. It is well-known that M-cotype p implies cotype (q,p), and the converse is true if $q < p$. It is also known that cotype (p, p) implies Λ_r-ideal property for all $r \in [p, 2]$ (see Linde [6]).

Main results

Theorem 1 (Kato and Takahashi [5]). A Banach space X is super-reflexive if and only if $\widetilde{C}_{NJ}(X) < 2$.

Theorem 2 (Kato and Takahashi [5]). Let X be a Banach space with $\widetilde{C}_{NJ}(X) = 2^{2/t-1}$, $1 \leqq t \leqq 2$. Then X and X' are of type r for all $r < t$.

Corollary 1. If X is super-reflexive, then X is of type r for some $r > 1$.

Corollary 2. $\widetilde{C}_{NJ}(L_p) = 2^{2/t-1}$, where $1 \leqq p \leqq \infty$ and $t = \min\{p, p'\}$.

Theorem 3. Let X be a Banach space with $\widetilde{C}_{NJ}(X) \leqq 2^{2/p-1}$, $1 \leqq p < 2$. Suppose that X has the Λ_s-ideal property for all $s \in [p, 2]$. Then we have

$$(*) \qquad \sup\{ r \,;\, X \text{ is of type } r\} = 2/(1 + \log_2 \widetilde{C}_{NJ}(X)).$$

Remark 3. Theorem 3 is false if X does not have the Λ_s-ideal property. In fact, there exists a non-reflexive Banach space X of type 2 (James [3]), and then $\widetilde{C}_{NJ}(X) = 2$ by Theorem 1. Since every Banach space has the Λ_2-ideal property, Theorem 3 is valid for $p = 2$ without any additional assumption on X.

Theorem 4. Let X be a Banach space with $\widetilde{C}_{NJ}(X) \leqq 2^{2/p-1}$, $1 \leqq p < 2$. Suppose that X is of cotype (q, p) with $q \leqq p$. Then we have the equality (*) in Theorem 3.

Corollary 3. Suppose that X has the Λ_s-ideal property for all $s \in [1,2]$, or, X is of cotype $(q,1)$ with $q \leqq 1$. Then we have the equality (*) in Theorem 3.

Corollary 5. Suppose that X is isomorphic to a subspace of L_1. Then we have the equality (*) in Theorem 3.

Remark 4. If X is an S-space (Sazonov-space), then it is of cotype (p,p) for all $p \in (0, 2]$ (see Linde [6]), and so we have the equality (*) in Theorem 3. We note that if X is a Banach lattice, then we have the following equality (**) (see Kato and Takahashi [5]).

$$(**) \qquad \sup\{ r \; ; X \text{ and } X' \text{ are of type } r\} = 2/(1 + \log_2 \widetilde{C}_{NJ}(X))$$

References

[1] J. A. Clarkson, The von Neumann-Jordan constant for the Lebesgue spaces, Ann. of Math. 38 (1937), 114-115.

[2] P. Enflo, Banach spaces which can be given an equivalent uniformly convex norm, Israel J. Math. 13 (1972), 281-288.

[3] R. C. James, Nonreflexive spaces of type 2, Israel J. Math. 30 (1978), 1-13.

[4] P. Jordan and J. von Neumann, On inner products in linear metric spaces, Ann. of Math. 36 (1935), 719-723.

[5] M. Kato and Y. Takahashi, On the von Neumann-Jordan constant for Banach spaces, to appear in Proc. Amer. Math. Soc.

[6] W. Linde, Probability in Banach Spaces - Stable and Infinitely Divisible Distributions, John Wiley & Sons, Chichester·New York·Brisbane·Toronto·Singapore, 1986.

[7] B. Maurey and G. Pisier, Series de variables aleatoires vectorielles independantes et proprietes geometriques des espaces de Banach, Studia Math. 58 (1976), 45-90.

Normed partially ordered vector space with Lebesgue property

S. Koshi   Hokkaido Institute of Technology

Abstract

Let E be a normed Riesz space with Lebesgue property. Then E is complete (E is a Banach space). This fact is proved by I.Amemiya and this is a generalization of the Riesz-Fisher's theorem.

We shall consider same problem in more general normed partially ordered vector space and show that Lebesgue property is not sufficient to be complete. The necessary and sufficient condition to be complete will be discussed iln this note.

1 partial order

Let E be a normed vector spaco with real coefficient. We assume that there is a convex cone P with

(1) $\lambda P \subset P$ for every nonnegative scalar $\lambda \geqq 0$

(2) $P + P \subset P$

(3) $P \cap (-P) = \{0\}$

(4) $P - P = E$.

P is called a positive convex cone. A set P satisfying (1), (2), is a conve cone.

We shall define a partial order relation $\geqq$ s.t.

$$x \geqq y \Leftrightarrow x - y \in P \Leftrightarrow y \leqq x.$$

Then the relation "$\geqq$" have the following property.

(a) $x \geqq y$ and $y \geqq x \Rightarrow x = y$.

(b) $x \geqq y,\ y \geqq z \Rightarrow x \geqq z$.

(c) $x \geqq y \Rightarrow x+z \geqq y+z$ for all $z \in E$

(d) $x \geqq y \Rightarrow \lambda x \geqq \lambda y$ for all non-negative scalar λ

(e) $\forall x \ \exists x_1, x_2 \in P$ with $x = x_1 - x_2$

If there is a partial order "$\geqq$" satisfying (a)—(e), then, $P = \{x, \ x \geqq 0\}$ is a positive conve cone, i.e. P satisfies (1) — (4).

We call a normed vector space E partially ordered if there exists a subset P satisfying (1) — (4).

2 Lebesgue property

A norm $\|\cdot\|$ on a partially ordered vector space E is called an order norm if

$$0 \leqq x \leqq y \Rightarrow \|x\| \leqq \|y\|$$

We shall assume that every norm is an order norm in this note.

A norm $\|\cdot\|$ on E is called to have Lebesgue property if

$0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n \leqq \cdots$ and $\sup \|x_n\| < +\infty$,

then there exists upper bound in E in the sense of order i.e. there exists $x = \sup x_n \in E$, where $\sup x_n$ means upper bound for $\{x_1, x_2, \cdots, x_n, \cdots\}$.

Theorem Let E be a normed partially ordered vector space with Lebesgue property. Then, there exists a positive number $A > 0$ such that

for every $0 \leqq a_n \uparrow a$ (i.e. $0 \leqq a_1 \leqq \cdots \leqq a_n \leqq \cdots$ and $a = \sup a_n$)

it follows :

$$\sup_n \|a_n\| \geqq A\|a\|.$$

The last property is called a weak Fatou property.

3 completeness

A norm of a partially ordered vector space E is called well situated if there exists a positive number α such that for every $x \in E$, there exists $x_1, x_2 \in P$ with

$$x = x_1 - x_2$$

and

$$\|x_1\|, \|x_2\| \leq \alpha \|x\|.$$

Theorem Let E be a normed partially ordered vector space in which the positive cone P is closed. We assume that norm of E has Lebesgue property. norm is complete iff norm is well situated.

Now we shall consider when P is not necessary to be closed.

norm $\|\cdot\|$ of E is called weak well situated if there exists a positive number $\alpha > 0$ such that for every $x \in E$ and for every positive number $\varepsilon > 0$, there exist $x_1, x_2 \in P$ with

$$x = x_1 - x_2,$$

$$x = x_1 - x_2 + y.$$

$$\|x_1\|, \|x_2\| \leq \alpha \|x\| \text{ and } \|y\| < \varepsilon$$

for some $y \in E$.

Theorem Let E be a normed partially ordered vector space and norm of E has Lebesgue property.

norm is complete iff norm is weak well situated.

There exist many examples of normed partially ordered vector space with Lebesgue property. but not being well sitated, even if positive cone is closed.

In the case of Riesz space, norm is always well situated if $|x| \leqq |y|$ imply $\|x\| \leqq \|y\|$. Hence, norm is complete if E is a Riesz space and E has Lebesgue property.