



# HOKKAIDO UNIVERSITY

Title	NONLINEAR WAVES Proceedings of the Fourth MSJ International Research Institute Vol II
Author(s)	Agemi, R.; Giga, Y.; Ozawa, T.
Description	Sapporo, July 10-21, 1995
Citation	Hokkaido University technical report series in mathematics, 44, 1
Issue Date	1996-01-01
DOI	<a href="https://doi.org/10.14943/5170">https://doi.org/10.14943/5170</a>
Doc URL	<a href="https://hdl.handle.net/2115/5484">https://hdl.handle.net/2115/5484</a>
Type	departmental bulletin paper
File Information	44.pdf



NONLINEAR WAVES

Proceedings of the Fourth MSJ  
International Research Institute Vol II

Sapporo, July 10-21, 1995

Edited by R. Agemi, Y. Giga and T. Ozawa

Series #44. March, 1996

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NONLINEAR WAVES

Proceedings of the Fourth MSJ International Research Institute

Sapporo

July 10-21, 1995

VOLUME II

Edited by

Rentaro AGEMI

Yoshikazu GIGA

Tohru OZAWA

Supported by

Commemorative Association for the Japan World Exposition (1970)

Hokkaido University

Inoue Foundation for Sciences

Japan Society of Promotion of Sciences

Ministry of Education, Science and Culture, Japan

Sapporo International Communication Plaza Foundation

Suhara Memorial Foundation

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Yamada Science Foundation

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International Research Institute on Nonlinear Waves  
PROGRAMME

- July 10**      Opening address by T.Ozawa, Chief Organizer  
 Congratulatory address by K.Hikichi, Dean of Faculty of Science, Hokkaido University  
 9:10-10:30 J.Ginibre(Orsay)  
                   An Introduction to Nonlinear Schrödinger Equations  
 10:40-12:00 W.Strauss(Brown)  
                   Expository Lectures on Nonlinear Waves
- Session A**    2:00-3:00 M.Flato(Bourgogne)  
                   Nonlinear Relativistic Wave Equations in General Dimensions  
 3:10-4:10 Y.Tsutsumi(University of Tokyo)  
                   Nonlinear Klein-Gordon Equations and Hörmander's Conjectures  
 4:30-5:30 A.Inoue(Tokyo Institute of Technology)  
                   On a Construction of the Fundamental Solution for the Free Weyl  
                   Equation by Hamiltonian Path-Integral Method
- Session B**    2:00-3:00 F.Asakura(Osaka Electro-Communication University)  
                   Global Solutions with a Single Transonic Shock Wave for Quasi-  
                   linear Hyperbolic Systems  
 3:10-4:10 P.T.Kan(Indiana)  
                   Hyperbolic Systems of Conservation Laws with Umbilic  
                   Degeneracy and Two Dimensional Transonic Gas Systems  
 4:30-5:30 Li Tatsien(Fudan)  
                   Global Regularity and Breakdown of Nonlinear Hyperbolic Waves
- July 11**      9:00-10:20 J.Ginibre(Orsay)  
                   An Introduction to Nonlinear Schrödinger Equations  
 10:40-12:00 W.Strauss(Brown)  
                   Expository Lectures on Nonlinear Waves
- Session A**    2:00-2:50 V.Georgiev(Sophia)  
                   Weighted Decay Estimate for the Wave Equation  
 3:00-3:50 W.Dan(University of Tsukuba)  
                   Low-Frequency Asymptotic Expansion for Second Order Elliptic  
                   Equation Systems in two Dimensions  
 4:00-4:50 K.Tsutaya(Hokkaido University)  
                   Global Existence of Small Amplitude Solutions for the Klein-  
                   Gordon-Zakharov Equations  
 5:00-5:50 S.Katayama(Kyoto University)  
                   Global Existence for Nonlinear Wave Equations in Two Space  
                   Dimensions
- Session B**    2:00-2:50 K.Taniguchi(University of Osaka Prefecture)  
                   Gevrey Regularizing Effect for a Nonlinear Schrödinger Equation  
 3:00-3:50 L.Vega(Pais Vasco)  
                   A Semilinear Dirac Equation  
 4:00-4:50 H.Chihara(Kyoto University)  
                   The Initial Value Problem Semilinear Schrödinger Equations  
 5:00-5:50 P.I.Naumkin(Moscow)  
                   Asymptotics for Large Time of Solutions to Nonlinear Schrödinger

- Equation
- July 12** 9:10-10:20 J.Ginibre(Orsay)  
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- 10:40-12:00 W.Strauss(Brown)  
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- Session A** 2:00-2:40 T.Yamazaki(Senshu University)  
Scattering for a Quasilinear Hyperbolic Equation of Kirchhoff Type
- 2:50-3:30 T.Mizumachi(University of Tokyo)  
The Asymptotic Behavior of Solutions to the Kirchhoff Equation with Viscous Damping Term
- 3:40-4:20 T.Matsuyama(Hakodate National College of Technology)  
Global Existence and Energy Decay for a Class of Quasilinear Wave Equations with Linear Damping Terms
- 4:30-5:10 H.Uesaka(Nihon University)  
An Oscillation Property of Solutions of Nonlinear Wave Equations
- 5:20-6:00 K.Furuya(Ochanomizu University)  
Mixed Problem for Wave Equations with Oblique Boundary Condition
- Session B** 2:00-2:50 J.C.Saut(Orsay)  
On the Solitary Waves of Kadomtsev-Petviashvili Type Equations
- 3:00-3:50 Y.Shibata(University of Tsukuba)  
Global in Time Existence of Small Solutions of Nonlinear Thermo-viscoelastic Equations
- 4:00-4:50 C.M.Dafermos(Brown)  
Hyperbolic Systems of Conservation Laws in Several Space Dimensions
- 5:00-5:50 K.Kato(Osaka University)  
Interaction of Analytic Singularities for Semilinear Wave Equations
- July 13** 9:00-9:50 J.Hoppe(ETH)  
Conservation Laws and Formation of Singularities in Relativistic Theories of Extended Objects
- 10:00-10:50 G.Velo(Universita di Bologna)  
Generalized Strichartz Inequalities for the Wave Equation in Besov Spaces
- 11:10-12:00 N.Hayashi(Science University of Tokyo)  
On the Davey-Stewartson and the Ishimori Systems
- Session A** 2:00-2:40 K.Hidano(Waseda University)  
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- 2:50-3:30 R.Ikehata(Tokyo National College of Technology)  
The Global Behavior of Weak Solutions to Some Nonlinear Wave Equations
- 3:40-4:20 H.Kubo(Hokkaido University)  
Asymptotic Behaviors of Radially Symmetric Solutions of  $\square u = |u|^p$  for Super Critical Values  $p$  in High Dimensions
- 4:30-5:10 H.Takamura(University of Tsukuba)  
Nonexistence of Global Solutions to Semilinear Wave Equations

- 5:20-6:00 T.Motai(Tokyo University of Foreign Studies)  
Asymptotics of Energy for Wave Equations with a Nonlinear  
Dissipative Term in  $R^N$
- Session B 2:00-2:30 Zhang Jian(University of Tokyo)  
Finite-time Behaviour for Nonlinear Schrödinger Equations
- 2:50-3:40 C.K.Lin(National Cheng Kung University)  
Incompressible Limit of the Quantum Hydrodynamic Equations
- 4:00-4:50 F.Linares(IMPA)  
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- 5:00-5:50 G.Ponce(UCSB)  
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- July 14 9:00-9:50 T.Suzuki(Osaka University)  
 $-\Delta u = \lambda \exp(\frac{\alpha u}{\alpha+u})$  v.s.  $-\Delta u = \lambda e^u$
- 10:00-10:50 C.S.Lin(Chung Cheng University)  
Estimates of the Conformal Scalar Curvature Equation Via the  
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- 11:10-12:00 H.Matano(University of Tokyo)  
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Cylinder with Periodic Wiggles
- Session A 2:00-2:40 M.Ohta(University of Tokyo)  
Stability of Standing Waves for Coupled Nonlinear Klein-Gordon  
Equations
- 2:50-3:30 T.Shibata(Aichi Prefectural University)  
Two-Parameter Nonlinear Eigenvalue Problems
- 3:40-4:20 V.Georgiev(Sophia)  
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- 4:30-5:10 P.Aviles(Universidad Católica at Santiago)  
Semilinear Wave Equations
- 5:20-6:00 K.Moriyama(University of Tokyo)  
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Cubic Nonlinear Klein-Gordon Equations in One Space Dimension
- Session B 2:00-2:40 M.Kuwamura(Wakayama University)  
The Stability of Roll Solutions of the 2-D Swift-Hopenberg  
Equation and the Phase Diffusion Equation
- 2:50-3:30 K.M.Hui(Academia Sinica)  
Singular Limit of Solutions of Some Degenerate Parabolic  
Equation
- 3:40-4:20 C.Dohmen(University of Bonn)  
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- 4:30-5:10 T.Tassa(UCLA)  
On the Homogenization of Oscillatory Solutions to Nonlinear  
Convection-Diffusion Equations
- 5:20-6:00 S.T.Kusimov(Ufa State Aviation Technical University)  
The Phase Function of the Wave-front of Coherent Optical  
Radiation Determination: Modern Methods and Means
- July 15 Open Symposium for Citizens at Sapporo International Communication  
Plaza Foundation - Frontiers of Mathematical Sciences -  
M.Tsutsumi(Waseda University)

- Nonlinearity and Wave  
A.Matsumura(Osaka University)  
Let's Enjoy PDE on Personal Computer  
C.Dafermos(Brown)  
Exploration in Mathematics
- July 18** 9:00-10:20 T.Miyakawa(Kyushu University)  
On Decay Properties of Exterior Stationary Navier-Stokes Flows  
10:40-12:00 T.Nishida(Kyoto University)  
Fluid Dynamical Equations with a Free Surface and Bifurcation Problems
- Session A** 2:00-2:40 H.Kozono(Nagoya University)  
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2:50-3:30 Y.Shibata(University of Tsukuba)  
On an Exterior Initial Boundary Value Problem for Navier-Stokes Equations  
3:40-4:20 P.D'Ancona(L'Aquila)  
On Degenerate Parabolic Systems  
4:30-5:10 M.Mei(Kanazawa University)  
Nonlinear Stability of Viscous Shock Profile for a Non-Convex System of Viscoelasticity  
5:20-6:00 M.Yamazaki(Hitotsubashi University)  
The Navier-Stokes Exterior Problem in the Space  $L^{n,\infty}$
- Session B** 2:00-2:40 N.Koiso(Osaka University)  
Vortex Filament Equation and Semilinear Schrödinger Equation  
2:50-3:30 T.Yoshinaga(Osaka University)  
Solitary Wave Solutions and Integrability of the Long and Short Wave Interaction Equation  
3:40-4:20 Y.Fukumoto(Nagoya University)  
Stationary Configurations of a Vortex Filament in Background
- July 19** 9:00-10:20 T.Miyakawa(Kyushu University)  
On Decay Properties of Exterior Stationary Navier-Stokes Flows  
10:40-12:00 T.Nishida(Kyoto University)  
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- Session A** 2:00-2:50 M.Struwe(ETH)  
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2:50-3:30 A.Tachikawa(Shizuoka University)  
Weak Solutions for the Evolution Problems of Harmonic Maps on Nondecreasing Domains  
3:40-4:20 F.Nakajima(Iwate University)  
Nonlinear Mathieu's Equations  
4:30-5:10 Hi Jun Choe(POSTECH)  
Removable Singularity for Stationary Navier-Stokes System  
5:20-6:00 G.Weiss(University of Bonn)  
Partial Regularity for a Minimum Problem with Free Boundary
- Session B** 2:00-2:50 A.S.Fokas(Loughborough)  
A New Class of Physically Important Integrable Equations

- 3:00-3:50 F.Calogero(Roma)  
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- 4:00-4:50 K.Hayata(Hokkaido University)  
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- July 20** 9:00-10:20 T.Miyakawa(Kyushu University)  
On Decay Properties of Exterior Stationary Navier-Stokes Flows
- 10:40-12:00 T.Nishida(Kyoto University)  
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- Session A** 2:00-2:40 T.Nakaki(Hiroshima University)  
On Numerical Computations to the Oil-Reservoir Problems in Two-Dimensional Rectangular Regions
- 2:50-3:30 I.A.Shishmarev(Moscow)  
The Asymptotic Representation of the Surface Waves
- 3:40-4:20 N.Ishimura(University of Tokyo)  
A Note on Uniqueness for the Classical Solution of the MHD Equations in  $R^3$
- 4:30-5:10 H.Morimoto(Meiji University)  
A Remark on the Existence of the Navier-Stokes Flow with Non-Vanishing Outflow Condition
- 5:20-6:00 S.Takahashi(Tokyo Denki University)  
On Decay Rate Estimates for the Navier-Stokes Equations
- Session B** 2:00-2:30 K.Matsuuchi(University of Tsukuba)  
Instability of Pipe-Poiseuille Flow and the Global Nature of Disturbances
- 2:40-3:10 Y.Mizuta(Hokkaido University)  
Analysis on Large Free Surface Deformation of Magnetic Fluid
- 3:20-3:50 D.Summers(Kyoto University)  
Nonlinear Waves and Chaos in the Pierce Beam-Plasma System
- 4:00-4:30 G.A.Barsegian(Academy of Sciences of Armenia)  
The Theory of  $G$ -Lines and Its Applications in Physics
- 4:40-5:10 S.Makhmutov(Hokkaido University)  
Estimates of Spherical Derivative
- July 21** 9:00-9:50 S.Klainerman(Princeton)  
On the Regularity Properties of Nonlinear Wave Equations
- 10:10-11:00 M.Tsutsumi(Waseda)  
Regional Blow-up of Solutions to the Initial-Boundary Value Problem
- 11:10-12:00 H.Okamoto(Kyoto University)  
New Family of Exact Solutions of the Navier-Stokes Equations
- Closing address by Y. Giga, Organizer

# GLOBAL REGULARITY AND BREAKDOWN OF NONLINEAR HYPERBOLIC WAVES

Li Ta-tsien

**Abstract:** The global existence and the life-span of  $C^1$  solutions to the Cauchy problem for general first order quasilinear hyperbolic systems with small decay initial data are considered and some applications with physical interest are given.

## 1. Introduction

Consider the following Cauchy problem for general first order quasilinear hyperbolic systems

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0, \quad (1.1)$$

$$t = 0 : u = \phi(x), \quad (1.2)$$

where  $u = (u_1, \dots, u_n)^T$  is the unknown vector function of  $(t, x)$ ,  $A(u) = (a_{ij}(u))$  is an  $n \times n$  matrix with suitably smooth entries  $a_{ij}(u)$  ( $i, j = 1, \dots, n$ ), and  $\phi(x)$  is a  $C^1$  vector function with bounded  $C^1$  norm.

By hyperbolicity, for any given  $u$  on the domain under consideration,  $A(u)$  has  $n$  real eigenvalues  $\lambda_1(u), \dots, \lambda_n(u)$  and a complete set of left (resp. right) eigenvectors  $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$  (resp.  $r_i(u) = (r_{i1}(u), \dots, r_{in}(u))^T$ ) ( $i = 1, \dots, n$ ):

$$l_i(u) A(u) = \lambda_i(u) l_i(u) \quad (\text{resp. } A(u) r_i(u) = \lambda_i(u) r_i(u)). \quad (1.3)$$

Our aim is to study the following two kinds of problems which are of great importance in the propagation of nonlinear waves:

(1) Under what conditions does Cauchy problem (1.1)-(1.2) admit a unique global  $C^1$  solution  $u = u(t, x)$  on  $t \geq 0$ ?

(2) Under what conditions does the  $C^1$  solution  $u = u(t, x)$  to Cauchy problem (1.1)-(1.2) blow up in a finite time? Can we get a sharp estimate on the life-span of  $C^1$  solutions?

When  $n = 1$  or  $2$ , through the efforts of many authors these two problems have been almost completely solved (see Li Ta-t sien [6]).

For the general quasilinear hyperbolic system of  $n (\geq 1)$  equations, the first result was given by F. John [5]. Suppose that in a neighbourhood of  $u = 0$ ,  $A(u) \in C^2$ , system (1.1) is strictly hyperbolic:

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) \tag{1.4}$$

and genuinely nonlinear in the sense of P.D. Lax: for  $i = 1, 2, \dots, n$ ,

$$\nabla \lambda_i(u) r_i(u) \neq 0. \tag{1.5}$$

Suppose furthermore that  $\phi(x) \in C^2$  have a compact support:

$$\text{supp } \phi \subseteq [\alpha_0, \beta_0]. \tag{1.6}$$

F. John proved that if

$$\theta \triangleq (\beta_0 - \alpha_0)^2 \cdot \sup_{x \in \mathbb{R}} |\phi''(x)| \tag{1.7}$$

is small enough, then the first order derivatives of the  $C^2$  solution  $u = u(t, x)$  to Cauchy problem (1.1)-(1.2) must blow up in a finite time.

L. Hörmander [4] reproved F. John's result and gave a sharp estimate on the life-span of  $C^2$  solutions.

T. P. Liu [12] generalized F. John's result to the case that in a neighbourhood of  $u = 0$ , a part of characteristics is genuinely nonlinear, while the other part of characteristics is linearly degenerate in the sense of P.D. Lax. Precisely speaking, let  $J \subseteq \{1, 2, \dots, n\}$  be a nonempty set such that  $\lambda_i(u)$  is genuinely nonlinear if and only if  $i \in J$ . Suppose that for  $i \notin J$ , in a neighbourhood of  $u = 0$ ,  $\lambda_i(u)$  is linearly degenerate:

$$\nabla \lambda_i(u) r_i(u) \equiv 0. \tag{1.8}$$

T.P.Liu proved the same result as in F.John [5] for a quite large class of initial data, however, in his proof he imposed an additional hypothesis “linear waves do not generate nonlinear waves”. To illustrate this hypothesis, using the expansion of  $u_x$  with respect to the right eigenvectors:

$$u_x = \sum_{k=1}^n w_k r_k(u), \quad (1.9)$$

we have

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k \quad (i = 1, \dots, n), \quad (1.10)$$

where  $\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x}$  denotes the directional derivative along the  $i$ -th characteristic and

$$\gamma_{ijk}(u) = (\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla \lambda_k(u) r_j(u) \delta_{ik}. \quad (1.11)$$

The hypothesis “linear waves do not generate nonlinear waves” means that in a neighbourhood of  $u = 0$

$$\gamma_{ijk}(u) \equiv 0, \quad \forall i \in J, \quad \forall j, k \notin J. \quad (1.12)$$

T.P.Liu’s result can be applied to the system of one-dimensional gas dynamics with convexity.

By means of the concept of weak linear degeneracy, Li Ta-tsien, Zhou Yi & Kong De-xing [9] presented a complete result on the global existence and the life-span of  $C^1$  solutions to Cauchy problem (1.1)-(1.2) is which system (1.1) is strictly hyperbolic and  $\phi(x)$  is a small  $C^1$  vector function with compact support. In this talk, as a joint work with Zhou Yi and Kong De-xing, the result in [9] will be generalized to the case that system (1.1) might be non-strictly hyperbolic and  $\phi(x)$  is a small  $C^1$  vector function satisfying certain decay properties as  $|x| \rightarrow +\infty$  (also see Li Ta-tsien, Zhou Yi and Kong De-xing [10,11]). Moreover, these results will be applied to the system of the motion of an elastic string and the system of plane elastic waves for hyperelastic materials. This application answers the open problem “Investigate shock formation in non-genuinely nonlinear systems for initial data of compact support” proposed in A.Majda [13].  $\square$

## 2. Main results

Suppose that in a neighbourhood of  $u = 0$ , each eigenvalue of  $A(u)$  has a constant multiplicity. Without loss of generality, we suppose that

$$\lambda(u) \triangleq \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u). \quad (2.1)$$

When  $p = 1$ , system (1.1) is strictly hyperbolic; while, when  $p > 1$ , (1.1) is a non-strictly hyperbolic system. In the case  $p > 1$ , for the sake of simplicity we only consider system (1.1) with  $A = \nabla f$ , i.e., suppose that (1.1) is a system of conservation laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad (2.2)$$

where  $f(u) = (f_1(u), \cdots, f_n(u))^T$ . Thus, by G.Boillat [1] and H.Freistühler [3], the characteristic  $\lambda(u)$  with constant multiplicity  $p (> 1)$  must be linearly degenerate:

$$\nabla \lambda(u) r_i(u) \equiv 0 \quad (i = 1, 2, \cdots, p). \quad \square \quad (2.3)$$

As defined in Li Ta-tsien, Zhou Yi & Kong De-xing [9], each simple eigenvalue  $\lambda_i(u)$  of  $A(u)$  is called to be weakly linearly degenerate, if along the  $i$ -th characteristic trajectory  $u = u^{(i)}(s)$  passing through  $u = 0$ , defined by

$$\begin{cases} \frac{du}{ds} = r_i(u), \\ s = 0 : u = 0, \end{cases} \quad (2.4)$$

we have

$$\nabla \lambda_i(u) r_i(u) \equiv 0, \quad \forall |u| \text{ small}, \quad (2.5)$$

namely,

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(0), \quad \forall |s| \text{ small}. \quad (2.6)$$

Obviously, if  $\lambda_i$  is linearly degenerate, then  $\lambda_i$  is weakly linearly degenerate. On the other hand, if  $\lambda_i(u)$  is not weakly linearly degenerate, then either there exists an integer  $\alpha_i \geq 0$  such that

$$\frac{d^l \lambda_i(u^{(i)}(s))}{ds^l} \Big|_{s=0} = 0 \quad (l = 1, \cdots, \alpha_i) \quad \text{but} \quad \frac{d^{\alpha_i+1} \lambda_i(u^{(i)}(s))}{ds^{\alpha_i+1}} \Big|_{s=0} \neq 0 \quad (2.7)$$

or

$$\frac{d^l \lambda_i(u^{(i)}(s))}{ds^l} \Big|_{s=0} = 0 \quad (l = 1, 2, \cdots), \quad \text{denoted by } \alpha_i = +\infty. \quad \square \quad (2.8)$$

**Theorem 2.1 (Global existence).** *Under the assumptions mentioned at the beginning of this paragraph, suppose that in a neighbourhood of  $u = 0$ ,  $A(u) \in C^2$  and all simple eigenvalues of  $A(u)$  are weakly linearly degenerate. Suppose furthermore that  $\phi(x)$  is a  $C^1$  function satisfying the following decay property: there exists a constant  $\mu > 0$  such that*

$$\theta \triangleq \sup_{x \in \mathbf{R}} \left\{ (1 + |x|)^{1+\mu} (|\phi(x)| + |\phi'(x)|) \right\} < \infty. \quad (2.9)$$

Then there exists  $\theta_0 > 0$  so small that for any given  $\theta \in [0, \theta_0]$ , Cauchy problem (1.1)-(1.2) admits a unique global  $C^1$  solution  $u = u(t, x)$  on  $t \geq 0$ .  $\square$

**Theorem 2.2 (Blow up phenomenon).** *Under the assumptions mentioned at the beginning of this paragraph, suppose that in a neighbourhood of  $u = 0$ ,  $A(u)$  is suitably smooth and there exists a nonempty index set  $J \subseteq \{1, 2, \dots, n\}$  such that a simple eigenvalue  $\lambda_i(u)$  is not weakly linearly degenerate if and only if  $i \in J$ . Suppose furthermore that*

$$\alpha = \min \{ \alpha_i, i \in J \} < \infty, \quad (2.10)$$

where  $\alpha_i$  is defined by (2.7)-(2.8). Suppose finally that  $\phi(x) = \varepsilon \psi(x)$ , where  $\varepsilon > 0$  is a small parameter and  $\psi(x)$  is a  $C^1$  vector function such that

$$\sup_{x \in \mathbf{R}} \left\{ (1 + |x|)^{1+\mu} (|\psi(x)| + |\psi'(x)|) \right\} < \infty, \quad (2.11)$$

where  $\mu$  is a positive constant. Let

$$J_1 = \{ i \mid i \in J, \alpha_i = \alpha \}. \quad (2.12)$$

If there exists  $k \in J_1$  such that

$$l_k(0) \psi(x) \not\equiv 0, \quad (2.13)$$

then there exists  $\varepsilon_0 > 0$  so small that for any fixed  $\varepsilon \in (0, \varepsilon_0]$ , the first order derivatives of the  $C^1$  solution  $u = u(t, x)$  to Cauchy problem (1.1)-(1.2) must blow up in a finite time, and there exist two positive constants  $c$  and  $C$  independent of  $\varepsilon$ , such that the life-span  $\tilde{T}(\varepsilon)$  of  $u = u(t, x)$  satisfies

$$c\varepsilon^{-(1+\alpha)} \leq \tilde{T}(\varepsilon) \leq C\varepsilon^{-(1+\alpha)}, \quad (2.14)$$

denoted by

$$\tilde{T}(\varepsilon) \approx \varepsilon^{-(1+\alpha)}. \quad \square \tag{2.15}$$

### 3. Applications

#### 3.1 System of the motion of an elastic string

Consider the Cauchy problem for the system of the motion of an elastic string (see [7]-[8]):

$$\begin{cases} u_t - v_x = 0, \\ v_t - \left(\frac{T(r)}{r}u\right)_x = 0, \end{cases} \tag{3.1}$$

$$t = 0 : u = \tilde{u}^0 + \varepsilon u^0(x), v = \varepsilon v^0(x), \tag{3.2}$$

where  $u = (u_1, \dots, u_n)^T$ ,  $v = (v_1, \dots, v_n)^T$ ,  $r = |u| = \sqrt{u_1^2 + \dots + u_n^2}$  ( $n \geq 2$ ),  $T(r)$  is a suitably smooth function of  $r > 0$  such that

$$T'(\tilde{r}_0) > \frac{T(\tilde{r}_0)}{\tilde{r}_0} > 0, \tag{3.3}$$

where  $\tilde{r}_0 = |\tilde{u}^0| = \sqrt{(\tilde{u}_1^0)^2 + \dots + (\tilde{u}_n^0)^2} > 1$ ,  $\tilde{u}^0 = (\tilde{u}_1^0, \dots, \tilde{u}_n^0)$  is a constant vector and  $(u^0(x), v^0(x)) \in C^1$  satisfies the same decay property as (2.11).

In a neighbourhood of  $(u, v) = (\tilde{u}^0, 0)$ , (3.1) is a hyperbolic system with the real eigenvalues

$$\lambda_1 \triangleq -\sqrt{T'(r)} < \lambda_2 \triangleq -\sqrt{\frac{T(r)}{r}} < \lambda_3 \triangleq \sqrt{\frac{T(r)}{r}} < \lambda_4 \triangleq \sqrt{T'(r)}, \tag{3.4}$$

where both  $\lambda_2$  and  $\lambda_3$  are  $(n - 1)$  multiple eigenvalues which are linearly degenerate, while both  $\lambda_1$  and  $\lambda_4$  are single eigenvalues which are neither genuinely nonlinear nor linear degenerate in general. Except  $n = 2$ , (3.1) is a non-strictly hyperbolic system of conservation laws with constant multiple eigenvalues. Moreover, T.P.Liu's hypothesis "linear waves do not generate nonlinear waves" is not satisfied.

By Theorem 2.2, we get

**Theorem 3.1.** *Suppose that there exists an integer  $\alpha \geq 0$  such that*

$$T''(\tilde{r}_0) = \dots = T^{(1+\alpha)}(\tilde{r}_0) = 0 \quad \text{but} \quad T^{(2+\alpha)}(\tilde{r}_0) \neq 0. \tag{3.5}$$

If  $\left( \sum_{i=1}^n \tilde{u}_i^0 u_i^0(x), \sum_{i=1}^n \tilde{v}_i^0 v_i^0(x) \right)$  is not identically equal to zero, namely,  $\tilde{u}^0$  is not always simultaneously orthogonal to  $u^0(x)$  and  $v^0(x)$  for all  $x \in \mathbf{R}$ , then there exists  $\varepsilon_0 > 0$  so small that for any  $\varepsilon \in (0, \varepsilon_0]$ , the first order derivatives of the  $C^1$  solution  $(u, v) = (u(t, x), v(t, x))$  to Cauchy problem (3.1)-(3.2) must blow up in a finite time and the life-span

$$\tilde{T}(\varepsilon) \approx \varepsilon^{-(1+\alpha)}. \quad \square \quad (3.6)$$

### 3.2 System of plane elastic waves for hyperelastic materials

For any given  $\omega = (\omega_1, \omega_2, \omega_3)$  with  $|\omega| = 1$ , the solution of plane elastic waves can be written as

$$Y = \pi X + f(t, x), \quad (3.7)$$

where  $X = (X_1, X_2, X_3)^T$ ,  $Y = (Y_1, Y_2, Y_3)^T$ ,  $f = (f_1, f_2, f_3)^T$ ,  $x = \omega X$  and  $\pi$  is a nonsingular matrix of order 3. Without loss of generality, we may suppose that  $\pi = I$ .

$f$  satisfies the following system (see [5])

$$\frac{\partial^2 f}{\partial t^2} - V''(f_x) \frac{\partial^2 f}{\partial x^2} = 0, \quad (3.8)$$

where  $V'' = (V_{ij})$  with

$$V_{ij}(\eta) = V_{ji}(\eta) = \frac{\partial^2 V(\eta)}{\partial \eta_i \partial \eta_j} \quad (i, j = 1, 2, 3) \quad (3.9)$$

and

$$V(\eta) = W(I + \eta\omega) \quad (3.10)$$

for any given  $\eta = (\eta_1, \eta_2, \eta_3)^T$ , where  $W = W(P)$  is the stored energy function, in which  $P = (p_{ik}) = \left( \frac{\partial Y_i}{\partial X_k} \right)$  denotes the strain tensor.

Let

$$u_i = \frac{\partial f_i}{\partial x}, \quad u_{i+3} = \frac{\partial f_i}{\partial t} \quad (i = 1, 2, 3). \quad (3.11)$$

From (3.8) we get

$$u_t + A(u)u_x = 0, \quad (3.12)$$

where  $u = (u_1, \dots, u_6)^T$  and

$$A = \begin{pmatrix} 0 & -I \\ -V'' & 0 \end{pmatrix}. \quad \square \quad (3.13)$$

As an example, we consider the material of Ciarlet-Geymonat (see P.G.Ciarlet [2]):

$$W(p) = a \|P\|^2 + b \|\text{Cof}P\|^2 + \Gamma(\det P) + e, \quad (3.14)$$

where

$$\Gamma(\delta) = c\delta^2 - d \log \delta, \quad \forall \delta > 0, \quad (3.15)$$

$a, b, c, d$  are positive constants,  $e$  is a real number,

$$\|P\| = (\text{tr } P^T P)^{\frac{1}{2}} \quad (3.16)$$

and

$$\text{Cof}P = (\det P) (P^{-1})^T. \quad (3.17)$$

The system has six real eigenvalues:

$$\lambda_1 \triangleq -\lambda < \lambda_{2,3} \triangleq -\lambda_0 < \lambda_{4,5} \triangleq \lambda_0 < \lambda_6 \triangleq \lambda, \quad (3.18)$$

where

$$\lambda = \sqrt{2(a+b) + 2(b+c) + d(1+\omega u)^{-2}}, \quad \lambda_0 = \sqrt{2(a+b)}. \quad (3.19)$$

$\lambda_{2,3}$  and  $\lambda_{4,5}$  are linearly degenerate eigenvalues with constant multiplicity, while  $\lambda_1$  and  $\lambda_6$  are genuinely nonlinear.

Consider the Cauchy problem for system (3.8) with the initial data

$$t = 0: f = \tilde{f}^0 + \varepsilon \phi(x), \quad f_t = \varepsilon \psi(x), \quad (3.20)$$

where  $\tilde{f}^0 = (\tilde{f}_1^0, \tilde{f}_2^0, \tilde{f}_3^0)$  is a constant vector,  $\varepsilon > 0$  is a small parameter,  $\phi(x) \in C^2$ ,  $\psi(x) \in C^1$  and  $(\phi'(x), \psi(x))$  satisfies the same decay property as (2.11). By Theorem 2.2 we have

**Theorem 3.2.** *If  $(\omega \phi'(x), \omega \psi(x))$  is not identically equal to zero, namely,  $\omega$  is not always simultaneously orthogonal to  $\phi'(x)$  and  $\psi(x)$  for all  $x \in \mathbb{R}$ , then there exists  $\varepsilon_0 > 0$  so small that for any  $\varepsilon \in (0, \varepsilon_0]$ , the second order derivatives of the  $C^2$  solution  $f = f(t, x)$  to Cauchy problem (3.8) and (3.20) must blow up in a finite time and the life-span*

$$\tilde{T}(\varepsilon) \approx \varepsilon^{-1}. \quad \square$$

For some other classical hyperelastic materials such as the neo-Hookean material

$$W(P) = a \|P\|^2 + \Gamma(\det P) \quad (a > 0), \quad (3.21)$$

the material of Burgess and Levinson

$$W(P) = \frac{1}{2} \|P\|^2 + \frac{1}{\sigma} (\det P)^{-\sigma} \quad (\sigma > 0), \quad (3.22)$$

the material of Hadamard-Green

$$W(P) = \frac{\alpha}{2} \|P\|^2 + \frac{\beta}{4} (\|P\|^4 - \|PP^T\|^2) + \Gamma(\det P) \quad (\alpha, \beta > 0) \quad (3.23)$$

etc. (see [2]), the conclusion of Theorem 3.3 still holds.  $\square$

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ON THE INCOMPRESSIBLE LIMIT  
OF  
THE SLIGHTLY COMPRESSIBLE VISCOUS FLUID FLOWS

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**Abstract:** In this article we study the weak solutions of the compressible viscous fluid dynamics equations and its incompressible limit. Using the concept of entropy we prove the singular limit system for the viscous compressible fluid dynamics equations is a system resembling the incompressible nonhomogeneous viscous fluid equations.

### §1 Preliminaries

The nondimensional unsteady anisentropic viscous flow of a polytropic gas neglecting the effect of conduction and radiation is represented by the following equations

$$\epsilon \partial_t \rho_\epsilon + \nabla \cdot \boldsymbol{\mu}_\epsilon = 0 \quad (1.1)$$

$$\epsilon \partial_t \boldsymbol{\mu}_\epsilon + \nabla \cdot \left( \frac{\boldsymbol{\mu}_\epsilon \otimes \boldsymbol{\mu}_\epsilon}{\rho_\epsilon} \right) + \nabla P(\rho_\epsilon, S_\epsilon) = \epsilon \nabla \cdot (\nu \sum_\epsilon) \quad (1.2)$$

$$\epsilon \partial_t S_\epsilon + \frac{\boldsymbol{\mu}_\epsilon}{\rho_\epsilon} \cdot \nabla S_\epsilon = 0, \quad (1.3)$$

where

$$\sum_\epsilon \equiv \left( \nabla \frac{\boldsymbol{\mu}_\epsilon}{\rho_\epsilon} \right) + \left( \nabla \frac{\boldsymbol{\mu}_\epsilon}{\rho_\epsilon} \right)^t - \frac{2}{D} \left( \nabla \cdot \frac{\boldsymbol{\mu}_\epsilon}{\rho_\epsilon} \right) \mathbf{I}. \quad (1.4)$$

The equation of state is given by

$$P(\rho, S) = [\rho \Phi'(\rho) - \Phi(\rho)] e^{A(S)}. \quad (1.5)$$

which will be reduced to the barotropic gas when the exponent  $A$  is a constant, i.e.,  $A'(S) = 0$ . The parameter  $\epsilon$  is definitely related to the Mach number. Here,  $\rho_\epsilon$ ,  $\boldsymbol{\mu}_\epsilon$ ,  $S_\epsilon$ ,  $P_\epsilon$  represent the nondimensional density, momentum, specific entropy, and pressure of the

gas, respectively. To avoid complications at the boundary, we concentrate below on the case where  $\mathbf{x} \in \mathbb{T}^D$ , the  $D$ -dimensional torus.

It is obvious that  $(1, 0, 1)$  is a trivial solution of (1.1) – (1.5). We consider the perturbation near  $(1, 0, 1)$ , i.e., looking at data near the equilibrium  $(\rho, \boldsymbol{\mu}, S) = (1, 0, 1)$ . Let

$$\rho_\epsilon = 1 + \epsilon^2 \tilde{\rho}_\epsilon, \quad \boldsymbol{\mu}_\epsilon = \epsilon \tilde{\boldsymbol{\mu}}_\epsilon, \quad S_\epsilon = 1 + \epsilon^2 \tilde{S}_\epsilon. \quad (1.6)$$

where  $\tilde{\rho}_\epsilon, \tilde{\boldsymbol{\mu}}_\epsilon, \tilde{S}_\epsilon$  play the roles of the density fluctuation, velocity fluctuation and entropy fluctuation, respectively. Then the formal asymptotic expansion suggests that formally letting  $\epsilon \rightarrow 0$  in (1.1) – (1.5) produces the incompressible stratified fluid equations given below

$$\nabla \cdot \mathbf{v} = 0, \quad \Pi = \Pi(\rho) \quad (1.7)$$

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] + \nabla \Pi = \nu \Delta \mathbf{v} \quad (1.8)$$

$$\partial_t \rho + (\mathbf{v} \cdot \nabla) \rho = 0. \quad (1.9)$$

Note that equation (1.9) follows from  $\partial_t S + \frac{\boldsymbol{\mu}}{\rho} \nabla S = 0$ , since  $\rho = \rho(S)$  in the limit  $\epsilon \rightarrow 0$ .

## §2 Entropy and energy estimate

For our system, a natural entropy is given by the mechanical energy function

$$H(\rho, \boldsymbol{\mu}, S) = \frac{1}{2} \frac{|\boldsymbol{\mu}|^2}{\rho} + U(\rho, S) \equiv \frac{1}{2} \frac{|\boldsymbol{\mu}|^2}{\rho} + \Phi(\rho) e^{A(S)} \quad (2.1)$$

which will be shown to be convex. We assume

$$\partial_\rho P(\rho, S) = \rho \partial_{\rho\rho} U(\rho, S) = \rho \Phi''(\rho) e^{A(S)} > 0, \quad (2.2)$$

i.e., no phase transition, then we have

**(2.1) Lemma** *For strict convex functions  $\Phi(\rho) = \rho^\gamma$  and  $A(S)$  with  $A''(S) \geq \frac{\gamma}{\gamma-1} (A'(S))^2$  then the mechanical energy function  $H(\rho, \boldsymbol{\mu}, S)$  defined by (2.1) is a strict convex entropy of (1.1) – (1.5).*

**Proof :** It follows immediately the fact that the Hessian matrix of  $H$  is positive definite.

Furthermore, let

$$V(t) = (\rho(t), \boldsymbol{\mu}(t), S(t))^t, \quad (2.3)$$

then the standard energy estimate gives

$$\frac{d}{dt} \mathcal{H}(V(t)) + \mathcal{R}(V(t)) = 0, \quad (2.4)$$

where

$$\mathcal{H}(V(t)) = \int_\Omega H(V(t)) \, d\mathbf{x}, \quad (2.5)$$

is the entropy functional and

$$\mathcal{R}(V(t)) = \nu \int_\Omega |\boldsymbol{\Sigma}|^2 \, d\mathbf{x} = \nu \int_\Omega \boldsymbol{\Sigma} : \boldsymbol{\Sigma} \, d\mathbf{x}, \quad (2.6)$$

is the entropy dissipation rate functional. However, for weak solutions we only have the energy inequality.

**(2.2) Theorem** *Let  $(\rho, \mu, S)$  be a weak solution of (1.1)–(1.5) then we have the energy inequality*

$$\mathcal{H}(V(T)) + \int_0^T \mathcal{R}(V(t)) dt \leq \mathcal{H}(V(0)). \quad (2.7)$$

From (2.2) Theorem it is nature to consider a sequence of solutions  $V_\epsilon$  indexed by a vanishing positive sequence  $\epsilon$  such that for some constant  $C > 0$ , the initial data  $V_\epsilon^{in} = (\rho_\epsilon(0), \mu_\epsilon(0), S_\epsilon(0))^t$  satisfies the entropy bound

$$\int_\Omega \tilde{H}(V_\epsilon^{in}) dx \leq C\epsilon^2, \quad 0 < \epsilon \ll 1. \quad (2.8)$$

Since  $\tilde{H}$  is strictly convex then we have (see [6,7])

**(2.3) Lemma** *For  $(\rho_\epsilon, \frac{\mu_\epsilon}{\epsilon}, S_\epsilon)$  satisfying the same hypothesis as (2.2) Theorem then*

$$\left(\rho_\epsilon(t), \frac{\mu_\epsilon(t)}{\epsilon}, S_\epsilon(t)\right) \text{ is compact in } w\text{-}L^1(\Omega). \quad (2.9)$$

for all  $t$  fixed but arbitrary.

Moreover, applying the Arzela-Ascoli theorem yields

**(2.4) Theorem**  *$(\rho_\epsilon, \frac{\mu_\epsilon}{\epsilon}, S_\epsilon)$  is compact in  $C([0, \infty), w\text{-}L^1(\Omega))$ .*

It follows directly from the strict convexity of  $\Psi$  and the entropy bound (2.9) that

**(2.5) Lemma**  *$\left\| \frac{\rho_\epsilon(\cdot, t) - 1}{\epsilon^2} \right\|_{L^1(\Omega)}$  and  $\left\| \frac{S_\epsilon(\cdot, t) - 1}{\epsilon^2} \right\|_{L^1(\Omega)}$  are uniformly bounded for all  $t \in \mathbf{R}^+$  fixed but arbitrary. Hence,  $\rho_\epsilon \rightarrow 1$  in  $s\text{-}L^1(\Omega)$ . By passing to subsequence we also have  $\rho_\epsilon \rightarrow 1$  a.e. in  $\Omega$ . Similar result is also true for  $S_\epsilon$ .*

Moreover, by the energy estimate, Young's inequality and the structure of the entropy we can show that

$$\left\{ \frac{\rho_\epsilon(\cdot, t) - 1}{\epsilon^2} \right\}_\epsilon, \quad \left\{ \frac{S_\epsilon(\cdot, t) - 1}{\epsilon^2} \right\}_\epsilon \text{ are equiintegrable.} \quad (2.10)$$

Therefore, by Dunford-Pettis theorem we deduce

**(2.6) Theorem** *For all  $t \in (0, \infty)$  fixed but arbitrary one has*

$$\left\{ \frac{\rho_\epsilon(\cdot, t) - 1}{\epsilon^2} \right\}_\epsilon, \quad \left\{ \frac{S_\epsilon(\cdot, t) - 1}{\epsilon^2} \right\}_\epsilon \text{ are compact in } w\text{-}L^1(\Omega). \quad (2.11)$$

### §3 Incompressible Limit

The scaled time discretized viscous compressible fluid dynamics equations are given by

$$\epsilon \frac{\rho_\epsilon - \rho_\epsilon^{in}}{\Delta t} + \nabla \cdot \boldsymbol{\mu}_\epsilon = 0, \quad (3.1)$$

$$\epsilon \frac{\boldsymbol{\mu}_\epsilon - \boldsymbol{\mu}_\epsilon^{in}}{\Delta t} + \nabla \cdot \left( \frac{\boldsymbol{\mu}_\epsilon \otimes \boldsymbol{\mu}_\epsilon}{\rho_\epsilon} \right) + \nabla P(\rho_\epsilon, S_\epsilon) = \epsilon \nabla \cdot (\nu \boldsymbol{\Sigma}_\epsilon), \quad (3.2)$$

$$\epsilon \frac{S_\epsilon - S_\epsilon^{in}}{\Delta t} + \frac{\boldsymbol{\mu}_\epsilon}{\rho_\epsilon} \cdot \nabla S_\epsilon = 0. \quad (3.3)$$

It is an implicit time discretization of the scaled compressible viscous fluid dynamics equations (1.1) – (1.3). Throughout this article, we shall always set the time step  $\Delta t = 1$ . The theory in section 2 can be transposed to this new problem without any significant change. The form of the the entropy inequality is however somewhat different;

$$\tilde{\mathcal{H}}(V_\epsilon) + J(V_\epsilon^{in}, V_\epsilon) + \tilde{\mathcal{R}}(V_\epsilon) \leq \tilde{\mathcal{H}}(V_\epsilon^{in}) \quad (3.4)$$

where  $J(V_\epsilon^{in}, V_\epsilon)$  is the **relative entropy** of  $V_\epsilon^{in} = (\rho_\epsilon^{in}, \boldsymbol{\mu}_\epsilon^{in}, S_\epsilon^{in})^t$  with respect to  $V_\epsilon = (\rho_\epsilon, \boldsymbol{\mu}_\epsilon, S_\epsilon)^t$  which is given by

$$\begin{aligned} J(V_\epsilon^{in}, V_\epsilon) &= \int_\Omega \left[ \left( \frac{1}{2} \frac{|\boldsymbol{\mu}_\epsilon^{in}|^2}{\rho_\epsilon^{in}} + \Psi(\rho_\epsilon^{in}) e^{A(S_\epsilon^{in})} \right) - \left( \frac{1}{2} \frac{|\boldsymbol{\mu}_\epsilon|^2}{\rho_\epsilon} + \Psi(\rho_\epsilon) e^{A(S_\epsilon)} \right) \right. \\ &\quad - \left( -\frac{1}{2} \frac{|\boldsymbol{\mu}_\epsilon|^2}{\rho_\epsilon^2} + \Psi'(\rho_\epsilon) e^{A(S_\epsilon)} \right) (\rho_\epsilon^{in} - \rho_\epsilon) \\ &\quad \left. - \left( \frac{\boldsymbol{\mu}_\epsilon}{\rho_\epsilon} \right) \cdot (\boldsymbol{\mu}_\epsilon^{in} - \boldsymbol{\mu}_\epsilon) - \Psi(\rho_\epsilon) A'(S_\epsilon) e^{A(S_\epsilon)} (S_\epsilon^{in} - S_\epsilon) \right] dx, \end{aligned} \quad (3.5)$$

with

$$\Psi(\rho) = \Phi(\rho) - \Phi(1) - \Phi'(1)(\rho - 1). \quad (3.6)$$

The integrand of (3.5) is easily understood to be a convex function of  $V_\epsilon^{in}$ . In fact, it is a jointly convex function of both of its arguments,  $V_\epsilon^{in}$  and  $V_\epsilon$ .

**(3.1) Theorem** ([7]) *For  $\gamma > \frac{D}{2}$ , the following is true*

$$\left\{ \frac{1}{\rho_\epsilon} \left( \frac{\boldsymbol{\mu}_\epsilon}{\epsilon} \otimes \frac{\boldsymbol{\mu}_\epsilon}{\epsilon} \right) \right\} \quad \text{and} \quad \left\{ \frac{S_\epsilon - 1}{\epsilon^2} \frac{\boldsymbol{\mu}_\epsilon}{\epsilon \rho_\epsilon} \right\} \quad \text{are compact in } w\text{-}L^1(\Omega). \quad (3.7)$$

Now we are in a position to discuss the incompressible limit. From the equation of continuity and (2.6) Theorem we can conclude

**(3.2) Lemma** ([6,7])  *$\bar{\mathbf{v}}$  is weakly divergence free, i.e.,  $\nabla \cdot \bar{\mathbf{v}} = 0$  in the sense of distribution.*

From the weak formulation of the momentum equation (3.2)

$$\begin{aligned} \int_\Omega \left( \frac{\boldsymbol{\mu}_\epsilon}{\epsilon} - \frac{\boldsymbol{\mu}_\epsilon^{in}}{\epsilon} \right) \cdot \boldsymbol{\psi} \, dx &= \int_\Omega \nabla \boldsymbol{\psi} : \frac{1}{\rho_\epsilon} \left( \frac{\boldsymbol{\mu}_\epsilon}{\epsilon} \otimes \frac{\boldsymbol{\mu}_\epsilon}{\epsilon} \right) \, dx \\ &\quad + \int_\Omega \nabla \cdot \boldsymbol{\psi} \frac{1}{\epsilon^2} P(\rho_\epsilon, S_\epsilon) \, dx - \frac{\nu}{\epsilon} \int_\Omega \nabla \boldsymbol{\psi} : \boldsymbol{\Sigma}_\epsilon \, dx, \end{aligned} \quad (3.8)$$

for all  $\psi \in C_c^\infty(\Omega; \mathbf{R}^D)$ , if we choose the test function  $\psi$  in the space of divergence free, i.e.,

$$\mathcal{V} \equiv \left\{ \psi \in C_c^\infty(\Omega; \mathbf{R}^D), \quad \nabla \cdot \psi = 0 \right\}$$

then the second term of the RHS of (3.8) will eliminate. Since

$$\begin{aligned} \rho_\epsilon &\rightarrow 1 \quad \text{a.e. in } \Omega, \\ \frac{\mu_\epsilon}{\epsilon} &\rightarrow \bar{\nu} \quad \text{a.e. in } \Omega. \end{aligned} \quad (3.9)$$

So we have

$$\frac{1}{\rho_\epsilon} \left( \frac{\mu_\epsilon}{\epsilon} \otimes \frac{\mu_\epsilon}{\epsilon} \right) \rightarrow \bar{\nu} \otimes \bar{\nu} \quad \text{a.e. in } \Omega. \quad (3.10)$$

But  $\left\{ \frac{1}{\rho_\epsilon} \left( \frac{\mu_\epsilon}{\epsilon} \otimes \frac{\mu_\epsilon}{\epsilon} \right) \right\}$  is compact in  $w\text{-}L^1(\Omega)$ . Thus

$$\frac{1}{\rho_\epsilon} \left( \frac{\mu_\epsilon}{\epsilon} \otimes \frac{\mu_\epsilon}{\epsilon} \right) \rightharpoonup \bar{\nu} \otimes \bar{\nu} \quad \text{in } w\text{-}L^1(\Omega), \quad (3.11)$$

therefore we derive the convergence of the nonlinear term

$$\int_\Omega \nabla \psi : \frac{1}{\rho_\epsilon} \left( \frac{\mu_\epsilon}{\epsilon} \otimes \frac{\mu_\epsilon}{\epsilon} \right) dx \rightarrow \int_\Omega \nabla \psi : \bar{\nu} \otimes \bar{\nu} dx. \quad (3.12)$$

Taking all the convergence results into account we obtain

$$\int_\Omega \psi \cdot \bar{\nu} dx - \int_\Omega \psi \cdot \bar{\nu}^{in} dx - \int_\Omega \nabla \psi : (\bar{\nu} \otimes \bar{\nu}) dx = \nu \int_\Omega \bar{\nu} \cdot \Delta \psi dx. \quad (3.13)$$

Note that the vector field  $\psi$  is of divergence free. Hence by (3.13) we have

$$\mathcal{P}(\bar{\nu} - \bar{\nu}^{in} + \bar{\nu} \cdot \nabla \bar{\nu} - \nu \Delta \bar{\nu}) = 0 \quad \text{in } \Omega, \quad (3.14)$$

where  $\mathcal{P}$  is the orthogonal projection. We shall refer to it as the Leray Projector. From Helmholtz Decomposition theorem and (3.14) it follows that there exists  $\Pi$  such that

$$\bar{\nu} - \bar{\nu}^{in} + (\bar{\nu} \cdot \nabla) \bar{\nu} + \nabla \Pi = \nu \Delta \bar{\nu}. \quad (3.15)$$

Moreover, due to the special structure of the equation of state  $\frac{1}{\epsilon^2} P(\rho_\epsilon, S_\epsilon)$ , we see that the limit of the entropy is a function of the density  $\rho$  only hence  $\bar{\Pi}$  is also a function of  $\rho$  only. It follows from (3.1) Theorem that there exists  $\bar{S}$  such that

$$\bar{S} - \bar{S}^{in} + (\bar{\nu} \cdot \nabla) \bar{S} = 0 \quad (3.16)$$

On the other hand, from the equation of state,  $\frac{1}{\epsilon^2} P(\rho_\epsilon, S_\epsilon)$ , we see that  $\bar{S} = \bar{S}(\bar{\rho})$ , i.e.,  $\bar{S}$  is a function of  $\bar{\rho}$  only. Therefore the equation of entropy (3.16) is equivalent to the equation of continuity

$$\bar{\rho} - \bar{\rho}^{in} + (\bar{\nu} \cdot \nabla) \bar{\rho} = 0 \quad (3.17)$$

in the limiting case.

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**Acknowledgements:** The author would like to thank the National Science Council of Taiwan under the grant NSC83-0208-M006-021.

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## Estimates of spherical derivative

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**Abstract.** We estimate the growth of spherical derivative of meromorphic solutions of algebraic differential equations of the first order. It essentially improve earlier known results. For meromorphic functions with given growth of spherical derivative we establish new results about distribution and mutual arrangement of  $a$ - and  $b$ - points of functions.

Consider algebraic differential equation of the first order

$$(1) \quad (w')^n + \sum_{k=1}^n P_k(z, w) (w')^{n-k} = 0$$

where

$$P_k(z, w) = \sum_{j=0}^{m_k} a_{kj}(z) w^j, \quad 1 \leq k \leq n.$$

Our goals are:

- to estimate the growth of the spherical derivative of meromorphic solutions of the differential equation (1);
- to describe value distribution of meromorphic functions with given growth of spherical derivative in the unit disc  $D$  and in the complex plane  $\mathbf{C}$ .

There are several approaches to study this problem: Wiman-Valiron's method (e.g. Malmquist); from the position of Nevanlinna's theory of meromorphic functions (see [1], I.Laine, K.Yosida); from the point of view Ahlfors's theory of covering surfaces (e.g. [2]).

Wiman-Valiron's method gives good results in the case of integer solutions and cannot be used in the case of meromorphic solutions. Approach from the position of the Ahlfors-Nevanlinna's theory gives sufficiently big information about value distribution

of meromorphic solutions of equation (1), about the growth of the Nevanlinna's characteristic function  $T(r, w)$  of meromorphic function  $w(z)$

$$T(r, w) = \frac{1}{\pi} \iint_{|z| \leq r} (w^\#(z))^2 \ln \frac{r}{|z|} dx dy \quad ,$$

where  $w^\#(z) = |w'(z)|(1 + |w(z)|^2)^{-1}$  is a spherical derivative of meromorphic function  $w(z)$ .

Here we note that for any two meromorphic in  $\mathbb{C}$  (or in  $D$ ) functions  $f(z)$  and  $g(z)$

$$T(r, f + g) \leq T(r, f) + T(r, g)$$

$$T(r, fg) \leq T(r, f) + T(r, g)$$

but there is not the same in the case of spherical derivative.

We also know that each meromorphic function  $f(z)$  with growth of spherical derivative  $f^\#(z) = O((1 - |z|)^{-p})$ ,  $|z| \rightarrow 1$ , has the growth of Nevanlinna characteristic function  $T(r, f) = O((1 - r)^{p-1})$  as  $r \rightarrow 1$  and converse is not true. There exist such functions that  $T(r, f) = O((1 - r)^{p-1})$  as  $r \rightarrow 1$  but  $\limsup_{|z| \rightarrow 1} (1 - |z|)^{-p} f^\#(z) = \infty$  and the set

of "explosion" of the growth of spherical derivative must be very small. Thus we can see the importance of estimate of the growth of spherical derivative of meromorphic solutions of equation (1).

Consider the differential equation (1). Let  $M = \max_{1 \leq k \leq n} \frac{m_k}{k} - 1$ . Transform equation (1)

$$(w')^n w^{nM} + \dots + P_k(z, w)(w')^{n-k} w^{nM} + \dots + P_n(z, w)w^{nM} = 0$$

$$[(w^{M+1})']^n + \dots + \tilde{P}_k(z, w)[(w^{M+1})']^{n-k} + \dots + \tilde{P}_n(z, w) = 0 \quad ,$$

where  $\tilde{P}_k(z, w) = (M + 1)^k P_k(z, w) w^{kM}$ . Then

$$|(w^{M+1})'|^n \leq (n + 1) \max |(w^{M+1})'|^{n-k} |\tilde{P}_k|$$

$$|(w^{M+1})'| \leq C \max |\tilde{P}_k|^{1/k} |w|^M.$$

Let  $|u|^+ = \max\{1, |u|\}$ . Then  $|P_k(z, w)| \leq \sum_{j=0}^{m_k} |a_{kj}(z)| (|w|^+)^j$ .

$$\begin{aligned} \frac{|(w^{M+1})'|}{1 + |w^{2M+2}|} &\leq C \max |\tilde{P}_k|^{1/k} \frac{|w^M|}{1 + |w^{2M+2}|} \\ &\leq C \max \left( \sum_{j=0}^{m_k} |a_{kj}(z)| \right)^{1/k} (|w|^+)^{\frac{m_k}{k}} \frac{|w^M|}{1 + |w^{2M+2}|} \\ &\leq C_1 \max \left( \sum_{j=0}^{m_k} |a_{kj}(z)| \right)^{1/k} \end{aligned}$$

Thus we obtain

$$(2) \quad (w^{M+1})^\# \leq C_1 \max \left( \sum_{j=0}^{m_k} |a_{kj}(z)| \right)^{1/k}.$$

1. If  $a_{kj}(z)$  are polynomials or rational functions

$$|a_{kj}(z)| = |z|^{\alpha_{kj}} (c_{kj} + o(1)), \quad |z| \rightarrow \infty,$$

then meromorphic solutions of equation (1) satisfy to condition

$$|z|^{2-p} (w^{M+1})^\# \leq C_1$$

where  $p = 2 + \max_{1 \leq k \leq n} \max_{0 \leq j \leq m_k} \frac{\alpha_{kj}}{k}$ .

The last inequality is equivalent to

$$(3) \quad \lim_{|z| \rightarrow \infty} |z|^{2-p} w^\#(z) \leq C.$$

It is known that for any function  $f(z)$  which satisfy to condition (3) family of functions  $\{f(a_n + |a_n|^{2-p}z)\}$ ,  $\lim_{n \rightarrow \infty} |a_n| = \infty$ , ( $p > 1$ ) is normal in the sense of Montel in  $\mathbf{C}$ , i.e. in any sequence of functions  $\{f_n(z)\}$  there is some subsequence  $\{f_{n_k}(z)\}$  which convergence uniformly on compact subsets of  $\mathbf{C}$  in chordal metric to some meromorphic function including infinity. In the case  $p = 1$  condition (3) is equivalent to normality of family of functions  $\{f(a_n z)\}$  in  $\mathbf{C} \setminus \{0\}$ .

By the formula of Ahlfors-Shimizu we can estimate the growth of Nevanlinna characteristic function  $T(r, f)$ .

If  $p = 1$  and the constant  $C_1 < \frac{1}{2}$  then meromorphic solution  $w(z)$  is a rational function (see [3]).

2. If  $a_{kj}(z)$  are holomorphic functions in the unit disk  $D$  and belong to the Hardy classes  $H_{p_{kj}}$ , respectively, then  $|a_{kj}(z)| \leq c_{kj}(1 - |z|)^{-\frac{1}{p_{kj}}}$  and hence

$$w^\#(z) \leq \frac{C}{(1 - |z|)^p},$$

where  $p = \max_{1 \leq k \leq n} \max_{0 \leq j \leq m_k} \frac{1}{k p_{kj}}$ .

Now we describe the value distribution and mutual arrangement of  $a$ - and  $b$ - points of meromorphic functions with given growth of spherical derivative.

**Theorem 1.** Meromorphic in  $\mathbb{C}$  function  $f(z)$  has the growth of spherical derivative

$$\lim_{|z| \rightarrow \infty} |z|^{2-p} f^\#(z) < \infty \quad p \geq 1,$$

if and only if for any two values  $a$  and  $b$  from  $\overline{\mathbb{C}}$  ( $a$  and  $b$  are not Picard's exceptional values of  $f(z)$ )

$$\inf_{\nu, \mu} \left| 1 - \frac{b_\mu}{a_\nu} \right| |a_\nu|^{p-1} > 0$$

where  $\{a_\nu\}$  and  $\{b_\mu\}$  are solutions of equations  $f(z) = a$  and  $f(z) = b$ , respectively.

Let  $\sigma(a, b)$  be hyperbolic metric in the unit disk  $D$ ,  $f(z)$  be meromorphic function in  $D$  and  $a_j$ ,  $j = 1, 2, 3, 4$  are arbitrary values of  $\overline{\mathbb{C}}$ . Let  $\{a_j(k)\}$  be roots of equations  $f(z) = a_j$ .

**Theorem 2.** Meromorphic in  $D$  function  $f(z)$  satisfy to condition

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) f^\#(z) < \infty \quad p \geq 1,$$

if and only if

$$\inf_{\substack{a \in \{a_j(k)\} \\ b \in \{a_l(k)\}}} \frac{\sigma(a, b)}{(1 - |b|)^{p-1}} > 0$$

for any  $j, l$ ,  $j \neq l$ .

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# INSTABILITY OF PIPE-POISEUILLE FLOW AND THE GLOBAL NATURE OF DISTURBANCES

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**Abstract:** According to the stability theory of laminar flows, infinitesimal disturbances in a pipe always decay, while the real flows undergo a transition to turbulent ones in a certain Reynolds number. We discuss here the arbitrary deformation of disturbances. The development is governed by an equation including the Weyl and Riemann-Liouville fractional derivatives of order  $1/2$ . Both the fractional derivatives denote non-local properties, but the role of the two derivatives is quite different from each other.

An extended equation with an additional nonlinear dispersion term was solved numerically. It was found that the non-local properties described by the fractional derivatives combined with the nonlinearity leads to some kind of instability.

## 1. Introduction

In 1883 O. Reynolds found that when a dimensionless parameter called after his name exceeds a certain value the flow undergoes a change from laminar to turbulent flows. Since then much theoretical attention has been paid to the determination of the value[5]. However, up to this time the determination has not yet been made, while for other flows such as plane-Poiseuille flow and boundary layer flow it gives successful result.

As is well known, the disturbances may be divided into two modes called the wall and center modes(Corcus and Sellars[1]). These names originate from the disturbance location. We confine ourselves to the center mode, because the deformation due to nonlinearity is easier than the wall mode. The complex phase velocity  $c$  for the center mode is written in the form,

$$c = 1 - (A_m - iB_m)\alpha^{-\frac{1}{2}}, \quad (1)$$

where the values  $A_m$  and  $B_m$  were already given(see for example, [4]).

For an analytical convenience, we divide it into two components such as

$$c = 1 + \left[ \frac{\Delta_+}{\sqrt{2}}(-1 + i) + \frac{\Delta_-}{\sqrt{2}}(-1 - i) \right] \alpha^{-\frac{1}{2}}. \quad (2)$$

This decomposition is made not only for an analytical convenience but also for having an insight into the physics included. We refer the first component to as the downstream-diffusion component and the other to as the upstream-diffusion one, although they are both dispersive and dissipative. The coefficient of the upstream-diffusion component  $\Delta_-$  is always positive, while the coefficient  $\Delta_+$  of the downstream-diffusion one allows to have both signs. The latter component becomes unstable for  $\Delta_+ > 0$ . It should be noted that the downstream-diffusion component can not appear independently in physical systems. On the other hand, the upstream-diffusion component appears frequently in real phenomena[2, 6].

We consider the center mode of an arbitrary shape. The form depending on time and space  $\eta(x, t)$  is written as

$$\eta(x, t) = \int_{-\infty}^{\infty} A(\alpha) e^{i\alpha(x-ct)} d\alpha, \quad (3)$$

where  $A(\alpha)$  is an arbitrary function of the wavenumber  $\alpha$ . The convolution theorem leads to

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} = \Delta_+ I_x^{-1/2} \eta - \Delta_- K_x^{-1/2} \eta, \quad (4)$$

where the two operators on the right-hand side  $K_x^{-1/2}$  and  $I_x^{-1/2}$  are defined as

$$K_x^{-1/2} \eta = -\frac{1}{\sqrt{\pi}} \int_x^{\infty} \frac{\partial \eta}{\partial x'} \frac{dx'}{\sqrt{x' - x}}, \quad I_x^{-1/2} \eta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^x \frac{\partial \eta}{\partial x'} \frac{dx'}{\sqrt{x - x'}}. \quad (5)$$

The former is called the Riemann-Liouville and the latter the Weyl integrals of order 1/2. It is seen from eq.(5) that the downstream-diffusion component is affected from the upstream condition while the upstream-diffusion one from the downstream condition.

## 2. Asymptotic solution-summary of linear theory

We first consider the upstream-diffusion component. The initial value problem can easily be solved using the Laplace transform. To apply the method, we assume,

$$\eta(x, t) = \frac{1}{2\pi i} \int_L \eta_0(s) e^{i\alpha(s)x} e^{st} ds, \quad (6)$$

where the function  $\eta_0(s)$  is an arbitrary function of  $s$  and the integral path  $L$  is a vertical line lying to the right of all singularities of  $\eta_0(s)$ , and the integration is from  $-\infty$  and  $i\infty$ . The function  $\alpha(s)$  will be found as the dispersion relation. Assuming that there is no disturbance when  $t < 0$ , and choosing a boundary condition at  $x = 0$  such that

$$\eta(0, t) = \exp(i\Omega t) \quad \text{for } t \geq 0,$$

we will seek the solution. When  $x$  and  $t$  are large and also  $m(=x/t) > 1$ , modification of the integration path  $L$  to round around the right half plane gives

$$\eta(x, t) = 0, \quad (7)$$

because there are no singularities inside the closed path. When  $m < 1$ , the method of steepest descent gives the asymptotic solution for  $0 \leq m < 1$ ,

$$\begin{aligned} \eta(x, t) = & \frac{1}{\sqrt{2\pi x h_0''}} \frac{1}{s_0 - i\Omega} \exp[s_0 t - (s_0 + \frac{\Delta^2}{2} \frac{1}{1-m})x] \\ & + \exp[i\Omega(t-x) - \Delta(i\Omega + \frac{\Delta^2}{4})^{1/2}x - \frac{\Delta^2}{2}x], \end{aligned} \quad (8)$$

where  $s_0$  is the saddle point and  $h_0'' = 2(1-m)^3/\Delta^2 m^3$ . In the above equation the subscript attached to the diffusion coefficient have been omitted. The second term on the right-hand side is valid only for  $0 \leq m < m_c$ , where  $m_c$  is the smaller root of the equation,

$$\Omega = \frac{\Delta^2 m \sqrt{2m-1}}{2(1-m)^2}.$$

The first term on the right-hand side of eq.(8) is significant only near the wavefront  $m \approx 1$  and the second term is the contribution from the residue which describes the wave behavior.

Next, we consider the downstream-diffusion component. In this case the coefficient  $\Delta_+$  is positive or negative. We first take up the solution of the stable case,  $\Delta_+ < 0$ . The solution for  $m > 1$  from the saddle point is written as

$$\eta(x, t) = \frac{1}{\sqrt{2\pi h_0'' x}} \frac{1}{s_0 - i\Omega} \exp[\frac{\Delta^2}{4} \frac{t}{1-m}]. \quad (9)$$

For  $0 < m < m_c (> 1)$  we have an additional contribution from the residue,

$$\eta(x, t) = \exp[i\Omega(t-x) - i\Delta(i\Omega - \frac{\Delta^2}{4})^{1/2}x + \frac{\Delta^2}{2}x]. \quad (10)$$

It is noted that  $\eta(x, t) \neq 0$  even when  $m > 1$ . In other words, the disturbance can diffuse far downstream. This point is quite different from the case of the upstream-diffusion component. Noting that the argument of exponent function is simply written as  $\Delta^2/4m(1-m)$  in eq.(9), we have the diffusion property expanding downstream as  $1/\sqrt{x}$ . This downstream-diffusion component has never appeared individually in physical systems. In this sense it is reasonable that the component may be referred to as the anomalous component. On the other hand, the upstream-diffusion component appears frequently in usual systems[2, 6].

Next, we consider the unstable case,  $\Delta_+ > 0$ . When  $m < 1$  the solution is easily obtained, by integrating so as to turn round the left half plane, as

$$\eta(x, t) = \exp[i\Omega(t-x) - i\Delta(i\Omega - \frac{\Delta^2}{4})^{1/2}x + \frac{\Delta^2}{2}x], \quad (11)$$

which is due to the residue contribution. The way of integration which has been done cannot be applied to the region  $m > 1$ . The integration there can be carried out by taking two fans in the right half plane and hence it gives

$$\eta(x, t) = -\frac{1}{\pi} \exp\left(\frac{\Delta^2}{4} \frac{1+m}{m} x\right) \int_0^\infty \exp[(1-m)\lambda t] \sin \Delta \sqrt{\lambda} x \frac{d\lambda}{\lambda - i\Omega + \Delta^2/4}. \quad (12)$$

The final form includes an integral whose integrand oscillates more rapidly as  $m \rightarrow 1$  and cannot be described using the well-known functions.

### 3. Nonlinear theory

We extend the linear equation for infinitesimal disturbances to the nonlinear one for finite amplitude ones such that

$$\frac{\partial \eta}{\partial t} + (1 - \eta^2) \frac{\partial \eta}{\partial x} + \eta^3 = \Delta_+ I^{-1/2} \eta - \Delta_-^{-1/2} \eta. \quad (13)$$

The linear terms are exact but two nonlinear terms are artificial. The nonlinear convection term can explain partly a real amplitude effect. The above nonlinear equation was replaced by the following finite difference scheme,

$$\begin{aligned} \eta_{i+1,j} + \eta_{i-1,j} = 2\eta_{ij} &+ \frac{k^2}{h^2} (1 - 2\eta_{ij}^2) (\eta_{i,j+1} - 2\eta_{ij} + \eta_{i,j-1}) - \frac{k^2}{h^2} \eta_{ij} (\eta_{i,j+1} - \eta_{i,j-1})^2 \\ &- \delta \frac{k^2}{h} (f_{i,j+1} - f_{i,j-1}) + \delta k (f_{ij} - f_{i-1,j}), \end{aligned} \quad (14)$$

where we have dropped the nonlinear dissipation term for simplicity. The function  $f(t, x)$  stands for the integral corresponding to the Weyl or the Riemann-Liouville fraction derivative and  $\delta = \Delta/\sqrt{\pi}$ . To investigate the interaction between long and short waves, the boundary condition at  $x = 0$  is chosen as

$$\eta(0, t) = -0.3 \exp(-0.05(t-5)^2) - a_1 \sin \omega_1 t \quad \text{for } t \geq 0.$$

This is a simple model for a puff observed in transitional pipe flow[3].

We first mention about the result for the upstream-diffusion component. For  $\delta = 0.02$ , computation was carried out. It was found that when the high frequency disturbance is strong, our numerical computation fails even under the influence of the nonlinear dissipation. Since this nonlinear diffusion term does not affect critically the development, we will not discuss it any more. To see how the computation fails, in Fig.1 we plotted the temporal variations of energy defined by

$$\int_0^l \eta(x, t)^2 dx, \quad (15)$$

where  $l=102.4$ , being the length of pipe. The variations were calculated for five different  $a_1$ 's when the frequency  $\omega_1$  is fixed, say  $\omega_1 = 5$ . For  $a_1 = 0.167$ , the energy diverges very rapidly, hence the computation could not be proceeded. If the amplitude is slightly less

than the above value, say  $a_1 = 0.165$ , the energy did not diverge indefinitely. It increases up to  $t = 52$ , and then decreases. The profiles at this time together at  $t = 60$  are shown in Fig.2. It was clarified that the bottom of the averaged profile did not move up to  $t = 52$ , then it travelled at almost the same velocity as unity. It should here be noted that the time at which energy has the maximum value is near the breaking time  $t_B$  predicted in the inviscid limit of eq.(13) with the boundary condition without the high frequency part. For weaker disturbances the delay is very small or is not almost all discernible. It is thus conjectured that when the nonlinear effect balances with or dominates over the non-local effect such instability plays an important role. This unstable phenomenon is very sensitive to the frequency  $\omega_1$ . The energy variation for  $\omega_1 = 6$  and  $a_1 = 0.14$  is also plotted in Fig.1. The difference between  $\omega_1 = 5$  and 6 is remarkable.

Similar computation was also carried out for the downstream-diffusion component, setting  $\delta = -0.01$ . The results are shown in Fig.3 for five different  $a_1$ 's. For larger amplitude than 0.11, we could not compute the long time development of the disturbance. Near  $t = 52$ , the energy has its maximum when  $a_1 = 0.108$  as well as for the upstream-diffusion component. Contrary to the case of the upstream-diffusion component, an energy peak always appears near or after  $t = 52$ . Furthermore, there exists another difference between the two components. As is shown in Fig.3, the dependence of  $\omega_1$  is not so strong as the upstream-diffusion component. The solid line denotes the variation for  $\omega_1=6$  and  $a_1=0.05$ .

## Concluding remarks

It was found that non-local properties in terms of the fractional derivatives combined with nonlinearity give rise to a certain kind of instability. However, the physical mechanism could not be clarified. This problem will be remained in the future.

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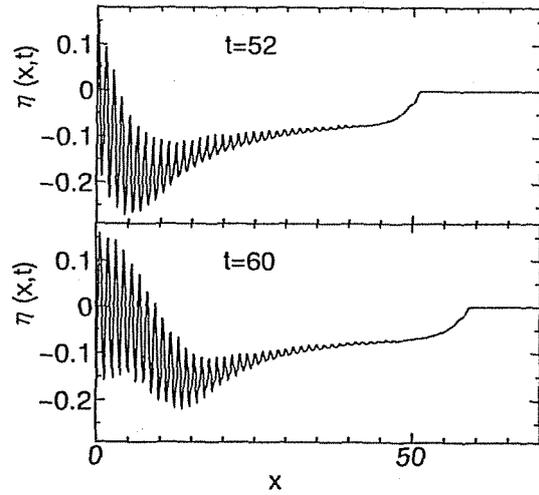
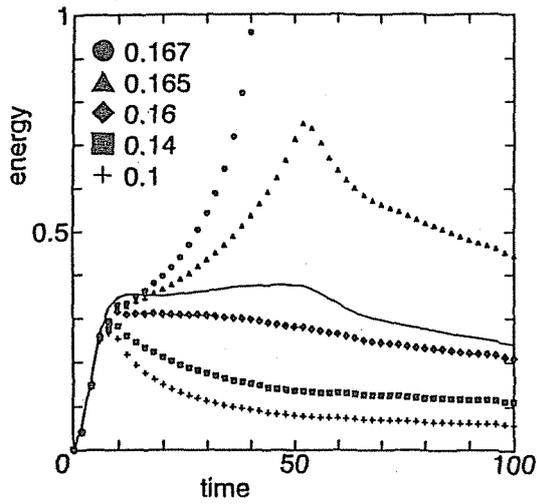


Fig.1 Temporal variation of energy for various  $a_1$  Fig.2 Disturbance profiles at  $t = 52$  and  $60$ .  
The solid line is for  $\omega_1 = 6$  and  $a_1 = 0.14$ .

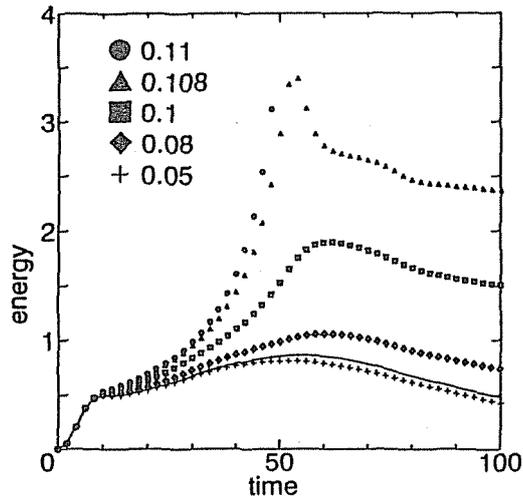


Fig.3 Temporal variation of energy for various  $a_1$ .  
The solid line is for  $\omega_1 = 6$  and  $a_1 = 0.05$ .

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# GLOBAL EXISTENCE AND ENERGY DECAY FOR A CLASS OF QUASILINEAR WAVE EQUATIONS WITH LINEAR DAMPING TERMS

TOKIO MATSUYAMA

**Abstract.** Local existence and energy decay of global solutions are investigated to the Cauchy problem for the wave equations of Kirchhoff type with linear damping terms. In this paper we present a special method, based on the construction of “potential well” in order to estimate the energy decay and improve the results of our previous work [3].

## 1. Introduction.

In this paper we are concerned with the Cauchy problem

$$u_{tt} - M(\|\nabla u(t)\|_2^2)\Delta u + \delta u_t = \mu|u|^{p-1}u, \quad t > 0, \quad x \in \mathbf{R}^N, \quad (1.1)$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbf{R}^N, \quad (1.2)$$

where  $\delta > 0$  and  $\mu \in \mathbf{R}$  are given constants and

$$u_t = \frac{\partial u}{\partial t}, \quad \|\nabla u(t)\|_2^2 = \sum_{j=1}^N \int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_j}(t, x) \right|^2 dx, \quad \Delta u = \sum_{j=1}^N \frac{\partial^2 u}{\partial x_j^2}.$$

Here  $M(r)$  is a  $C^1[0, \infty)$ -class function satisfying  $M(r) \geq m_0 > 0$  ( $r \geq 0$ ) with a constant  $m_0$ .

For the Cauchy problem (1.1) – (1.2) with  $\delta > 0$  and  $f(t)$  instead of  $\mu|u|^{p-1}u$ , Yamada [9] proved the global existence and energy decay for sufficiently small initial data. His method of proof in the existence theorems is based on the regularization by the mollifiers. In the case of  $\delta = 0$  and  $\mu = 0$ , D’Ancona and Spagnolo [1] proved the global existence in the class  $H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$  with  $L^1$  decay conditions. Recently, in [2] they constructed the global smooth solutions for small initial data with  $\delta \in \mathbf{R}$  and  $\mu \in \mathbf{R}$ . If,

in particular,  $M(s) \equiv 1$  and  $\mu < 0$ , it is known that there exist global classical solutions without any smallness conditions on the initial data. See Reed [6], Struwe [7] and the references therein. With regard to the Cauchy problem with more general power nonlinearities Matsuyama and Ikehata [3] obtained the global existence in the class  $H^2(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$  and energy decay under the restricted conditions.

Our purpose in this paper is to improve the results of the previous work [3]. First we discuss the local existence of solutions to the Cauchy problem (1.1) – (1.2) in more general space dimensions. Secondly, we obtain the global solvability in the class  $H^2(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$  and energy decay for the typical Cauchy problem of the form

$$u_{tt} - \left( \alpha + \beta \|\nabla u(t)\|_2^2 \right) \Delta u + \delta u_t = \mu |u|^{p-1} u, \quad t > 0, \quad x \in \mathbf{R}^N, \quad (1.3)$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbf{R}^N, \quad (1.4)$$

where  $\alpha > 0$ ,  $\beta \geq 0$ ,  $\delta > 0$  and  $\mu > 0$  are given constants. In discussing the global existence, we shall construct the modified potential well which comes from the idea of Nakao and Ono [4]. If  $\beta = 0$ , then Nakao and Ono [4] obtained the global weak solutions in the class  $H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$  and energy decay by constructing the modified potential well. Roughly speaking, the modified potential well is the extended  $\mathbf{R}^N$  version of Sattinger's potential well.

Throughout this paper the functions considered are all real valued. Let  $p$  be a number with  $1 \leq p \leq \infty$ .  $\|u\|_p$  stands for the usual  $L^p(\mathbf{R}^N)$  norm of  $u \in L^p(\mathbf{R}^N)$ . Let  $k$  be a nonnegative integer and  $H^k(\mathbf{R}^N)$  denote the usual Sobolev space of order  $k$ . Then the norm  $\|u\|_{H^k}$  of  $u \in H^k(\mathbf{R}^N)$  is defined by  $\|u\|_{H^k}^2 = \sum_{j=0}^k \|\nabla^j u\|_2^2$ .

## 2. Local existence.

In this section we state the local existence results to the problem (1.1) – (1.2).

First we impose the assumptions on  $M(r)$  and  $p$  as follows:

$$(A.1) \quad M(r) \text{ belongs to } C^1[0, \infty)\text{-class with } M(r) \geq m_0 > 0 \text{ for } r \geq 0,$$

$$(A.2) \quad 2 \leq p \leq \frac{N-2}{N-4} \quad (2 \leq p < \infty \text{ if } 1 \leq N \leq 4).$$

Then we can state the local existence and uniqueness to the problem (1.1) – (1.2).

**Theorem 2.1 (Local existence)** *Let  $N$  be an integer with  $1 \leq N \leq 6$ ,  $\delta > 0$  and  $\mu \in \mathbf{R}$ . Suppose (A.1) and (A.2) and let  $(u_0, u_1) \in H^2(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$  be an arbitrary initial data. Then there exists a number  $T_m$  ( $0 < T_m \leq +\infty$ ) such that the problem (1.1) – (1.2) admits a unique solution  $u(t, x)$  on  $[0, T_m)$  which belongs to the class*

$$C_w([0, T_m); H^2(\mathbf{R}^N)) \cap C_w^1([0, T_m); H^1(\mathbf{R}^N)) \cap C_w^2([0, T_m); L^2(\mathbf{R}^N)),$$

where the subscript “w” means the weak continuity with respect to  $t$ . Furthermore, if  $T_m < +\infty$ , then  $\lim_{t \nearrow T_m} [\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1}] = +\infty$ .

**Corollary 2.2 (Strong solutions)** *Let  $1 \leq N \leq 3$ . Then, under the assumptions of Theorem 2.1, the solution  $u$  to the Cauchy problem (1.1) – (1.2) satisfies*

$$u \in C([0, T_m]; H^2(\mathbf{R}^N)) \cap C^1([0, T_m]; H^1(\mathbf{R}^N)) \cap C^2([0, T_m]; L^2(\mathbf{R}^N)).$$

In order to prove Theorem 2.1 and Corollary 2.2 we need the following lemma concerning the local Lipschitz continuity of the nonlinear term. In what follows, we put  $f(u) \equiv \mu|u|^{p-1}u$ .

**Lemma 2.3** (i) *Let  $2 \leq p \leq (N-2)/(N-4)$  ( $2 \leq p < \infty$  if  $1 \leq N \leq 4$ ). Then we have*

$$\|f(u)\|_{H^1} \leq C_S \|u\|_{H^2}^p \quad (u \in H^2(\mathbf{R}^N)) \quad (2.1)$$

for some  $C_S > 0$ .

(ii) *Let  $1 \leq N \leq 3$ . Assume  $2 \leq p < \infty$ . Then we have*

$$\|f(u) - f(v)\|_{H^1} \leq L \left( \|u\|_{H^2}^{p-1} + \|u\|_{H^2}^{p-2} \|v\|_{H^2} + \|v\|_{H^2}^{p-1} \right) \|u - v\|_{H^1} \quad (u, v \in H^2(\mathbf{R}^N)) \quad (2.2)$$

for some  $L > 0$ .

(iii) *Let  $1 \leq N \leq 3$ . Assume  $2 \leq p < \infty$ . Then we have*

$$\|\nabla f(u)\|_2 \leq C_K \|u\|_{H^2}^{p-1} \|\nabla u\|_2 \quad (u \in H^2(\mathbf{R}^N)) \quad (2.3)$$

for some  $C_K > 0$ .

**Proof.** (i) Note that the imbeddings  $H^2(\mathbf{R}^N) \subset L^{2p}(\mathbf{R}^N)$  and  $H^2(\mathbf{R}^N) \subset L^{N(p-1)}(\mathbf{R}^N)$  hold because of  $2 \leq p \leq (N-2)/(N-4)$ . Then we have

$$\begin{aligned} \|f(u)\|_{H^1}^2 &= \|f(u)\|_2^2 + \|\nabla f(u)\|_2^2 \\ &\leq \mu^2 \|u\|_{2p}^{2p} + \mu^2 p^2 \int_{\mathbf{R}^N} |u|^{2(p-1)} |\nabla u|^2 dx \\ &\leq C \|u\|_{H^2}^{2p} + \mu^2 p^2 \|u\|_{N(p-1)}^{2(p-1)} \|\nabla u\|_{2N/(N-2)}^2 \\ &\leq C \|u\|_{H^2}^{2p} + C \|u\|_{H^2}^{2(p-1)} \|\Delta u\|_2^2 \\ &\leq C_S^2 \|u\|_{H^2}^{2p}. \end{aligned}$$

Hence we get (2.1).

(ii) By the mean value theorem and Hölder's inequality we have

$$\begin{aligned} \|f(u) - f(v)\|_2^2 &\leq C \int_{\mathbf{R}^N} \left\{ |u|^{2(p-1)} + |v|^{2(p-1)} \right\} |u - v|^2 dx \\ &\leq C \left( \|u\|_{N(p-1)}^{2(p-1)} + \|v\|_{N(p-1)}^{2(p-1)} \right) \|u - v\|_{2N/(N-2)}^2 \\ &\leq C \left( \|u\|_{H^2}^{2(p-1)} + \|v\|_{H^2}^{2(p-1)} \right) \|\nabla u - \nabla v\|_2^2 \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} &\|\nabla f(u) - \nabla f(v)\|_2 \\ &\leq p|\mu| \left\| |u|^{p-1} \nabla u - \nabla v \right\|_2 + p(p-1)|\mu| \left\| (|u|^{p-2} + |v|^{p-2}) |\nabla v| |u - v| \right\|_2 \\ &\leq C \|u\|_{\infty}^{p-1} \|\nabla u - \nabla v\|_2 + C \left( \|u\|_{\infty}^{p-2} + \|v\|_{\infty}^{p-2} \right) \left\| |\nabla v| |u - v| \right\|_2. \end{aligned} \quad (2.5)$$

In the case when  $N = 1$ , it follows from (2.5) and the imbedding  $H^2(\mathbf{R}) \subset W^{1,\infty}(\mathbf{R})$  that

$$\begin{aligned} \|\nabla f(u) - \nabla f(v)\|_2 &\leq C\|u\|_{H^1}^{p-1}\|u - v\|_{H^1} + C\left(\|u\|_{H^1}^{p-2} + \|v\|_{H^1}^{p-2}\right)\|\nabla v\|_\infty\|u - v\|_2 \\ &\leq C\|u\|_{H^1}^{p-1}\|u - v\|_{H^1} + C\left(\|u\|_{H^1}^{p-2} + \|v\|_{H^1}^{p-2}\right)\|v\|_{H^2}\|u - v\|_{H^1} \\ &\leq C\left(\|u\|_{H^1}^{p-1} + \|u\|_{H^1}^{p-2}\|v\|_{H^2} + \|v\|_{H^1}^{p-2}\|v\|_{H^2}\right)\|u - v\|_{H^1}. \end{aligned} \quad (2.6)$$

In the case when  $N = 2$  or  $3$ , from (2.5) and the imbeddings  $H^2(\mathbf{R}^N) \subset L^\infty(\mathbf{R}^N)$ ,  $H^2(\mathbf{R}^N) \subset W^{1,N}(\mathbf{R}^N)$  that

$$\begin{aligned} &\|\nabla f(u) - \nabla f(v)\|_2 \\ &\leq C\|u\|_{H^2}^{p-1}\|\nabla u - \nabla v\|_2 + C\left(\|u\|_{H^2}^{p-2} + \|v\|_{H^2}^{p-2}\right)\|\nabla v\|_N\|u - v\|_{2N/(N-2)} \\ &\leq C\|u\|_{H^2}^{p-1}\|\nabla u - \nabla v\|_2 + C\left(\|u\|_{H^2}^{p-2} + \|v\|_{H^2}^{p-2}\right)\|v\|_{H^2}\|\nabla u - \nabla v\|_2 \\ &\leq C\left(\|u\|_{H^2}^{p-1} + \|u\|_{H^2}^{p-2}\|v\|_{H^2} + \|v\|_{H^2}^{p-1}\right)\|u - v\|_{H^1}. \end{aligned} \quad (2.7)$$

Thus, (2.4), (2.6) and (2.7) imply (2.2).

(iii) Since  $H^2(\mathbf{R}^N) \subset L^\infty(\mathbf{R}^N)$  ( $1 \leq N \leq 3$ ), we get

$$\begin{aligned} \|\nabla f(u)\|_2 &\leq p|\mu|\left(\int_{\mathbf{R}^N} |u|^{2(p-1)}|\nabla u|^2 dx\right)^{1/2} \\ &\leq p|\mu|\|u\|_\infty^{p-1}\|\nabla u\|_2 \\ &\leq C_K\|u\|_{H^2}^{p-1}\|\nabla u\|_2 \end{aligned}$$

for some  $C_K > 0$ . The proof of Lemma 2.3 is now completed. Q.E.D.

(2.1) is used to construct the local solutions to the problem (1.1) – (1.2). (2.2) plays an essential role in deriving the strong continuity of solutions with respect to  $t$ . Furthermore, (2.3) is used to obtain  $H^2$  bounds in the proof of global existence. Combining Lemma 2.3 with the method of [3], we can prove Theorem 2.1 and Corollary 2.2. So we shall omit the details.

### 3. Global existence and decay.

In this section we state the global existence theorem and decay property to the problem (1.3) – (1.4). To do so we introduce the notion of the modified potential well (see Nakao and Ono [4]). Let

$$\tilde{I}(u) \equiv \alpha\|\nabla u\|_2^2 - \mu\|u\|_{p+1}^{p+1}.$$

Then we define the modified potential well  $\tilde{\mathcal{W}}$  by

$$\tilde{\mathcal{W}} = \{u \in H^1(\mathbf{R}^N); \tilde{I}(u) > 0\} \cup \{0\}.$$

Next, let  $J(u)$  and  $E(u, v)$  be the potential and energy associated with the equation (1.3), respectively:

$$J(u) \equiv \frac{\alpha}{2}\|\nabla u\|_2^2 + \frac{\beta}{4}\|\nabla u\|_2^4 - \frac{\mu}{p+1}\|u\|_{p+1}^{p+1},$$

$$E(u, v) \equiv \frac{1}{2} \|v\|_2^2 + J(u).$$

We further assume that

$$(A.3) \quad \frac{7}{3} \leq p \leq 5 \quad \text{if } N = 3 \quad \text{or} \quad 1 + \frac{4}{N} \leq p < \infty \quad \text{if } N = 1, 2.$$

In the course of proof, we shall often use the Gagliardo-Nirenberg inequality:

$$\|u\|_{\frac{p+1}{p}}^{p+1} \leq K \|u\|_2^{(p+1)(1-\theta)} \|\nabla u\|_2^{(p+1)\theta} \quad \text{with} \quad \theta = \frac{N(p-1)}{2(p+1)}.$$

Setting

$$C_0 \equiv \frac{\mu K}{\alpha} \left\{ \frac{2(p+1)}{\alpha(p-1)} \right\}^{(Np-N-4)/2},$$

$$I_0 \equiv \left\{ 2\|u_0\|_2^2 + \frac{4}{\delta}(u_0, u_1) + \frac{8}{\delta^2} E(u_0, u_1) \right\}^{1/2},$$

we impose the assumption on the initial data  $\{u_0, u_1\}$  as follows:

$$(A.4) \quad C_0 I_0^{(2p+2-Np+N)/2} E(u_0, u_1)^{(Np-N-4)/4} < 1.$$

Now, since Theorem 2.1 assures the local solvability to the problem (1.3) – (1.4), we can state our main theorem.

**Theorem 3.1 (Global existence and energy decay)** *Let  $1 \leq N \leq 3$ ,  $\delta > 0$  and  $\mu > 0$ . Let  $T_m > 0$  be a maximal existence time of solutions to the problem (1.3) – (1.4). Suppose (A.3) and (A.4). Then there exists a number  $\varepsilon_0 > 0$  (depending on  $\|u_0\|_{H^1}$  and  $\|u_1\|_2$ ) such that if the initial data  $u_0 \in \widetilde{W} \cap H^2(\mathbf{R}^N)$  and  $u_1 \in H^1(\mathbf{R}^N)$  satisfy  $\|\Delta u_0\|_2 + \|\nabla u_1\|_2 \leq \varepsilon_0$ , it holds that  $T_m = +\infty$ . Furthermore, it follows that*

$$E(u(t), u_t(t)) \leq \frac{C}{1+t} \quad \text{on } [0, \infty)$$

for some positive constant  $C$ .

The proof of Theorem 3.1 is completely analogous to that of Matsuyama and Ikehata [3] which is treated in three space dimensions. Since we use the imbedding  $H^2(\mathbf{R}^N) \subset L^\infty(\mathbf{R}^N)$  ( $1 \leq N \leq 3$ ) to obtain  $H^2$  bounds, we must restrict the space dimensions to  $1 \leq N \leq 3$ . We shall omit the details.

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**Acknowledgement.** The author would like to express sincere gratitude to Professor K. Mochizuki for his constant encouragement.

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# Remarks on Decay Properties of Exterior Stationary Navier-Stokes Flows

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## 1. Introduction and Results

Let  $\Omega$  be a smooth exterior domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , with  $0 \notin \bar{\Omega}$ . We discuss decay properties of solutions to the exterior stationary problem for the incompressible Navier-Stokes equations of the following form :

$$(S) \quad \begin{aligned} -\Delta \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w} &= \mathbf{f} - \nabla p && \text{in } \Omega, \\ \nabla \cdot \mathbf{w} &= 0 && \text{in } \Omega, \\ \mathbf{w}|_{\partial\Omega} &= \mathbf{w}^*, && \lim_{|x| \rightarrow \infty} \mathbf{w} = 0. \end{aligned}$$

Here,  $\partial\Omega$  is the (smooth) boundary of  $\Omega$ ;  $\mathbf{w}^* \in C^2(\partial\Omega)$ ; and the external force  $\mathbf{f}$  is assumed to be of the form  $\mathbf{f} = (f_1, \dots, f_n)$  with

$$f_j = (\nabla \cdot \mathbf{F})_j \equiv \sum_{k=1}^n \partial_k F_{jk}, \quad (\partial_k = \partial/\partial x_k, \quad k = 1, \dots, n),$$

where  $\mathbf{F} = (F_{jk})$  are given smooth functions satisfying

$$(1.1) \quad |F_{jk}(x)| \leq C|x|^{1-n}, \quad |\nabla F_{jk}(x)| \leq C|x|^{-n}.$$

In this paper we use the standard notation of vector analysis :

$$\begin{aligned} \nabla &= (\partial_1, \dots, \partial_n), & \partial_j &= \partial/\partial x_j \quad (j = 1, \dots, n), \\ \Delta \mathbf{u} &= \sum_{j=1}^n \partial_j^2 \mathbf{u}, & \nabla \cdot \mathbf{u} &= \sum_{j=1}^n \partial_j u_j, & \mathbf{u} \cdot \nabla \mathbf{u} &= \sum_{j=1}^n u_j \partial_j \mathbf{u}. \end{aligned}$$

Leray [18] proved the existence of a solution to (S) with finite Dirichlet integral for an arbitrary  $\mathbf{F}$  and  $\mathbf{w}^*$ . However, the decay properties of solutions given in [18] are not yet understood so well. It has recently been proved in [5, 25] that if  $\mathbf{w}^*$  and  $\mathbf{F}$  are small enough in an appropriate sense, then problem (S) admits a unique regular solution  $\mathbf{w}$  such that

$$(1.2) \quad |\mathbf{w}(x)| \leq C|x|^{2-n}, \quad |\nabla \mathbf{w}(x)| \leq C|x|^{1-n}.$$

Novotny and Padula [25] discussed problem (S) in case  $n = 3$  with  $\mathbf{w}^* \equiv 0$  and deduced the existence of a solution  $\mathbf{w}$  satisfying (1.2) by finding new estimates for the volume potential associated with the Stokes system. Borchers and Miyakawa [5] extended the result of [25] to the case  $n \geq 3$  with nonvanishing  $\mathbf{w}^*$  with the aid of the Schauder estimates for the boundary layer potentials as developed by Wiegner [37]. The decay estimate (1.2) improves a result of Finn [9, 10]. Miyakawa [21] discussed the roles played by conditions like (1.2) in the uniqueness question of exterior stationary flows, applying an argument given in [14].

Condition (1.2) implies

$$(1.2') \quad \mathbf{w} \in L_w^{n/(n-2)}(\Omega) \cap L^\infty(\Omega), \quad \nabla \mathbf{w} \in L_w^{n/(n-1)}(\Omega) \cap L^\infty(\Omega),$$

where and in what follows  $L_w^r = L^{r, \infty}$  stands for the weak  $L^r$  spaces (see [33]). Kozono and Yamazaki [16] discussed the existence problem within the framework of weak  $L^r$  spaces under the boundary condition  $\mathbf{w}^* = 0$ . On the other hand, we know by [5, Sect. 2] that if  $\mathbf{F} \in L^{n/(n-1)}(\Omega) \cap L^\infty(\Omega)$  and if  $\mathbf{w}$  satisfies the condition

$$(1.3) \quad \mathbf{w} \in L^{n/(n-2)}(\Omega) \cap L^\infty(\Omega), \quad \nabla \mathbf{w} \in L^{n/(n-1)}(\Omega) \cap L^\infty(\Omega),$$

which is a little more stringent than (1.2) or (1.2'), then the following *vanishing flux condition* is deduced :

$$(1.4) \quad \int_{\partial\Omega} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F}) dS = 0.$$

Here and in what follows  $\nu$  is the unit outward normal to  $\partial\Omega$ , and  $\mathbf{T}[\mathbf{w}, p] = (T_{jk}[\mathbf{w}, p])_{j,k=1}^n$  is the stress tensor associated to the flow  $\{\mathbf{w}, p\}$ , with components

$$T_{jk}[\mathbf{w}, p] = -\delta_{jk}p + (\partial_j w_k + \partial_k w_j) = -\delta_{jk}p + 2\varepsilon_{jk}(\mathbf{w}).$$

In this paper we are mainly interested in the converse of the above statement, as well as in some related questions. To be more precise, we shall prove the following results.

**Theorem A.** (i). *Let  $n \geq 4$ ,  $\mathbf{F} \in L^{n/(n-1)}(\Omega)$ , and let  $\mathbf{w}$  satisfy (1.2). Then condition (1.4) implies (1.3).*

(ii). *Let  $n = 3$  and let  $\mathbf{w}$  satisfy (1.2). If  $\mathbf{F} \in L^{3/2}(\Omega)$  and if the functions  $\mathbf{w}^*$ ,  $\mathbf{w}$  and  $\mathbf{F}$  are small in the sense as specified in the proof, then condition (1.4) implies (1.3).*

**Theorem B.** Let  $n \geq 3$  and  $\mathbf{F} \in \mathbf{L}^r(\Omega)$  for all  $r$  with  $1 < r \leq \infty$ . If  $\mathbf{w} \in \mathbf{L}^{n/(n-2)}(\Omega)$  and  $\nabla \mathbf{w} \in \mathbf{L}^{n/(n-1)}(\Omega)$ , then  $\nabla \mathbf{w} \in \mathbf{L}^r(\Omega)$  for all  $1 < r \leq \infty$ . If, in addition,  $\mathbf{F} \in \mathbf{L}^1(\Omega)$ , then  $\nabla \mathbf{w} \in \mathbf{L}_w^1(\Omega)$ .

**Theorem C.** Let  $n = 3$ ,  $\mathbf{F} \in \mathbf{L}^{3/2}$ , and let  $\mathbf{w}$  satisfy (1.2). If  $\nabla \mathbf{w}$  is in the closure in the norm  $\|\nabla \varphi\|_{3/2, \mathbf{w}}$  of the set of smooth solenoidal fields  $\varphi$  with compact support in  $\bar{\Omega}$ , then  $\mathbf{w}$  satisfies (1.4); furthermore, (1.4) implies (1.3). Here  $\|\cdot\|_{r, \mathbf{w}}$  is the norm of the space  $\mathbf{L}_w^r(\Omega)$ .

**Theorem D.** Let  $n = 3$ ,  $\mathbf{F} \in \mathbf{L}^{3/2}(\Omega)$ , and let  $\mathbf{w}$  satisfy (1.2). Then  $\mathbf{w}$  satisfies (1.4) if and only if

$$(1.5) \quad 2\langle \varepsilon(\mathbf{w}), \varepsilon(\varphi) \rangle = -\langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla \varphi \rangle,$$

for all  $\varphi$  with  $\nabla \varphi \in \mathbf{L}^{3,1}(\Omega)$ ,  $\nabla \cdot \varphi = 0$  and  $\varphi|_{\partial\Omega} = 0$ . Here  $\mathbf{L}^{r,q}$  stands for the Lorentz spaces.

**Corollary E.** Let  $n = 3$  and let  $\mathbf{w}$  satisfy (1.2). For each  $\varphi$  with  $\nabla \varphi \in \mathbf{L}^{3,1}(\Omega)$ ,  $\nabla \cdot \varphi = 0$  and  $\varphi|_{\partial\Omega} = 0$ , there is a constant vector  $c_\varphi \in \mathbb{R}^3$  such that

$$(1.6) \quad \begin{aligned} 2\langle \varepsilon(\mathbf{w}), \varepsilon(\varphi) \rangle &= -\langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla \varphi \rangle \\ &\quad - \int_{\partial\Omega} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F}) \cdot c_\varphi dS. \end{aligned}$$

The constant vector  $c_\varphi$  is characterized by the Sobolev inequality

$$\|\varphi - c_\varphi\|_\infty \leq \frac{1}{3} \|\nabla \varphi\|_{3,1}.$$

In Theorem A, it is of course desirable to remove the smallness assumptions in case  $n = 3$ . Theorem B with  $n = 3$  improves [5, Theorem 2.5 (ii)]. A related result was proved in [15]. In Corollary E, the constant vector  $c_\varphi$  can be chosen arbitrarily. Indeed, as will be shown in the proof of Theorem C, for any given  $c \in \mathbb{R}^3$  there is a function  $\varphi$  with  $\nabla \varphi \in \mathbf{L}^{3,1}(\mathbb{R}^3)$ ,  $\nabla \cdot \varphi = 0$ , and  $\varphi|_{\partial\Omega} = 0$ , such that  $c = c_\varphi$ .

In the case of the stationary problem on the entire space  $\mathbb{R}^n$ ,  $n \geq 3$ , condition (1.4) is always satisfied, with  $\partial\Omega$  replaced by an arbitrary sphere, by an arbitrary solution  $\mathbf{w}$  such that  $\mathbf{w} \in \mathbf{L}^{n/(n-2)}$  and  $\nabla \mathbf{w} \in \mathbf{L}^{n/(n-1)}$ . This fact is effectively applied in [22] to the study of decay properties of solutions on  $\mathbb{R}^n$ .

We next consider the perturbation problem for an exterior stationary flow  $\mathbf{w}$  satisfying the decay property :

$$(1.7) \quad |\mathbf{w}| \leq C/|x|, \quad |\nabla \mathbf{w}| \leq C/|x|^2.$$

Let  $\mathbf{a}$  be an initial perturbation to the stationary flow  $\mathbf{w}$ . The time-evolution  $\mathbf{u}$  of  $\mathbf{a}$  is then governed by the equations

$$(1.8) \quad \begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{w} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{w} + \mathbf{u} \cdot \nabla \mathbf{u} &= \Delta \mathbf{u} - \nabla p \quad (x \in \Omega, \quad t > 0) \\ \nabla \cdot \mathbf{u} &= 0 \quad (x \in \Omega, \quad t \geq 0) \\ \mathbf{u}|_{\partial\Omega} &= 0, \quad \lim_{|x| \rightarrow \infty} \mathbf{u}(x, t) = 0 \\ \mathbf{u}|_{t=0} &= \mathbf{a}. \end{aligned}$$

As in the case of the standard nonstationary Navier-Stokes problem, *i.e.*, the case where  $\mathbf{w} = 0$ , one can consider two notions of solution of (1.8), the *weak solution* and the *strong solution*. We shall discuss the stability properties of a given stationary flow within the frameworks of weak and strong solutions, respectively. To this end, let  $C_{0,\sigma}^\infty = C_{0,\sigma}^\infty(\Omega)$  be the set of compactly supported smooth solenoidal vector fields in  $\Omega$ , and for  $1 < r < \infty$  we denote by  $L_\sigma^r = L_\sigma^r(\Omega)$  the  $L^r$ -closure of  $C_{0,\sigma}^\infty$ . Then we have the following *Helmholtz decomposition* ([20, 30]) :

$$(1.9) \quad L^r(\Omega) = L_\sigma^r \oplus G^r$$

with

$$(1.10) \quad \begin{aligned} L_\sigma^r &= \{\mathbf{u} \in L^r(\Omega) : \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} \cdot \nu|_{\partial\Omega} = 0\}, \\ G^r &= \{\nabla p \in L^r(\Omega) : p \in L_{\text{loc}}^r(\bar{\Omega})\}. \end{aligned}$$

Using the (bounded) projection  $P = P_r : L^r(\Omega) \rightarrow L_\sigma^r$  associated to decomposition (1.8), we can rewrite (1.7) in the form

$$(1.11) \quad \frac{d\mathbf{u}}{dt} + A\mathbf{u} + B\mathbf{u} = 0, \quad \mathbf{u}(0) = \mathbf{a},$$

where

$$A\mathbf{u} = -P\Delta\mathbf{u}$$

is the *Stokes operator* on  $\Omega$  defined on

$$D(A) = W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L_\sigma^r$$

and

$$B\mathbf{u} = P(\mathbf{w} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{w}).$$

Problem (1.11) is then rewritten in the form of the integral equation

$$(1.11') \quad \mathbf{u}(t) = e^{-tL}\mathbf{a} - \int_0^t e^{-(t-s)L} P(\mathbf{u} \cdot \nabla \mathbf{u})(s) ds$$

where

$$L = A + B, \quad D(L) = D(A)$$

and  $\{e^{-tL}\}_{t \geq 0}$  is the semigroup generated by  $-L$ .

Given an  $\mathbf{a} \in \mathbf{L}_\sigma^2$ , a weakly continuous function  $\mathbf{u}$  from  $[0, \infty)$  to  $\mathbf{L}_\sigma^2$  is called a weak solution of (1.11) if  $\mathbf{u}(0) = \mathbf{a}$  ;

$$(1.12) \quad \mathbf{u} \in L^\infty(0, \infty ; \mathbf{L}_\sigma^2), \quad \nabla \mathbf{u} \in L^2(0, \infty ; \mathbf{L}^2) ;$$

and, with  $0 \leq s \leq t < \infty$ ,

$$(1.13) \quad \langle \mathbf{u}(t), \varphi(t) \rangle - \langle \mathbf{u}(s), \varphi(s) \rangle + \int_s^t [\langle \nabla \mathbf{u}, \nabla \varphi \rangle + \langle B\mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}, \varphi \rangle] d\tau = \int_s^t \langle \mathbf{u}, \varphi' \rangle d\tau,$$

for all  $\varphi \in C_0^1([0, \infty) ; \mathbf{L}_\sigma^2 \cap \mathbf{L}_\sigma^n)$  such that  $\nabla \varphi \in C_0([0, \infty) ; \mathbf{L}^2)$ . The condition  $\varphi \in \mathbf{L}_\sigma^n$  is needed for the term  $\langle \mathbf{u} \cdot \nabla \mathbf{u}, \varphi \rangle$  to be finite. Indeed, the Hölder and Sobolev inequalities then imply that

$$|\langle \mathbf{u} \cdot \nabla \mathbf{u}, \varphi \rangle| \leq \|\mathbf{u}\|_{2n/(n-2)} \|\nabla \mathbf{u}\|_2 \|\varphi\|_n \leq C \|\nabla \mathbf{u}\|_2^2 \|\varphi\|_n < \infty.$$

See [17, 19] for the details of the theory of weak solutions. Since we are dealing only with the case  $n \geq 3$ , the uniqueness and the regularity of the weak solutions are open problems.

To state our  $L^2$ -stability results, we use the notation :

$$\|\mathbf{w}\| = \sup(|x| \cdot |\mathbf{w}(x)|), \quad \|\nabla \mathbf{w}\| = \sup(|x|^2 \cdot |\nabla \mathbf{w}(x)|)$$

and we write  $f \in L_w^q(\Omega)$  for  $1 \leq q < \infty$  if

$$\|f\|_{q,w} = \sup_{t>0} t |\{x \in \Omega : |f(x)| > t\}|^{1/q} < \infty,$$

where  $|E|$  is the  $n$ -dimensional Lebesgue measure of a measurable set  $E \subset \Omega$ .

**Theorem F.** *Let  $\mathbf{w}$  be an exterior stationary flow satisfying (1.7). There is a constant  $C_n$  with  $0 < C_n < (n-2)/2$  such that if*

$$\|\mathbf{w}\| < C_n$$

*then  $\mathbf{w}$  is  $L^2$ -stable in the following sense :*

(i) *For each  $\mathbf{a} \in \mathbf{L}_\sigma^2$ , problem (1.11) admits a weak solution  $\mathbf{u}$  defined for all  $t \geq 0$  such that*

$$(1.13) \quad \lim_{t \rightarrow \infty} \|\mathbf{u}(t)\|_2 = 0.$$

(ii) *For each  $0 < \delta < 1/4$  there is a positive number  $\eta = \eta(\delta)$  so that if*

$$\|\mathbf{w}\| < \min(C_n, \eta)$$

and if the initial perturbation  $\mathbf{a}$  satisfies

$$\|e^{-tL}\mathbf{a}\|_2 = O(t^{-\alpha}) \quad \text{as } t \rightarrow \infty$$

for some  $\alpha > 0$ , then as  $t \rightarrow \infty$ ,

$$(1.14) \quad \|\mathbf{u}(t)\|_2 = O(t^{-\beta}), \quad \beta = \min(\alpha, n/4 - \delta).$$

(iii) Suppose further that  $\nabla \mathbf{w} \in \mathbf{L}_w^q$  for some  $n' \leq q < n/2$  in case  $n \geq 4$ , and that  $\nabla \mathbf{w} \in \mathbf{L}^q$  for some  $1 < q < 3/2$  in case  $n = 3$ . Then there is a  $\mu = \mu(n) > 0$  so that if

$$\|\mathbf{w}\| + \|\nabla \mathbf{w}\|_{q,w} + \|\nabla \mathbf{w}\|_\infty \leq \mu \quad (n \geq 4)$$

or if

$$\|\mathbf{w}\| + \|\nabla \mathbf{w}\|_q + \|\nabla \mathbf{w}\|_\infty \leq \mu \quad (n = 3),$$

then the following results hold: Let  $\mathbf{a} \in \mathbf{L}_\sigma^2$  satisfy

$$\|e^{-tL}\mathbf{a}\|_2 = O(t^{-\alpha}) \quad \text{as } t \rightarrow \infty$$

for some  $\alpha > 0$ . Then, as  $t \rightarrow \infty$ ,

$$(1.15) \quad \|\mathbf{u}(t)\|_2 = O(t^{-\gamma}), \quad \gamma = \min(\alpha, n/4).$$

Observe that (1.15) slightly improves (1.14). This is because the assumptions on  $\mathbf{w}$  is more stringent in (iii) than those in (ii). Indeed, from (1.2) and Theorems A–D and Corollary E, we see that if  $n = 3$ , the assumptions of (iii) imply the vanishing total flux condition (1.4).

One can also discuss the time-evolution of the initial perturbation  $\mathbf{a} \in \mathbf{L}_\sigma^2 \cap \mathbf{L}_\sigma^r$  in the Banach space  $\mathbf{L}_\sigma^r$ ,  $1 < r \leq n/(n-1)$ . Namely, as noticed in [5], it is possible to apply the techniques given in [2] and show that if  $\mathbf{a} \in \mathbf{L}_\sigma^r \cap \mathbf{L}_\sigma^2$ , then the time-evolution  $\mathbf{u}(t)$  of  $\mathbf{a}$  given in Theorem F belongs to  $\mathbf{L}_\sigma^r$  for all  $t \geq 0$  and tends to 0 in  $L^r$  norm as  $t \rightarrow \infty$ . In the case where  $r = 1$ , however, nothing is known on the exterior problem. In order to see what happens in this limit case  $r = 1$ , we finally treat stationary flows  $\mathbf{w}$  in the whole space  $\mathbb{R}^n$ , with  $n \geq 3$ , satisfying

$$(1.16) \quad \mathbf{w} \in \mathbf{L}^n(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n), \quad \nabla \mathbf{w} \in \mathbf{L}^{n/2}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n).$$

Assuming that  $\mathbf{F}$  is small, one can construct stationary flows with property (1.16) by means of hydrodynamic potentials, as given in [22]. Kozono and Ogawa [13] discusses stability of stationary flows satisfying (1.16) in an exterior domain. Note, however, that in the case of the exterior problem, (1.16) automatically implies the vanishing flux condition (1.4). In the case of the problem in  $\mathbb{R}^n$ , one can discuss the stability of  $\mathbf{w}$  in the Hardy space  $\mathbf{H}^1(\mathbb{R}^n)$ ,

by employing the methods as developed in [22]. It is now well known (see [32]) that the Hardy space  $\mathbf{H}^1(\mathbb{R}^n)$  is a good substitute for the usual  $L^1$  space with respect to the duality properties and the action of singular integrals and Riesz potentials. Indeed, we shall show that the techniques used in proving Theorem F can be applied with no essential change to the stability analysis, in the Hardy space, of stationary flows satisfying (1.6).

We now recall the definition of the Hardy space  $\mathbf{H}^1(\mathbb{R}^n)$  ([32]). Let

$$\varphi \in \mathcal{S} \quad \text{with} \quad \int \varphi dx = 1$$

and set

$$\varphi_t(x) = t^{-n} \varphi(x/t) \quad (t > 0).$$

A tempered distribution  $f$  belongs to the Hardy space  $\mathbf{H}^1(\mathbb{R}^n)$  if and only if

$$f^+ \equiv \sup_{t>0} |\varphi_t * f| \in L^1(\mathbb{R}^n).$$

The space  $\mathbf{H}^1(\mathbb{R}^n)$  is a Banach space with norm  $\|f\|_{\mathbf{H}^1} = \|f^+\|_1$  and (see [32]) any change of the function  $\varphi$  gives rise to an equivalent norm. We also know ([32]) that  $f$  is in  $\mathbf{H}^1(\mathbb{R}^n)$  if and only if

$$f \in L^1 \quad \text{and} \quad R_j f \in L^1,$$

where  $R_j$  ( $j = 1, \dots, n$ ) denotes the Riesz transforms ([31, 32]). Furthermore,  $\|f\|_{\mathbf{H}^1}$  is equivalent to the norm  $\|f\|_1 + \sum_j \|R_j f\|_1$  and

$$(1.17) \quad f \in \mathbf{H}^1(\mathbb{R}^n) \quad \text{implies} \quad \int f(x) dx = 0.$$

**Theorem G.** *Let  $n \geq 3$  and let a stationary flow  $\mathbf{w}$  on  $\mathbb{R}^n$ .*

(i) *Suppose  $\mathbf{w}$  satisfies (1.16). Then, there is a constant  $C_n > 0$  so that if*

$$\|\mathbf{w}\|_n + \|\nabla \mathbf{w}\|_{n/2} < C_n,$$

*then for each  $\mathbf{a} \in \mathbf{H}^1(\mathbb{R}^n) \cap \mathbf{L}_\sigma^2(\mathbb{R}^n)$ , problem (1.11) on the whole space  $\mathbb{R}^n$  admits a weak solution  $\mathbf{u}$  such that*

$$(1.18) \quad \mathbf{u}(t) \in \mathbf{H}^1(\mathbb{R}^n) \quad \text{for all } t \geq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\mathbf{u}(t)\|_{\mathbf{H}^1} = 0.$$

(ii) *Suppose that*

$$(1.16') \quad \mathbf{w} \in \mathbf{L}^{(n,1)}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n), \quad \nabla \mathbf{w} \in \mathbf{L}^{(n/2,1)}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n),$$

*where  $\mathbf{L}^{(p,q)}$  denotes the Lorentz spaces [33, 35]. Then there is a constant  $C'_n > 0$  so that if*

$$\|\mathbf{w}\|_{n,1} + \|\nabla \mathbf{w}\|_{n/2,1} < C'_n,$$

then for each  $\mathbf{a} \in \mathbf{L}_\sigma^1(\mathbb{R}^n) \cap \mathbf{L}_\sigma^2(\mathbb{R}^n)$ , problem (1.11) on the space  $\mathbb{R}^n$  admits a weak solution  $\mathbf{u}$  such that

$$(1.18') \quad \mathbf{u}(t) \in \mathbf{L}_\sigma^1(\mathbb{R}^n) \quad \text{for all } t \geq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\mathbf{u}(t)\|_1 = 0.$$

Theorem G is of course valid when  $\mathbf{w} = 0$ . This case was discussed [22], with the aid of the results of [8], and explicit decay rates were deduced for some specific initial perturbations. However, in general case we have no results on the decay rates. We shall prove Theorem G, applying a perturbation technique to the results given in [22]. The details are given in [23].

## 2. Preliminaries

Let  $D$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , with local Lipschitz boundary  $\partial D$ . To prove our main results, we frequently need the following, which is due to Bogovski [1].

**Lemma 2.1.** *There exists a linear operator  $S_D$  from  $C_c^\infty(D)$  to  $C_c^\infty(D)$  such that :*

(i) *For  $m = 0, 1, 2, \dots$ , and  $1 < r < \infty$ ,*

$$(2.1) \quad \|\nabla^{m+1} S_D f\|_{r,D} \leq C \|\nabla^m f\|_{r,D}, \quad (f \in C_c^\infty(D)),$$

*with some constant  $C = C(r, m, D) > 0$ .*

(ii) *The vector field  $S_D f$  satisfies*

$$(2.2) \quad \nabla \cdot S_D f = f \quad \text{in } D \quad \text{if} \quad \int_D f dx = 0.$$

(iii) *If  $t \neq 0$ ,  $y \in \mathbb{R}^n$  and*

$$D_t = \{(1-t)y + tx ; x \in D\},$$

*then  $C(r, m, D_t) = C(r, m, D)$ , where  $C(r, m, D)$  is the constant in estimate (2.1).*

See [7] for a full proof of Lemma 2.1. We will call  $S_D$  the *Bogovski operator associated to the domain  $D$* . Applying the real interpolation theory of Banach spaces to Lemma 2.1 gives

**Corollary 2.2.** *Let  $1 < r < \infty$ ,  $1 \leq q \leq \infty$ , and*

$$\mathbf{X}_{r,q}(D) = \{\varphi \in \mathbf{L}^{r,q}(D) ; \nabla \varphi \in \mathbf{L}^{r,q}(D), \varphi|_{\partial D} = 0\}.$$

(i) The Bogovski operator  $S_D$  associated to  $D$  maps  $L^{r,q}(D)$  to  $\mathbf{X}_{r,q}(D)$ , and satisfies the estimate

$$(2.3) \quad \|\nabla S_D f\|_{r,q,D} \leq C \|f\|_{r,q,D},$$

with some constant  $C = C(r, q, D) > 0$ .

(ii) The vector field  $S_D f$ ,  $f \in L^{r,q}(D)$ , satisfies (2.2).

(iii) If  $t \neq 0$ ,  $y \in \mathbb{R}^n$  and

$$D_t = \{(1-t)y + tx; \quad x \in D\},$$

then  $C(r, q, D) = C(r, q, D_t)$ , where  $C(r, q, D)$  is the constant in estimate (2.3).

In the next section we will systematically apply Corollary 2.2 with  $n = 3$ ,  $r = 3$  and  $q = 1$ , in order to prove Theorems C, D and Corollary E.

### 3. Decay Properties of Exterior Stationary Flows

This section proves Theorems A–D and Corollary E. To this end we introduce the Stokes fundamental solution  $\mathbf{E} = (E_{jk})$ ,  $\mathbf{Q} = (Q_j)$ , on  $\mathbb{R}^n$ ,  $n \geq 3$ :

$$(3.1) \quad \begin{aligned} E_{jk}(x) &= \frac{1}{2(n-2)\omega_{n-1}|x|^{n-2}} \left[ \delta_{jk} + (n-2) \frac{x_j x_k}{|x|^2} \right], \\ Q_j(x) &= \frac{x_j}{\omega_{n-1}|x|^n}, \end{aligned}$$

where  $\omega_{n-1}$  is the area of the unit sphere  $\{|x| = 1\}$  in  $\mathbb{R}^n$ . Formula (3.1) is easily deduced via simple calculation involving the Fourier transform; see [17, 26]. The decay property (1.2) allows us for invoking  $\mathbf{E}$  and  $\mathbf{Q}$  to represent the solution  $\mathbf{w}$  in the form

$$(3.2) \quad \begin{aligned} \mathbf{w} &= \mathbf{E} \cdot (\nabla \cdot (\mathbf{F} - \mathbf{w} \otimes \mathbf{w})) + \int_{\partial\Omega} \mathbf{E} \cdot \nu \cdot \mathbf{T}[\mathbf{w}, p] dS \\ &\quad + \int_{\partial\Omega} \mathbf{w}^* \cdot \mathbf{T}[\mathbf{E}, \mathbf{Q}] \cdot \nu dS \\ &= (\nabla \mathbf{E}) \cdot (\mathbf{F} - \mathbf{w} \otimes \mathbf{w}) + \int_{\partial\Omega} \mathbf{E} \cdot \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F}) dS \\ &\quad + \int_{\partial\Omega} \mathbf{w}^* \cdot \mathbf{T}[\mathbf{E}, \mathbf{Q}] \cdot \nu dS \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Furthermore, without loss of generality we may assume that  $\bar{\Omega} \subset \{x ; |x| > 1\}$ .

*Proof of Theorem A.* (i) We easily see from the definition of  $T[\mathbf{E}, \mathbf{Q}]$  that

$$|I_3| = O(|x|^{1-n}), \quad |\nabla I_3| = O(|x|^{-n}), \quad \text{as } |x| \rightarrow \infty,$$

so  $I_3 \in L_w^{n/(n-1)}(\Omega) \cap L^\infty(\Omega)$  and  $\nabla I_3 \in L_w^1(\Omega) \cap L^\infty(\Omega)$ . We next invoke (1.4) to write the single layer potential  $I_2$  in the form

$$I_2 = \int_{\partial\Omega} \widetilde{\mathbf{E}}(x, y) \cdot \nu_y \cdot (T[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F})(y) dS_y,$$

where

$$\widetilde{\mathbf{E}}(x, y) = \mathbf{E}(x - y) - \mathbf{E}(x) = \int_0^1 \frac{d}{d\theta} \mathbf{E}(x - \theta y) d\theta.$$

Since

$$|\widetilde{\mathbf{E}}(x, y)| \leq C|x|^{1-n}, \quad |\nabla_x \widetilde{\mathbf{E}}(x, y)| \leq C|x|^{-n} \quad \text{for large } |x| \text{ and } y \in \partial\Omega,$$

with  $C$  independent of  $x$  and  $y$ , we conclude that  $I_2 \in L_w^{n/(n-1)}(\Omega) \cap L^\infty(\Omega)$  and  $\nabla I_2 \in L_w^1(\Omega) \cap L^\infty(\Omega)$ . It thus suffices to examine the integrability properties of

$$(3.3) \quad \nabla I_1 = (\nabla^2 \mathbf{E}) \cdot (\mathbf{F} - \mathbf{w} \otimes \mathbf{w}).$$

Here and in what follows we define  $\mathbf{F} \equiv 0$  and  $\mathbf{w} \equiv 0$  outside  $\Omega$ , so that the right-hand side of (3.3) is the convolution of  $\mathbf{F} - \mathbf{w} \otimes \mathbf{w}$  and the Calderón–Zygmund kernel  $\nabla^2 \mathbf{E}$  ([31, 32]). Now, the assumption (1.2) implies

$$\mathbf{w} \in L^\infty(\Omega) \cap L_w^{n/(n-2)}(\Omega) \subset L^\infty(\Omega) \cap L_w^2(\Omega) \quad \text{if } n \geq 4.$$

So we see that

$$(3.4) \quad \mathbf{w} \otimes \mathbf{w} \in L_w^1(\Omega) \cap L^\infty(\Omega) \subset L^{n/(n-1)}(\Omega) \quad \text{provided } n \geq 4.$$

Since  $\nabla^2 \mathbf{E}$  is the Calderón–Zygmund kernel, it follows from (3.4) that if  $\mathbf{F} \in L^{n/(n-1)}(\Omega)$ , then

$$(\nabla^2 \mathbf{E}) \cdot (\mathbf{F} - \mathbf{w} \otimes \mathbf{w}) \in L^{n/(n-1)}(\Omega) \quad \text{provided } n \geq 4.$$

This implies  $\nabla \mathbf{w} \in L^{n/(n-1)}(\Omega)$  and therefore  $\mathbf{w} \in L^{n/(n-2)}(\Omega)$ . The proof of (i) is complete.

(ii) Suppose  $n = 3$  so that  $n/(n-1) = n/2 = 3/2$ . We again invoke formula (3.2) :

$$\mathbf{w} = (\nabla \mathbf{E}) \cdot (\mathbf{F} - \mathbf{w} \otimes \mathbf{w}) + \mathbf{w}_0 \quad (\mathbf{w}_0 = I_2 + I_3),$$

and the corresponding formula for the derivatives :

$$(3.2') \quad \nabla \mathbf{w} = (\nabla^2 \mathbf{E}) \cdot (\mathbf{F} - \mathbf{w} \otimes \mathbf{w}) + \nabla \mathbf{w}_0.$$

Recall that (1.4) implies

$$|\mathbf{w}_0| = O(|x|^{-2}), \quad |\nabla \mathbf{w}_0| = O(|x|^{-3}) \quad \text{as } |x| \rightarrow \infty,$$

and so

$$(3.5) \quad \mathbf{w}_0 \in \mathbf{L}_w^{3/2}(\Omega) \cap \mathbf{L}^\infty(\Omega), \quad \nabla \mathbf{w}_0 \in \mathbf{L}_w^1(\Omega) \cap \mathbf{L}^\infty(\Omega).$$

Consider next the iteration scheme

$$(3.6) \quad \mathbf{w}_{k+1} = (\nabla \mathbf{E}) \cdot (\mathbf{F} - \mathbf{w}_k \otimes \mathbf{w}_k) + \mathbf{w}_0 \quad (k = 0, 1, 2, \dots).$$

Since  $|\nabla \mathbf{E}| \leq C|x|^{-2}$  if  $n = 3$ , the Hardy–Littlewood–Sobolev inequality ([31, 32]) for the fractional integration yields

$$\|\mathbf{w}_{k+1}\|_{3,\Omega} \leq M(\|\mathbf{F}\|_{3/2,\Omega} + \|\mathbf{w}_k\|_{3,\Omega}^2) + \|\mathbf{w}_0\|_{3,\Omega}$$

with  $M > 0$  depending only on the fundamental solution  $\mathbf{E}$ . Thus, if

$$(3.7) \quad \|\mathbf{w}_0\|_{3,\Omega} + M\|\mathbf{F}\|_{3/2,\Omega} < \frac{1}{4M^2},$$

then we get

$$(3.8) \quad \|\mathbf{w}_k\|_{3,\Omega} \leq K \equiv \frac{1 - \sqrt{1 - 4M^2(\|\mathbf{w}_0\|_{3,\Omega} + M\|\mathbf{F}\|_{3/2,\Omega})}}{2M} < \frac{1}{2M}$$

for all  $k \geq 1$ . Furthermore, from

$$\begin{aligned} \mathbf{v}_k \equiv \mathbf{w}_{k+1} - \mathbf{w}_k &= -(\nabla \mathbf{E}) \cdot (\mathbf{w}_k \otimes \mathbf{w}_k - \mathbf{w}_{k-1} \otimes \mathbf{w}_{k-1}) \\ &= -(\nabla \mathbf{E}) \cdot (\mathbf{v}_{k-1} \otimes \mathbf{w}_k + \mathbf{w}_{k-1} \otimes \mathbf{v}_{k-1}), \quad (\mathbf{v}_0 = \mathbf{w}_0), \end{aligned}$$

we see that if  $\mathbf{w}_0$  and  $\mathbf{F}$  satisfy (3.7), then

$$\|\mathbf{v}_k\|_{3,\Omega} \leq M(\|\mathbf{w}_k\|_{3,\Omega} + \|\mathbf{w}_{k-1}\|_{3,\Omega})\|\mathbf{v}_{k-1}\|_{3,\Omega} \leq 2KM\|\mathbf{v}_{k-1}\|_{3,\Omega},$$

so that

$$(3.9) \quad \|\mathbf{v}_k\|_{3,\Omega} \leq (2KM)^k \|\mathbf{w}_0\|_{3,\Omega} \quad (k = 1, 2, \dots).$$

Since  $2KM < 1$  by (3.8), it follows that

$$\sum_{k=0}^{\infty} \|\mathbf{v}_k\|_{3,\Omega} < +\infty.$$

Hence, there exists a function  $\mathbf{v} \in \mathbf{L}^3(\Omega)$  such that, by (3.8) and (3.9),

$$(3.10) \quad \lim_{k \rightarrow \infty} \|\mathbf{w}_k - \mathbf{v}\|_{3,\Omega} = 0 \quad \text{and} \quad \|\mathbf{v}\|_{3,\Omega} \leq K.$$

It is easy to verify that

$$(3.11) \quad \mathbf{v} = (\nabla \mathbf{E}) \cdot (\mathbf{F} - \mathbf{v} \otimes \mathbf{v}) + \mathbf{w}_0,$$

and

$$(3.11') \quad \nabla \mathbf{v} = (\nabla^2 \mathbf{E}) \cdot (\mathbf{F} - \mathbf{v} \otimes \mathbf{v}) + \nabla \mathbf{w}_0.$$

Subtracting (3.11) from (3.2) gives

$$(3.12) \quad \mathbf{w} - \mathbf{v} = (\nabla \mathbf{E}) \cdot (\mathbf{w} \otimes \mathbf{w} - \mathbf{v} \otimes \mathbf{v}) = (\nabla \mathbf{E}) \cdot [(\mathbf{w} - \mathbf{v}) \otimes \mathbf{w} + \mathbf{v} \otimes (\mathbf{w} - \mathbf{v})].$$

Since  $\|\mathbf{v}\|_{3,w,\Omega} \leq \|\mathbf{v}\|_{3,\Omega} \leq K$  by (3.10), applying the weak Hölder inequality ([3, 4]) :

$$\|fg\|_{3/2,w,\Omega} \leq C\|f\|_{3,w,\Omega}\|g\|_{3,w,\Omega}$$

and the weak Young inequality ([3, 4]) :

$$\|f * g\|_{3,w} \leq C\|f\|_{3,w}\|g\|_{3/2,w},$$

we get from (3.12) the estimate

$$\|\mathbf{w} - \mathbf{v}\|_{3,w,\Omega} \leq M_1(\|\mathbf{w}\|_{3,w,\Omega} + K)\|\mathbf{w} - \mathbf{v}\|_{3,w,\Omega},$$

with another constant  $M_1 > 0$  depending only on the fundamental solution  $\mathbf{E}$ . Thus, if

$$M_1(\|\mathbf{w}\|_{3,w,\Omega} + K) < 1,$$

then  $\mathbf{w} = \mathbf{v} \in \mathbf{L}^3(\Omega) \cap \mathbf{L}^\infty(\Omega)$ . The relation (3.11') and the Calderón–Zygmund inequality then imply  $\nabla \mathbf{w} \in \mathbf{L}^{3/2}(\Omega) \cap \mathbf{L}^\infty(\Omega)$ , and this proves (ii).

*Proof of Theorem B.* If  $n \geq 4$ , the assumption implies

$$\mathbf{w} \otimes \mathbf{w} \in \mathbf{L}^1(\Omega) \cap \mathbf{L}^\infty(\Omega).$$

The proof of Theorem A (i) then shows that

$$(\nabla^2 \mathbf{E}) \cdot (\mathbf{w} \otimes \mathbf{w}) \in \mathbf{L}_w^1(\Omega) \cap \mathbf{L}^r(\Omega) \quad \text{for all } 1 < r < \infty.$$

This implies the result in case  $n \geq 4$ .

Suppose next  $n = 3$  and  $\mathbf{F} \in \mathbf{L}^r(\Omega)$  for all  $1 < r \leq \infty$ . We take  $R > 0$  to be fixed later and consider on the domain  $D = \{x \in \Omega ; |x| > R\}$  with  $\bar{D} \subset \Omega$  the formula (3.2) :

$$(3.2'') \quad \mathbf{w} = (\nabla \mathbf{E}) \cdot (\widetilde{\mathbf{F}} - \widetilde{\mathbf{w}} \otimes \widetilde{\mathbf{w}}) + \mathbf{w}_0,$$

where

$$\mathbf{w}_0 = \int_{|x|=R} \mathbf{E} \cdot \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w} \otimes \mathbf{w} + \mathbf{F}) dS + \int_{|x|=R} \mathbf{w} \cdot \mathbf{T}[\mathbf{E}, \mathbf{Q}] \cdot \nu dS.$$

Here and in what follows  $\tilde{u}$  denotes the zero-extension to  $\mathbb{R}^3$  of the function  $u$  defined on  $D$ . Since the divergence theorem and (1.4) together imply

$$\int_{|x|=R} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w} \otimes \mathbf{w} + \mathbf{F}) dS = 0,$$

the same argument as in the proof of Theorem A gives

$$\mathbf{w}_0 \in \mathbf{L}_w^{3/2}(D) \cap \mathbf{L}^\infty(D) \quad \text{and} \quad \nabla \mathbf{w}_0 \in \mathbf{L}_w^1(D) \cap \mathbf{L}^\infty(D).$$

Consider now the linear operator

$$T_w \mathbf{v} = -(\nabla \mathbf{E}) \cdot (\tilde{\mathbf{w}} \otimes \tilde{\mathbf{v}}).$$

Applying the Hardy–Littlewood–Sobolev inequality for the fractional integration gives

$$\|T_w \mathbf{v}\|_{q,D} \leq M_q \|\mathbf{w}\|_{3,D} \|\mathbf{v}\|_{q,D} \quad \text{for all } 3/2 < q < \infty.$$

Now fix  $r$  with  $3/2 < r < 3$  and take  $R > 0$  sufficiently large so that

$$(3.13) \quad M_r \|\mathbf{w}\|_{3,D} < \frac{1}{2} \quad \text{and} \quad M_3 \|\mathbf{w}\|_{3,D} < \frac{1}{2},$$

which is possible since  $\mathbf{w} \in \mathbf{L}^3(\Omega)$  by assumption. Using (3.13), we can solve the *linear* iteration scheme

$$\mathbf{w}_{k+1} = (\nabla \mathbf{E}) \cdot (\tilde{\mathbf{F}} - \tilde{\mathbf{w}} \otimes \tilde{\mathbf{w}}_k) + \mathbf{w}_0, \quad (k = 0, 1, 2, \dots),$$

to get a sequence  $\{\mathbf{w}_k\}$  of functions  $\mathbf{w}_k$  defined on  $D$ , which is bounded and converges in  $\mathbf{L}^3(D) \cap \mathbf{L}^r(D)$  to a function  $\mathbf{v}$  satisfying

$$(3.14) \quad \mathbf{v} = (\nabla \mathbf{E}) \cdot (\tilde{\mathbf{F}} - \tilde{\mathbf{w}} \otimes \tilde{\mathbf{v}}) + \mathbf{w}_0.$$

This, together with (3.2''), implies  $\mathbf{w} - \mathbf{v} = T_w(\mathbf{w} - \mathbf{v})$  and so (3.13) yields

$$\|\mathbf{w} - \mathbf{v}\|_{3,D} \leq \frac{1}{2} \|\mathbf{w} - \mathbf{v}\|_{3,D}.$$

Hence,  $\mathbf{v} = \mathbf{w}$  on  $D$ , and so  $\mathbf{w} \in \mathbf{L}^r(\Omega)$ . On the other hand, (3.14) implies

$$\nabla \mathbf{v} = (\nabla^2 \mathbf{E}) \cdot (\tilde{\mathbf{F}} - \tilde{\mathbf{w}} \otimes \tilde{\mathbf{v}}) + \nabla \mathbf{w}_0.$$

Since  $\mathbf{F}$  and  $\nabla \mathbf{w}_0$  are in  $\mathbf{L}^r(D)$  for all  $1 < r \leq \infty$ , the Calderón–Zygmund theory on singular integrals yields

$$\nabla \mathbf{v} = \nabla \mathbf{w} \in \mathbf{L}^q(D) \quad \text{with } 1/q = 1/r + 1/3,$$

and therefore

$$\nabla \mathbf{w} \in \mathbf{L}^q(\Omega) \quad \text{with } 1/q = 1/r + 1/3.$$

But  $1 < q < 3/2$ ; and so  $\nabla \mathbf{w} \in L^r(\Omega)$  for all  $1 < r \leq \infty$  as shown in [5, Theorem 2.5 (ii)]. Furthermore, if  $\mathbf{F} \in L^1(\Omega)$ , then the Hardy–Littlewood–Sobolev inequality implies that  $\mathbf{w} \in L_w^{3/2}(\Omega)$  and the Calderón–Zygmund theory shows that  $\nabla \mathbf{w} \in L_w^1(\Omega)$ . This completes the proof of Theorem B in case  $n = 3$ .

*Proof of Theorem C.* We first prove (1.4)  $\Rightarrow$  (1.3), assuming that  $\nabla \mathbf{w}$  is in the  $L_w^{3/2}$ -closure of the set of smooth solenoidal fields with compact support in  $\bar{\Omega}$ . To do so, we again invoke the representation

$$(3.2'') \quad \mathbf{w} = (\nabla \mathbf{E}) \cdot (\tilde{\mathbf{F}} - \tilde{\mathbf{w}} \otimes \tilde{\mathbf{w}}) + \mathbf{w}_0,$$

with

$$\mathbf{w}_0 = \int_{|x|=R} \mathbf{E} \cdot \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w} \otimes \mathbf{w} + \mathbf{F}) dS + \int_{|x|=R} \mathbf{w} \cdot \mathbf{T}[\mathbf{E}, \mathbf{Q}] \cdot \nu dS.$$

The assumption on  $\mathbf{w}$  implies that if we set  $D_R = \{x; |x| > R\}$ , then

$$(3.15) \quad \lim_{R \rightarrow \infty} \|\nabla \mathbf{w}\|_{3/2, w, D_R} = 0 \quad \text{and} \quad \lim_{R \rightarrow \infty} \|\mathbf{w}\|_{3, w, D_R} = 0.$$

On the other hand, by the weak Young inequality the linear operator

$$T_w \mathbf{v} = (\nabla \mathbf{E}) \cdot (\tilde{\mathbf{w}} \otimes \tilde{\mathbf{v}})$$

satisfies the estimates

$$\|T_w \mathbf{v}\|_{r, w, D_R} \leq C_r \|\mathbf{w}\|_{3, w, D_R} \|\mathbf{v}\|_{r, w, D_R} \quad (1 < r < \infty)$$

with  $C_r$  depending only on  $r$  and  $\mathbf{E}$ , and so the Marcinkiewicz interpolation theorem ([31, 32]) gives

$$(3.16) \quad \|T_w \mathbf{v}\|_{r, D_R} \leq C'_r \|\mathbf{w}\|_{3, w, D_R} \|\mathbf{v}\|_{r, D_R} \quad (1 < r < \infty)$$

with another constant  $C'_r$  depending only on  $r$  and  $\mathbf{E}$ . Choosing  $R$  sufficiently large so that

$$(3.17) \quad C_3 \|\mathbf{w}\|_{3, w, D_R} < \frac{1}{2} \quad \text{and} \quad C'_3 \|\mathbf{w}\|_{3, w, D_R} < \frac{1}{2},$$

we invoke (3.16) to solve the linear iteration scheme

$$\mathbf{w}_{k+1} = (\nabla \mathbf{E}) \cdot (\tilde{\mathbf{F}} - \tilde{\mathbf{w}} \otimes \tilde{\mathbf{w}}_k) + \mathbf{w}_0 \quad (k = 0, 1, 2, \dots),$$

to get a sequence  $\{\mathbf{w}_k\}$  which converges in  $L^3(D_R)$  to a function  $\mathbf{v}$ , satisfying

$$\mathbf{v} = (\nabla \mathbf{E}) \cdot (\tilde{\mathbf{F}} - \tilde{\mathbf{w}} \otimes \tilde{\mathbf{v}}) + \mathbf{w}_0.$$

As in the proof of Theorem B, we conclude from (3.17) that

$$\|\mathbf{w} - \mathbf{v}\|_{3, w, D_R} \leq \frac{1}{2} \|\mathbf{w} - \mathbf{v}\|_{3, w, D_R}$$

and therefore  $\mathbf{w} = \mathbf{v} \in \mathbf{L}^3(D_R)$ . But, since  $\mathbf{w}$  is a smooth function on  $\bar{\Omega}$ , it follows that  $\mathbf{w} \in \mathbf{L}^3(\Omega)$ . It is now easy to verify that  $\nabla \mathbf{w} \in \mathbf{L}^{3/2}(\Omega)$  by applying the Calderón–Zygmund inequality to the relation

$$\nabla \mathbf{w} = (\nabla^2 \mathbf{E}) \cdot (\mathbf{F} - \mathbf{w} \otimes \mathbf{w}) + \nabla \mathbf{w}_0$$

where

$$\mathbf{w}_0 = \int_{\partial\Omega} \mathbf{E} \cdot \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F}) dS + \int_{\partial\Omega} \mathbf{w}^* \cdot \mathbf{T}[\mathbf{E}, \mathbf{Q}] \cdot \nu dS.$$

This completes the proof of the implication (1.4)  $\Rightarrow$  (1.3).

We next show that if  $\nabla \mathbf{w}$  is in the  $\mathbf{L}_w^{3/2}$ -closure of the set of smooth solenoidal fields with compact support in  $\bar{\Omega}$ , then  $\mathbf{w}$  satisfies (1.4). To do so, let  $\varphi$  be any solenoidal vector field such that  $\nabla \varphi$  is in the Lorentz space  $\mathbf{L}^{3,1}(\Omega)$  and  $\varphi|_{\partial\Omega} = 0$ . We then take  $\psi \in C_c^\infty(\mathbb{R}^3)$  so that  $\psi = 1$  for  $|x| \leq 1$  and  $\psi = 0$  for  $|x| \geq 2$  and set  $\psi_N(x) = \psi(x/N)$ . If we set

$$\varphi_N = \psi_N \varphi - S_N(\varphi \cdot \nabla \psi_N)$$

by means of the Bogovski operator  $S_N$  associated to the domain  $\{N < |x| < 2N\}$ , then Corollary 2.2 (ii) ensures  $\nabla \cdot \varphi_N = 0$ . An integration by parts thus gives

$$(3.18) \quad 2\langle \varepsilon(\mathbf{w}), \varepsilon(\varphi_N) \rangle = -\langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla \varphi_N \rangle.$$

But, due to (i) and (iii) of Corollary 2.2, we have

$$\begin{aligned} |\langle \varepsilon(\mathbf{w}), \varepsilon(\varphi_N - \varphi) \rangle| &\leq |\langle \varepsilon(\mathbf{w}), (1 - \psi_N)\varepsilon(\varphi) \rangle| + |\langle \varepsilon(\mathbf{w}), \varphi \nabla \psi_N \rangle| \\ &\quad + |\langle \varepsilon(\mathbf{w}), \varepsilon(S_N(\varphi \cdot \nabla \psi_N)) \rangle| \\ &\leq |\langle \varepsilon(\mathbf{w}), (1 - \psi_N)\varepsilon(\varphi) \rangle| + C \|\varepsilon(\mathbf{w})\|_{3/2, w, \{N < |x| < 2N\}} \|\varphi\|_\infty \|\nabla \psi_N\|_{3,1} \\ &\leq |\langle \varepsilon(\mathbf{w}), (1 - \psi_N)\varepsilon(\varphi) \rangle| + C \|\varepsilon(\mathbf{w})\|_{3/2, w, \{|x| > N\}} \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ ; and similarly (since  $\mathbf{L}^{3/2} \subset \mathbf{L}_w^{3/2}$ )

$$|\langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla(\varphi_N - \varphi) \rangle| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Note that we have here used the Sobolev inequality

$$(3.19) \quad \|\varphi - c_\varphi\|_\infty \leq \frac{1}{3} \|\nabla \varphi\|_{3,1},$$

with an appropriate  $c_\varphi \in \mathbb{R}^3$ , as well as the fact that

$$\lim_{N \rightarrow \infty} \|\varepsilon(\mathbf{w})\|_{3/2, w, \{|x| > N\}} = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \|\mathbf{w}\|_{3, w, \{|x| > N\}} = 0.$$

This last property follows, via the Sobolev-type inequality (see [5, Sect. 5]) :

$$\|\mathbf{w}\|_{3, w, \{|x| > N\}} \leq C \|\nabla \mathbf{w}\|_{3/2, w, \{|x| > N\}}$$

with  $C > 0$  independent of  $N$ , from the fact that  $\nabla \mathbf{w}$  is in the  $\mathbf{L}_w^{3/2}$ -closure of the set of smooth and compactly supported vector fields in  $\bar{\Omega}$ . Passing to the limit  $N \rightarrow \infty$  in (3.18) now gives

$$(3.20) \quad 2\langle \varepsilon(\mathbf{w}), \varepsilon(\varphi) \rangle = -\langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla \varphi \rangle$$

whenever  $\nabla \varphi \in \mathbf{L}^{3,1}(\Omega)$ ,  $\nabla \cdot \varphi = 0$ , and  $\varphi|_{\partial\Omega} = 0$ .

Now fix an arbitrary  $c \in \mathbb{R}^3$  and take an  $R > 0$  so that  $\{|x| \geq R\} \subset \Omega$ . The argument below is borrowed from [5, Sect. 2]. We take  $\psi \in C_c^\infty(\mathbb{R}^3)$  so that  $\psi(x) = 1$  for  $|x| \geq 2R$  and  $\psi(x) = 0$  for  $|x| \leq R$ , and define

$$\varphi = c\psi - S_R(c \cdot \nabla \psi),$$

in terms of the Bogovski operator  $S_R$  associated to the domain  $\{R < |x| < 2R\}$ . By Corollary 2.2 we have  $\nabla \varphi \in \mathbf{L}^{3,1}(\Omega)$ ,  $\nabla \cdot \varphi = 0$ , and  $\varphi|_{\partial\Omega} = 0$ , which enables us to insert this  $\varphi$  in (3.20). Keeping this fact in mind, we proceed as follows : since

$$\nabla \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w} \otimes \mathbf{w} + \mathbf{F}) = 0,$$

applying the divergence theorem and (3.20) gives

$$\begin{aligned} 0 &= \int_{\Omega \cap \{|x| < 2R\}} \nabla \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w} \otimes \mathbf{w} + \mathbf{F}) \cdot \varphi dx \\ &= \int_{|x|=2R} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w} \otimes \mathbf{w} + \mathbf{F}) \cdot c dS \\ &\quad - 2\langle \varepsilon(\mathbf{w}), \varepsilon(\varphi) \rangle_{\Omega \cap \{|x| < 2R\}} - \langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla \varphi \rangle_{\Omega \cap \{|x| < 2R\}} \\ &= \int_{|x|=2R} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w} \otimes \mathbf{w} + \mathbf{F}) \cdot c dS \\ &\quad - 2\langle \varepsilon(\mathbf{w}), \varepsilon(\varphi) \rangle - \langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla \varphi \rangle \\ &= \int_{|x|=2R} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w} \otimes \mathbf{w} + \mathbf{F}) \cdot c dS. \end{aligned}$$

Since  $c \in \mathbb{R}^3$  is arbitrary, it follows that

$$\int_{|x|=2R} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w} \otimes \mathbf{w} + \mathbf{F}) dS = 0.$$

Applying again the divergence theorem gives (1.4). This completes the proof of Theorem C.

*Proof of Theorem D.* That (1.5)  $\Rightarrow$  (1.4) is shown in the proof of Theorem C ; so we need only show (1.4)  $\Rightarrow$  (1.5). We first show that if  $\psi$  is a scalar function satisfying  $\nabla \psi \in \mathbf{L}^{3,1}(\Omega)$  and  $\psi|_{\partial\Omega} = 0$ , and if  $\tilde{\psi} = \psi$  in  $\Omega$  and  $\tilde{\psi} = 0$  outside  $\Omega$ , then there is a sequence  $\psi_m \in C_c^\infty(\mathbb{R}^3)$  so that

$$(3.21) \quad \lim_{m \rightarrow \infty} \|\nabla(\tilde{\psi} - \psi_m)\|_{3,1} = 0.$$

Here and in what follows  $\|\cdot\|_{r,q}$  is the norm of the Lorentz space  $L^{r,q}(\mathbb{R}^3)$ . Indeed, the relation  $-\Delta\tilde{\psi} = -\nabla \cdot (\nabla\tilde{\psi})$  implies

$$\nabla\tilde{\psi} = -R[R \cdot (\nabla\tilde{\psi})] \quad \text{with} \quad R \cdot (\nabla\tilde{\psi}) \in L^{3,1}(\mathbb{R}^3),$$

where  $R = (R_1, R_2, R_3)$  are the Riesz transforms ([31, 32, 33]). We approximate  $R \cdot (\nabla\tilde{\psi})$  in  $L^{3,1}(\mathbb{R}^3)$  by functions  $h_\ell \in C_c^\infty(\mathbb{R}^3)$  to see that if we set

$$v_\ell = -(-\Delta)^{-1/2}h_\ell,$$

then

$$(3.22) \quad \lim_{\ell \rightarrow \infty} \|\nabla(\tilde{\psi} - v_\ell)\|_{3,1} = 0.$$

Note that  $v_\ell$  is smooth, but its support is not compact. It thus suffices to show that each  $\nabla v_\ell$  is approximated in  $L^{3,1}(\mathbb{R}^3)$  by functions of the form  $\nabla\varphi$  with  $\varphi \in C_c^\infty(\mathbb{R}^3)$ . Take  $\zeta \in C_c^\infty$  so that  $\zeta(x) = 1$  if  $|x| \leq 1$  and  $\zeta(x) = 0$  if  $|x| \geq 2$ , and set  $\zeta_m(x) = \zeta(x/m)$ . Then for any fixed  $\ell$ , the function  $\varphi_m = \zeta_m v_\ell$  is in  $C_c^\infty(\mathbb{R}^3)$  and direct calculation gives

$$\|\nabla(v_\ell - \varphi_m)\|_{3,1} \leq \|(1 - \zeta_m)\nabla v_\ell\|_{3,1} + \|(\nabla\zeta_m)v_\ell\|_{3,1}.$$

Due to the interpolation inequality ([33]) :

$$\|f\|_{3,1} \leq C\|f\|_2^{1/3}\|f\|_4^{2/3},$$

we need only show that

$$(3.23) \quad \lim_{m \rightarrow \infty} \|(1 - \zeta_m)\nabla v_\ell\|_r = 0; \quad \lim_{m \rightarrow \infty} \|(\nabla\zeta_m)v_\ell\|_r = 0$$

for any fixed  $r$  with  $1 < r < \infty$ . The first assertion of (3.23) is obvious from the dominated convergence theorem. As for the second, take  $M > 0$  so that  $\text{supp } h_\ell \subset \{|x| \leq M\}$ . Direct calculation then gives

$$\begin{aligned} \|(\nabla\zeta_m)v_\ell\|_r^r &\leq Cm^{-r} \int_{m \leq |x| \leq 2m} \left( \int_{|y| \leq M} |x-y|^{-2} |h_\ell(y)| dy \right)^r dx \\ &\leq C\|h_\ell\|_1^r m^{-r} (m-M)^{-2r} \int_{m \leq |x| \leq 2m} dx \\ &\leq C'\|h_\ell\|_1^r m^{3(1-r)} \rightarrow 0 \end{aligned}$$

as  $m \rightarrow \infty$ , since  $1 < r < \infty$ . This proves (3.23), and so (3.21) is proved.

Now, given  $\varphi$  with  $\nabla\varphi \in L^{3,1}(\Omega)$ ,  $\nabla \cdot \varphi = 0$  and  $\varphi|_{\partial\Omega} = 0$ , we take  $\psi_m \in C_c^\infty(\mathbb{R}^3)$  so that  $\nabla\psi_m \rightarrow \nabla\tilde{\varphi}$  in  $L^{3,1}(\mathbb{R}^3)$  and so  $\nabla \cdot \psi_m \rightarrow 0$  in  $L^{3,1}(\mathbb{R}^3)$  as  $m \rightarrow \infty$ . Taking numbers  $\varrho_m > 0$

so that  $\text{supp } \psi_m \subset \{|x| < \varrho_m\}$ , and then employing the Bogovski operator  $S_m$  associated to the domain  $\{|x| < \varrho_m\}$ , we define

$$\varphi_m = \psi_m - S_m(\nabla \cdot \psi_m).$$

Due to Lemma 2.1, we then easily see that  $\varphi_m \in C_{0,\sigma}^\infty(\mathbb{R}^3)$  and

$$\begin{aligned} \|\nabla(\tilde{\varphi} - \varphi_m)\|_{3,1} &\leq \|\nabla(\tilde{\varphi} - \psi_m)\|_{3,1} + \|\nabla S_m(\nabla \cdot \psi_m)\|_{3,1} \\ &\leq \|\nabla(\tilde{\varphi} - \psi_m)\|_{3,1} + C\|\nabla \cdot \psi_m\|_{3,1} \rightarrow 0 \end{aligned}$$

as  $m \rightarrow \infty$ . An integration by parts over  $\Omega$  gives

$$2\langle \varepsilon(\mathbf{w}), \varepsilon(\varphi_m) \rangle = \int_{\partial\Omega} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F}) \cdot \varphi_m dS - \langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla \varphi_m \rangle.$$

Since

$$\langle \varepsilon(\mathbf{w}), \varepsilon(\varphi_m) \rangle \rightarrow \langle \varepsilon(\mathbf{w}), \varepsilon(\varphi) \rangle \quad \text{and} \quad \langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla \varphi_m \rangle \rightarrow \langle \mathbf{F} - \mathbf{w} \otimes \mathbf{w}, \nabla \varphi \rangle$$

as  $m \rightarrow \infty$ , it suffices to show that

$$(3.24) \quad \lim_{m \rightarrow \infty} \int_{\partial\Omega} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F}) \cdot \varphi_m dS = 0.$$

To deduce (3.24) we write

$$D = \mathbb{R}^3 \setminus \bar{\Omega}, \quad c_m = |D|^{-1} \int_D \varphi_m dx \quad \text{and} \quad \bar{\varphi}_m = \varphi_m - c_m.$$

Note that (1.4) implies

$$\int_{\partial\Omega} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F}) \cdot \varphi_m dS = \int_{\partial\Omega} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F}) \cdot \bar{\varphi}_m dS.$$

By the Poincaré–Sobolev inequality as given in [5, Sect. 5], we see that, as  $m \rightarrow \infty$ ,

$$(3.25) \quad \|\bar{\varphi}_m\|_{\infty, \partial\Omega} \leq \|\bar{\varphi}_m\|_{\infty, D} \leq C\|\nabla \varphi_m\|_{3,1,D} \rightarrow C\|\nabla \tilde{\varphi}\|_{3,1,D} = 0,$$

where  $\|\cdot\|_{r,q,D}$  is the norm of  $L^{r,q}(D)$ . This implies (3.24), and so Theorem D is proved.

*Proof of Corollary E.* Let  $\varphi$  and  $\varphi_m$  be as in the proof of Theorem D. Since

$$|\varphi_m(x)| \leq \frac{1}{4\pi} \int |x-y|^{-2} |\nabla \varphi_m(y)| dy$$

and since  $|x|^{-2} \in L_w^{3/2}(\mathbb{R}^3)$ , the duality relation between  $L^{3,1}$  and  $L_w^{3/2} = L^{3/2,\infty}$  as given in [33] yields

$$\|\varphi_m\|_\infty \leq \frac{1}{3} \|\nabla \varphi_m\|_{3,1}.$$

It follows that  $\{\varphi_m\}$  converges uniformly on  $\mathbb{R}^3$  to some function  $\varphi'$  satisfying  $\nabla\varphi' = \nabla\tilde{\varphi}$ . Hence,  $\tilde{\varphi} = \varphi' + c_\varphi$  with some  $c_\varphi \in \mathbb{R}^3$ , and so we get the Sobolev inequality (3.19) :

$$\|\varphi - c_\varphi\|_\infty \leq \frac{1}{3}\|\nabla\varphi\|_{3,1}.$$

On the other hand, (3.25) shows that  $\varphi_m - c_m \rightarrow 0$  in  $L^\infty(D)$  ; so  $c_m \rightarrow \varphi' = \tilde{\varphi} - c_\varphi = -c_\varphi$  as  $m \rightarrow \infty$ , since  $\tilde{\varphi} \equiv 0$  on  $D$ . The limit in (3.24) therefore equals

$$-\int_{\partial\Omega} \nu \cdot (\mathbf{T}[\mathbf{w}, p] - \mathbf{w}^* \otimes \mathbf{w}^* + \mathbf{F}) \cdot c_\varphi dS.$$

We thus get equality (1.6) in the same way as in the proof of Theorem D. This completes the proof of Corollary E.

#### 4. Large Time Behavior of $L^2$ Perturbations

In this section we describe an outline of the proof of Theorem F. A detailed proof and related results are given in [5]. Let  $L^* = A + B^*$  be the dual of  $L$ , which is well defined as a closed linear operator on  $\mathbf{L}_\sigma^{r'}$  with  $r' = r/(r-1)$ . The associated semigroups  $\{e^{-tL}\}_{t \geq 0}$  and  $\{e^{-tL^*}\}_{t \geq 0}$  are formally defined by means of the Dunford integrals :

$$e^{-tL} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda + L)^{-1} d\lambda, \quad e^{-tL^*} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda + L^*)^{-1} d\lambda,$$

via an appropriate choice of the path  $\Gamma$  in the complex plane. When  $B = 0$ , *i.e.*, in the case of the Stokes operator  $A$ , it is shown in [6, 11] that the semigroup  $\{e^{-tA}\}_{t \geq 0}$  is defined to be bounded-analytic and satisfies the so-called  $L^q$ - $L^r$  estimates :

$$(4.1) \quad \begin{aligned} \|e^{-tA} \mathbf{a}\|_r &\leq Ct^{-(n/q-n/r)/2} \|\mathbf{a}\|_q \quad (1 < q \leq r < \infty, \quad 1 \leq q < r \leq \infty) \\ \|\nabla e^{-tA} \mathbf{a}\|_r &\leq Ct^{-1/2-(n/q-n/r)/2} \|\mathbf{a}\|_q \quad (1 < q \leq r \leq n, \quad 1 \leq q < r \leq n) \end{aligned}$$

Applying perturbation techniques to the resolvents  $(\lambda + L)^{-1}$  and  $(\lambda + L^*)^{-1}$ , we get

**Lemma 4.1.** (i) *Under the assumption of Theorem F (ii), the semigroup  $\{e^{-tL}\}_{t \geq 0}$  is bounded-analytic and satisfies*

$$(4.2) \quad \begin{aligned} \|e^{-tL} \mathbf{a}\|_r &\leq Ct^{-(n/q-n/r)/2} \|\mathbf{a}\|_q \quad (1 < q \leq r < \infty) \\ \|\nabla e^{-tL} \mathbf{a}\|_r &\leq Ct^{-1/2-(n/q-n/r)/2} \|\mathbf{a}\|_q \quad (1 < q \leq r < n). \end{aligned}$$

*The same is true of the dual semigroup  $\{e^{-tL^*}\}_{t \geq 0}$ .*

(ii) Under the assumption of Theorem F (iii), the semigroup  $\{e^{-tL}\}_{t \geq 0}$  is bounded-analytic and satisfies

$$(4.3) \quad \begin{aligned} \|e^{-tL}\mathbf{a}\|_r &\leq Ct^{-(n/q-n/r)/2}\|\mathbf{a}\|_q \quad (1 < q \leq r < \infty, \quad 1 \leq q < r \leq \infty) \\ \|\nabla e^{-tL}\mathbf{a}\|_r &\leq Ct^{-1/2-(n/q-n/r)/2}\|\mathbf{a}\|_q \quad (1 < q \leq r \leq n, \quad 1 \leq q < r \leq n). \end{aligned}$$

The same is true of  $\{e^{-tL^*}\}_{t \geq 0}$ .

A proof is given in [5]. Applying Lemma 4.1 and properties of fractional powers of  $A$  as given in [3], we immediately obtain

**Lemma 4.2.** Let  $A = \int_0^\infty \lambda dE_\lambda$  be the spectral decomposition of the positive self-adjoint operator  $A$  in  $\mathbf{L}_\sigma^2$ .

(i) Under the assumptions of Theorem F (ii), we have

$$(4.4) \quad \|E_\lambda e^{-tL} P(\mathbf{u} \cdot \nabla \mathbf{v})\|_2 \leq Ct^{-1/2} \lambda^{n/4-1/2-\delta} \|\mathbf{u}\|_2^{1-2\delta} \|\nabla \mathbf{v}\|_2^{1+2\delta}.$$

Here,  $0 < \delta < 1/4$  and the constant  $C$  depends on  $\delta$ .

(ii) Under the assumptions of Theorem F (iii), we have

$$(4.5) \quad \|E_\lambda e^{-tL} P(\mathbf{u} \cdot \nabla \mathbf{v})\|_2 \leq Ct^{-3/4} \lambda^{(n-3)/4} \|\mathbf{u}\|_2 \|\nabla \mathbf{v}\|_2.$$

*Proof of Theorem F.* For simplicity we assume that our solution  $\mathbf{u}$  satisfies (1.11) or (1.11') in the usual sense. We give an outline of the proof of (i) and (ii); (iii) is proved more simply in almost the same way as given in [3]. Now, since  $\langle P(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{v}) \rangle = \langle \mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{v} \rangle = 0$ , we get from (1.11)

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_2^2 + \|\nabla \mathbf{u}\|_2^2 + \langle \mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{u} \rangle = 0.$$

Applying the well-known inequality

$$\left\| \frac{\mathbf{u}}{|\mathbf{x}|} \right\|_2 \leq \frac{2}{n-2} \|\nabla \mathbf{u}\|_2$$

we have

$$|\langle \mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{u} \rangle| = |\langle \mathbf{w}, \mathbf{u} \cdot \nabla \mathbf{u} \rangle| \leq \|\mathbf{w}\| \cdot \left\| |\mathbf{x}|^{-1} \mathbf{u} \right\|_2 \|\nabla \mathbf{u}\|_2 \leq \frac{2}{n-2} \|\mathbf{w}\| \cdot \|\nabla \mathbf{u}\|_2^2$$

so that

$$(4.6) \quad \langle L\mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{u}, L^*\mathbf{u} \rangle = \|\nabla \mathbf{u}\|_2^2 + \langle \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{u} \rangle \geq \left(1 - \frac{2}{n-2} \|\mathbf{w}\|\right) \|\nabla \mathbf{u}\|_2^2.$$

Thus, if  $\|\mathbf{w}\| < (n-2)/2$ , then both  $L$  and  $L^*$  are regularly accretive in the sense of [27, 34];  $N(L) = N(L^*) = \{0\}$  in  $\mathbf{L}_\sigma^2$ ; so  $R(L)$  is dense in  $\mathbf{L}_\sigma^2$ ; and therefore

$$(4.7) \quad \lim_{t \rightarrow \infty} \|e^{-tL} \mathbf{a}\|_2 = 0 \quad \text{for all } \mathbf{a} \in \mathbf{L}_\sigma^2.$$

Furthermore, from (4.6) we get

$$(4.8) \quad \frac{d}{dt} \|\mathbf{u}\|_2^2 + 2C_0 \|\nabla \mathbf{u}\|_2^2 \leq 0$$

for some  $C_0 > 0$ , and so

$$(4.9) \quad \|\mathbf{u}(t)\|_2 \leq \|\mathbf{a}\|_2, \quad 2C_0 \int_0^\infty \|\nabla \mathbf{u}\|_2^2 ds \leq \|\mathbf{a}\|_2^2.$$

Here we apply

$$\|\nabla \mathbf{u}\|_2^2 = \|A^{1/2} \mathbf{u}\|_2^2 \geq \int_\rho^\infty \lambda d\|E_\lambda \mathbf{u}\|_2^2 \geq \rho (\|\mathbf{u}\|_2^2 - \|E_\rho \mathbf{u}\|_2^2)$$

for any fixed  $\rho > 0$ , to get from (4.8)

$$2\|\mathbf{u}\|_2 \frac{d}{dt} \|\mathbf{u}\|_2 + C_0 \rho \|\mathbf{u}\|_2 \leq C_0 \rho \|E_\rho \mathbf{u}\|_2^2.$$

But, since  $\|E_\rho \mathbf{u}\|_2 \leq \|\mathbf{u}\|_2$ , it follows that

$$(4.10) \quad \frac{d}{dt} \|\mathbf{u}\|_2 + C_0 \rho \|\mathbf{u}\|_2 \leq C_0 \rho \|E_\rho \mathbf{u}\|_2.$$

On the other hand, applying  $E_\rho$  to (1.11') gives

$$\|E_\rho \mathbf{u}\|_2 \leq \|e^{-tL} \mathbf{a}\|_2 + \int_0^t \|E_\rho e^{-(t-\tau)L} P(\mathbf{u} \cdot \nabla \mathbf{u})\|_2 d\tau.$$

By Lemma 4.2 (i) we have

$$\begin{aligned} \|E_\rho \mathbf{u}\|_2 &\leq \|e^{-tL} \mathbf{a}\|_2 + C \rho^{n/4-1/2-\delta} \int_0^t (t-\tau)^{-1/2} \|\mathbf{u}\|_2^{1-2\delta} \|\nabla \mathbf{u}\|_2^{1+2\delta} d\tau \\ &\leq \|e^{-tL} \mathbf{a}\|_2 + C \rho^{n/4-1/2-\delta} F_1(t)^{1/2-\delta} F_2(t)^{1/2+\delta} \end{aligned}$$

where

$$F_1(t) = \int_0^t (t-\tau)^{-1/2} \|\mathbf{u}\|_2^2 d\tau, \quad F_2(t) = \int_0^t (t-\tau)^{-1/2} \|\nabla \mathbf{u}\|_2^2 d\tau.$$

We thus obtain from (4.10)

$$\frac{d}{dt} \|\mathbf{u}\|_2 + C_0 \rho \|\mathbf{u}\|_2 \leq C_0 \rho (\|e^{-tL} \mathbf{a}\|_2 + C \rho^{n/4-1/2-\delta} F_1^{1/2-\delta} F_2^{1/2+\delta}).$$

Here we set  $\rho = m(C_0 t)^{-1}$  with a sufficiently large integer  $m > 0$  to be chosen later and then multiply both sides above by  $t^m$ , to obtain

$$\frac{d}{dt} (t^m \|\mathbf{u}\|_2) \leq m t^{m-1} (\|e^{-tL} \mathbf{a}\|_2 + C t^{1/2+\delta-n/4} F_1^{1/2-\delta} F_2^{1/2+\delta})$$

so that

$$\begin{aligned} \|\mathbf{u}(t)\|_2 &\leq t^{-m} \int_0^t m\tau^{m-1} \|e^{-\tau L} \mathbf{a}\|_2 d\tau \\ &\quad + Ct^{1/2+\delta-n/4} \left(\frac{1}{t} \int_0^t F_1 d\tau\right)^{1/2-\delta} \left(\frac{1}{t} \int_0^t F_2 d\tau\right)^{1/2+\delta}. \end{aligned}$$

Since  $\|\nabla \mathbf{u}\|_2^2$  is integrable on  $[0, \infty)$  by (4.9), we see that  $t^{-1} \int_0^t F_2 d\tau \leq Ct^{-1/2}$ ; so we get

$$(4.11) \quad \|\mathbf{u}(t)\|_2 \leq t^{-m} \int_0^t m\tau^{m-1} \|e^{-\tau L} \mathbf{a}\|_2 d\tau + Ct^{1/4+\delta/2-n/4} \left(\frac{1}{t} \int_0^t F_1 d\tau\right)^{1/2-\delta}.$$

Since  $\|\mathbf{u}\|_2$  is bounded on  $[0, \infty)$  by (4.9), it follows from (4.7) and (4.11) that

$$\|\mathbf{u}(t)\|_2 \leq t^{-m} \int_0^t m\tau^{m-1} \|e^{-\tau L} \mathbf{a}\|_2 d\tau + Ct^{1/2-n/4} \rightarrow 0$$

as  $t \rightarrow \infty$ , and this proves (i). Furthermore, (4.11) shows that

$$(4.12) \quad \|\mathbf{u}(t)\|_2 \leq C(t^{-\alpha} + t^{1/2-n/4})$$

provided  $\|e^{-tL} \mathbf{a}\|_2 = O(t^{-\alpha})$ , and this proves (ii) in case  $\alpha \leq n/4 - 1/2$ . In the opposite case we have  $\|\mathbf{u}(t)\|_2^2 = O(t^{-1/2})$  by (4.12); so  $F_1$  is bounded in  $t \geq 0$ , and therefore  $t^{-1} \int_0^t F_1 d\tau \leq C$ . It thus follows from (4.11) that

$$\|\mathbf{u}(t)\|_2 \leq C(t^{-\alpha} + t^{1/4+\delta/2-n/4}) \leq C(t^{-\alpha} + t^{3/8-n/4}).$$

This proves (ii) in case  $\alpha \leq n/4 - 3/8$ . In the opposite case we have  $\|\mathbf{u}(t)\|_2^2 = O(t^{3/4})$ ; so

$$\left(\frac{1}{t} \int_0^t F_1 d\tau\right)^{1/2-\delta} \leq Ct^{-1/8+\delta/4} \leq Ct^{-1/16}$$

and therefore (4.11) gives

$$\|\mathbf{u}(t)\|_2 \leq C(t^{-\alpha} + t^{3/16+\delta/2-n/4}) \leq C(t^{-\alpha} + t^{5/16-n/4})$$

which proves (ii) in case  $\alpha \leq n/4 - 5/16$ .

Repeating these processes eventually gives

$$(4.13) \quad \begin{aligned} \|\mathbf{u}(t)\|_2^2 &= O(t^{-2\alpha}) & (\alpha \leq n/4 - (2^{\ell-1} + 1)/2^{\ell+1}) \\ \|\mathbf{u}(t)\|_2^2 &= O(t^{-2\alpha} + t^{-1+1/2^\ell}) & (\alpha \geq n/4 - (2^{\ell-1} + 1)/2^{\ell+1}) \end{aligned}$$

for an arbitrary integer  $\ell > 0$ . Since  $n/4 - (2^{\ell-1} + 1)/2^\ell \geq 1 - 2^{-\ell}$ , we have  $\|\mathbf{u}\|_2^2 = O(t^{-1+1/2^\ell})$  in the latter case of (4.13), which implies

$$\left(\frac{1}{t} \int_0^t F_1 d\tau\right)^{1/2-\delta} \leq Ct^{(1/2^\ell - 1/2)/(1/2-\delta)}$$

and so we have

$$\|\mathbf{u}(t)\|_2 \leq C(t^{-\alpha} + t^{\delta-n/4+\mu})$$

where  $\mu = (1/2 - \delta)/2^\ell$ . We can take  $\ell$  so that  $n/4 - \delta - \mu \geq 3/4 - \delta - \mu > 1/2$ ; so we can set

$$\frac{n}{4} - \delta - \mu \geq \frac{1}{2} + \kappa$$

with  $\kappa > 0$ . We thus obtain

$$(4.14) \quad \|\mathbf{u}(t)\|_2^2 \leq C(t^{-2\alpha} + t^{-1-2\kappa}).$$

Suppose  $n \geq 4$ . By (4.13) and (4.14) we may assume

$$2\alpha > n/2 - (2^{\ell-1} + 1)/2^\ell \leq 3/2 - 1/2^\ell$$

for some large  $\ell$ . Hence  $\|\mathbf{u}\|_2^2$  is integrable on  $[0, \infty)$  and so

$$\left(\frac{1}{t} \int_0^t F_1 d\tau\right)^{1/2-\delta} \leq Ct^{-1/4+\delta/2}.$$

This, together with (4.11), gives

$$(4.15) \quad \|\mathbf{u}(t)\|_2 \leq C(t^{-\alpha} + t^{\delta-n/4})$$

and the proof is complete. Suppose next that  $n = 3$ . If  $2\alpha > 1 - 1/2^\ell$ , then (4.13) gives  $\|\mathbf{u}\|_2^2 = O(t^{-1+1/2^\ell})$ ; so the above argument yields  $\|\mathbf{u}(t)\|_2 = O(t^{-\alpha})$  in case  $\alpha < 1/2$ . When  $\alpha > 1/2$ , the same argument as above gives  $\|\mathbf{u}\|_2^2 = O(t^{-1+1/2^\ell})$  for *all*  $\ell$ , so we get (4.14) for some  $\kappa > 0$ . This implies  $\|\mathbf{u}\|_2^2 = O(t^{-1-\eta})$  for some  $\eta > 0$  since  $2\alpha > 1$ . Hence  $\|\mathbf{u}\|_2^2$  is integrable on  $[0, \infty)$  and we arrive at the desired result (4.15). Finally, if  $\alpha = 1/2$ , then

$$\|\mathbf{u}(t)\|_2^2 \leq C((t+1)^{-1} + (t+1)^{-1-2\kappa})$$

for some  $\kappa > 0$ . This implies that

$$\left(\frac{1}{t} \int_0^t F_1 d\tau\right)^{1/2-\delta} \leq C([t^{-1/2} \log(t+1)]^{1/2-\delta} + t^{-1/4+\delta/2})$$

and so (4.11) gives

$$\|\mathbf{u}(t)\|_2 \leq C(t^{-1/2} + t^{-1+\delta/2}(\log t)^{1/2-\delta} + t^{\delta-3/4}) = O(t^{-1/2}).$$

The proof is complete.

**Remark.** The idea of the above proof goes back to Schonbek [28, 29] and Wiegner [36]. They treated the case of the Cauchy problem for the Navier-Stokes equations and directly applied the Fourier transformation instead of the spectral measures associated to the Stokes operator. The operator-theoretic reformulation of their idea, based on Lemmas 4.1 and 4.2, was carried out in [2, 3, 4, 12].

## 5. Large Time Behavior of Perturbations in $H^1(\mathbb{R}^n)$ or $L^1(\mathbb{R}^n)$

We now consider the stationary flows  $\mathbf{w}$  on  $\mathbb{R}^n$ ,  $n \geq 3$ , satisfying (1.16) or (1.16'), and give an outline of the proof of Theorem G. The idea is almost the same as in Section 4, but in this case the linearized operator will be analyzed in the Hardy spaces by applying the result of [8] with slight modification. We begin with

**Lemma 5.1.** *Let  $A = -\Delta$  be the Laplacian defined on  $\mathbb{R}^n$ ,  $n \geq 3$ .*

(i) *If  $\mathbf{w}$  satisfies (1.16), then*

$$(5.1) \quad \|\mathbf{w} \cdot \nabla \mathbf{u}\|_{H^1} \leq C \|\mathbf{w}\|_n \|\nabla^2 \mathbf{u}\|_{H^1}, \quad \|\mathbf{u} \cdot \nabla \mathbf{w}\|_{H^1} \leq C \|\nabla \mathbf{w}\|_{n/2} \|\nabla^2 \mathbf{u}\|_{H^1},$$

and so

$$(5.2) \quad \|B\mathbf{u}\|_{H^1} \leq C(\|\mathbf{w}\|_n + \|\nabla \mathbf{w}\|_{n/2}) \|\nabla^2 \mathbf{u}\|_{H^1}.$$

(ii) *If  $\mathbf{w}$  satisfies (1.16'), then for*

$$(5.3) \quad \|\mathbf{w} \cdot \nabla \mathbf{u}\|_{H^1} \leq C \|\mathbf{w}\|_{n,1} \|A\mathbf{u}\|_1, \quad \|\mathbf{u} \cdot \nabla \mathbf{w}\|_{H^1} \leq C \|\nabla \mathbf{w}\|_{n/2,1} \|A\mathbf{u}\|_1$$

and so

$$(5.4) \quad \|B\mathbf{u}\|_1 \leq C \|B\mathbf{u}\|_{H^1} \leq C(\|\mathbf{w}\|_{n,1} + \|\nabla \mathbf{w}\|_{n/2,1}) \|A\mathbf{u}\|_1.$$

We give a detailed proof of Lemma 5.1, which is essentially due to [8]. Take  $\varphi \in C_c^\infty(\mathbb{R}^n)$  with  $\text{supp } \varphi = \{|x| \leq 1\}$  and  $\int \varphi dx = 1$ ; and set  $\varphi_t(x) = t^{-n} \varphi(x/t)$  for  $t > 0$ . Since  $\nabla \cdot \mathbf{w} = 0$ , we have  $\mathbf{w} \cdot \nabla \mathbf{u} = \nabla \cdot (\mathbf{w} \otimes (\mathbf{u} - \mathbf{c}))$  for any constant vector  $\mathbf{c}$ ; so

$$[\varphi_t * (\mathbf{w} \cdot \nabla \mathbf{u})](x) = \frac{1}{t^{n+1}} \sum_{j=1}^n \int (\partial_j \varphi)((x-y)/t) w_j(y) [\mathbf{u}(y) - \bar{\mathbf{u}}_t] dy,$$

where  $\bar{\mathbf{u}}$  denotes the average of  $\mathbf{u}$  over the open ball  $B_t(x)$  of radius  $t$  centered at  $x$ . Taking  $1 < \alpha < n$  and  $1 < \beta < n/(n-1)$  so that  $1/\alpha + 1/\beta = 1 + 1/n$ , we apply the Hölder and the Poincaré–Sobolev inequality to get

$$\begin{aligned} |\varphi_t * (\mathbf{w} \cdot \nabla \mathbf{u})| &\leq \frac{C}{t^{n+1}} \int |\mathbf{w}| \cdot |\mathbf{u} - \bar{\mathbf{u}}_t| dy \\ &\leq \frac{C}{t^{n+1}} \left( \int_{B_t(x)} |\mathbf{w}|^\alpha dy \right)^{1/\alpha} \left( \int_{B_t(x)} |\mathbf{u} - \bar{\mathbf{u}}_t|^{\beta^*} dy \right)^{1/\beta^*} \\ &\leq \frac{C}{t^{n+1}} \left( \int_{B_t(x)} |\mathbf{w}|^\alpha dy \right)^{1/\alpha} \left( \int_{B_t(x)} |\nabla \mathbf{u}|^\beta dy \right)^{1/\beta} \end{aligned}$$

$$\begin{aligned}
&= C \left( \frac{1}{|B_t|} \int_{B_t(x)} |\mathbf{w}|^\alpha dy \right)^{1/\alpha} \left( \frac{1}{|B_t|} \int_{B_t(x)} |\nabla \mathbf{u}|^\beta dy \right)^{1/\beta} \\
&\leq CM(|\mathbf{w}|^\alpha)^{1/\alpha} M(|\nabla \mathbf{u}|^\beta)^{1/\beta}
\end{aligned}$$

where  $M(|f|)$  is the Hardy–Littlewood maximal function of  $f$  [31]. Using the assumption, we estimate the right-hand side as

$$\|M(|\mathbf{w}|^\alpha)^{1/\alpha} M(|\nabla \mathbf{u}|^\beta)^{1/\beta}\|_1 \leq \|M(|\mathbf{w}|^\alpha)^{1/\alpha}\|_n \|M(|\nabla \mathbf{u}|^\beta)^{1/\beta}\|_{n/(n-1)}.$$

But, the Hardy–Littlewood maximal theorem [31] shows

$$\|M(|\mathbf{w}|^\alpha)^{1/\alpha}\|_n \leq C\|\mathbf{w}\|_n, \quad \|M(|\nabla \mathbf{u}|^\beta)^{1/\beta}\|_{n/(n-1)} \leq C\|\nabla \mathbf{u}\|_{n/(n-1)}.$$

Applying the Sobolev inequality

$$\|\nabla \mathbf{u}\|_{n/(n-1)} \leq \|\nabla \mathbf{u}\|_{n/(n-1),1} \leq C\|\nabla^2 \mathbf{u}\|_{H^1},$$

we obtain

$$\sup_{t>0} |\varphi_t * (\mathbf{w} \cdot \nabla \mathbf{u})| \in L^1$$

and

$$\|\mathbf{w} \cdot \nabla \mathbf{u}\|_{H^1} \equiv \left\| \sup_{t>0} |\varphi_t * (\mathbf{w} \cdot \nabla \mathbf{u})| \right\|_1 \leq C\|\mathbf{w}\|_n \|\nabla^2 \mathbf{u}\|_{H^1}.$$

This shows the first estimate of (5.1). The second estimate is deduced similarly ; and (5.2) follows from (5.1) and the boundedness of  $P$  on  $\mathbf{H}^1(\mathbb{R}^n)$ .

(ii) We proceed as above starting from

$$\sup_{t>0} |\varphi_t * (\mathbf{w} \cdot \nabla \mathbf{u})| \leq CM(|\mathbf{w}|^\alpha)^{1/\alpha} M(|\nabla \mathbf{u}|^\beta)^{1/\beta}.$$

This time, we apply the duality relation

$$\|M(|\mathbf{w}|^\alpha)^{1/\alpha} M(|\nabla \mathbf{u}|^\beta)^{1/\beta}\|_1 \leq \|M(|\mathbf{w}|^\alpha)^{1/\alpha}\|_{n,1} \|M(|\nabla \mathbf{u}|^\beta)^{1/\beta}\|_{n/(n-1),w}$$

and the interpolated form of the maximal theorem :

$$\|M(|\mathbf{w}|^\alpha)^{1/\alpha}\|_{n,1} \leq C\|\mathbf{w}\|_{n,1}, \quad \|M(|\nabla \mathbf{u}|^\beta)^{1/\beta}\|_{n/(n-1),w} \leq C\|\nabla \mathbf{u}\|_{n/(n-1),w}$$

as well as the Sobolev inequality

$$\|\nabla \mathbf{u}\|_{n/(n-1),w} \leq C\|A\mathbf{u}\|_1.$$

The proof is complete.

*Proof of Theorem G.* Fix  $0 < \omega < \pi/2$ . We first note that, for  $j = 0, 1, 2$ ,

$$(5.5) \quad \|\nabla^j (\lambda + A)^{-1} \mathbf{u}\|_{H^1} \leq C\|\mathbf{u}\|_{H^1} / |\lambda|^{1-j/2} \quad (|\arg \lambda| \leq \pi - \omega),$$

and

$$(5.6) \quad \|A^{j/2}(\lambda + A)^{-1}\mathbf{u}\|_1 \leq C\|\mathbf{u}\|_1/|\lambda|^{1-j/2} \quad (|\arg \lambda| \leq \pi - \omega).$$

Estimate (5.5) follows from the Mihlin multiplier theorem (see [31]), while (5.6) is obvious. Applying this and Lemma 5.1 to the expansion

$$(\lambda + L)^{-1} = (\lambda + A)^{-1} \sum_{k=0}^{\infty} [-B(\lambda + A)^{-1}]^k,$$

we obtain

**Lemma 5.2.** (i) *If  $\|\mathbf{w}\|_n + \|\nabla \mathbf{w}\|_{n/2}$  is small enough, then for  $j = 0, 1, 2$ ,*

$$(5.7) \quad \|\nabla^j(\lambda + L)^{-1}\mathbf{u}\|_{H^1} \leq C\|\mathbf{u}\|_{H^1}/|\lambda|^{1-j/2} \quad (|\arg \lambda| \leq \pi - \omega).$$

(ii) *If  $\|\mathbf{w}\|_{n,1} + \|\nabla \mathbf{w}\|_{n/2,1}$  is small enough, then for  $j = 0, 1, 2$ ,*

$$(5.8) \quad \|A^{j/2}(\lambda + L)^{-1}\mathbf{u}\|_1 \leq C\|\mathbf{u}\|_1/|\lambda|^{1-j/2} \quad (|\arg \lambda| \leq \pi - \omega).$$

(iii) *Under the assumption of (i) or (ii), the operator  $L$  is injective.*

The injectivity asserted in (iii) follows, respectively, from

$$C_1\|\nabla^2\mathbf{u}\|_{H^1} \leq \|L\mathbf{u}\|_{H^1} \leq C_2\|\nabla^2\mathbf{u}\|_{H^1}$$

which is a consequence of Lemma 5.1 (i) and Lemma 5.2 (i), and

$$C_1\|A\mathbf{u}\|_1 \leq \|L\mathbf{u}\|_1 \leq C_2\|A\mathbf{u}\|_1$$

which is obtained from Lemma 5.1 (ii) and Lemma 5.2 (ii).

Applying the standard argument, we immediately obtain

**Lemma 5.3.** (i) *Under the assumption of Lemma 5.2 (i), the semigroup  $\{e^{-tL}\}_{t \geq 0}$  is bounded-analytic on  $\mathbf{H}_\sigma^1$ , and we have*

$$(5.9) \quad \|\nabla^j e^{-tL}\mathbf{a}\|_{H^1} \leq Ct^{-j/2}\|\mathbf{a}\|_{H^1}$$

for  $j = 0, 1, 2$ .

(ii) *Under the assumption of Lemma 5.2 (ii), the semigroup  $\{e^{-tL}\}_{t \geq 0}$  is bounded-analytic on  $\mathbf{L}_\sigma^1$ , and*

$$(5.10) \quad \|A^{j/2}e^{-tL}\mathbf{a}\|_1 \leq Ct^{-j/2}\|\mathbf{a}\|_1$$

for  $j = 0, 1, 2$ .

(iii) In each of the above two cases, we have

$$(5.11) \quad \lim_{t \rightarrow \infty} \|e^{-tL} \mathbf{a}\|_{H^1} = 0 \quad \text{for all } \mathbf{a} \in \mathbf{H}^1_\sigma,$$

or

$$(5.12) \quad \lim_{t \rightarrow \infty} \|e^{-tL} \mathbf{a}\|_1 = 0 \quad \text{for all } \mathbf{a} \in \mathbf{L}^1_\sigma.$$

To proceed further, we recall the following duality relations (see [22]) :

$$(5.13) \quad (\mathbf{H}^1_\sigma)^* = \text{BMO}_\sigma, \quad (\text{VMO}_\sigma)^* = \mathbf{H}^1_\sigma$$

and consider the integral equation

$$(5.14) \quad \langle \mathbf{u}(t), \psi \rangle = \langle e^{-(t-s)L} \mathbf{u}(s), \psi \rangle - \int_s^t \langle \mathbf{u} \cdot \nabla \mathbf{u}, e^{-(t-\tau)L^*} P\psi \rangle d\tau, \quad (0 \leq s \leq t).$$

Assume first (1.16). Since  $\{e^{-tL^*}\}_{t \geq 0}$  is bounded on  $\text{VMO}_\sigma$ , taking  $s = 0$  in (5.14) and applying (5.13) we get

$$\begin{aligned} |\langle \mathbf{u}(t), \psi \rangle| &\leq \|e^{-tL} \mathbf{a}\|_{H^1} \|\psi\|_{\text{BMO}} + \int_0^t \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{H^1} d\tau \times \|\psi\|_{\text{BMO}} \\ &\leq \|e^{-tL} \mathbf{a}\|_{H^1} \|\psi\|_{\text{BMO}} + C \left( \int_0^t \|\mathbf{u}\|_2 \|\nabla \mathbf{u}\|_2 d\tau \right) \|\psi\|_{\text{BMO}}. \end{aligned}$$

Note that we have used the following, which is due to [8] :

$$\|\mathbf{u} \cdot \nabla \mathbf{u}\|_{H^1} \leq C \|\mathbf{u}\|_2 \|\nabla \mathbf{u}\|_2.$$

Since  $C_{0,\sigma}^\infty(\mathbb{R}^n)$  is dense in  $\text{VMO}_\sigma$ , it follows from (5.13) that  $\mathbf{u}(t) \in \mathbf{H}^1_\sigma$  for all  $t \geq 0$ . The same calculation for general  $s \geq 0$  gives

$$\begin{aligned} \|\mathbf{u}(t)\|_{H^1} &\leq \|e^{-(t-s)L} \mathbf{u}(s)\|_{H^1} + C \int_s^t \|\mathbf{u}\|_2 \|\nabla \mathbf{u}\|_2 d\tau \\ &\leq \|e^{-(t-s)L} \mathbf{u}(s)\|_{H^1} + C \left( \int_s^t \|\mathbf{u}\|_2^2 d\tau \right)^{1/2} \left( \int_s^\infty \|\nabla \mathbf{u}\|_2^2 d\tau \right)^{1/2}. \end{aligned}$$

On the other hand, (5.9) implies

$$\|e^{-tL} \mathbf{a}\|_2 \leq C t^{-n/4} \|\mathbf{a}\|_{H^1} \quad \text{and} \quad \|e^{-tL^*} \mathbf{a}\|_{\text{BMO}} \leq C t^{-n/4} \|\mathbf{a}\|_2,$$

so, in the same way as in Section 4, we conclude that

$$\|\mathbf{u}(t)\|_2 \leq C(t+1)^{-n/4}.$$

Hence,

$$\int_0^\infty \|\mathbf{u}\|_2^2 d\tau < \infty,$$

and the above calculation gives

$$\|\mathbf{u}(t)\|_{H^1} \leq \|e^{-tL}\mathbf{u}(s)\|_{H^1} + C \left( \int_s^\infty \|\nabla \mathbf{u}\|_2^2 d\tau \right)^{1/2}$$

for  $t \geq s$ . Since the last term can be made as small as we please, (5.11) implies

$$\lim_{t \rightarrow \infty} \|\mathbf{u}(t)\|_{H^1} = 0.$$

Assume next (1.16') and let  $\mathbf{a} \in L_\sigma^1$ . We take this time  $\psi \in C_c^\infty(\mathbb{R}^n)$  to get

$$\begin{aligned} |\langle \mathbf{u}(t), \psi \rangle| &\leq \|e^{-(t-s)L}\mathbf{u}(s)\|_1 \|\psi\|_\infty + C \left( \int_s^t \|\mathbf{u}\|_2 \|\nabla \mathbf{u}\|_2 d\tau \right) \|P\psi\|_{\text{BMO}} \\ &\leq \|e^{-(t-s)L}\mathbf{u}(s)\|_1 \|\psi\|_\infty + C \left( \int_s^t \|\mathbf{u}\|_2 \|\nabla \mathbf{u}\|_2 d\tau \right) \|\psi\|_{\text{BMO}} \\ &\leq \left( \|e^{-(t-s)L}\mathbf{u}(s)\|_1 + C \int_s^t \|\mathbf{u}\|_2 \|\nabla \mathbf{u}\|_2 d\tau \right) \|\psi\|_\infty \end{aligned}$$

for all  $0 \leq s \leq t$ . Here we have invoked the continuous embedding  $L^\infty \subset \text{BMO}$  as well as the boundedness of the operator  $P$  on BMO. Setting  $s = 0$ , we see that  $\mathbf{u}(t)$  is a finite Borel measure on  $\mathbb{R}^n$ , so  $\mathbf{u}(t) \in L_\sigma^1$  for all  $t$ , and therefore

$$\|\mathbf{u}(t)\|_1 \leq \|e^{-tL}\mathbf{u}(s)\|_1 + C \left( \int_s^t \|\mathbf{u}\|_2^2 d\tau \right)^{1/2} \left( \int_s^\infty \|\nabla \mathbf{u}\|_2^2 d\tau \right)^{1/2}.$$

for  $0 \leq s \leq t$ . On the other hand, (5.10) gives

$$\|e^{-tL}\mathbf{a}\|_2 \leq Ct^{-n/4} \|\mathbf{a}\|_1,$$

so by an argument similar to the foregoing case and (5.12), we arrive at

$$\lim_{t \rightarrow \infty} \|\mathbf{u}(t)\|_1 = 0.$$

This completes the proof of Theorem G.

*Remark.* It is possible to deduce the decay rate of  $\|\mathbf{u}(t)\|_1$  or  $\|\mathbf{u}(t)\|_{H^1}$  under some assumptions on  $\mathbf{a}$ . The details are discussed in [23].

*Acknowledgment.* The present work is partially supported by Grant-in-Aid for Scientific Research from the Japan Ministry of Education, Science and Culture.

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# The Asymptotic Behavior of Solutions to Some Degenerate Kirchhoff Type Equations

TETSU MIZUMACHI

**Abstract:** We investigate some degenerate quasilinear wave equations which have their origin in nonlinear vibration of a string. In the case when equation describes a vibrating string with viscosity, we determine the decay order of all solutions by investigating the dynamics near an infinite dimensional center manifold. Moreover, we classify the asymptotic behavior of solutions from a dynamical systems point of view.

We also deal with the case when equation models a vibrating string with the resistance proportional to the velocity.

## 1. Introduction.

In this article, we investigate the asymptotic behavior of solutions to the following initial-boundary value problems:

$$(1.1a) \quad u_{tt} - \|\nabla u\|_{L^2(\Omega)}^2 \Delta u - \alpha \Delta u_t = 0 \quad \text{in } \Omega \times (0, +\infty),$$

$$(1.1b) \quad u = 0 \quad \text{on } \partial\Omega \times (0, +\infty),$$

$$(1.1c) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega,$$

and

$$(1.2a) \quad u_{tt} - \|\nabla u\|_{L^2(\Omega)}^2 \Delta u + \alpha u_t = 0 \quad \text{in } \Omega \times (0, +\infty),$$

$$(1.2b) \quad u = 0 \quad \text{on } \partial\Omega \times (0, +\infty),$$

$$(1.2c) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega,$$

where  $\alpha$  is a positive constant and  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . Equations (1.1) and (1.2) have their origin in the so-called Kirchhoff equation which describes transversal oscillations of an elastic string (see [5]).

We first consider (1.1). The global existence and the uniqueness have been established in the energy class (see [14]). Once global existence is known, one easily sees that solutions to (1.1) decay as  $t \rightarrow \infty$ . Much of the efforts have been focused on upper estimates of rate of decay (see, e.g., [8,9]). But it is difficult to obtain the lower estimates. In fact, except for some special cases ([10,11]), little has been known about the estimates from below.

In [6], we obtain exact decay estimates of *all* solutions and study the limiting profiles of solutions. By a *limiting profile*, we mean the limit of a solution divided by its rate of decay as  $t \rightarrow \infty$ .

It should be noted that, unlike most earlier works, our method makes use of the dynamical systems point of view involving the theory of center manifolds. The advantage of such an approach is that one can obtain the exact decay estimates out of relatively simple computations. Moreover, our method also tells us which decay rate occurs ‘generically’.

In order to state our results explicitly, let  $\lambda_1 < \lambda_2 < \lambda_3 < \dots$  be the eigenvalues of  $-\Delta$ , let  $\{\varphi_j\}_{j=1}^\infty$  be the corresponding sequence of eigenfunctions normalized to satisfy  $\|\varphi_j\|_{L^2(\Omega)} = 1$ , and let  $\{m_k\}_{k=1}^\infty$  be a sequence of integers satisfying

$$\lambda_{m_1} < \lambda_{m_2} = \lambda_{m_2+1} = \dots = \lambda_{m_3-1} < \lambda_{m_3} = \lambda_{m_3+1} = \dots = \lambda_{m_4-1} < \lambda_{m_4} = \dots.$$

For each  $k \in \mathbb{N}$ , we put  $\tilde{\lambda}_k = \lambda_{m_k}$ . Now set  $\mathcal{Z} = \{w \in H_0^1(\Omega) \mid \|\nabla w\|_{L^2(\Omega)}^2 = \frac{\alpha}{2}\}$ . One of our main results is the following:

**Theorem A1** ([6]). *The set  $H_0^1(\Omega) \times L^2(\Omega) \setminus \{(0,0)\}$  is decomposed into a disjoint union of the two sets  $S_1, S_2$  satisfying the following:*

- (i)  $S_1$  is a dense open set and  $S_2 \cup \{(0,0)\}$  is a closed set with no interior.
- (ii) Let  $(u_0, u_1) \in S_1$ . Then there exists  $u_\infty \in \mathcal{Z}$  such that the solution  $u(t)$  to (1.1) satisfies, as  $t \rightarrow \infty$ ,

$$(1.3a) \quad \left\| u(t) - (1+t)^{-\frac{1}{2}} u_\infty \right\|_{H_0^1} = O((1+t)^{-\frac{3}{2}} \log(1+t)),$$

$$(1.3b) \quad \left\| u_t(t) + \frac{1}{2}(1+t)^{-\frac{3}{2}} u_\infty \right\|_{L^2(\Omega)} = O((1+t)^{-\frac{5}{2}} \log(1+t)),$$

$$(1.3c) \quad \left\| u_{tt}(t) - \frac{3}{4}(1+t)^{-\frac{5}{2}} u_\infty \right\|_{L^2(\Omega)} = O((1+t)^{-\frac{7}{2}} \log(1+t)).$$

- (iii) Let  $(u_0, u_1) \in S_2$ . Then there exists  $k \in \mathbb{N}$  such that the solution to (1.1) satisfies, as  $t \rightarrow \infty$ ,

$$\left\| \begin{pmatrix} u(t) \\ u_t(t) \\ u_{tt}(t) \end{pmatrix} - e^{-\alpha \tilde{\lambda}_k t} \sum_{m_k \leq j \leq m_{k+1}-1} c_j \begin{pmatrix} -\alpha^{-1} \tilde{\lambda}_k^{-1} \\ 1 \\ -\tilde{\lambda}_k \end{pmatrix} \varphi_j \right\|_{H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)} = O(e^{-\gamma t})$$

for every  $\gamma$  satisfying  $\alpha \tilde{\lambda}_k < \gamma < \min\{\alpha \tilde{\lambda}_{k+1}, 3\alpha \tilde{\lambda}_k\}$ , where

$$c_j = (u_1, \varphi_j)_{L^2(\Omega)} - \lambda_j \int_0^\infty e^{\alpha \lambda_j s} \|\nabla u(s)\|_{L^2(\Omega)}^2 (u(s), \varphi_j)_{L^2(\Omega)} ds.$$

Roughly speaking, solutions lying on  $S_1$  decay polynomially, and those on  $S_2$  decay exponentially. Theorem A1 shows that all the limiting profiles of the polynomially decaying solutions belong to  $\mathcal{Z}$ . Conversely, we have the following:

**Theorem A2** ([6]). *For every  $u_\infty \in \mathcal{Z}$ , there exists a solution  $u(t)$  of (1.1) that satisfies (1.3a)–(1.3c).*

**Remark 1.** Suppose that a solution  $u(t)$  to (1.1) satisfies (1.3a)–(1.3c). Then the term  $u_{tt}$  decays faster than the other terms of (1.1a). If the term  $u_{tt}$  is absent in (1.1), then the equation can be written as

$$(1.4) \quad \|v\|_{L^2(\Omega)}^2 v + \alpha \frac{dv}{dt} = 0,$$

where  $v = A^{\frac{1}{2}}u$ . Here  $A$  is the self-adjoint closure in  $L^2(\Omega)$  of the operator defined on  $C_0^\infty(\Omega)$  by  $Au = -\Delta u$ . Now let  $\tilde{u}(t)$  be a solution to (1.4) satisfying  $\tilde{u}(0) = u_\infty$ . We see from (1.3a) that

$$\|u(t) - \tilde{u}(t)\|_{H_0^1} = o(\|\tilde{u}(t)\|_{H_0^1}).$$

This tells us that if a solution  $u(t)$  of (1.1) decays polynomially, it behaves, asymptotically, like a solution to the ordinary differential equation (1.4).

On the other hand, if  $u(t)$  is an exponentially decaying solution to (1.1), it behaves, asymptotically, like a solution to the classical heat equation.

Next we consider the behavior of solutions to (1.2). For analytic initial data, equation (1.2) is globally solvable (see [1]). On the other hand, for non-analytic initial data, [12] shows that solutions to (1.2) globally exist when  $u_0 (\neq 0)$  is small and  $u_1$  is much smaller than  $u_0$  in appropriate Sobolev spaces.

For solutions which exist globally, the upper estimates of rate of decay can be obtained by using the method of [8,15]. But to the best of our knowledge, there is no results concerning lower estimates. In [7], we obtain an asymptotic expansion of solutions to (1.2) for some class of initial data.

**Theorem B** ([7]). *Suppose  $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ ,  $u_0 \neq 0$ , and*

$$(H) \quad \frac{\|\nabla u_1\|_{L^2(\Omega)}^2}{\|\nabla u_0\|_{L^2(\Omega)}^2} + \|\Delta u_0\|_{L^2(\Omega)}^2 < \alpha^2.$$

*Then the solution to (1.2) satisfies, as  $t \rightarrow \infty$ ,*

$$\left\| t^{\frac{1}{2}} u(t) - \varphi \right\|_{H_0^1(\Omega)} = o(1),$$

*where  $\varphi$  is an eigenfunction of  $A$  belonging to  $\tilde{\lambda}_j$  and satisfying  $\|\varphi\|_{L^2(\Omega)} = 1/(\sqrt{2\tilde{\lambda}_j})$  for some  $j \in \mathbb{N}$ .*

The condition (H) has been used in [12] to show the global existence of solutions to (1.2). We find in [7] that (H) with  $u_0 \neq 0$  is a sufficient condition for solutions to (1.2) to be approximated by nontrivial solutions to the degenerate parabolic equation

$$\alpha u_t - \|\nabla u\|_{L^2(\Omega)}^2 \Delta u = 0.$$

Using this fact, we prove Theorem B in [7].

**2. Center manifold analysis of (1.1).**

Making use of the change of variables similar to what is used in [2,13], namely,

$$(2.1) \quad p = A^{-\frac{1}{2}}u_t, \quad q = -\alpha A^{\frac{1}{2}}u - p,$$

we can rewrite (1.1) into a coupled system of a heat equation and an ordinary differential equation:

$$(2.2a) \quad p_t = \alpha \Delta p + \frac{1}{\alpha^3} \|p + q\|_{L^2(\Omega)}^2 (p + q) \quad \text{in } \Omega \times (0, +\infty),$$

$$(2.2b) \quad q_t = -\frac{1}{\alpha^3} \|p + q\|_{L^2(\Omega)}^2 (p + q) \quad \text{in } \Omega \times (0, +\infty),$$

$$(2.2c) \quad p(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty),$$

$$(2.2d) \quad p(x, 0) = p_0(x), \quad q(x, 0) = q_0(x) \quad \text{in } \Omega,$$

We see that  $(p_0, q_0) \in H_0^1(\Omega) \times L^2(\Omega)$  if and only if  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ . For such a class of initial data, solutions to (1.1) and (2.2) are unique. Hence the formula (2.1) gives a one-to-one correspondence between solutions to (1.1) and solutions to (2.2). From now on, we will analyze (2.2) instead of (1.1).

Since  $(p, q) \rightarrow (0, 0)$  as  $t \rightarrow \infty$ , it suffices to analyze (1.1) in some neighborhood of the origin. Now, let us linearize (2.2) about the origin. Then

$$\frac{d}{dt} \begin{pmatrix} p \\ q \end{pmatrix} + L \begin{pmatrix} p \\ q \end{pmatrix} = 0,$$

where

$$L = \begin{pmatrix} \alpha A & 0 \\ 0 & 0 \end{pmatrix}.$$

The spectrum of the linear operator  $L$ ,  $\sigma(L)$ , satisfies

$$\sigma(L) = \{0\} \cup \{\alpha\lambda_j\}_{j=1}^\infty.$$

This means that the origin  $(p, q) = (0, 0)$  is not linearly stable, but simply neutrally stable. In our problem, we see that  $\inf\{\text{Re } \eta \mid \eta \in \sigma(L) \setminus \{0\}\} > 0$  and that the Lipschitz coefficient of the nonlinear term of (2.2) is small about the origin. Under such circumstances, the so-called center manifold theory often proves exceedingly useful in determining the nonlinear dynamics around the origin.

Let  $X_0$  be an eigenspace belonging to the eigenvalue 0 and let  $X_+$  be the direct sum of eigenspaces belonging to  $\{\alpha\lambda_j\}_{j=1}^\infty$ . Then we have

$$X_0 = \{{}^t(0, q) \mid q \in L^2(\Omega)\} \quad \text{and} \quad X_+ = \{{}^t(p, 0) \mid p \in H_0^1(\Omega)\}.$$

Hereafter, we will identify  $X_0$  and  $X_+$  with  $L^2(\Omega)$  and  $H_0^1(\Omega)$ , respectively.

By applying Theorem 6.1 in [3], we have the following:

**Lemma 2.1.** For some neighborhood  $U$  of 0 in  $H_0^1(\Omega) \times L^2(\Omega)$ , equation (2,2) has a local center manifold  $W_{loc}^c(0)$  satisfying the following:

(i)

$$W_{loc}^c(0) = \{(p, q) \in U \mid p = h^c(q)\},$$

where  $h^c: X_0 \cap U \rightarrow X_1$  is a  $C^5$ -mapping with  $h^c(0) = 0$  and  $Dh^c(0) = 0$ .

(ii) For each  $(p_0, q_0) \in W_{loc}^c(0)$ ,  $\{(p(t), q(t)) \mid t_1 \leq t \leq t_2\} \subset U$  implies  $\{(p(t), q(t)) \mid t_1 \leq t \leq t_2\} \subset W_{loc}^c(0)$  for any  $t_1, t_2$  with  $t_1 < 0 < t_2$ , where  $(p(t), q(t))$  is the solution to (2,2).

Moreover, the dynamical systems theory tells us that the center manifold  $W_{loc}^c(0)$  attracts neighboring orbits (solution curves) exponentially and solutions on it decay at most polynomially (see [3]). Thus the asymptotic behavior of solutions near the center manifold is much the same as that of solutions on the center manifold. So except exponentially decaying solutions, we have only to analyze solutions on the center manifold to obtain asymptotic expansions of solutions.

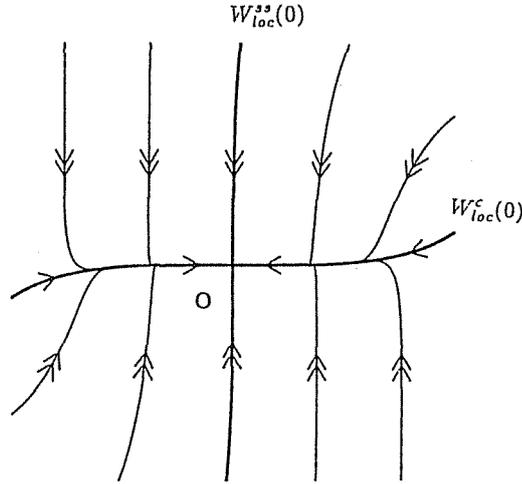


Figure A schematic picture of  $W_{loc}^c(0)$ .

By Lemma 2.1, solutions on the center manifold satisfy

$$(2.3) \quad p(t) = h^c(q(t))$$

Substituting (2,3) into (2,2) and eliminating  $q_t$ , we obtain

$$(2.4) \quad \frac{1}{\alpha^3} Dh^c(q) \|h^c(q) + q\|_{L^2}^2 (h^c(q) + q) + \alpha \Delta h^c(q) + \frac{1}{\alpha^3} \|h^c(q) + q\|_{L^2}^2 (h^c(q) + q) = 0,$$

where  $Dh^c(q)$  stands for the Fréchet derivative of  $h^c$ . So  $h^c$  must satisfy (2,4), together with the condition  $h^c(0) = 0$ ,  $Dh^c(0) = 0$ . Now we see from (2,4) that the center manifold is approximated as

$$(2.5) \quad \left\| h^c(q) - \frac{1}{\alpha^4} \|q\|_{L^2(\Omega)}^2 A^{-1} q \right\|_{H_0^1(\Omega)} = O(\|q\|_{L^2(\Omega)}^5).$$

Hence, solutions on the center manifold are described by

$$(2.6) \quad \begin{aligned} q_t &= -\frac{1}{\alpha^3} \|h^c(q) + q\|_{L^2(\Omega)}^2 (h^c(q) + q) \\ &= -\frac{1}{\alpha^3} \|q\|_{L^2(\Omega)}^2 q + O(\|q\|_{L^2(\Omega)}^5). \end{aligned}$$

with (2,3). By analyzing (2,6), we can obtain asymptotic expansions of solutions on the center manifold. For the details of the proof of Theorems A1 and A2, see [8].

**Acknowledgements:** The author would like to express his gratitude to Professor Hiroshi Matano for his helpful advice and continuous encouragement, as well as many stimulating discussions. He would also like to express his gratitude to Professor Yoshio Tsutsumi for drawing the attention of the author to the present problem and giving valuable suggestions.

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# ANALYSIS ON LARGE FREE SURFACE DEFORMATION OF MAGNETIC FLUID

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**Abstract.** This paper analyzes the free surface deformation of magnetic fluid under intense magnetic fields. The analysis in a vertical plane allows for full nonlinearity, multi-valuedness and the inhomogeneous magnetic field which changes in reaction to the free surface deformation. The method is based on the conformal mapping as used for the analysis of stationary capillary-gravity waves. The source of the magnetic field is assumed a dipole as in actual cases, and the magnetic potential determined in the flat space with the straight free surface is mapped onto the real space. The equation for the inclination angle of the free surface  $\theta$  is solved numerically by the spectral collocation method, and the solution is used to express the free surface shape parametrically. The result shows the first bifurcations of the shape with the increase of the magnetic field.

## 1. Introduction

One of the features peculiar to magnetic fluid is the shape of its free surface. As the external magnetic field increases, the surface deforms gradually first, but changes abruptly to a stationary shape just like a set of cones. This is the result of the interaction between the surface deformation and the magnetic field. The surface is moved by the magnetic force acting on it to the position where the magnetic force balances with the capillary and gravity forces. However, the magnetic field itself is disturbed by the large deformation of the surface, and its effect is reflected back on the magnetic force.

Since the magnetic fluid of the present type was devised in 1960's, linear and weakly nonlinear analyses of this phenomenon has been made. Cowley and Rosensweig [1] investigated its linear stability. Malik and Singh [3, 4, 5, 6] employed the multiple scales method to investigate one-dimensional harmonic wave propagation. Gailitis [2] and Twombly [8, 9] analyzed statically the two-dimensional surface wave which bifurcates from "trivial solution" to rectangles or hexagons under uniform magnetic field normal to the surface.

This paper presents an analysis for the free surface deformation of magnetic fluid in a vertical plane, allowing for full nonlinearity, multi-valuedness and inhomogeneous magnetic fields. The formulation is based on the method of conformal mapping as employed by Okamoto [7] and the previous authors for the analysis on stationary capillary-gravity waves. In both problems, the “flat space” with the straight free surface is mapped onto the “real space” with the curved free surface. In the present problem, however, this method is especially useful to obtain the harmonic magnetic field which is originally inhomogeneous, and changes in reaction to the free surface deformation.

## 2. Basic equation — dynamic boundary condition

On the free surface of magnetic fluid, we consider the Bernoulli equation

$$\rho \left( \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + g\eta \right) + p + p_t - \frac{1}{2} \left( \left[ \frac{1}{\mu} \right] b_n^2 - [\mu] h_s^2 \right) = \text{const.} \quad (1)$$

with the condition for the pressure  $p = 0$ , where  $\rho$ ,  $g$ ,  $\eta$ ,  $p_t$  are the fluid density, gravity acceleration, surface elevation and surface tension, respectively. The terms including the velocity potential  $\phi$  and the fluid velocity  $v = -\nabla\phi$  are dropped in stationary cases. The last term of l.h.s. is the magnetic pressure characteristic of magnetic fluid, where  $b_n$  is the component of the magnetic flux density normal to the surface,  $h_s$  is the component of the magnetic field tangential to the surface,  $\mu$  is the permeability of either magnetic fluid or vacuum, and  $[\dots]$  is the jump quantity across the surface.

## 3. Method — conformal mapping

Here we consider the conformal mapping  $z = z(Z)$  from the flat space described by  $Z = X + iY$  to the real space by  $z = x + iy$ . When an infinitesimal element on the real space  $dz$  is inclined by an angle  $\theta$  to its correspondence on the flat space  $dZ$ , we find  $dz = ce^{i\theta}dZ$  with the real positive constant  $c$ . After replacing  $c$  by  $e^{-\tau}$  where we call  $\tau$  logarithmic contraction rate here, we obtain

$$\frac{dz}{dZ} = e^{i[\theta(Z)] + i\tau(Z)}. \quad (2)$$

The function  $\theta(Z) + i\tau(Z)$  is analytic in  $Z$  as far as  $0 < |dz/dZ| < \infty$ , and tends to 0 as  $Z \rightarrow \infty$ . These conditions lead to the Hilbert transform between  $\theta$  and  $\tau$  as

$$\tau(X) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\theta(X')}{X' - X} dX'. \quad (3)$$

In the limit  $dz/dZ \rightarrow 0$  or  $\tau \rightarrow \infty$ , the free surface has a branch point or cusp, and  $\theta$  changes discontinuously there.

After  $\theta$  and  $\tau$  are obtained, the free surface shape is determined by integrating each of the real and imaginary part of eq. (2) on  $Y = 0$ , that is,

$$\partial x / \partial X = e^{-\tau} \cos \theta, \quad \partial y / \partial X = e^{-\tau} \sin \theta. \quad (4)$$

The result  $x = x(X)$ ,  $y = y(X)$  expresses the shape parametrically allowing for multi-valuedness. The equation for  $\theta$  and  $\tau$  is derived from eq. (1), but we will discuss the magnetic field before that.

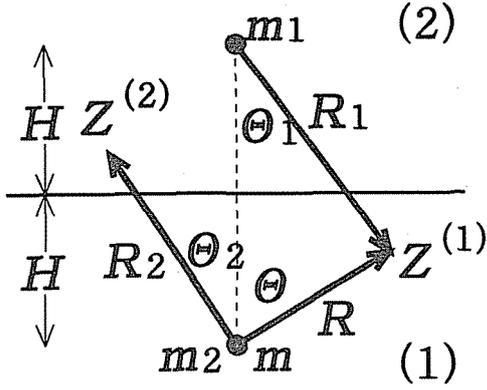


Fig. 1: Method of images to determine magnetic potentials in (1) (right) and in (2) (left)

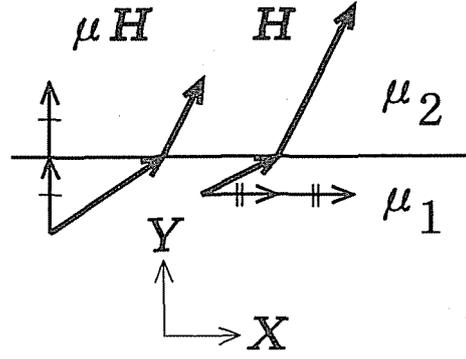


Fig. 2: Interface conditions for magnetic flux density with normal component continuous (left) and magnetic field with tangential component continuous (right)

#### 4. Determination of magnetic field — method of images

We introduce here the complex magnetic potential  $W = \Phi - iA$  where  $\Phi$  and  $A$  are the scalar potential and (the  $z$ -component of) the vector potential. Both of them are harmonic, and the ones determined in the flat space can be mapped onto the real space conformally. They give the magnetic field as  $\mathbf{H} = (H_X, H_Y) = -\nabla\Phi = \text{rot} A$ . Then,  $dW/dZ = (\partial\Phi/\partial X) - i(\partial A/\partial X) = -(\partial A/\partial Y) - i(\partial\Phi/\partial Y) = -H_X + iH_Y$  is derived.

In an infinite medium of the permeability  $\mu$ , the potential at  $Z$  due to a monopole  $m$  at  $Z_0$  is  $W = m \ln(Z - Z_0)/\mu = m(\ln R + i\Theta)/\mu$  at  $Z$  where  $Z - Z_0 = Re^{i\Theta}$ . We can determine the potential due to a monopole  $m$  at  $Z_1$  near an interface between two media with the permeability  $\mu_1$  and  $\mu_2$  through the “method of images”, as Fig. 1 shows. In the medium (1), the potential  $W^{(1)}$  at  $Z^{(1)}$  (1) is the overlap of those due to  $m$  at  $Z_1$  and its image  $m_1$  at  $Z_2$ . In (2),  $W^{(2)}$  at  $Z^{(2)}$  is due to  $m_2$  at  $Z_1$ . Therefore,

$$\begin{cases} W^{(1)} = \frac{m \ln(Z^{(1)} - Z_1) + m_1 \ln(Z^{(1)} - Z_2)}{\mu_1} = \frac{m \ln R + m_1 \ln R_1}{\mu_1} + i \frac{m\Theta + m_1\Theta_1}{\mu_1}, \\ W^{(2)} = \frac{m_2 \ln(Z^{(2)} - Z_1)}{\mu_2} = \frac{m_2 \ln R_2}{\mu_2} + i \frac{m_2\Theta_2}{\mu_2} \end{cases} \quad (5)$$

follows, where  $Z^{(1)} - Z_1 = Re^{i\Theta}$ ,  $Z^{(2)} - Z_1 = R_2e^{i\Theta_2}$ ,  $Z^{(1)} - Z_2 = R_1e^{i\Theta_1}$  and  $-Y_1 = Y_2 \equiv H$ . Newly introduced  $m_1$  and  $m_2$  are determined from the interface conditions  $[-\mu H_Y] = 0$  and  $[-H_X] = 0$  (Fig. 2), or equivalently,

$$\begin{cases} [\mu A] = (m - m_1 - m_2)\Theta = 0, \\ [\Phi] = \{(m + m_1)/\mu_1 - m_2/\mu_2\} \ln R = 0 \end{cases} \quad \text{on } Y = 0, \quad (6)$$

since  $Z^{(1)} = Z^{(2)}$  and  $R_1 = R_2 = R$ ,  $-\Theta_1 = \Theta_2 = \Theta$ . They are solved for  $m_1$  and  $m_2$ , and the scalar potential is obtained as:

$$\begin{cases} \Phi^{(1)} = \frac{m + m_1}{\mu_1} \ln R, \\ \Phi^{(2)} = \frac{m_2}{\mu_2} \ln R \end{cases} \quad \text{with} \quad \begin{cases} m_1 = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} m, \\ m_2 = \frac{2\mu_2}{\mu_1 + \mu_2} m. \end{cases} \quad (7)$$

In the case due to a dipole  $\pm m$  with the separation  $\Delta Y$  between the poles, the potential  $\varphi^{(2)}$  is derived easily from eq. (7) as

$$\varphi^{(2)} = \Phi^{(2)} = -\Delta Y \frac{\partial}{\partial Y} \frac{m_2}{\mu_2} \ln R = -\frac{m_2^* Y}{\mu_2 R^2} \quad \text{with} \quad m_2^* \equiv m_2 \Delta Y. \quad (8)$$

The components  $b_n$  and  $h_s$ , continuous across the surface, are obtained from  $\varphi^{(2)}$  as

$$\begin{cases} b_n = -\mu_2 \frac{\partial \varphi^{(2)}}{\partial n} = e^\tau m_2^* \frac{X^2 - Y^2}{R^4}, \\ h_s = -\frac{\partial \varphi^{(2)}}{\partial s} = e^\tau \frac{m_2^* 2XY}{\mu_2 R^4} \end{cases} \quad (9)$$

with the normal and tangential derivative  $\partial/\partial n = e^\tau \partial/\partial Y$ ,  $\partial/\partial s = -e^\tau \partial/\partial X$  in mind.

### 5. Equation for $\theta$ and $\tau$

Now that the magnetic field is determined, the equation for  $\theta$  and  $\tau$  can be derived from eq. (1). If the both sides of eq. (1) is differentiated by  $X$ ,

$$e^{-\tau} \sin \theta - \frac{\gamma}{\rho g} \frac{\partial}{\partial X} \left( e^\tau \frac{\partial \theta}{\partial X} \right) - \frac{(m_2^*)^2}{2\rho g} \left[ \frac{1}{\mu} \right] \frac{\partial}{\partial X} e^{2\tau} \left\{ \left( \frac{X^2 - Y^2}{R^4} \right)^2 + \frac{\mu_1}{\mu_2} \left( \frac{2XY}{R^4} \right)^2 \right\} = 0 \quad (10)$$

follows, where, if  $\dot{A}$  denotes  $\partial A/\partial X$ ,  $\dot{\eta} = e^{-\tau} \sin \theta$  and  $p_t = \gamma(\ddot{x}\dot{y} - \dot{x}\ddot{y})/(\dot{x}^2 + \dot{y}^2)^{3/2} = -\gamma e^\tau \dot{\theta}$  ( $\gamma$ : coefficient of surface tension), due to eq. (4). Equation (10) is to be solved in the domain  $0 \leq X < \infty$  with the boundary conditions  $\theta(X=0) = 0$  and  $\theta(X \rightarrow \infty) \rightarrow 0$ .

Each term in eq. (10) is characterized by the factor  $e^{-\tau}$ , which means the contraction in the real space as seen from eq. (4). When the space contracts, the first gravity term becomes unimportant, but the second tension term grows, and the third magnetic term more. The curvature of the free surface increases to compete with the magnetic pressure and keep the balance among forces.

For the advantage of numerical analysis, both sides of eq. (10) is divided by  $e^{2\tau}$ , and  $X$  is scaled by  $H$ , in addition. Then, eq. (10) is rewritten as

$$e^{-3\tau} \sin \theta - \Gamma e^{-\tau} (\tau_1 \theta_1 + \theta_2) - M(2\tau_1 G + G_1) = 0, \quad (11)$$

where subscripts 1 and 2 are the first and second derivative with respect to the scaled  $X$ , and  $G \equiv 1/(X^2 + 1)^2 + dX^2/(X^2 + 1)^4$  with  $d \equiv -4[\mu]/\mu_2$ . Equation (11) is characterized by the tension factor  $\Gamma \equiv \gamma/\rho g H^2$  and the magnetic factor  $M \equiv m_2^2[1/\mu]/2\rho g H^5$ .

### 6. Numerical analysis — Spectral collocation method

Though the domain of eq. (11) is semi-infinite, we bound the upper limit at  $l$  where the magnetic field is weak enough, and impose the boundary condition  $\theta(X=l) = 0$  there. In  $0 \leq X \leq l$ , we expand  $\theta(X)$  into a Fourier series as

$$\theta(X) = \sum_{n=1}^{\infty} a_n \sin(n\pi X/l). \quad (12)$$

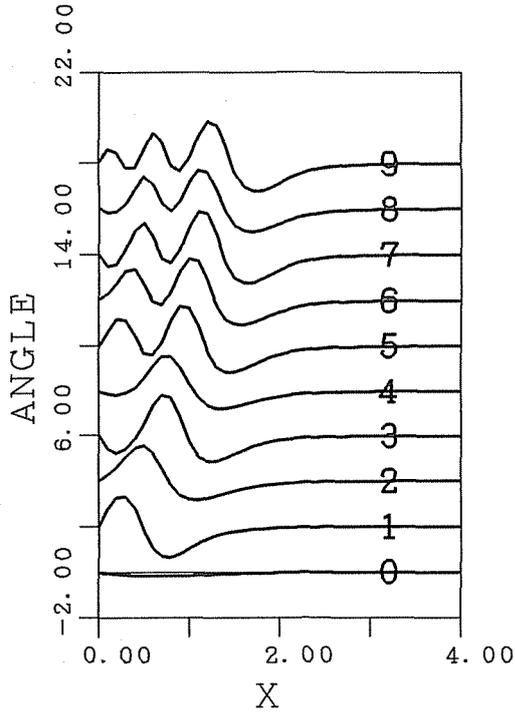


Fig. 3: Calculated angle function  $\theta$  for several magnetic factors  $M = 0.2$  (0), 0.4 (1), 0.6 (2), 0.8 (3), 1.0 (4), 1.2 (5), 1.4 (6), 1.6 (7), 1.8 (8), 2.0 (9) and common tension factor  $\Gamma = 0.2$  and  $d = 1.6$

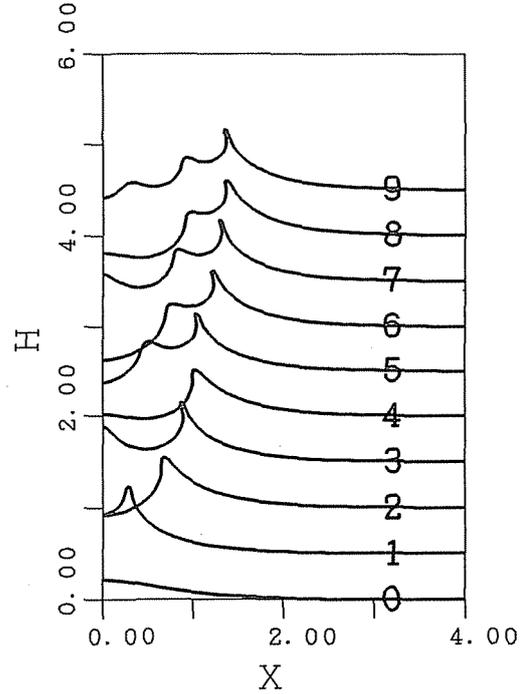


Fig. 4: Free surface shape  $x = x(X)$ ,  $y = y(X)$  calculated from  $\theta$  as shown in Fig. 3

Then, eq. (3) leads us to

$$\tau(X) = - \sum_{n=1}^{\infty} a_n \cos(n\pi X/l). \quad (13)$$

When the terms higher than the  $N$ -th are truncated, the coefficients  $a_n$  ( $1 \leq n \leq N$ ) are determined so that eq. (11) is satisfied at the collocation points  $X_i = il/(N+1)$  ( $1 \leq i \leq N$ ) in the domain. The simultaneous nonlinear equations thus obtained were solved by the Newton-Raphson method with the deceleration technique.

Figure 3 shows the angle functions  $\theta(X)$  as the solution of eq. (10) or (11) for several values of  $M$ , and common  $\Gamma$  and  $d$ . Numerical parameters are  $l = 5.0$  and  $N = 49$ . The convergence in solving eq. (11) was not necessarily slow for large  $M$ , but perturbations to  $a_n$ 's were occasionally given on the way to convergence, if necessary. In all cases, the errors in eq. (11) relative to each of its terms reached under the level of 2 percents within 50 iterations. The angle functions  $\theta(X)$  thus obtained were used for integrating eq. (4) by the fourth Runge-Kutta method with the increment  $\Delta X = 0.02$ . The resultant free surface shapes are shown in Fig. 4.

We observe from Fig. 4 that the waves generated near the center one after another shift outwards as  $M$  increases, but the extension of the domain containing waves has a

tendency of saturation, which means that the wavelength of the waves decreases as  $M$  increases. The outermost wave sometimes becomes three-valued or close to a cusp.

## 7. Discussions

For detailed discussions, Fig. 3 will be more convenient rather than Fig. 4. We observe the increase of the number of waves and the decrease of their wavelength more clearly on the angle function. The amplitude of the outer wave is larger than that of the inner, and the free surface shape turns to multi-valued when it exceeds  $\pi/2$ .

Since the magnetic field depends on  $X$  now, precise linear analysis on eq. (11) is difficult. But near the center where  $X \sim 0$  and  $G \sim 1$ , eq. (11) is linearized as  $\epsilon \equiv \theta - \Gamma\theta_2 - 2M\tau_1 = 0$ . Then, if  $\theta \sim \sin kX$  and  $\tau \sim -\cos kX$  is assumed,  $k = (M \pm \sqrt{M^2 - \Gamma})/\Gamma$  follows. There exists real  $k$  satisfying  $\epsilon = 0$  when  $M > \sqrt{\Gamma}$  ( $=0.45$  now), and it increases with  $M$ .

Figures 3 and 4 are the results reached through the iteration from the initial value  $a_n = 0$  for all  $n$ . There can be other branches of solution in the nonlinear equation (11), and actually do, which has the self-crossing free surface shape. They must be investigated in detail on the basis of the bifurcation theory.

Another problem is the existence of cusps. Since  $\tau \rightarrow \infty$  and  $e^{-\tau} \rightarrow 0$  there, the first and second term in eq. (11) vanish, and  $\tau \sim \ln(X^2 + 1)$  follows. Thus, cusps are unlikely to exist as far as  $X$  is finite.

**Acknowledgement.** This research is supported by a Grant-in-aid for Scientific Research No. 05650060 from the Ministry of Education, Science and Culture of Japan.

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# NORMAL FORMS AND GLOBAL EXISTENCE OF SOLUTIONS TO A CLASS OF CUBIC NONLINEAR KLEIN-GORDON EQUATIONS IN ONE SPACE DIMENSION

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**Abstract.** We prove that for small initial data there exist the unique global solutions to a certain special class of nonlinear Klein-Gordon equations with cubic nonlinearity in one space dimension, which asymptotically approach the free solutions of the linear Klein-Gordon equation as  $t \rightarrow +\infty$ .

## 1. Introduction

In the present paper we consider the global existence of solution for the Cauchy problem of the nonlinear Klein-Gordon equation with cubic nonlinearity in one space dimension, which asymptotically approaches the free solution of the linear Klein-Gordon equation as  $t \rightarrow +\infty$  :

$$\partial_t^2 u - \Delta u + u = F(u, \partial u, \partial \partial_x u), \quad t > 0, \quad x \in \mathbb{R} \quad . \quad (1.1)$$

$$u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad x \in \mathbb{R} \quad . \quad (1.2)$$

where  $u = u(t, x)$  is a real valued function and  $\partial_t = \frac{\partial}{\partial t}$ ,  $\partial_x = \frac{\partial}{\partial x}$ ,  $\Delta = \partial_x^2$ ,  $u_t = \partial_t u$ ,  $u_x = \partial_x u$ ,  $u_{tx} = \partial_t \partial_x u$ ,  $u_{xx} = \partial_x^2 u$ ,  $\partial u = (u_t, u_x)$ . Here,  $F(u, v, p)$  is a smooth function of  $(u, v, p) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ , and

$$F(u, v, p) = O(|u|^3 + |v|^3 + |p|^3) \quad \text{near } (u, v, p) = (0, 0, 0). \quad (1.3a)$$

Furthermore, we assume that  $F$  is linear with respect to  $p$ .

There are many papers concerning the global existence and the asymptotic behavior of solutions for nonlinear Klein-Gordon equations (see. e.g., [1]-[5], [7]-[11], [13]). Let  $N$  be the spatial dimensions. Klainerman and Ponce [3] and Shatah [9] showed that problem

(1.1)-(1.2) has the unique global solution for small initial data under the following condition on the nonlinear function  $F$ : When  $N = 1$ ,

$$F(u, v, p) = O(|u|^4 + |v|^4 + |p|^4) \quad \text{near } (u, v, p) = (0, 0, 0). \quad (1.4)$$

When  $N = 2, 3, 4$ ,

$$F(u, v, p) = O(|u|^3 + |v|^3 + |p|^3) \quad \text{near } (u, v, p) = (0, 0, 0).$$

When  $N \geq 5$ ,

$$F(u, v, p) = O(|u|^2 + |v|^2 + |p|^2) \quad \text{near } (u, v, p) = (0, 0, 0). \quad (1.5)$$

They also showed that the above solution asymptotically approaches the free solution of the linear Klein-Gordon equation with  $F \equiv 0$  as  $t \rightarrow +\infty$ . When  $N = 2, 3, 4$  and  $F$  is quadratic, and when  $N = 1$  and  $F$  is cubic, however, the usual  $L^p - L^q$  estimate does not provide us with a sufficient time decay estimate. To overcome this difficulty, Klainerman [4] and Shatah [10] separately developed two new techniques. In [4] Klainerman uses the invariant Sobolev space with respect to the generators of the Lorentz group in order to prove the global existence of solution of (1.1)-(1.2) under (1.5) for small initial data, when  $N = 3, 4$ . On the other hand, in [9] Shatah extends Poincaré's theory of normal forms for the ordinary differential equations to the nonlinear Klein-Gordon equations and proves the global existence of solution of (1.1)-(1.2) under (1.5) for small initial data, when  $N = 3, 4$  (see also Simon and Taffin [11] and its references). When  $N = 2$  and the initial data are small, Georgiev and Popivanov [1] and Kosecki [2] prove the global existence of solution of (1.1)-(1.2) for a certain special form of quadratic nonlinearity by using Klainerman's technique and by combining Klainerman's and Shatah's techniques, respectively. In [1] and [2] they study a new sufficient condition on the quadratic nonlinearity for the global existence of solution of (1.1)-(1.2) for small initial data, which is a variant of the null condition introduced by Klainerman [5] for the case of the nonlinear wave equation. In the two space dimensions, however, the global existence of solution of (1.1)-(1.2) for small initial data has been recently proved for all quadratic nonlinearity satisfying (1.5) by Simon and Taffin [11] and Ozawa, Tsutaya and Tsutsumi [7]. In [11] and [7], it is also showed that the solution constructed in their papers approaches a free solution of the linear Klein-Gordon equation as  $t \rightarrow +\infty$ . If we compare the nonlinear Klein-Gordon equation in  $N$  space dimensions with the nonlinear wave equation in  $N + 1$  space dimensions, from the results in [11] and [7] we have to say that the nonlinear Klein-Gordon equation is quite different from the nonlinear wave equation, although they have the same rate of time decay. In contrast to the two space dimensional case, however, in one space dimension we can not expect that for all cubic nonlinearity satisfying (1.3a) the non-trivial global solutions of (1.1)-(1.2) approach the free solutions of the linear Klein-Gordon equation in a usual sense as  $t \rightarrow +\infty$ . In fact, although we can show by using energy methods the global existence of solution of the following simple example of the Klein-Gordon equation with cubic nonlinearity in one space dimension :

$$\partial_t^2 u - \Delta u + u + u^3 = 0, \quad t > 0, \quad x \in \mathbb{R}.$$

but it is thought that the nontrivial solution of the above equation does not have a free profile as  $t \rightarrow +\infty$  (see Strauss [12, Theorem 2 in §6]). For some other nonlinear functions  $F$  in (1.1), we can also show the global existence of solutions of (1.1)-(1.2) by using energy methods (see Concluding Remark). However, it seems difficult that we investigate by using only energy methods whether such solutions have free profiles near  $t = +\infty$  or not. Therefore, in the one space dimensional case we need consider a new condition for the global existence of solution of (1.1)-(1.2) under (1.3a) with a free profile near  $t = +\infty$ . Recently, Yagi [13] has showed by using Shatah's technique that when

$$F = F_1 = 3uu_t^2 - 3uu_x^2 - u^3 \quad (1.6)$$

in (1.1), there is the unique global solution of (1.1)-(1.2) for small initial data. It is easy to see that the solution approaches a free solution of the linear Klein-Gordon equation as  $t \rightarrow +\infty$ . It is, however, of great interest to look for other nonlinear functions such as (1.6), for which (1.1)-(1.2) has a global solution with a free profile near  $t = +\infty$  for small initial data, and to study the analytical properties of them. In this paper, for simplicity of the exposition we assume that the nonlinear function  $F$  in (1.1) is a homogeneous polynomial of degree three. That is, we assume

$$\begin{aligned} F(u, v, p) & \text{ is a homogeneous polynomial of degree 3} \\ & \text{with respect to } u, v \text{ and } p. \end{aligned} \quad (1.3b)$$

By using the method of normal forms due to Shatah [10] in the same way as Yagi [13], we find, in addition to (1.6), the following six nonlinear functions leading to the global existence of solution of (1.1)-(1.2) with a free profile near  $t = +\infty$  for small initial data:

$$F_2 = 3u_t^2 u_x - u_x^3 - 3u^2 u_x + 6uu_t u_{tx} \quad (1.7)$$

$$F_3 = uu_x u_{xx} - u^2 u_x + u_t^2 u_x + 2uu_t u_{tx} \quad (1.8)$$

$$F_4 = (u_t^2 - u_x^2 - u^2) u_{xx} - 2uu_x^2 \quad (1.9)$$

$$F_5 = (u_t^2 - u_x^2 - u^2) u_{tx} + 2uu_t u_x \quad (1.10)$$

$$F_6 = u_t^3 - 3u_x^2 u_t - 3u^2 u_t - 6uu_x u_{tx} \quad (1.11)$$

$$F_7 = u_t u_x^2 + uu_t u_{xx} + 2uu_x u_{tx} \quad (1.12)$$

That is, we prove that when

$$F = \sum_{i=1}^7 c_i F_i \quad (1.13)$$

where  $c_i, i = 1, 2, \dots, 7$ , are real constants, for small initial data there exists the unique global solution of (1.1)-(1.2) which asymptotically approaches the free solution of the linear Klein-Gordon equation as  $t \rightarrow +\infty$ . We note that  $F_i, i = 1, 2, \dots, 7$ , are of course linearly independent as polynomials with variables  $u, u_t, u_x, u_{tx}$  and  $u_{xx}$ . Because  $F_2, F_3$  are the "odd-derivative-( $uuu, uu_t u_t$ )" type, that is, of the form

$$\sum_{a_1+a_2+a_3=1,3} (C_{a_1 a_2 a_3}^{uu_t u_t} \partial_x^{a_1} u \partial_x^{a_2} u_t \partial_x^{a_3} u_t + C_{a_1 a_2 a_3}^{uuu} \partial_x^{a_1} u \partial_x^{a_2} u \partial_x^{a_3} u) \quad (1.14)$$

$$a_1, a_2, a_3 \in \mathbb{N} \cup \{0\}, \quad C_{a_1 a_2 a_3}^{u u_t u_t} \cdot C_{a_1 a_2 a_3}^{u u u} \in \mathbb{R}.$$

and  $F_1, F_4$  are the "even-derivative- $(uuu, uu_t u_t)$ " type, that is, of the form (1.14) with  $a_1 + a_2 + a_3 = 0, 2, 4$ , and  $F_5$  is the "odd-derivative- $(u_t u_t u_t, uu u_t)$ " type, that is, of the form

$$\sum_{a_1 + a_2 + a_3 = 1, 3} (C_{a_1 a_2 a_3}^{u_t u_t u_t} \partial_x^{a_1} u_t \partial_x^{a_2} u_t \partial_x^{a_3} u_t + C_{a_1 a_2 a_3}^{u u u_t} \partial_x^{a_1} u \partial_x^{a_2} u \partial_x^{a_3} u_t). \quad (1.15)$$

$$a_1, a_2, a_3 \in \mathbb{N} \cup \{0\}, \quad C_{a_1 a_2 a_3}^{u_t u_t u_t} \cdot C_{a_1 a_2 a_3}^{u u u_t} \in \mathbb{R}.$$

and  $F_6, F_7$  are the "even-derivative- $(u_t u_t u_t, uu u_t)$ " type, that is, of the form (1.15) with  $a_1 + a_2 + a_3 = 0, 2$ . When  $F(u, v, p)$  is linear with respect to  $p$  and satisfies (1.3b), the dimensions of the space of cubic polynomials  $F(u, v, p)$  are 22. Our assumption (1.13) covers the subspace of 7 dimensions.

Before we state the main results in the present paper, we give several notations. For  $k \in \mathbb{N}$  and  $1 \leq p \leq \infty$ , let  $W^{k,p}$  denote the standard Sobolev space on  $\mathbb{R}$ . We choose  $a$  as an integer larger than or equal to 4. We put

$$\begin{aligned} X_1 &= W^{k+1,2} \oplus W^{k,2}, \\ X_2 &= W^{k,2} \oplus W^{k-1,2}, \quad k > (2a+3) + (2a+6), \\ \tilde{X}_a &= W^{k-(2a+1),2} \oplus W^{k-(2a+2),2}, \\ Y_a &= W^{2a+3,\infty} \oplus W^{2a+2,\infty}, \\ Z_a &= W^{2a+6,1} \oplus W^{2a+5,1}, \end{aligned}$$

with norm  $\|\cdot\|_{X_i}$ ,  $i = 1, 2$ ,  $\|\cdot\|_{\tilde{X}_a}$ ,  $\|\cdot\|_{Y_a}$  and  $\|\cdot\|_{Z_a}$ , respectively. For a matrix  $B$ ,  ${}^t B$  denotes the transpose matrix. We put  $\mathbf{u} = {}^t(u, \partial_t u)$ . Then we can write equation (1.1) as a first order system:

$$\mathbf{u}_t + A\mathbf{u} = \mathbf{F}(\mathbf{u}). \quad (1.16)$$

$$\mathbf{u}_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad (1.17)$$

where

$$A = - \begin{pmatrix} 0 & 1 \\ \partial_x^2 - 1 & 0 \end{pmatrix}, \quad \mathbf{F}(\mathbf{u}) = \begin{pmatrix} 0 \\ F(u, \partial u, \partial \partial_x u) \end{pmatrix}.$$

We have the following theorem concerning the global existence and asymptotic behavior of solution of (1.1)-(1.2).

**Theorem 1.1.** *Assume that the nonlinear function  $F$  in (1.1) is given by (1.13). Let  $\mathbf{u}_0 \in X_1 \cap Z_a$  and  $0 < \lambda < \frac{1}{4}$ . Let  $a$  be an integer not less than 4. Then, there exists a  $\delta > 0$  such that if*

$$\|\mathbf{u}_0\|_{X_1} + \|\mathbf{u}_0\|_{Z_a} + \|\mathbf{u}_0\|_{Y_a} < \delta \quad (1.18)$$

then (1.16)-(1.17) (i.e. (1.1)-(1.2)) has the unique global solution  $u$  satisfying

$$\mathbf{u}(\cdot) \in \bigcap_{i=0}^1 C^i([0, +\infty), X_{i+1}) \quad (1.19)$$

$$||| \mathbf{u} ||| = \sup_{t \geq 0} \left( (1+t)^{-\lambda} \| \mathbf{u}(t) \|_{X_1} + (1+t)^{\frac{1}{2}} \| \mathbf{u}(t) \|_{Y_a} \right) < +\infty. \quad (1.20)$$

Furthermore, the above solution  $u$  has a free profile  $\mathbf{u}_{+0} = {}^t(u_{+0}, u_{+1}) \in \tilde{X}_a$  such that

$$\| \mathbf{u}(t) - \mathbf{u}_+(t) \|_{\tilde{X}_a} \longrightarrow 0 \quad \text{as } t \rightarrow +\infty. \quad (1.21)$$

where  $\mathbf{u}_+ = (u_+, \partial_t u_+)$  is a free solution of the linear Klein-Gordon equation

$$\partial_t^2 u_+ - \Delta u_+ + u_+ = 0, \quad t > 0, \quad x \in \mathbb{R}$$

with the initial condition

$$u_+(0, x) = u_{+0}, \quad \partial_t u_+(0, x) = u_{+1}, \quad x \in \mathbb{R}.$$

*Remark 1.1.* We do not know the algebraic condition representing  $F_i, i = 1, 2, \dots, 7$  such as the null condition (see [5]) and the unit condition (see [2]).

The unique existence of local solutions of (1.1)-(1.2) follows from the standard iteration argument (see, e.g., [3] and [9]). The crucial part of the proof of Theorem 1.1 is to establish the a priori estimates of the solution for (1.1)-(1.2) in order to extend the local solution globally in time. The global behavior of the local solution of (1.1)-(1.2) can not be controlled directly, since the cubic nonlinear term in (1.1) does not provide a sufficient decay property for the one dimensional case. Here we use the argument of normal forms of Shatah [10] to transform the cubic nonlinearity into the nonlinearity of degree five. In general, however, in our problem the transformed nonlinearity is represented in terms of the integral operator with singular kernel. The singularity of the integral operator makes it difficult to solve (1.1)-(1.2). Therefore, our main task in the proof of Theorem 1.1 is to find the class of the cubic nonlinear function  $F$  in (1.1) whose transformed nonlinearity is represented in terms of the integral operator with regular kernel. This enables us to apply the usual  $L^\infty - L^1$  estimate to the transformed equation, which provides us with the sufficient decay properties of solution of (1.1)-(1.2) for the proof of Theorem 1.1.

**Concluding Remark.** We can show by using the energy method that (1.1)-(1.2) has the global solution, if the nonlinear function  $F$  independent of  $u_{tx}, u_{xx}$  satisfies the following two conditions (see, e.g., Sattinger [8]):

There exist positive constants  $C_1, C_2$  such that

$$(1) \quad \int_0^t \int_{\mathbb{R}} F(u, u_t, u_x) u_t dx dt \\ \leq C_1 \int_{\mathbb{R}} (u^4 + u_x^2 u^2 + u_t^2 u^2 + u_x u^3 + u_t u^3) dx, \quad u \in C_0^1((0, +\infty) \times \mathbb{R}),$$

$$(2) \quad \| F(u, u_t, u_x) \|_{H^1} \leq C_2 (\| u \|_{H^1} + \| u_t \|_{L^2})^2 \\ \times (\| u \|_{H^1} + \| u_t \|_{L^2} + \| u_{xx} \|_{L^2} + \| u_{tx} \|_{L^2}), \quad u \in C_0^1((0, +\infty) \times \mathbb{R}).$$

It is easily verified that the following cubic nonlinear function  $F = \alpha u^3 + \beta u^2 u_x + \gamma u_t^3 + \lambda u^2 u_t$  satisfies the above two conditions, where  $\alpha, \beta \in \mathbb{R}, \gamma, \lambda < 0$ .

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**Acknowledgement.** I wish to express my gratitude to Professor Y. Tsutsumi for his constant encouragement and the interest in my work and for very helpful suggestions. I also wish to thank the referee for reading this paper carefully and suggesting the concluding remark.

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# ASYMPTOTICS OF ENERGY FOR WAVE EQUATIONS WITH A NONLINEAR DISSIPATIVE TERM IN $\mathbb{R}^N$

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**Abstract.** We study the asymptotic behavior of solutions to the wave equations with a nonlinear dissipative term  $b(x, t)|w_t(t)|^{\rho-1}w_t(t)$ , where  $\rho > 1$  and  $0 \leq b(x, t) \leq C(1 + |x|)^{-\delta}$  with some constants  $C > 0$  and  $0 \leq \delta \leq 1$ . We obtain sufficient conditions on  $\delta$  and  $\rho$  under which every solution behaves like a free solution as  $t \rightarrow \infty$  in the energy space.

## 1. Introduction

We consider the Cauchy problem

$$(1.1) \quad w_{tt}(t) - \Delta w(t) + \lambda w(t) + b(x, t)|w_t(t)|^{\rho-1}w_t(t) = 0,$$

$$(1.2) \quad w(x, 0) = w_1(x) \quad \text{and} \quad w_t(x, 0) = w_2(x)$$

for  $(x, t) \in \mathbb{R}^N \times (0, \infty)$ , where  $\lambda \geq 0, \rho > 1$  and

$$(1.3) \quad |b_t(x, t)| + |\nabla b(x, t)| \leq C_1 b(x, t)$$

for a.e.  $(x, t) \in \mathbb{R}^N \times (0, \infty)$  with some constants  $C_1 > 0$ . As it is well-known, the existence of a unique weak global solution is established under a weak condition on  $b(x, t)$  (See Lions - Strauss [2]). It becomes a strong solution if we require (1.3) (See Motai - Mochizuki [4]). In [3] we have investigated energy decay and nondecay problems to (1.1) and (1.2). Here energy to (1.1) is defined by  $\|W(t)\|_E^2$ , where  $W(t) = {}^t(w(t), w_t(t))$  and

$$\|f\|_E^2 = \frac{1}{2} \{ \|\nabla f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2 + \lambda \|f_1\|_{L^2}^2 \}$$

for  $f = {}^t(f_1, f_2)$ . We denote by  $E$  the space of pairs  $f$  of functions such that  $\|f\|_E < \infty$  and by  $U_0(t)$  the unitary operator in  $E$  which represents the solution of the free wave

equations. The following results is obtained in [3]: The energy decay occurs provided that (1.4) is satisfied;

(1.4)

$$\lambda \geq 0, \quad N \geq 1, \quad 1 < \rho \leq 1 + \frac{2(1-\delta)}{N} \quad \text{and} \quad b_1(1+|x|+t)^{-\delta} \leq b(x,t) \leq b_2$$

for some  $0 \leq \delta < 1$  and  $b_1, b_2 > 0$ . If  $\lambda = 0$ , we additionally require  $b(x,t)$  is nonincreasing in  $t \geq 0$ .

On the other hand, the energy does not decay for a class of small initial data provided that (1.5) is satisfied;

(1.5)

$$\text{If } \lambda = 0, \quad N \geq 2, \quad \rho > 1 + \frac{2(1-\delta)}{N-1} \quad \text{and} \quad 0 \leq b(x,t) \leq b_3(1+|x|)^{-\delta}$$

$$\text{and if } \lambda > 0, \quad N \geq 1, \quad \rho > 1 + \frac{2(1-\delta)}{N} \quad \text{and} \quad 0 \leq b(x,t) \leq b_3(1+|x|)^{-\delta}$$

for some  $0 \leq \delta \leq 1$  and  $b_3 > 0$ .

Our purpose is to investigate the asymptotics of the energy to (1.1) and (1.2) when it does not decay as  $t \rightarrow \infty$ . Then we obtain the following

**Theorem.** *Let  $w(t)$  be a strong solution to (1.1) and (1.2). Assume other than (1.5) the following (1.6) or (1.7):*

$$(1.6) \quad \text{if } \lambda = 0, \quad N \geq 2 \quad \text{and} \quad 1 + \frac{4(1-\delta)}{N-1} < \rho < \rho_N,$$

$$(1.7) \quad \text{if } \lambda > 0 \quad N \geq 1 \quad \text{and} \quad 1 + \frac{2(2-\delta)}{N} < \rho < \rho_N,$$

where  $\rho_N = \infty$  ( $1 \leq N \leq 6$ ),  $= N/(N-6)$  ( $N \geq 7$ ). Then there exists a  $W^+ = {}^t(w_1^+, w_2^+) \in E$  such that

$$(1.8) \quad \|U_0(-t)W(t) - W^+\|_E \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In order to prove this theorem, an weighted Strichartz estimate, which is deduced from the interpolation theory, will play an important role.

## 2. Outline of Proof

For simplicity we assume  $\lambda = 0$  and  $N \geq 3$ . An argument in the case  $\lambda > 0$  is as same as the one in the case  $\lambda = 0$ .

The following lemma is an extension of the well-known sufficient condition for which (1.8) holds.

**Lemma 2.1.** *Let  $w(t)$  be a solution to (1.1) and (1.2). Assume that there exists an  $r \geq 2$  and a Sobolev space  $Y$  such that*

$$(2.1) \quad \|[U_0(\cdot)g]_2\|_{L^r((0,\infty); Y)} \leq C\|g\|_E$$

for any  $g = {}^t(g_1, g_2) \in E$ . Here we denote by  $[U_0(t)g]_2$  the second component of  $U_0(t)g$ . In addition, assume that

$$(2.2) \quad b(x, t)|w_t(t)|^{\rho-1}w_t(t) \in L^{r'}((0, \infty); Y'),$$

where  $1/r + 1/r' = 1$  and  $Y'$  is the dual space of  $Y$  with respect to  $L^2$ . Then there exists a  $W^+ = {}^t(w_1^+, w_2^+) \in E$  such that (1.8) holds.

For a proof see Motai - Mochizuki [4] Proposition 2.1.

In order to apply Lemma 2.1, we need to show (2.1) and (2.2) hold. For the strong solution to (1.1) and (1.2) we can prove that

$$(2.3) \quad b(x, t)|w_t(t)|^{\rho-1}w_t(t) \in L^{(\rho+1)/\rho}((0, \infty); H_{(\rho+1)/\rho, \delta/(\rho+1)}^{1,0}).$$

Here  $H_{p, \mu}^{k, s}$  is a weighted Sobolev space with norm

$$\|u\|_{H_{p, \mu}^{k, s}} = \|(1 + |x|^2)^{\mu/2}(-\Delta)^{s/2}(1 - \Delta)^{k/2}u\|_{L^p}.$$

Especially we denote  $L_\mu^p = H_{p, \mu}^{0,0}$  and  $H_p^{k, s} = H_{p, 0}^{k, s}$ . As we consider the duality of the space in (2.3), it is necessary to establish an weighted Strichartz estimate. We have

**Lemma 2.2.** *Let  $N \geq 3$ . Suppose that*

$$(2.4) \quad \frac{N-1}{2(N+1)} \leq \frac{1}{r} \leq \frac{1}{2}, \quad \mu = \beta\left(\frac{N+1}{r} - \frac{N-1}{2}\right)$$

with  $\beta > \frac{1}{2}$  and  $e = \frac{(N+1)}{2}\left(\frac{1}{2} - \frac{1}{r}\right)$ .

Then we have

$$(2.5) \quad \|[U_0(\cdot)g]_2\|_{L^r((0,\infty); H_{r, -\mu}^{0, -e})} \leq C\|g\|_E$$

for any  $g \in E$ .

*Proof:* (2.5) can be obtained by the interpolation of the following two well-known estimates

$$(2.6) \quad \|[U_0(\cdot)g]_2\|_{L^{2(N+1)/(N-1)}((0,\infty); H_{2(N+1)/(N-1)}^{0, -1/2})} \leq C\|g\|_E$$

and

$$(2.7) \quad \|[U_0(\cdot)g]_2\|_{L^2((0,\infty); L_{-\beta}^2)} \leq C\|g\|_E \quad \text{with } \beta > \frac{1}{2}.$$

for any  $g \in E$ . ■

Now we are ready to apply Lemma 2.1. We put  $r = \rho + 1$  and  $Y = H_{\rho+1, -\mu}^{0, -e}$ . So we need to show

$$(2.8) \quad b(x, t)|w_t(t)|^{\rho-1}w_t(t) \in L^{(\rho+1)/\rho}((0, \infty); H_{(\rho+1)/\rho, \mu}^{0, e}).$$

Note (2.3), then  $0 \leq e \leq 1$  and  $\mu \leq \delta/(\rho + 1)$  imply (2.8). Calculating (2.4) and  $\mu \leq \delta/(\rho + 1)$ , we obtain the range

$$1 + \frac{4(1 - \delta)}{N - 1} < \rho \leq \frac{N + 3}{N - 1}.$$

Thus we can prove Theorem for this range. The proof in the case  $\frac{N+3}{N-1} < \rho < \rho_N$  can be found in Motai - Mochizuki [4].

Finally we mention about the case  $\lambda > 0$ . As we show the proof of the case  $\lambda = 0$ , one of the important estimates is (2.7). But in the case  $\lambda > 0$  the estimate of this type has not been known yet. If we follow the work of Ben-Artzi - Klainerman [1], we obtain the estimate

$$(2.9) \quad \|[U_0(\cdot)g]_2\|_{L^2((0, \infty); L^2_{-\beta})} \leq C\|g\|_E \quad \text{with } \beta = 1.$$

But this only implies (2.9) for  $\beta = 1$ . This is the difference between the case  $\lambda = 0$  and the case  $\lambda > 0$ . This is the reason why the infima of  $\rho$  in Theorem are different form between the case  $\lambda = 0$  and the case  $\lambda > 0$ . For details refer to our forthcoming paper [5].

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# NONLINEAR MATHIEU EQUATIONS I

Fumio Nakajima

## 1. Introduction

We shall consider the nonlinear Mathieu equation

$$\ddot{x} + k\dot{x} + a(t)x + x^3 = 0, \quad (\cdot = \frac{d}{dt}) \quad (1.1)$$

where  $k$  is a positive constant,  $a(t)$  is a continuous,  $2\pi$ -periodic function and  $a(t) \geq 0$  for  $0 \leq t \leq 2\pi$ . This equation describes the phenomenon of parametric excitations, and the existence of nontrivial periodic solutions was investigated by some sorts of approximation methods ([2],[5]). Then there arises another natural question whether every solution is bounded for  $t \geq 0$ .

To this problem we may only recall [1, Theorem 4] among many boundedness theorems, because the term of restoring force of (1.1) depends not only on  $x$ , but also on  $t$ ; by this result every solution  $x(t)$  of (1.1) satisfies that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  under the additional condition

$$k^2 > \max\{a(t); 0 \leq t \leq 2\pi\} - \min\{a(t); 0 \leq t \leq 2\pi\} \quad (1.2)$$

(cf.[3]). In Theorem 1 we shall prove the ultimate boundedness of solutions of (1.1) for any positive  $k$ , and hence (1.2) may be not assumed for the boundedness. Our

tool of the proof is M.Struwe's lemmas on the rapid oscillation property [4], whose use in the study of boundedness of solutions seems to be new.

## 2. Boundedness theorems

We consider the equation

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + a(t)x + g(x) = e(t), \quad (2.1)$$

where  $f(t, x, y)$  satisfies the Caratheodory condition, i.e.,  $f(t, x, y)$  is measurable in  $t \in R$  and continuous for  $(x, y) \in R^2$ ,  $a(t)$  and  $e(t)$  are measurable in  $t \in R$  and  $g(x)$  is continuous for  $x \in R$ . Then the solutions are understood in the term of absolutely continuous functions. Through this paper we assume the following conditions :

(I)  $a(t)$  and  $e(t)$  are bounded on  $R$  ;

(II) there exist positive constants  $k$  and  $K$ , where  $k < K$ , such that  $k \leq f(t, x, y) \leq K$  for all  $(t, x, y) \in R^3$  ;

(III)

$$\lim_{|x| \rightarrow \infty} \frac{g(x)}{x} = \infty. \quad (2.2)$$

Clearly (1.1) is a special case of (2.1). Our main theorem is the following

**Theorem 1.** *Under the situation above the solutions of (1.1) are ultimately bounded. Namely there exist a positive constant  $B$  and a function  $T(C)$  of positive number  $C$  such that if  $x(t)$  is any solution of (1.1) satisfying  $x(t_0) = \xi$  and  $\dot{x}(t_0) = \eta$  for an arbitrary  $t_0 \in R$  and for an arbitrary  $(\xi, \eta) \in R^2$ , where  $|\xi| + |\eta| \leq C$ , then  $x(t)$  is defined for all  $t \in R$  and satisfies that*

$$|x(t)| + |\dot{x}(t)| \leq B \quad \text{for} \quad t \geq t_0 + T(C). \quad (2.3)$$

For the proof we prepare two lemmas. Let consider a more general equation

$$\ddot{x} + F(t, x, \dot{x}) = 0, \quad (2.4)$$

where  $F(t, x, y)$  satisfies the Caratheodory condition and there exist a positive constant  $L$  and a continuous function  $g(x)$  with (2.2) such that

$$|F(t, x, y) - g(x)| \leq L(|x| + |y| + 1) \quad \text{for } (t, x, y) \in R^3. \quad (2.5)$$

Clearly equation (2.1) satisfies this condition. Let  $x(t, t_0, \xi, \eta)$  denote every solution  $x(t)$  of (2.4) satisfying  $x(t_0) = \xi$  and  $\dot{x}(t_0) = \eta$  for an arbitrary  $t_0 \in R$  and for an arbitrary  $(\xi, \eta) \in R^2$ . We restate [4, Lemmas 1,2,3] as follows :

**Lemma 1.** *Under the situation above the following facts hold for (2.4) :*

(i)  $x(t, t_0, \xi, \eta)$  exists for  $t \in R$  and for  $(t_0, \xi, \eta) \in R^3$  ;

(ii) for any positive constants  $C$  and  $\tau$  there exists a positive constant  $\alpha(C, \tau)$  such that if  $|\xi| + |\eta| \geq \alpha(C, \tau)$ , then

$$|x(t, t_0, \xi, \eta)| + |\dot{x}(t, t_0, \xi, \eta)| \geq C \quad \text{for } |t - t_0| \leq \tau \text{ and for } t_0 \in R;$$

(iii) for any positive constant  $\tau$  there exists a positive constant  $\beta(\tau)$  such that if  $|\xi| + |\eta| \geq \beta(\tau)$ , then  $x(t, t_0, \xi, \eta)$  has at least one zero point in the interval  $[t_0, t_0 + \tau]$ , where  $t_0$  is any number.

Using Lemma 1 we shall prove

**Lemma 2.** *For any positive constants  $N$  and  $\tau$  there exists a positive constant  $\gamma(N, \tau)$ , where  $\gamma(N, \tau) > \beta(\tau)$ , such that if  $|\xi| + |\eta| \geq \gamma(N, \tau)$ , then for any  $t_0 \in R$  we obtain*

$$\int_{t_0}^{t_0+\tau} \dot{x}^2(t, t_0, \xi, \eta) dt > N^2. \quad (2.6)$$

*Proof.* By (2.5) we choose a positive constant  $M$  such that if  $|x| \geq M$ , then

$$F(t, x, 0) \operatorname{sgn} x > 0 \quad \text{for } t \text{ in } R. \quad (2.7)$$

In the following we assume that  $N \geq M$  and set  $\gamma(N, \tau) = \alpha(\max\{N, \beta(\frac{\tau}{3})\}, \tau)$ . Suppose that  $|\xi| + |\eta| \geq \gamma(N, \tau)$ . Then  $x(t, t_0, \xi, \eta)$ , or simply  $x(t)$ , satisfies

$$|x(t)| + |\dot{x}(t)| \geq \max\{N, \beta(\frac{\tau}{3})\} \quad \text{for } [t_0, t_0 + \tau] \quad (2.8)$$

by (ii) of Lemma 1. Setting  $I = [t_0, t_0 + \tau]$ , by (iii) of Lemma 1 we see that  $x(t)$  has at least three zero points in  $I$ , and hence two critical points, say  $s_1$  and  $s_2$ , in  $I$ . By (2.7)  $s_1$  and  $s_2$  are locally either minimal or maximal. For the simplicity, we may suppose that  $s_1 < s_2$ , that  $s_1$  is locally minimal and  $s_2$  locally maximal and that  $\dot{x}(t) > 0$  for  $s_1 < t < s_2$ . Therefore the  $t$  of  $x(t)$  is defined as an increasing function of  $x$  on the interval  $J$ , where  $J = [x(s_1), x(s_2)]$ , and hence it follows from change of integral variable that

$$\int_{t_0}^{t_0+\tau} \dot{x}^2(t) dt \geq \int_{s_1}^{s_2} \dot{x}^2(t) dt = \int_{x(s_1)}^{x(s_2)} \dot{x}(t) dx.$$

Since the right-hand side is the area of the set  $\{(x, y) \in R^2 : x \in J, 0 \leq y \leq \dot{x}(t)\}$  and since this area is larger than  $N^2$  by (2.8), we obtain (2.6). The proof of Lemma 2 is complete.

We shall prove Theorem 1 by three steps.

Step 1. Let choose a positive constant  $\tau$  such that  $\tau < k/A$ , where  $A = \sup_{t \in R} |a(t)|$ , and set  $N_0$  to be the positive root of the quadratic equation of  $N$

$$(k - \tau A)N^2 - \sqrt{\tau}EN - 1 = 0,$$

where  $E = \sup_{t \in R} |e(t)|$ . Furthermore we set

$$V(x, y) = \frac{y^2}{2} + \int_0^x g(s) ds.$$

Because of (2.2) we see that  $V(x, y) \rightarrow \infty$  if and only if  $|x| + |y| \rightarrow \infty$  and that the set  $L(C) = \{(x, y) \in R^2 : V(x, y) = C\}$ , where  $C$  is a positive constant, is a

closed, bounded curve in  $R^2$ . Therefore we may take positive constants  $C_0$  such that if  $C \geq C_0$ , then every point  $(\xi, \eta)$  of  $L(C)$  satisfies

$$|\xi| + |\eta| \geq \gamma(N_0, \tau), \quad (2.9)$$

and moreover a constant  $C_1$  by (ii) of Lemma 1 such that if  $V(\xi, \eta) = C_1$ , then

$$V(x(t), \dot{x}(t)) \geq C_0 \quad \text{for } |t - t_0| \leq \tau \text{ and for any } t_0 \in R, \quad (2.10)$$

where  $x(t)$  is any solution of (2.1) satisfying  $x(t_0) = \xi$  and  $\dot{x}(t_0) = \eta$ .

Step 2. Setting  $v(t) = V(x(t), \dot{x}(t))$  for each solution  $x(t)$  of (2.1) we claim that if  $v(t_1) \geq C_0$  for some  $t_1 \in R$ , then

$$v(t_1 + \tau) \leq v(t_1) - 1. \quad (2.11)$$

To the contrary suppose that there is a  $t_1 \in R$  such that  $v(t_2) > v(t_1) - 1$ , where  $t_2 = t_1 + \tau$ . Then multiplying the both sides of (2.1) by  $\dot{x}(t)$  and integrating them on the interval  $[t_1, t_2]$  we obtain that

$$v(t_2) - v(t_1) + \int_{t_1}^{t_2} f(t, x(t), \dot{x}(t)) \dot{x}^2(t) dt + \int_{t_1}^{t_2} a(t)x(t)\dot{x}(t) dt = \int_{t_1}^{t_2} e(t)\dot{x}(t) dt$$

and hence by condition (II) and the assumption above that

$$-1 + k \int_{t_1}^{t_2} \dot{x}^2(t) dt - A \int_{t_1}^{t_2} |x(t)\dot{x}(t)| dt \leq \int_{t_1}^{t_2} |e(t)\dot{x}(t)| dt.$$

Setting

$$\|\varphi\| = \sqrt{\int_{t_1}^{t_2} \varphi^2(t) dt}$$

for any continuous function  $\varphi(t)$  on  $[t_1, t_2]$  and applying Schwartz's inequality to the inequality above we see that

$$k\|\dot{x}\|^2 - A\|x\| \cdot \|\dot{x}\| - \|e\| \cdot \|\dot{x}\| - 1 \leq 0, \quad (2.12)$$

Since  $v(t_1) \geq C_0$ , (2.9) implies that

$$|x(t_1)| + |\dot{x}(t_1)| \geq \gamma(N_0, \tau),$$

and since  $\gamma(N_0, \tau) \geq \beta(\tau)$ , there is a  $s_0 \in [t_1, t_2]$  such that  $x(s_0) = 0$  which implies

$$x(t) = \int_{s_0}^t \dot{x}(s) ds.$$

Therefore we obtain that  $\|x\| \leq \tau \|\dot{x}\|$ , and moreover since  $\|e\| \leq \sqrt{\tau}E$ , (2.12) implies that

$$(k - A\tau)\|\dot{x}\|^2 - \sqrt{\tau}E\|\dot{x}\| - 1 \leq 0.$$

Consequently it follows from the definition of  $N_0$  that  $\|\dot{x}\| \leq N_0$ . On the other hand Lemma 2 guarantees that  $\|\dot{x}\| > N_0$ . This contradiction proves (2.11).

Step 3. Let  $x(t)$  be any solution of (2.1) satisfying  $x(t_0) = \xi$  and  $\dot{x}(t_0) = \eta$  for an arbitrary  $t_0 \in R$  and for an arbitrary  $(\xi, \eta) \in L(C)$ , where  $C \geq C_0$ . We set  $n$  to be the least integer larger than  $C - C_0$ . Then by (2.11) there exists a positive constant  $T_1$ , where  $T_1 \leq n\tau$ , such that  $v(t) \leq C_0$  for  $t = t_0 + T_1$ . Now we claim that  $(x(t), \dot{x}(t))$  cannot meet the curve  $L(C_1)$  for all  $t \geq t_0 + T_1$ . In fact, if  $(x(t), \dot{x}(t))$  meets  $L(C_1)$  for some  $t_3 \geq t_0 + T_1$ , that is  $v(t_3) = C_1$ , then (2.10) implies that  $C_0 \leq v(t) \leq C_1$  for  $t_3 - \tau \leq t \leq t_3$ . Since  $v(t_3 - \tau) \geq C_0$ , (2.11) holds for  $t_1 = t_3 - \tau$ , and hence  $C_1 = v(t_3) \leq v(t_3 - \tau) - 1 \leq C_1 - 1$ , which is a contradiction. Therefore  $(x(t), \dot{x}(t))$  must remain in the inside of  $L(C_1)$  for  $t \geq t_0 + T_1$  and hence for  $t \geq t_0 + n\tau$ . This shows the conclusion (2.3). The proof is complete.

**Remark 1.** We cannot drop condition (III) from Theorem 1. Indeed we may construct a continuous, positive and periodic function  $a(t)$  such that the linear equation  $\ddot{x} + k\dot{x} + a(t)x = 0$  has a unbounded solution as  $t \rightarrow \infty$ .

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# ON NUMERICAL COMPUTATIONS TO THE OIL-RESERVOIR PROBLEMS IN TWO-DIMENSIONAL RECTANGULAR REGIONS

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**Abstract.** We shall consider a two-phase flow through a porous medium in the region  $\Omega(l) = (0, 1) \times (0, l)$ . It is already known that there appears an *interface* separating the two fluids, and that the interface is unstable when a capillary pressure of the two fluids is ignored. In this paper, we treat the case where the capillary pressure is effective. From the numerical points of view, we shall try to investigate the relation among the stability of the interface, the capillary pressure and the aspect ratio  $l$ . In certain situation, the existence of a steady-state solution is already known. We shall also suggest that this solution is stable.

## 1. Introduction

In order to recover part of remaining oil in a reservoir from wells (called *production wells*), it is used the way that one injects water into another wells (called *injection wells*), which are located around the reservoir, so that water pushes oil toward the production wells. In this process, two immiscible fluids — water and oil — flow through the porous medium, and can be regarded as separated by a *sharp interface* during penetration of water into oil. By laboratory studies, it is already known that the interface is unstable; small perturbations given to the interface grow up (see [4], [6] and [7], for examples).

We consider this problem in the rectangular region  $\Omega(l) = (0, 1) \times (0, l)$ , and impose boundary conditions which imply the following:  $\{0\} \times (0, l)$  is the injection wells;  $\{1\} \times (0, l)$  is the production wells;  $(0, 1) \times \{0\}$  and  $(0, 1) \times \{1\}$  are reflective wells. Under these conditions, there appear *planar wave solutions* (see (2.11)). When capillary pressure between the two fluids is ignored, Chorin [1] shows that the interface of the planar wave solution is unstable in a linearized sense. He also shows that the growth rate of the perturbation given to the interface is proportional to the wave number of the perturbation.

In this paper, we treat the case where the capillary pressure is effective. Using numerical simulations, we investigate the stability of the interface varying the magnitude of the capillary pressure and the aspect ratio  $l$ . We also suggest that a steady-state solution, which exists under some condition (see Section 5), is stable. Our numerical simulations are done by some finite difference method.

## 2. Basic equations

Let  $s_i(\mathbf{x}, t)$ ,  $v_i(\mathbf{x}, t)$  and  $p_i(\mathbf{x}, t)$  be the fractional amount, velocity and pressure of fluid  $i$ , respectively ( $i = \text{water}, \text{oil}$ ). Then the following equations hold [6].

$$(2.1) \quad s_{\text{water}} + s_{\text{oil}} = 1;$$

$$(2.2) \quad -\nabla \cdot (\rho_i v_i) = \frac{\partial}{\partial t} (n \rho_i s_i), \quad i = \text{water}, \text{oil};$$

$$(2.3) \quad v_i = -\frac{K k_i}{\mu_i} \nabla p_i, \quad i = \text{water}, \text{oil};$$

$$(2.4) \quad p_{\text{oil}} - p_{\text{water}} = p_c.$$

Here  $\rho_i$  and  $\mu_i$  are the density and viscosity of fluid  $i$  ( $i = \text{water}, \text{oil}$ ),  $n = n(\mathbf{x})$  and  $K = K(\mathbf{x})$  are the porosity and absolute permeability of the medium,  $k_{\text{water}} = k_{\text{water}}(s_{\text{oil}})$  and  $k_{\text{oil}} = k_{\text{oil}}(s_{\text{water}})$  are the relative permeabilities of fluid *water* and *oil*, and  $p_c = p_c(s_{\text{water}})$  is the capillary pressure between the two fluids. The equations (2.1) and (2.2) imply the conservation of the fluids, (2.3) is called the Darcy's law, and (2.4) describes the balance law of the pressure. Our numerical computations are done in the region  $\mathbf{x} = (x, y) \in \Omega(l) = (0, 1) \times (0, l)$ ,  $t > 0$  under the boundary conditions

$$(2.5) \quad s_{\text{water}}(0, y, t) = s^*, \quad p_{\text{water}}(0, y, t) = p^* \quad \text{on } 0 < y < l, \quad t > 0,$$

$$(2.6) \quad s_{\text{water}}(1, y, t) = 0, \quad p_{\text{water}}(1, y, t) = 0 \quad \text{on } 0 < y < l, \quad t > 0,$$

$$(2.7) \quad \frac{\partial}{\partial y} s_i(x, y, t) = \frac{\partial}{\partial y} p_i(x, y, t) = 0 \quad \text{on } (x, y) \in (0, 1) \times \{0, l\}, \quad t > 0, \quad i = \text{water}, \text{oil},$$

where  $l > 0$  is the aspect ratio of the rectangular region  $\Omega(l)$  and  $s^*$  and  $p^*$  are constants. The initial condition

$$(2.8) \quad s_{\text{water}}(x, y, 0) = s_0(x, y) \quad \text{on } (x, y) \in \Omega(l)$$

is also imposed. The condition (2.5) describes that fluid of the saturation  $s^*$  is injected into the injection wells with pressure  $p^*$ .

**Remark 1.** It is natural to put  $s^* = 1$ , because  $s^* = 1$  implies that water is injected into the injection wells. However, to compare our numerical simulations with the result by Chorin [1], we take  $s^* = 1/\sqrt{1+\mu}$  in Sections 3 and 4, where  $\mu = \mu_{\text{oil}}/\mu_{\text{water}}$ . In Section 5, we put  $s^* = 1$ .

**Remark 2.** The value  $1/\sqrt{1+\mu}$  in Remark 1 is a typical one appearing our problem; Roughly speaking, if  $p_c \equiv 0$ ,  $s_{\text{water}} = 1/\sqrt{1+\mu}$  and  $s_{\text{water}} = 0$  are connected by a shock wave at the interface (see Theorem 4 in [5]).

Following [1], [2], [3] and [5], we assume that  $\rho_i$ ,  $\mu_i$  ( $i = \text{water, oil}$ ),  $n$  and  $K$  are constants and

$$(2.9) \quad k_i(s) = s^2, \quad p_c(s) = \varepsilon(1 - s),$$

where  $\varepsilon$  is a constant which describes the magnitude of the capillary pressure. Then, after some normalization on  $t$ , we find that all constants except for  $\mu = \mu_{oil}/\mu_{water}$ ,  $\varepsilon$ ,  $l$ ,  $s^*$  and  $p^*$  can be normalized to unity. The interface at time  $t$  is defined by

$$(2.10) \quad I(t) = \partial\{(x, y); s_{water}(x, y, t) > 0\} \setminus \partial\Omega(l),$$

which separates the region where  $s_{water} > 0$  and the region where  $s_{water} = 0$ .

Under the boundary conditions (2.5)–(2.7), there are solutions (called planar wave solutions)  $s_i(x, y, t)$ ,  $v_i(x, y, t)$ ,  $p_i(x, y, t)$  of the following form:

$$(2.11) \quad s_i(x, y, t) = \tilde{s}_i(x, t), \quad v_i(x, y, t) = \tilde{v}_i(x, t), \quad p_i(x, y, t) = \tilde{p}_i(x, t),$$

where  $\tilde{s}_i(x, t)$ ,  $\tilde{v}_i(x, t)$  and  $\tilde{p}_i(x, t)$  ( $i = \text{water, oil}$ ) are solutions to a one-dimensional problem of (2.1)–(2.4).

### 3. Numerical computations when $\varepsilon = 0$

In this section we treat the case where the capillary pressure is ignored. In this case, Chorin [1] shows the following (adapted to our notation).

**Theorem 1** [1]. Assume that  $\varepsilon = 0$ ,  $s^* = 1/\sqrt{1 + \mu}$ ,  $p^* > 0$  and

$$(3.1) \quad s_0(x, y) = \begin{cases} 1/\sqrt{1 + \mu}, & 0 < x < \eta(y, 0), \\ 0, & \text{otherwise;} \end{cases}$$

$$(3.2) \quad \eta(y, t) = a \exp(\lambda t) \cos(\pi y/l) + b,$$

where  $\eta(y, t)$  satisfies  $I(t) = \{(\eta(y, t), y); 0 < y < l\}$ ,  $a$  is small positive number and  $b > 0$ . Then there exists constant  $c$  satisfying

$$(3.3) \quad \lambda = c/l + O(a) \quad \text{as } a \rightarrow 0.$$

Moreover, if  $\mu > 3$  (resp.  $\mu < 3$ ) then  $c > 0$  (resp.  $c < 0$ ).

This theorem implies that perturbation with small wave length grows up rapidly when  $\mu > 3$ . To illustrate this result, we give some numerical examples when  $\mu = 20$ ,  $s^* = 1/\sqrt{1 + \mu}$ ,  $p^* = 1$  and the initial function is given by (3.1)–(3.2) with  $a = 0.05$  and  $b = 0.1$ . The mesh points are  $200 \times 200$ .

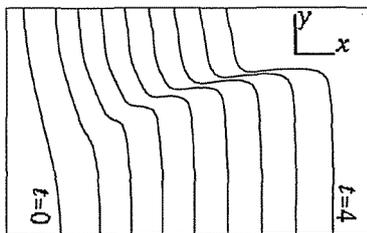


Fig. 1.  $l = 0.01$

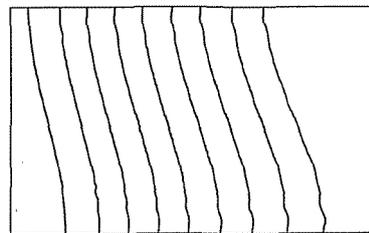


Fig. 2.  $l = 1$

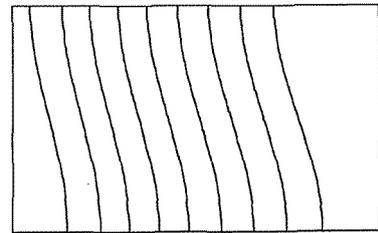


Fig. 3.  $l = 100$

Figs. 1–3 shows the numerical interface at  $t = 0, 0.5, 1, \dots, 4$ . One can find that the instability of the interface is strong for small  $l$ . Following figure illustrates this result clearly.

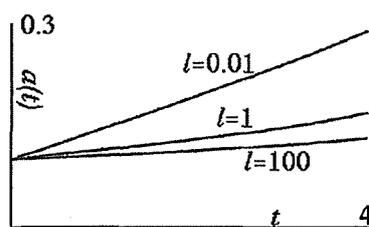


Fig. 4. The graph of  $a(t)$  with  $l = 0.01, 1, 100$ .

Here we define  $a(t)$  by

$$(3.4) \quad a(t) = \max\{x; (x, y) \in I(t)\} - \min\{x; (x, y) \in I(t)\},$$

and we may consider that the derivative of  $a(t)$  corresponds to the growth rate of the perturbation  $a \cos(\pi y/l)$  given to the initial interface  $x = b$ .

#### 4. Numerical computations when $\varepsilon > 0$

When the capillary pressure is effective, as far as we know, there is no theoretical result on the stability of the interface. We go on numerical simulations and try to investigate the stability. The initial function is same one in Figs. 1-4. The ratio of the viscosities of two fluids  $\mu$  is fixed to 20, and we use  $100 \times 100$  mesh points and put  $l = 0.01$ . Figs. 5-6 (resp. Figs. 7-8) are the case where  $\varepsilon = 0.0001$  (resp.  $\varepsilon = 0.01$ ).

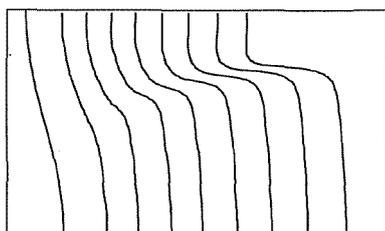


Fig. 5. Numerical interfaces with  $\varepsilon = 0.0001$

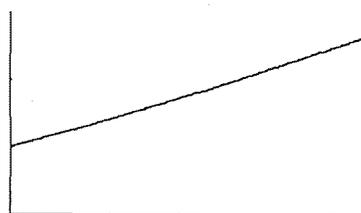


Fig. 6. The graph of  $a(t)$  with  $\varepsilon = 0.0001$

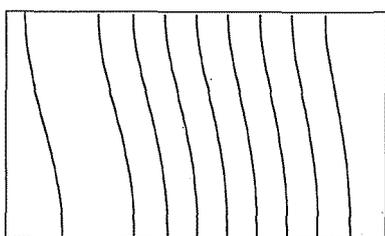


Fig. 7. Numerical interfaces with  $\varepsilon = 0.01$

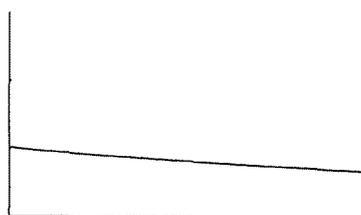


Fig. 8. The graph of  $a(t)$  with  $\varepsilon = 0.01$

When  $\varepsilon = 0.0001$ ,  $a(t)$  is increasing, and the interface is unstable. However the interface turns to be stable for  $\varepsilon = 0.01$ . Hence we can say that the capillary pressure makes the interface to be stable.

Next we show the relation among the stability of the interface,  $l$  and  $\varepsilon$ . The following figures display the graphs of  $a(t)$ .

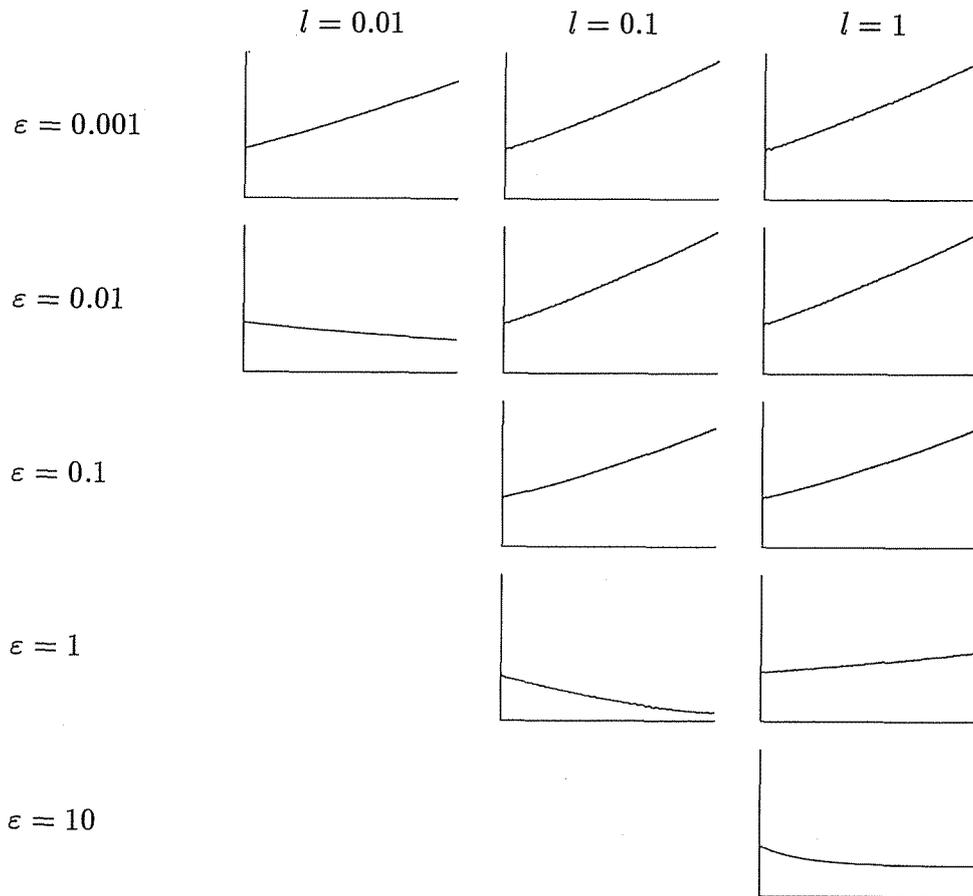


Fig. 9. The graphs of  $a(t)$ .

From these figures we can find that the capillary pressure works well for small  $l$ . In other word, when  $\varepsilon > 0$ , the perturbation with small wave length shall disappear and the instability of the interface causes by the perturbation with suitable wave length. This result is quite different from the one when  $\varepsilon = 0$  (see (3.3)).

### 5. Steady-state solutions

When  $\varepsilon > 0$  and  $p^* < 0$ , it is already known that the exists a steady-state solution of planar wave type (see Theorem 6 in [5]). In this section, by numerical simulations, we show the interface of this solution is stable.

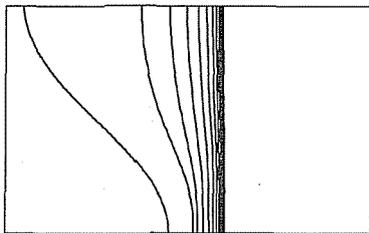


Fig. 10.  $l = 1$

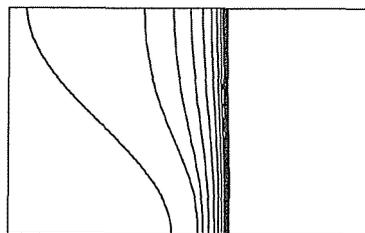


Fig. 11.  $l = 10$

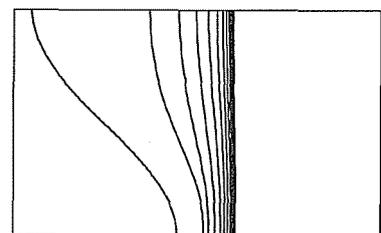


Fig. 12.  $l = 100$

Figs. 10–12 are numerical interfaces at  $t = 0, 0.05, 0.1, \dots, 1$  with  $\mu = 20$ ,  $\varepsilon = 100$ ,  $s^* = 1$ ,  $p^* = -50$  and  $50 \times 50$  mesh points. The initial function is

$$(5.1) \quad s_0(x, y) = \begin{cases} 1 - x/\eta(y), & 0 < x < \eta(y) \equiv 0.2 \cos(\pi y/l) + 0.25, \\ 0, & \text{otherwise.} \end{cases}$$

We also carry on the numerical simulations with another initial functions; In all cases numerical interfaces also converge to the flat interface as  $t \rightarrow \infty$ . Therefore, we can expect that the interface of the steady-state solution is stable. However, mathematical proof is not succeed yet.

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**Acknowledgments:** All numerical simulations in this paper are made on the DEC alpha workstations in Department of Mathematics, Faculty of Science, Hiroshima University.

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# ASYMPTOTICS FOR LARGE TIME FOR NONLINEAR SCHRÖDINGER EQUATION

P.I. NAUMKIN

**Abstract.** We consider asymptotics of the solution to the Cauchy problem for the nonlinear Schrödinger (*NS*) equation. Especially we are interested in the case when the nonlinearity decays in time with the same speed as the linear terms in the *NS* equation, and even slower. By suitable change of dependent variable we obtain an integral equation with rapidly decaying nonlinearity, so we can apply usual successive approximations method and get constructive algorithm for calculating asymptotics of the solution of the Cauchy problem for *NS* equation via the initial data.

## 1. Introduction

In this paper we concentrate our attention on the asymptotic behavior as  $t \rightarrow \infty$  of the solution of the Cauchy problem for one-dimensional nonlinear Schrödinger (*NS*) equation

$$(1.1) \quad u_t - ia|u|^2u - iu_{xx} = 0, \quad u(x, 1) = \bar{u}(x),$$

where the solution  $u(x, t)$  is a complex valued function of  $x \in R$ , and  $x \geq 1$ , the coefficient  $a(t)$  is real. In the case of a constant coefficient  $a(t)$  asymptotics of the solution to the Cauchy problem (1.1) is found by virtue of the IST method. If the coefficient  $a(t)$  “decays” in time  $a(t) \in \mathcal{L}_r(1, \infty)$ ,  $r \in [1, \infty]$  then the nonlinear term in *NS* equation decays in time faster than linear ones and asymptotics as  $t \rightarrow \infty$  of the solution has a quasilinear character in the sense, that the main term of the asymptotics is of the same form as for the linear Schrödinger equation and only the coefficient at the main term of asymptotics is responsible for the contribution of the nonlinearity. The following statement is valid.

**Theorem 1.** Let: 1)  $a(t) \in \mathcal{L}_r(1, \infty)$ ,  $r \in [1, \infty)$  2) initial data  $\bar{u}(x) \in H^{1,0}(R_1) \cap H^{0,1}(R_1)$  be sufficiently small

$$(1.2) \quad |\bar{u}|_1 \equiv \|\bar{u}\|_{H^{1,0}} + \|\bar{u}\|_{H^{0,1}} \leq \varepsilon,$$

where  $\varepsilon > 0$  is sufficiently small.

Then the asymptotics for  $t \rightarrow \infty$  of the solution  $u(x, t) \in C^0([1, \infty); H^{1,0}(R_1) \cap H^{0,1}(R_1))$  of the Cauchy problem (1.1) has the form

$$(1.3) \quad u(x, t) = \frac{1}{\sqrt{4\pi it}} V(\chi) e^{i\chi^2 t} + O(t^{-\frac{1}{2}-\delta}),$$

uniformly with respect to  $x \in R_1$ , where  $\delta > 0$  is some constant,  $\chi = \frac{x}{2t}$ , the coefficient  $V(\chi)$  is calculated via the recurrence relation.

If the coefficient  $a(t)$  is bounded or grows with respect to time then the nonlinearity in equation (1.1) exert more important influence on the character of the asymptotic behavior for large time. In paper [1, 2] the modified wave operator is constructed for the one dimensional nonlinear Schrödinger equation. In these papers authors "guessed" the asymptotic form  $U(\chi, t)$  of the wave  $u(x, t)$  as  $t \rightarrow +\infty$  and subtract it from the solution  $u(x, t)$ . Since the difference  $u(x, t) - U(\chi, t)$  decays faster authors succeeded to estimate it via an integral equation and proved that the solution  $u(x, t)$  exists and tends to the asymptotic  $U(\chi, t)$  as  $t \rightarrow \infty$  uniformly with respect to  $x \in R_1$ . Unfortunately it remains unclear how to calculate constructively the coefficients of the asymptotics form  $U(\chi, t)$  via initial data  $\bar{u}(x, t)$  of the Cauchy problem. The method of papers [1, 2] helps us to understand how to find the asymptotics of the solution of (1.1) in the case of nondecaying coefficient  $a(t)$ . First of all we note that the asymptotic form  $U(\chi, t)$  appears to be the asymptotics as  $t \rightarrow \infty$  of an integral

$$G_v(x, t) = \frac{1}{\sqrt{4\pi it}} \int_{R_1} dy v(y, t) e^{i(x-y)^2/4t}$$

of nondecreasing and rapidly oscillating function  $v(x, t)$ . So we first transform  $u(x, t)$  to  $v(x, t)$  by the formula

$$v(x, t) = G_u^{-1}(x, t) = \frac{1}{\sqrt{-4\pi it}} \int_{R_1} dy u(y, t) e^{-\frac{i(x-y)^2}{4t}}$$

and for the new function  $v(x, t)$  we derive an integral equation. The main term of asymptotics as  $t \rightarrow \infty$  of the nonlinearity in this integral equation is divergent and so is responsible for rapid oscillation of the solution. To get rid of this divergent term we make suitable change of dependent variable and derive a new integral equation with nonlinearity rapidly decaying. Thus we are able to apply usual method of successive approximations and obtain a constructive algorithm for evaluating asymptotics  $U(\chi, t)$  of the solution via initial data. The following statements are valid.

**Theorem 2.** Let 1) the real function  $a(t) \in C^1([1, \infty))$  be such that  $\sup_{t \geq 1} a(t) < \infty$ , and  $\sup_{t \geq 1} ta'(t) < \infty$ ; 2) the initial data  $\bar{u}(x)$  belong to  $H^{1,0}(R_1) \cap H^{0,1}(R_1)$  and be sufficiently small, so that (1, 2) be fulfilled.

Then the solution  $u(x, t) \in C^0([1, \infty); H^{1,0}(R_1) \cap H^{0,1}(R_1))$  of the Cauchy problem (1.1) has asymptotics

$$(1.4) \quad u(x, t) = \frac{W(\chi)}{\sqrt{4\pi it}} \exp(i\chi^2 t + |W(\chi)|^2 \int_1^t b(\tau) d\tau + i\Phi(\chi)) + O(t^{-\frac{1}{2}-\delta})$$

as  $t \rightarrow \infty$  uniformly in  $x \in R_1$ , where  $\delta > 0$  is a constant,  $\chi = \frac{x}{2t}$ ,  $b(t) = \frac{ia(t)}{4\pi t}$ . The amplitude  $W(\chi)$  and phase  $\Phi(\chi)$  are calculated via initial data by virtue of the recurrence relations.

Let us introduce Gevrey class

$$Z^M = \{\varphi(x) \in H^{\infty, \infty}(R_1) : |\varphi|_n \leq M_n, \forall n \geq 0\},$$

where  $|\varphi|_n = |\varphi|_{0,n} + |\varphi|_{n,0}$ ,  $|\varphi|_{k,l} = \|\varphi\|_{L_2(R_1)} + \|x^l \frac{d^k \varphi}{dx^k}\|_{L_2(R_1)}$ ,  $k, l \geq 0$ ,  $M_n = \varepsilon \theta^n (n+1)^{\sigma n}$ ,  $\forall n \geq 0$  the constants  $\tau > 1$ ,  $\varepsilon \in (0, 1)$ ,  $\sigma > 1$  are fixed.

**Theorem 3.** Let 1)  $a(t) \in C^0([1, \infty))$  be real and such that  $\sup_{t \geq 1} t^{-\beta} a(t) < \infty$ ,  $\beta \in (0, \frac{1}{16})$ , 2) initial data  $\bar{u}(x) \in Z^M$  with  $\varepsilon > 0$  small enough and  $\sigma \in (1, \frac{1-2\beta}{14\beta})$ . Then the solution  $u(x, t) \in C^0([1, \infty); Z^{K(t)})$ , ( $K_n(t) = (ct^\beta)^n M_n$ ,  $c > 0$  is a constant) of the Cauchy problem (1.1) has asymptotics (1.4).

*Remarks.* Heuristically the requirement  $\bar{u}(x) \in Z^M$  seems to be too strong for evaluating asymptotics for large time, since the change of dependent variable  $v(x, t) = B_{w,w}(x, t)$  which we carry out proving Theorem 3 is successful only in the domain  $|x| \ll \sqrt{t}$ . Also it is clear that the restriction  $\beta < \frac{1}{16}$  is not optimal. For larger values of the parameter  $\beta$  probably we should take in to account the next terms of the asymptotic expansion as  $t \rightarrow \infty$  of the nonlinearity in the integral equation for function  $v(x, t)$ .

### 2. Sketch of proof of Theorem 1.

Let us make change of variables

$$u(x, t) = G_v(x, t) \equiv \frac{1}{\sqrt{4\pi it}} \int_{R_1} dy v(y, t) e^{\frac{i(x-y)^2}{4t}}$$

then from (1.1) we obtain

$$(2.1) \quad v_t - iaG_f^{-1} = 0, \quad v(x, 1) = G^{-1}\bar{u}(x, 1)$$

where  $f = |u|^2 u = |G_v|^2 G_v$ . The nonlinearity in equation (2.1) can be written explicitly in the form:

$$G_f^{-1} = \frac{1}{4\pi t} \iint_{R_2} dy dz v(y, t) v(z, t) v^*(y+z-x, t) \exp(-\frac{i}{4t}(x^2 - y^2 - z^2 + (x-y-z)^2)),$$

(the asterisk refers to the complex conjugate function.) Substituting (2.2) in (2.1) and integrating with respect to  $t$  we find

$$(2.3) \quad v(x, t) = v(x, 1) + \int_1^t A_v(x, \tau) d\tau,$$

where

$$A_v(x, t) = b(t) \iint_{R_2} dy dz v(y, t) v(z, t) v^*(y + z - x, t) \exp\left(-\frac{i}{4t}(x^2 - y^2 - z^2 + (x - y - z)^2)\right),$$

$b(t) = \frac{ia(t)}{4\pi t}$ . We apply the successive approximations methods to solve (2.3). Let us denote  $v^0(x, t) = v(x, 1) = G_v^{-1}(x, 1)$  and the functions  $v^{(l+1)}(x, t)$  for all  $l \geq 0$  we define by the recurrence relation

$$(2.4) \quad v^{(l+1)}(x, t) = v(x, 1) + \int_1^t A_{v^{(l)}}(x, \tau) d\tau.$$

From (2.4) it is easy to prove estimates

$$(2.5) \quad |v^{(l)}|_1 \leq 4\varepsilon; \quad |v^{(l+1)} - v^{(l)}|_1 \leq \frac{1}{2} \sup_{t \geq 1} |v^{(l)} - v^{(l-1)}|_1.$$

Therefore, as  $l \rightarrow \infty$ , the sequence  $\{v^{(l)}\}$  tends to the solution

$$u(x, t) \in C^0([1, \infty); H^{0,1}(R_1) \cap H^{1,0}(R_1))$$

of the integral equation (2.3), corresponding to the solution

$$u(x, t) = G_v(x, t) \in C^0([1, \infty); H^{0,1}(R_1) \cap H^{1,0}(R_1))$$

of problem (1.1). Now we calculate asymptotics as  $t \rightarrow \infty$ :

$$(2.6) \quad u(x, t) = G_v(x, t) = \frac{1}{\sqrt{4\pi it}} \widehat{v}(\chi, t) e^{i\chi^2 t} + O(\|v\|_{H^{0,1}} t^{-\frac{1}{2} - \frac{\nu}{2}})$$

where  $\chi = \frac{x}{2t}$ ,  $\nu \in (0, \frac{1}{2})$ ,  $\widehat{v}(p, t) = \int_{R_1} e^{-ipx} v(x, t) dx$  is the Fourier transform. We denote  $V(\chi) = \lim_{t \rightarrow \infty} \widehat{v}(\chi, t)$  then we have estimate  $\|V - \widehat{v}\|_{\mathcal{L}_\infty} = O(t^{-\frac{1}{2}})$ , since  $a(t) \in \mathcal{L}_r(1, \infty)$ . So we get asymptotics (1.3). The coefficient  $V(\chi)$  can be calculated as the limit  $V(\chi) = \lim_{t \rightarrow \infty} \lim_{l \rightarrow \infty} \widehat{v}^{(l)}(\chi, t)$ , i.e. for given accuracy  $\varepsilon_1 > 0$  there exist  $T \geq 1$  and  $l \geq 0$  such that  $V(\chi) = \widehat{v}^{(l)}(\chi, T) + O(\varepsilon_1)$ , where  $\widehat{v}^{(l)}(\chi, T)$  is found from recurrence relation (2.4). Theorem 1 is proved.

### 3. Sketch of proof of Theorems 2 and 3.

Now we represent the nonlinear term  $A_v(x, t)$  of integral equation (2.3) in the form

$$A_v = \widetilde{A}_v + b(t) \iint_{R_2} dy dz v(y, t) v(z, t) v^*(y + z - x, t) E(x, y, z, t)$$

where

$$\begin{aligned}\tilde{A}_v &= b(t) \iint_{R_2} dydzv(y,t)v(z,t)v^*(y+z-x,t), \\ E(x,y,z,t) &= \exp\left(-\frac{i}{4t}(x^2 - y^2 - z^2 + (x-y-z)^2)\right) - 1.\end{aligned}$$

Now to get rid of the divergent summand  $\tilde{A}_v$  from the integral equation (2.3) we make change  $v(x,t) = B_{w,w}(x,t)$  where the Fourier multiplier  $B_{\varphi,\varphi}$  is equal:

$$\begin{aligned}B_{\varphi,\varphi}(x,t) &= \frac{1}{2\pi} \int_{R_1} dp e^{ipx} \widehat{B}_\varphi(p,t) \widehat{\varphi}(p,t), \\ \widehat{B}_\varphi(p,t) &= \exp\left(\int_1^t d\tau b(\tau) |\widehat{\varphi}(p,\tau)|^2\right); \\ \widehat{\varphi}(p,\tau) &= \int_{R_1} dx e^{-ipx} \varphi(x,t).\end{aligned}$$

The inverse transformation

$$w(x,t) = B_{v,v}^{-1}(x,t) = \frac{1}{2\pi} \int_{R_1} dp e^{ipx} \widehat{B}_v^*(p,t) \widehat{v}(p,t)$$

is determined by the conjugate symbol  $\widehat{B}_v^*(p,t) = \exp(-\int_1^t d\tau b(\tau) |\widehat{v}(p,\tau)|^2)$ . Since  $\widehat{B}_\varphi^*(p,1) = 1$  so that  $w(x,1) = v(x,1)$  we obtain from (2.3)  $w_t = b(t)B_{v,f}^{-1}$ , where

$$f(x,t) = \iint_{R_2} dydzv(y,t)v(z,t)v^*(y+z-x,t)E(x,y,z,t), \quad v(x,t) = B_{w,w}(x,t).$$

Thus after integration with respect to  $t$  we get

$$(3.1) \quad w(x,t) = v(x,1) + \int_1^t d\tau b(\tau) B_{w,f}^{-1}(x,\tau).$$

We apply the successive approximations method. Denoting  $w^{(0)}(x,t) = v(x,1)$  we define  $w^{(l)}$  via recurrence relation

$$(3.2) \quad w^{(l+1)}(x,t) = v(x,1) + \int_1^t d\tau b(\tau) B_{w^{(l)},f^{(l)}}^{-1}(x,\tau)$$

where

$$\begin{aligned}f^{(l)} &= \iint_{R_2} dydzv^{(l)}(y,t)v^{(l)}(z,t)v^{(l)*}(y+z-x,t)E(x,y,z,t) \\ v^{(l)} &= B_{w^{(l)},w^{(l)}}.\end{aligned}$$

In the case of Theorem 2 we prove the following estimates for functions  $w^{(l)}$  for all  $l \geq 1$ ,

$$(3.3) \quad \|w^{(l)}\| \leq 5\varepsilon$$

$$(3.4) \quad \sup_{t \geq 1} e^{-t} |w^{(l+1)} - w^{(l+1)}|_1 \leq \frac{1}{2} \sup_{t \geq 1} e^{-t} |w^{(l)} - w^{(l-1)}|_1$$

where we introduce a norm

$$\|\varphi\| \equiv \sup_{t \geq 1} (t^{1-4\gamma} |\varphi'_t|_1 + t^{-\gamma} |\varphi|_1 + t^{\frac{5}{4}-5\gamma} \|\varphi'_t\|_{\mathcal{L}_1}),$$

where  $\gamma \in (0, \frac{1}{20})$ . And in the case of Theorem 3 we prove estimate (3.4) and also the following estimates for all  $l \geq 0, n \geq 0, t \geq 1$

$$(3.5) \quad |w^{(l)}|_n \leq L_n(t)$$

where  $L_n(t) = \overline{M}_n t^{\beta(n-2)}$  for  $n \geq 2$ ,  $L_n = \overline{M}_n$  for  $n = 0, 1$ ,  $\overline{M}_n = 2\varepsilon \overline{\theta}^n (n+1)^{\sigma n}$  for all  $n \geq 0$ .  $\overline{\theta} = 20e\theta$ ,  $\sigma > 1$ ,  $\theta > 1$ ,  $\varepsilon \in (0, 1)$  are taken from the conditions of Theorem 3,  $\varepsilon > 0$  is small enough. From estimates (3.3) – (3.5) we conclude that the sequence  $w^{(l)}$  converges as  $l \rightarrow \infty$  in the norm  $C^0([0, T]; H^{1,0}(R_1) \cap H^{0,1}(R_1))$  for any fixed  $T > 1$  to the solution  $w(x, t) \in C^0([1, \infty); H^{1,0}(R_1) \cap H^{0,1}(R_1))$  in the case of Theorem 2 and  $w(x, t) \in C^0([1, \infty); Z^{k(t)})$  in the case of Theorem 3. Now let us prove asymptotics (1.4). Since  $\|v\|_{H^{0,1}(R_1)} = \|B_{w,w}\|_{H^{0,1}} = O(t^\beta + t^\gamma)$  from (2.8) we get

$$(3.6) \quad u(x, t) = \frac{e^{i\chi^2 t}}{\sqrt{4\pi i t}} \widehat{v}(\chi, t) + O(t^{-\frac{1}{2}-\delta})$$

as  $t \rightarrow \infty$  uniformly with respect to  $x \in R_1$ , where  $\delta > 0$ ,  $\chi = \frac{x}{2t}$ . We denote  $W(\chi) = \lim_{t \rightarrow \infty} \widehat{w}(\chi, t)$  and estimate the difference  $\widehat{w} - W$

$$(3.7) \quad \|\widehat{w} - W\|_{\mathcal{L}_\infty} = O(t^{-2\beta} + t^{5\gamma-1/4}).$$

Thus,  $\widehat{w}(\chi, t) = W(\chi) + O(t^{-\delta})$  with some  $\delta > 0$  and since  $\widehat{v}(\chi, t) = \widehat{B}_w(\chi, t) \widehat{w}(\chi, t)$  we find from (3.6)

$$(3.8) \quad u(x, t) = \frac{W(\chi)}{\sqrt{4\pi i t}} \exp(i\chi^2 t + \int_1^t d\tau b(\tau) |\widehat{w}(\chi, \tau)|^2) + O(t^{-\frac{1}{2}-\delta})$$

Now we denote  $\Phi(\chi) = \lim_{t \rightarrow \infty} \Psi(\chi, t)$ ,  $\Psi(\chi, t) = \int_1^t d\tau b(\tau) (|\widehat{w}(\chi, \tau)|^2 - |\widehat{w}(\chi, t)|^2)$  and represent the phase in (3.8) in the form

$$\begin{aligned} \int_1^t d\tau b(\tau) |\widehat{w}(\chi, \tau)|^2 &= |W(\chi)|^2 \int_1^t d\tau b(\tau) + \Phi(\chi) + \\ &(\Psi(\chi, t) - \Phi(\chi)) + (|\widehat{w}(\chi, t)|^2 - |W(\chi)|^2) \int_1^t b(\tau) d\tau. \end{aligned}$$

Using estimates  $\| |\widehat{w}|^2 - |W|^2 \|_{\mathcal{L}_\infty} \int_1^t b(\tau) d\tau = O(t^{-\delta})$  and  $\|\Psi(\chi, t) - \Phi(\chi)\|_{\mathcal{L}_\infty} = O(t^{-\beta})$  we obtain asymptotics (1.4). Let us make remark on evaluation of the amplitude  $W(\chi)$

and phase  $\Phi(\chi)$ . Due estimate (3.7) we can choose  $T > 1$  so large that  $W(\chi) = \widehat{w}(\chi, T) + O(\varepsilon_1)$ . After that in view of (3.4) we can choose a number  $l$  so large that

$$w^{(l)}(\chi, T) = \widehat{w}(\chi, T) + O(\varepsilon_2).$$

Thus for given accuracies  $\varepsilon_1$  and  $\varepsilon_2$  we get expression

$$W(\chi) = \widehat{w}^{(l)}(\chi, T) + O(\varepsilon_1) + O(\varepsilon_2).$$

Analogously for the phase  $\Phi(\chi)$  from (3.11) we obtain  $\Phi(\chi) = \Psi(\chi, T) + O(\varepsilon_3)$  if  $T > 1$  is large enough; and  $\Psi(\chi, T) = \Psi^{(l)}(\chi, T) + O(\varepsilon_4)$  if  $l$  is sufficiently large, where

$$\Psi^{(l)}(\chi, T) = \int_1^t (|\widehat{w}^{(l)}(\chi, \tau)|^2 - |\widehat{w}^{(l)}(\chi, T)|^2) b(\tau) d\tau.$$

Therefore  $\Phi(\chi) = \Psi^{(l)}(\chi, T) + O(\varepsilon_3) + O(\varepsilon_4)$ . Theorems 2 and 3 are proved.

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**Acknowledgments:** The research is partially supported by RFFI and ISF.

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# Similarity solutions of the Navier-Stokes equations

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## Abstract

Leray[5] considered a modified, stationary Navier-Stokes equations in the hope that it gives us an example of the finite-time blow-up of the three dimensional nonstationary Navier-Stokes equations. However, he showed no example of solutions. We list here some particular solutions and discuss their hydrodynamical properties.

## 1 Introduction

Leray[5] considers the following system of equations:

$$\Delta \vec{U} - \nabla P = \vec{U} + (\vec{\xi} \cdot \nabla) \vec{U} + (\vec{U} \cdot \nabla) \vec{U}, \quad (1)$$

$$\operatorname{div} \vec{U} = 0, \quad (2)$$

where,  $\vec{U} = (U_1, U_2, U_3)$  and  $P$  are functions of  $\vec{\xi} \in \mathbf{R}^3$ , only. If this system of equations are satisfied, then

$$\vec{u}(t, x) = \frac{\sqrt{\nu}}{\sqrt{2(T-t)}} \vec{U} \left( \frac{x}{\sqrt{2\nu(T-t)}} \right), \quad p(x) = \frac{\rho\nu}{2(T-t)} P \left( \frac{x}{\sqrt{2\nu(T-t)}} \right) \quad (3)$$

satisfy the Navier-Stokes equations

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = \nu \Delta \vec{u} - \frac{1}{\rho} \nabla p,$$

$$\operatorname{div} \vec{u} = 0,$$

in  $0 < t < T, x \in \mathbf{R}^3$ , and blows up at  $t = T$ .

Leray mentioned this scheme but he did not show any solution of (1). Many mathematicians including the author have quite a negative feeling about Leray's scheme. It may be that no solution of (1) with finite energy is possible. In this regard, we would like

to draw the reader's attention to Rosen[8], which proves that there is no solution of (1) which is smooth and decays sufficiently rapidly at infinity. However, much room seems to be left for the existence or non-existence problem, since Rosen assumes a very rapid decay at infinity.

One of the purposes of the present paper is to note that there are many unbounded solutions to (1) and (2). Because of (3), the domain in which  $\vec{\xi}$  runs must be a cone with the origin as its vertex. Whenever boundaries are present, we impose the adherence condition  $\vec{U} = 0$  on the boundaries.

Our method is also applicable to the following scheme:

$$\vec{u}(t, x) = \frac{\sqrt{\nu}}{\sqrt{2(t+T)}} \vec{U} \left( \frac{x}{\sqrt{2\nu(t+T)}} \right), \quad p(x) = \frac{\rho\nu}{2(t+T)} P \left( \frac{x}{\sqrt{2\nu(t+T)}} \right). \quad (4)$$

Here  $T$  is a positive constant. If  $\vec{U}$  and  $P$  satisfy

$$\Delta \vec{U} - \nabla P = -\vec{U} - (\vec{\xi} \cdot \nabla) \vec{U} + (\vec{U} \cdot \nabla) \vec{U}, \quad (5)$$

and

$$\operatorname{div} \vec{U} = 0, \quad (6)$$

then (4) defines a solution of the Navier-Stokes equations. This solution is a decaying solution, which is interesting when we wish to understand the mechanical relation between the viscous dissipation and the nonlinearity. The equations (1), (2), and (5), (6) are considered in Foias and Temam [2] in a context different from ours. We would like to show these equations admit solutions if some kind of singularities are allowed. However, due to the limitation of the size of the paper we omit the analysis and leave it to the forthcoming paper.

Acknowledgment: The author would like to express his thanks to Prof. K. Ohkitani, who informed the author of Rosen's paper [8]. The present paper is greatly influenced by his paper[7]. He is also delighted to express his deep gratitude to Prof. M. Yamada, whose comments improved this manuscript quite a lot. Prof. D. Chae directed the author's attention to Foias and Temam's paper [2]. The author is grateful to his advice.

## 2 Parallel flows

We first consider  $\vec{U} = (\phi(y), 0, 0)$ ,  $P = -p_0 \xi_1$ , where  $y = \xi_2$  and  $p_0$  is a constant. The equation (1) is reduced to

$$\phi''(y) = -p_0 + \phi + y\phi'(y).$$

General solutions of this equation are represented as follows:

$$p_0 + A \exp\left(\frac{y^2}{2}\right) + B \exp\left(\frac{y^2}{2}\right) \int_y^\infty \exp\left(-\frac{\eta^2}{2}\right) d\eta, \quad (7)$$

where  $A$  and  $B$  are constants.

We remark that  $e^{y^2/2}$  tends to infinity very rapidly as  $y \rightarrow \pm\infty$ . On the other hand,  $\phi(y) = \exp\left(\frac{y^2}{2}\right) \int_y^\infty \exp\left(-\frac{\eta^2}{2}\right) d\eta$  satisfies the following asymptotic relation:

$$\exp\left(\frac{y^2}{2}\right) \int_y^\infty \exp\left(-\frac{\eta^2}{2}\right) d\eta \sim \frac{1}{y} - \frac{1}{y^3} + \frac{3}{y^5} - \dots \quad \text{as } y \rightarrow +\infty.$$

It diverges very rapidly as  $y \rightarrow -\infty$ . The solution (7) makes a striking difference from stationary parallel Navier-Stokes flows, in which only quadratic functions such as the Poiseuille flow are permitted.

If we consider the equation in a half space  $\xi_2 > 0$ , then

$$\phi(y) = p_0 - p_0 \sqrt{\frac{2}{\pi}} \exp\left(\frac{y^2}{2}\right) \int_y^{+\infty} \exp\left(-\frac{\eta^2}{2}\right) d\eta \quad (8)$$

is a solution satisfying the adherence condition. This solution is bounded in the half space and satisfies the following asymptotic expansion:

$$\phi \sim p_0 - p_0 \sqrt{\frac{2}{\pi}} \frac{1}{y}, \quad \phi' \sim p_0 \sqrt{\frac{2}{\pi}} \frac{1}{y^2} \quad (y \rightarrow +\infty) \quad (9)$$

The velocity  $\vec{u}$  satisfies

$$u_1 = \frac{\sqrt{\nu} p_0}{\sqrt{2(T-t)}} - \sqrt{\frac{2}{\pi}} \frac{\nu p_0}{x_2} + O(|T-t|) \quad \text{as } t \rightarrow T.$$

This shows that the thickness of the boundary layer is of the order  $O\left(\sqrt{\nu(T-t)}\right)$ .

It is possible to obtain solutions of the form:  $\vec{U} = (\phi(r), 0, 0)$ ,  $P = -p_0 \xi_1$ , where  $r = \sqrt{\xi_2^2 + \xi_3^2}$ . The equation (1) is reduced to

$$\phi'' + \frac{1}{r} \phi' = -p_0 + \phi + r \phi' \quad (0 < r < \infty). \quad (10)$$

Again it suffices to consider the homogeneous equation.

$$\phi'' + \frac{1}{r} \phi' = \phi + r \phi' \quad (0 < r < \infty). \quad (11)$$

We transform the solution as follows:

$$\phi(r) = e^{r^2/4} u\left(\frac{r^2}{4}\right).$$

Then (11) is transformed to

$$u''(z) + \frac{1}{z} u'(z) - u(z) = 0,$$

where  $z = r^2/4$ . This is the differential equation for the modified Bessel function of order zero. Therefore the general solutions of the equation (11) are represented as

$$\phi(r) = e^{r^2/4} \left[ a_0 I_0 \left( \frac{r^2}{4} \right) + a_1 K_0 \left( \frac{r^2}{4} \right) \right],$$

where  $a_0$  and  $a_1$  are constants. The case where  $a_0 = 0$  is particularly interesting. Since

$$\phi(r) = e^{r^2/4} K_0(r^2/4) \sim \frac{\sqrt{2\pi}}{r} \quad \text{as } r \rightarrow \infty,$$

and

$$K_0(z) \sim \log \left( \frac{2}{z} \right) \quad \text{as } z \rightarrow 0,$$

the solution  $\phi(r) = e^{r^2/4} K_0(r^2/4)$  decays to zero with the order of  $1/r$  near infinity and is logarithmically unbounded near zero.

The velocity obtained by this function through (3) is bounded at  $r = \infty$  even as  $t \rightarrow T$ . Its velocity near  $r = 0$  becomes unbounded with a factor  $\log(T - t)/(T - t)$ .

### 3 Stagnation flows

We consider (1) and (2) in  $\Omega = \{(x, y); -\infty < x < \infty, 0 < y < \infty\}$ , where  $x$  and  $y$  denote  $\xi_1$  and  $\xi_2$ , respectively. We assume the following form of the solution:

$$U_1 = x f'(y), \quad U_2 = -f(y), \quad U_3 = 0. \quad (12)$$

This is an analogue from Hiemenz's solution for the 2D stationary Navier-Stokes equations (Hiemenz[3]). The divergence-free condition is satisfied automatically by the ansatz (12).

Substituting (12) into (1), we obtain

$$2f' + (f')^2 + yf'' - ff'' - f''' = \text{constant}.$$

At  $y = 0$ , we have the boundary condition:  $f(0) = f'(0) = 0$ . Since the equation is of third order, one more boundary condition is needed. This is supplied by the condition at infinity. At infinity, we assume that it converges to an well-known Euler flow. Namely we assume that

$$(U_1, U_2) \rightarrow k(x, -y) \quad \text{as } y \rightarrow +\infty,$$

where  $k$  is a constant. We interpret this as  $f'(\infty) = k$  and  $f''(\infty) = f'''(\infty) = 0$ . So, we get to the following boundary value problem:

$$2f' + (f')^2 + yf'' - ff'' - f''' = 2k + k^2 \quad (13)$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = k. \quad (14)$$

( In deriving (13), we have tacitly assumed that  $f''(y)$  tends to zero faster than  $1/y$ . )  
Once  $f$  is determined, then the pressure is represented as

$$P = -f'(y) + yf(y) - \frac{f(y)^2}{2} - \left( k + \frac{k^2}{2} \right) x^2.$$

We next consider an axisymmetric version of the above flow. If the domain is the half space  $z = \xi_3 > 0$ , we represent the flow in the cylindrical coordinates  $(r, \theta, z)$ .

$$\Delta U_r - \frac{U_r}{r^2} - \frac{2}{r^2} \frac{\partial U_\theta}{\partial \theta} - \frac{\partial P}{\partial r} = U_r + r \frac{\partial U_r}{\partial r} + z \frac{\partial U_r}{\partial z}, + (\vec{U} \cdot \nabla) U_r - \frac{U_\theta^2}{r} \quad (15)$$

$$\Delta U_\theta - \frac{U_\theta}{r^2} + \frac{2}{r^2} \frac{\partial U_r}{\partial \theta} - \frac{1}{r} \frac{\partial P}{\partial \theta} = U_\theta + r \frac{\partial U_\theta}{\partial r} + z \frac{\partial U_\theta}{\partial z}, + (\vec{U} \cdot \nabla) U_\theta + \frac{U_r U_\theta}{r} \quad (16)$$

$$\Delta U_z - \frac{\partial P}{\partial z} = U_z + r \frac{\partial U_z}{\partial r} + z \frac{\partial U_z}{\partial z}, + (\vec{U} \cdot \nabla) U_z \quad (17)$$

$$\frac{1}{r} \frac{\partial r U_r}{\partial r} + \frac{1}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{\partial U_z}{\partial z} = 0. \quad (18)$$

With a constant  $k$ , there is an Euler flow

$$(U_r, U_\theta, U_z) = k(r, 0, -2z), \quad P = -k^2 \left( \frac{r^2}{2} + 2z^2 \right).$$

Homann[4] considered an axisymmetric stationary solution of the regular Navier-Stokes equations which tends to this flow at infinity. We look for the solutions of (15) – (16) which tends to this Euler solution. Following Homann, we put

$$(U_r, U_\theta, U_z) = (r f'(z), 0, -2f(z)),$$

then we see that  $f$  satisfies

$$2f' + (f')^2 + z f'' - 2f f'' - f''' = 2k + k^2 \quad (0 < z < \infty). \quad (19)$$

With this  $f$ , the pressure is represented as

$$P = 2 \left( -f'(z) + z f(z) - f(z)^2 \right) - \left( k + \frac{k^2}{2} \right) r^2.$$

$f$  is required to satisfy  $f(0) = f'(0) = 0$ ,  $f'(\infty) = k$ . The equation (19) is the same as (13) except for the coefficient of  $f f''$ .

As for the existence of the solution of these boundary value problems, we can prove the following theorem:

**Theorem 1** *If  $k > 0$  and  $a \leq 2$ , then*

$$2f' + (f')^2 + (y - af) f'' - f''' = 2k + k^2 \quad (0 < y < \infty) \quad (20)$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = k \quad (21)$$

*has at least one solution.*

Proof will be given in the forthcoming paper and we give here some comments on this theorem:

1. When  $k < 0$ , the boundary value problem does not seem to have a solution, since Homann's equation does not have a solution in the corresponding case ( see von Mises [9] ).
2. We do not know whether the solution of (20) and (21) is unique or not. For a small  $k$ , however, we can prove the uniqueness, see the corollary below.

**Corollary 1** *If  $k > 0$  is sufficiently small, then the solution satisfying  $0 \leq f''$  everywhere is unique.*

## 4 Miscellaneous solutions

Leray's equations in the cylindrical coordinates ( (15) – (18) ) have the following solution in  $\mathbf{R}^3$ :

$$U_r = -\frac{Ar}{2} + \frac{K}{r}, \quad U_\theta = f(r), \quad U_z = Az + g(r), \quad (22)$$

$$P = -\frac{K^2}{2r^2} + \left(\frac{A}{2} - \frac{A^2}{8}\right)r^2 + \int^r \frac{f(\eta)^2}{\eta} d\eta + \frac{Ar^2}{2} - \left(A + \frac{A^2}{2}\right)z^2 + d_0z,$$

where  $A$ ,  $K$ , and  $d_0$  are constants.  $f$  and  $g$  are determined by

$$f'' + \left(\frac{A-2}{2}r + \frac{1-K}{r}\right)f' + \left(\frac{A-2}{2} - \frac{1+K}{r^2}\right)f = 0 \quad (23)$$

and

$$g'' + \left(\frac{A-2}{2}r + \frac{1-K}{r}\right)g' - (1+A)g = d_0, \quad (24)$$

respectively.

The equation (23) can be solved as follows:

$$f(r) = \frac{1}{r} \left( B_1 \int_1^r e^{-\frac{A-2}{4}y^2} y^{1+K} dy + B_2 \right),$$

where  $B_1$  and  $B_2$  are constants.

We can transform (24) as follows: When  $A - 2 \neq 0$ , we substitute

$$g(r) = e^{-(A-2)r^2/4} u \left( \frac{A-2}{4} r^2 \right).$$

Putting  $z = (A - 2)r^2/4$ , we have the following equation for  $u$ :

$$zu'' + \left(\frac{2-H}{2} - z\right)u' - \left(\frac{2-H}{2} + \frac{1+A}{A-2}\right)u = 0.$$

The solutions of this equations are the confluent hypergeometric functions. Two independent solutions are denoted by

$$\Phi \left( \frac{2-H}{2} + \frac{1+A}{A-2}, \frac{2-H}{2}; z \right), \quad \Psi \left( \frac{2-H}{2} + \frac{1+A}{A-2}, \frac{2-H}{2}; z \right).$$

When  $A - 2 = 0$ , we have to solve

$$g'' + \frac{1-H}{r}g' - 3g = 0,$$

We put

$$g(r) = r^{H/2} u(\sqrt{3}r).$$

Then  $u$  satisfies

$$u''(z) + \frac{1}{z}u'(z) - \left(1 + \frac{H^2}{4z^2}\right)u = 0,$$

where  $z = \sqrt{3}r$ . General solutions of this equations are linear combinations of

$$I_{H/2}(z) \quad \text{and} \quad K_{H/2}(z).$$

As a particular case, the solution (22) is reduced to

$$u_r = \frac{\nu K}{\sqrt{x_1^2 + x_2^2}}, \quad u_\theta = 0, \quad u_z = 0,$$

if  $A = 0$  and  $f \equiv g \equiv 0$ . This is a well-known stationary solution of the Navier-Stokes equations and Leray' device (1) implies nothing new. If  $A = K = 0$  and  $f \equiv 0$ , then the solution is reduced to those considered in section 2.

In the two dimensional sector  $0 < r < \infty$ ,  $-\alpha < \theta < +\alpha$ , we have the solutions of the following form:

$$U_r = \frac{f(\theta)}{r}, \quad U_\theta = 0, \quad U_z = 0.$$

This solution is an analogue to the Jeffery-Hamel flows. A good survey on the Jeffery-Hamel flows is found in Berker[1]. Under this form, the term

$$U_r + r \frac{\partial U_r}{\partial r} + z \frac{\partial U_r}{\partial z}$$

vanishes identically. So, the solutions are nothing but the famous Jeffery-Hamel flows. Again the Leray equation in this case does not serve in the way as Leray expected.

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**REMARKS ON THE KLEIN-GORDON EQUATION  
WITH QUADRATIC NONLINEARITY  
IN TWO SPACE DIMENSIONS**

T. OZAWA, K. TSUTAYA AND Y. TSUTSUMI

ABSTRACT. In this paper, we consider the global existence and the asymptotic behavior of solution for the Klein-Gordon equation with quadratic nonlinearity in two space dimensions. We first state the result concerning the global existence of solution for the Cauchy problem of the quasilinear Klein-Gordon equation with quadratic nonlinearity, which solves one of the two conjectures by Hörmander [8]. Second, we show the existence of wave operators and the asymptotic completeness in a neighborhood of zero for the Lorentz invariant quadratic semilinear Klein-Gordon equation, which is an alternative proof of the results by Simon and Taflin [18].

**§1. Introduction**

In the present paper, we consider the global existence and asymptotic behavior of small amplitude solution for the following Klein-Gordon equation with quadratic nonlinearity in two space dimensions:

$$(1.1) \quad \partial_t^2 u - \Delta u + u = f(u, \partial u, \partial^2 u), \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^2,$$

$$(1.2) \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad x \in \mathbf{R}^2,$$

where

$$\begin{aligned} \partial u &= (\partial_t u, \nabla u), \\ \partial^2 u &= (\partial_t \partial_j u, \partial_j \partial_k u; 1 \leq j, k \leq 2), \\ f(u, p, q) &\in C^\infty(\mathbf{R} \times \mathbf{R}^3 \times \mathbf{R}^5), \end{aligned}$$

and

$$(f.1) \quad f(u, p, q) = O(|u|^2 + |p|^2 + |q|^2) \quad \text{near } (u, p, q) = (0, 0, 0).$$

Later we shall additionally assume that

$$(f.2) \quad f(u, p, q) \text{ depends linearly on the variables } q$$

or that

$$(f.3) \quad f(u, p, q) = au^2 + b(p_1^2 - p_2^2 - p_3^2).$$

for some  $a, b \in \mathbf{R}$  with  $a^2 + b^2 \neq 0$ .

There are many papers concerning the global existence and the asymptotic behavior of solution for (1.1)-(1.2) with small initial data (see, e.g., [7], [8], [9], [10], [11], [13], [14], [15], and [18]). For the detailed reviews on these problems, see the introductions in [18] and [13], or see the lecture note by Strauss [19] which includes the historical background of these problems. In the present paper, we state the two results concerning the global existence and the asymptotic behavior of solution for (1.1)-(1.2) and illustrate the proofs of those results.

The first result is an extension of the global existence theorems in [18] and [13] to the quasilinear case. In [18] and [13], it is shown that if the nonlinear function  $f$  is semilinear, that is,  $f = f(u, p)$  and satisfies (f.1), then (1.1)-(1.2) has a unique global solution for small and smooth initial data (see also [7], [8] and [11]). After the paper [13] had been accepted, the authors knew the conjecture of Hörmander concerning the global existence of solution for (1.1)-(1.2) with small initial data. In [13] the global existence result is stated for the semilinear case, but the proof in [13] is also applicable to the quasilinear case without any essential change. In Section 2 we state the global existence result of (1.1)-(1.2) for small initial data under (f.1)-(f.2).

The second result is concerned with the construction of the scattering operator under (f.3). We first note that if the nonlinear function  $f$  is a homogeneous polynomial of degree 2 with respect to  $u$  and  $\partial u$  and is Lorentz invariant, then  $f$  satisfies (f.3). In [18] Simon and Taffin give the proofs of the construction of wave operators and the asymptotic completeness in a neighborhood of zero under (f.3) (they also state the related results for the more general case in the appendix of [18]). Their proofs in [18] are based on the nonlinear representation and the transformation canceling out the quadratic terms due to Simon [17]. The nonlinear representation in [18] is a kind of transformation of the original unknown functions to the other ones, which seems to correspond to the commutation technique between the D'Alembertian operator  $\square$  and the generators of the Poincaré group in the paper of Kleinerman [9]. In Section 3, we describe the alternative proofs for the construction of wave operators and the asymptotic completeness around zero without the nonlinear representation used in [18], which are based on the combination of the techniques by Klainerman [9] and Shatah [14].

We conclude this section by giving several notations. We put  $\partial_t = \partial/\partial t$  and  $\partial_j/\partial x_j$ ,  $j = 1, 2$ . Let  $\Gamma = (\Gamma_j; j = 1, \dots, 6)$  denote the generators of the Poincaré group  $(\partial_t, \partial_1, \partial_2, L_1, L_2, \Omega_{12})$ , where

$$L_j = x_j \partial_t + t \partial_j, \quad j = 1, 2,$$

$$\Omega_{12} = x_1 \partial_2 - x_2 \partial_1.$$

For a multi-index  $\alpha = (\alpha_1, \alpha_2)$ , we put  $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$ . For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_6)$ , we put  $\Gamma^\alpha = \Gamma_1^{\alpha_1} \cdots \Gamma_6^{\alpha_6}$ . For  $1 \leq p \leq \infty$ , let  $L^p$  denote the standard  $L^p$  space on  $\mathbf{R}^2$ . For  $m \in \mathbf{R}$  and  $s \in \mathbf{R}$ , we define the weighted Sobolev space  $H^{m,s}$  on  $\mathbf{R}^2$  as follows:

$$H^{m,s} = \{v \in \mathcal{S}'(\mathbf{R}^2); (1 + |x|^2)^{s/2} (1 - \Delta)^{m/2} v \in L^2\}$$

with the norm

$$\|v\|_{H^{m,s}} = \|(1 + |x|^2)^{s/2}(1 - \Delta)^{m/2}v\|_{L^2}.$$

Let  $\omega = (1 - \Delta)^{1/2}$ .

**§2. Global existence for the quasilinear case**

In this section, we state the global existence theorem for (1.1)-(1.2), provided that (f.1)-(f.2) are satisfied and the initial data are small and smooth. The global existence theorem was proved in [13] for the semilinear case (see also [18]). The proof in [13] is based on the normal form argument of Shatah [15] (see also Simon [17]) and the time decay estimate of the inhomogeneous linear Klein-Gordon equation due to Georgiev [6], and the proof in [13] is also applicable to the quasilinear case without any essential change. Because the normal form argument of Shatah [15] still works well for the quasilinear case (in fact, the fully nonlinear case was treated in [15]). By using the same argument as in [13], we have the following theorem concerning the global existence of solution for (1.1)-(1.2) under (f.1) and (f.2).

**Theorem 2.1.** *Assume that  $f$  satisfies (f.1) and (f.2). Let  $k \geq 40$  and let  $u_0 \in H^{k+1,k}$ ,  $u_1 \in H^{k,k}$ . Let  $0 < \varepsilon \leq 1/2$ . Then, there exists a  $\delta > 0$  such that if*

$$\|u_0\|_{H^{k+1,k}} + \|u_1\|_{H^{k,k}} \leq \delta,$$

then (1.1)-(1.2) has the unique global solution  $u$  satisfying

$$(2.1) \quad u \in \bigcap_{j=0}^{k+1} C^j(\mathbf{R}; H^{k+1-j}),$$

$$(2.2) \quad \sum_{|\alpha|=k} \sup_{t \in \mathbf{R}} (1+t)^{-\varepsilon} \{ \|\partial_t \Gamma^\alpha u(t)\|_{L^2} + \|\omega \Gamma^\alpha u(t)\|_{L^2} \} \\ + \sum_{|\alpha| \leq k} \sup_{t \in \mathbf{R}} (1+t)^{-\varepsilon} \|\Gamma^\alpha u(t)\|_{L^2} + \sum_{|\alpha| \leq k-8} \sup_{t \in \mathbf{R}} \{ \|\partial_t \Gamma^\alpha u(t)\|_{L^2} + \|\omega \Gamma^\alpha u(t)\|_{L^2} \} \\ + \sum_{|\alpha| \leq k-13} \sup_{\substack{t \in \mathbf{R} \\ x \in \mathbf{R}^2}} |(1+t+|x|)\Gamma^\alpha u(t,x)| < \infty.$$

Furthermore, the above solution  $u$  has the free profiles  $(u_{\pm 0}, u_{\pm 1}) \in H^{k-8} \oplus H^{k-9}$  such that

$$(2.3) \quad \sum_{j=0}^1 \|\partial_t^j \{u(t) - u_{\pm}(t)\}\|_{H^{k-8-j}} \rightarrow 0$$

as  $t \rightarrow \pm\infty$ , where

$$u_+(t) = (\cos \omega t)u_{\pm 0} + (\omega^{-1} \sin \omega t)u_{\pm 1}.$$

*Remark 2.1.* (i) Theorem 2.1 answers positively the conjecture of Hörmander for  $n = 2$  (see [8, page 179 in Section 7.5]). Recently, the conjecture of Hörmander for  $n = 1$  has been solved in the semilinear case (see [20]).

(ii) The argument by Simon and Taffin [18] seems to work for the construction of wave operators even in the quasilinear case, after some modifications. But their argument in [18] does not seem to lead to the asymptotic completeness in the quasilinear case, which is the important part in their paper [18].

### §3. Scattering operator for the Lorentz invariant case

In this section, we consider the construction of wave operators and the asymptotic completeness around zero in a certain weighted Sobolev space under (f.3). In [18] Simon and Taffin treated these problems and gave the proof for the existence of wave operators and the asymptotic completeness around zero under (f.3) (they also stated the related results for the more general nonlinearity in [18]). Here, we prove the following theorem concerning the scattering theory around zero for (1.1) under (f.3) in a different way from [18].

**Theorem 3.1.** *Assume that the nonlinear function  $f$  satisfies (f.3). Let  $m$  be an integer with  $m \geq 13$ .*

(i) *There exists a  $\delta > 0$  with the following property: if  $(u_{+0}, u_{+1}) \in H^{m, m-1} \oplus H^{m-1, m-1}$  and*

$$\|u_{+0}\|_{H^{m, m-1}} + \|u_{+1}\|_{H^{m-1, m-1}} < \delta,$$

*then there exists the interacting state  $(u_0, u_1)$  such that  $(u_0, u_1) \in H^{m, m-1} \oplus H^{m-1, m-1}$  and*

$$\begin{aligned} & \|u_+(t) - u(t)\|_{L^2} + \|\nabla(u_+(t) - u(t))\|_{L^2} \\ & + \|\partial_t(u_+(t) - u(t))\|_{L^2} \longrightarrow 0 \quad (t \rightarrow +\infty), \end{aligned}$$

*where  $u_+(t)$  is a solution of the linear problem (1.1)-(1.2) with  $a = b = 0$  and  $(u_+(0), \partial_t u_+(0)) = (u_{+0}, u_{+1})$ , and  $u(t)$  is a solution of the nonlinear problem (1.1)-(1.2) with  $(u(0), \partial_t u(0)) = (u_0, u_1)$ .*

(ii) *There exists a  $\delta > 0$  with the following property: if  $(u_{-0}, u_{-1}) \in H^{m, m-1} \oplus H^{m-1, m-1}$  and*

$$\|u_{-0}\|_{H^{m, m-1}} + \|u_{-1}\|_{H^{m-1, m-1}} < \delta,$$

*then there exists the interacting state  $(u_0, u_1)$  such that  $(u_0, u_1) \in H^{m, m-1} \oplus H^{m-1, m-1}$  and*

$$\begin{aligned} & \|u_-(t) - u(t)\|_{L^2} + \|\nabla(u_-(t) - u(t))\|_{L^2} \\ & + \|\partial_t(u_-(t) - u(t))\|_{L^2} \longrightarrow 0 \quad (t \rightarrow -\infty), \end{aligned}$$

*where  $u_-(t)$  is a solution of the linear problem (1.1)-(1.2) with  $a = b = 0$  and  $(u_-(0), \partial_t u_-(0)) = (u_{-0}, u_{-1})$ , and  $u(t)$  is a solution of the nonlinear problem (1.1)-(1.2) with  $(u(0), \partial_t u(0)) = (u_0, u_1)$ .*

(iii) *There exists a  $\delta > 0$  with the following property: if  $(u_0, u_1) \in H^{m, m-1} \oplus H^{m-1, m-1}$  and*

$$\|u_0\|_{H^{m, m-1}} + \|u_1\|_{H^{m-1, m-1}} < \delta,$$

then there exist the scattered states  $(u_{\pm 0}, u_{\pm 1})$  such that  $(u_{\pm 0}, u_{\pm 1}) \in H^{m, m-1} \oplus H^{m-1, m-1}$  and

$$\begin{aligned} & \|u_{\pm}(t) - u(t)\|_{L^2} + \|\nabla(u_{\pm}(t) - u(t))\|_{L^2} \\ & + \|\partial_t(u_{\pm}(t) - u(t))\|_{L^2} \longrightarrow 0 \quad (t \rightarrow \pm\infty), \end{aligned}$$

where  $u_{\pm}(t)$  are solutions of the linear problems (1.1)-(1.2) with  $a = b = 0$  and  $(u_{\pm}(0), \partial_t u_{\pm}(0)) = (u_{\pm 0}, u_{\pm 1})$ , respectively, and  $u(t)$  is a solution of the nonlinear problem (1.1)-(1.2) with  $(u(0), \partial_t u(0)) = (u_0, u_1)$ .

*Remark 3.1.* (i) Theorem 3.1 holds, even if the cubic nonlinearity dependent only on  $u$  and its first derivatives is added to the right hand side of (1.1). In that case, the cubic nonlinearity need not be Lorentz invariant.

(ii) Our proof of Theorem 3.1 is different from that in [18] in the following two respects: First, we do not use the nonlinear representation, while it plays an important role in the paper [18]. Second, our estimation of the transformation canceling out the quadratic terms is different from that in [18], although this kind of transformations are crucial in both their proof of [18] and our proof. It does not seem clear how much regularity of the data is necessary in [18] because of the usage of the  $L^2$  boundedness theorem for the pseudodifferential operator, and our theorem 3.1 seems to require less regularity on the data than that in [18].

Now we state a sketch of the proof of Theorem 3.1.

*Sketch of Proof of Theorem 3.1.* We proceed parallel to the proof of Theorem 1.1 in [13], and so we describe only the difference between the proof of Theorem 3.1 and the proof of Theorem 1.1 in [13].

Let us introduce the new unknown function  $v$  to consider the normal form of (1), following Shatah [15] (see also Simon [17]):

$$\begin{aligned} v &= u - [u, B_1, u] - [\partial_t u, B_2, \partial_t u], \\ \hat{B}_1(p, q) &= \hat{B}_{11}(p, q) + \hat{B}_{12}(p, q), \\ \hat{B}_{11}(p, q) &= \frac{(2a+b)(1-2p \cdot q)}{8\{|p|^2|q|^2 - (p \cdot q)^2 + |p|^2 + |q|^2 + p \cdot q + \frac{3}{4}\}}, \\ \hat{B}_{12}(p, q) &= \frac{1}{2}b, \\ \hat{B}_2(p, q) &= \frac{2a+b}{8\{|p|^2|q|^2 - (p \cdot q)^2 + |p|^2 + |q|^2 + p \cdot q + \frac{3}{4}\}}. \end{aligned}$$

The new function  $v$  satisfies the new cubic nonlinear equation:

$$(3.1) \quad \partial_t^2 v - \Delta v + v = F(u, \partial u, \partial^2 u, \partial_t^2 u), \quad t > 0, \quad x \in \mathbf{R}^2,$$

where

$$F(u, \partial u, \partial^2 u) = -[f, B_1, u] - [u, B_1, f]$$

$$\begin{aligned}
 & - [\partial_t f, B_2, \partial_t u] - 2[f, B_2, \Delta u - u] \\
 & - 2[\Delta u - u, B_2, f] - [\partial_t u, B_2, \partial_t f]
 \end{aligned}$$

(for (3.1), see [13, §3]).

The drawback of the result in [13] is that the invariant Sobolev norm of the highest order for the solution is not bounded in time (see, e.g., (2.2) in Section 2). This is the major reason why it is not shown in [13] that the free profile of the solution belongs to the same class as the initial data. But we can prove that when (f.3) is satisfied, the invariant Sobolev norm of the highest order for the solution is bounded in time.

The following proposition concerning the estimates of the bilinear integral operators  $B_{11}$  and  $B_2$  plays an essential role in our proof.

**Proposition 3.2.** *Let  $\alpha$  and  $\beta$  be any two multi-indices. We put*

$$D_{11}(x, y) = x^\alpha y^\beta B_{11}(x, y), \quad D_2(x, y) = x^\alpha y^\beta B_2(x, y).$$

Then, we have the following estimates for  $D_{11}$  and  $D_2$ :

$$\begin{aligned}
 \| [u, D_{11}, v] \|_{L^2} & \leq C \min \{ \|u\|_{L^2} \|v\|_{W^{4,4}}, \|u\|_{W^{4,4}} \|v\|_{L^2} \}, \\
 \| [u, D_2, v] \|_{L^2} & \leq C \min \{ \|u\|_{H^{-1}} \|v\|_{W^{3,4}}, \|u\|_{W^{3,4}} \|v\|_{H^{-1}} \},
 \end{aligned}$$

where  $C$  is a positive constant dependent only on  $\alpha$  and  $\beta$ .

In [13] we evaluate the original equation (1.1) to obtain Lemma 3.4 of [13], which is the energy estimate of higher order derivatives. Because we encounter the loss of derivative, if we use the normal form to obtain the energy estimate of higher order derivatives. But, if Proposition 3.2 has been proved, then we can use the normal form (3.1) to obtain the energy estimate of higher order derivatives. This implies that Lemma 3.4 in [13] holds with  $\varepsilon = 0$  and so Theorem 1.1 in [13] also holds with  $\varepsilon = 0$ . Therefore, we can obtain the boundedness in time of the invariant Sobolev norm of the highest order for the solution, which leads to the fact that the scattered data belong to the same class as the initial data. Moreover, we do not have to divide the energy estimate into two cases of higher order derivatives and lower order derivatives in contrast to [13], and so less regularity on the initial data is sufficient for the proof of Theorem 3.1. Since it is easily verified, we omit the details.

Now, it remains only to prove Proposition 3.2. For the proof of Proposition 3.2, we need the following three lemmas.

**Lemma 3.3.** *Let  $D_{11}$  and  $D_2$  be defined as in Proposition 3.2. We have the following estimates for  $D_{11}$  and  $D_2$ :*

$$\begin{aligned}
 \| [u, D_{11}, v] \|_{H^1} & \leq C \min \{ \|u\|_{H^2} \|v\|_{W^{3,\infty}}, \|u\|_{W^{3,\infty}} \|v\|_{H^2} \}, \\
 \| [u, D_2, v] \|_{H^1} & \leq C \min \{ \|u\|_{H^1} \|v\|_{W^{2,\infty}}, \|u\|_{W^{2,\infty}} \|v\|_{H^1} \},
 \end{aligned}$$

where  $C$  is a positive constant dependent only on  $\alpha$  and  $\beta$ .

*Proof.* We prove Lemma 3.3 only for the case of  $\alpha = \beta = 0$ , that is, for  $B_{11}$  and  $B_2$ , since Lemma 3.3 for the general case can be proved in the same way. Here and hereafter,

we use the notation  $\langle p \rangle = (1 + |p|^2)^{1/2}$ . A simple calculation yields

$$(3.2) \quad (|p| + |q|)^{|\gamma|} |\partial_p^\zeta \partial_q^\eta [\langle p \rangle^{-1} \langle q \rangle^{-1} B_{11}(p, q)]| \leq C, \\ \gamma = \zeta + \eta, \quad |\gamma| \leq 2,$$

where

$$\partial_p^\zeta = \frac{\partial^{|\zeta|}}{\partial_{p_1}^{\zeta_1} \partial_{p_2}^{\zeta_2}}, \quad \partial_q^\eta = \frac{\partial^{|\eta|}}{\partial_{q_1}^{\eta_1} \partial_{q_2}^{\eta_2}}$$

For (3.2), see Lemmas 2.2 and 2.5 in [13]. The Fourier multiplier theorem by Coifman and Meyer [5, Theorem [5] on page 22] and (3.3) give us

$$\|[u, B_{11}, v]\|_{L^2} \leq C \min\{\|u\|_{H^1} \|\omega v\|_{L^\infty}, \|\omega u\|_{L^\infty} \|v\|_{H^1}\}.$$

This shows the desired estimate for  $B_{11}$ , since we have

$$\|\omega u\|_{L^\infty} \leq C \|u\|_{W^{2,\infty}}.$$

Lemma 3.3 for  $B_2$  can be proved in the same way as above.  $\square$

**Lemma 3.4.** *Let  $u(x, y) \in H^2(\mathbf{R}_x^2 \times \mathbf{R}_y^2)$ . Then,*

$$\|u(\cdot, \cdot)\|_{L^2(\mathbf{R}^2)}^2 = \int_{\mathbf{R}^2} |u(x, x)|^2 dx \\ \leq C \|u\|_{L^2(\mathbf{R}^4)}^{1/2} \|\nabla_y u\|_{L^2(\mathbf{R}^4)} \|\Delta_y u\|_{L^2(\mathbf{R}^4)}^{1/2}$$

*Proof.* We may assume  $u \in C_0^\infty(\mathbf{R}^4)$  without loss of generality.

We first have

$$u^2(x, x) = \int_{-\infty}^{x_j} \partial_{y_j} u^2(x, y) dy_j \\ \leq 2 \int_{-\infty}^{x_j} |u(x, y)| |\partial_{y_j} u(x, y)| dy_j, \quad j = 1, 2.$$

On the other hand,

$$(\partial_{y_1} u)^2(x, x) = 2 \int_{-\infty}^{x_2} \partial_{y_1} u(x, y) \cdot \partial_{y_1} \partial_{y_2} u(x, y) dy_2.$$

Therefore, we have

$$u^2(x, x) \leq C \int_{-\infty}^{x_1} \left( \int_{-\infty}^{x_2} |u(x, y)| |\partial_{y_2} u(x, y)| dy_2 \right)^{1/2} \\ \times \left( \int_{-\infty}^{x_2} |\partial_{y_1} u(x, y)| |\partial_{y_1} \partial_{y_2} u(x, y)| dy_2 \right)^{1/2} dy_1$$

$$\begin{aligned} &\leq C \left( \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} |u(x, y)| |\partial_{y_2} u(x, y)| dy_2 dy_1 \right)^{1/2} \\ &\quad \times \left( \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} |\partial_{y_1} u(x, y)| |\partial_{y_1} \partial_{y_2} u(x, y)| dy_2 dy_1 \right)^{1/2} \end{aligned}$$

Accordingly, integrating the both sides of the above inequality over  $\mathbf{R}^2$ , we obtain by Schwarz' inequality

$$\begin{aligned} \int_{\mathbf{R}^2} u^2(x, x) dx &\leq C \left( \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x, y)| |\partial_{y_2} u(x, y)| dy_2 dy_1 dx \right)^{1/2} \\ &\quad \times \left( \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_{y_1} u(x, y)| |\partial_{y_1} \partial_{y_2} u(x, y)| dy_1 dy_2 dx \right)^{1/2} \\ &\leq C \|u\|_{L^2(\mathbf{R}^4)}^{1/2} \|\nabla_y u\|_{L^2(\mathbf{R}^4)} \|\Delta_y u\|_{L^2(\mathbf{R}^4)}^{1/2}. \quad \square \end{aligned}$$

**Lemma 3.5.** *Let  $D_{11}$  and  $D_2$  be defined as in Proposition 3.2. Then, we have the following estimates for  $D_{11}$  and  $D_2$ :*

$$\begin{aligned} \|[u, D_{11}, v]\|_{H^{-1}} &\leq C \min\{\|u\|_{H^{-2}} \|v\|_{H^4}, \|u\|_{H^4} \|v\|_{H^{-2}}\}, \\ \|[u, D_2, v]\|_{H^{-1}} &\leq C \min\{\|u\|_{H^{-3}} \|v\|_{H^3}, \|u\|_{H^3} \|v\|_{H^{-3}}\}, \end{aligned}$$

where  $C$  is a positive constant dependent only on  $\alpha$  and  $\beta$ .

*Proof.* We prove Lemma 3.5 only for the case of  $\alpha = \beta = 0$ , that is, for  $B_{11}$  and  $B_2$ , since Lemma 3.5 for the general case can be proved in the same way.

We first note that if  $|p + q| \geq \frac{1}{2}|p|$ ,

$$\langle p + q \rangle^{-1} |p \cdot q| \leq \frac{|p \cdot q|}{(1 + |p|^2/4)^{1/2}} \leq 2|q|$$

and if  $|p + q| \leq \frac{1}{2}|p|$ ,

$$\langle p + q \rangle^{-1} |p + q| \leq \frac{2|q|^2}{(1 + |p + q|^2)^{1/2}} \leq 2|q|^2.$$

Therefore, we have

$$(3.3) \quad \langle p + q \rangle^{-1} |\hat{B}_{11}(p, q)| \langle p \rangle^2 \langle q \rangle^{-2} \leq C$$

Inequality (3.3) and Lemma 3.4 give us

$$\begin{aligned} (3.4) \quad &\|[u, \tilde{B}_{11}, v]\|_{L^2(\mathbf{R}^2)} \\ &\leq C \left\| \int_{\mathbf{R}^4} \tilde{B}_{11}(x - z_1, y - z_2) u(z_1) \cdot (1 - \Delta_{z_2}) v(z_2) dz_1 dz_2 \right\|_{L^2(\mathbf{R}^4)} \\ &\leq C \|u\|_{H^{-2}} \|v\|_{H^4}, \end{aligned}$$

where

$$\hat{\hat{B}}_{11}(p, q) = \langle p + q \rangle^{-1} B_{11}(p, q).$$

Since  $\omega^{-1}[u, B_{11}, v] = [u, \hat{\hat{B}}_{11}, v]$  (see [12, Proof of Lemma 2.4(ii)]), inequality (3.4) and the symmetry in  $p$  and  $q$  of  $\hat{\hat{B}}_{11}$  imply that

$$\|[u, B_{11}, v]\|_{H^{-1}} \leq C \min\{\|u\|_{H^{-2}}\|v\|_{H^4}, \|u\|_{H^4}\|v\|_{H^{-2}}\},$$

which shows the desired inequality for  $B_{11}$ .

For the estimate of  $B_2$ , we note that

$$\langle p + q \rangle^{-1} \leq C \min\{\langle p \rangle^{-1}\langle q \rangle, \langle p \rangle\langle q \rangle^{-1}\}.$$

By using this inequality, we can show Lemma 3.5 for  $B_2$  in the same way as above.  $\square$

We are in a position to prove Proposition 3.2. Interpolating the two inequalities in Lemmas 3.3 and 3.5 for  $D_{11}$ , we have

$$(3.5) \quad \|[u, D_{11}, v]\|_{L^2} \leq C \min\{\|u\|_{L^2}\|v\|_{W^{4,4}}, \|u\|_{W^{4,4}}\|u\|_{L^2}\}.$$

(for the interpolation theorem of bilinear operator, see, e.g., Exercise 5 of §3.13 and Theorem 4.4.1 of §4.4 in [2]).

We can similarly prove Proposition 3.2 for  $D_2$ .

**Acknowledgement:** The authors would like to thank Professors Simon and Taffin for answering the questions on their paper [18].

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# ON THE WELL POSEDNESS OF SOME NONLINEAR EVOLUTION EQUATIONS

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## §1. Introduction.

In this paper we consider the initial value problem (IVP)

$$(1.1) \quad \begin{cases} \partial_t u + Au = F(u), & x \in \mathbb{R}, t \in \mathbb{R}^+, (\text{ or } \mathbb{R}) \\ u(x, 0) = u_0(x), \end{cases}$$

where  $A = A(\nabla_x)$  is a linear differential operator and  $F(\cdot)$  represents the nonlinearity.

Our main interest is to study well posedness of the IVP (1.1). Following [Kt2] the notion of well posedness used here includes existence, uniqueness, persistence, i.e. if  $u_0 \in \mathcal{X}$ , with  $\mathcal{X}$  a function space, then the corresponding solution describes a continuous curve in  $\mathcal{X}$ , and lastly continuous dependence of the solution upon the data. As it was remarked in [Kt2] this notion of well posedness is rather strong and is not always proved in its full strength in the literature.

For the IVP (1.1) we will assume that  $A = A(\nabla_x)$  is a homogeneous operator of order  $k$  with constant coefficients, i.e. the associated symbol  $A(\xi)$  satisfies

$$(1.2) \quad A(\lambda\xi) = \lambda^k A(\xi), \quad \lambda \in \mathbb{R}^+,$$

and that the nonlinearity  $F$  satisfies that

$$(1.3) \quad F(\lambda^\theta u(\lambda x, \lambda^k t)) = \lambda^{k+\theta} F(u(\lambda x, \lambda^k t)), \quad \lambda \in \mathbb{R}^+,$$

for some  $\theta \in \mathbb{R}$ .

This guarantees that the following scaling argument, or dimensional analysis, works : if  $u(x, t)$  is a solution of the IVP (1.1) then  $\lambda^\theta u(\lambda x, \lambda^k t)$  solves the same equation with initial data  $\lambda^\theta u_0(\lambda x)$ .

For the well posedness of the IVP (1.1) we will restrict the function spaces to the generalized Sobolev spaces, inhomogeneous and homogeneous, i.e. the function space  $\mathcal{X}$  is

$$(1.4) \quad L^{s,p}(\mathbb{R}^n) = (1 + D^2)^{-s/2} L^p(\mathbb{R}^n), \quad \dot{L}^{s,p}(\mathbb{R}^n) = D^{-s} L^p(\mathbb{R}^n), \quad D = (-\Delta)^{1/2},$$

respectively, and  $s \in \mathbb{R}$ ,  $p \in [1, \infty)$ .

Although these spaces are general enough for our purpose it may be remarked that other choices of function spaces are also possible. In particular, we observe that “self-similar” solutions of the equation (1.1), i.e. solutions of the form  $u(x, t) = t^{-\theta/k} w(t^{-1/k} x)$ , if they exist do not correspond to data in these generalized Sobolev spaces except for the case of zero data.

As examples of (1.1) we consider equations or systems of parabolic, dispersive and hyperbolic type. The cases of systems of mixed type, for example, the equations for compressible viscous fluid flows which is a parabolic-hyperbolic system, or the Zakharov system [Za] a hyperbolic-dispersive system, will not be studied here.

In general, we will be concerned with the problem of minimal regularity of the data which guarantees well posedness of (1.1). In our setting, one first restricts the values of  $p$  for which  $e^{-A(i\xi)}$  is an  $L^p$  multiplier. This guarantees that the associated linear problem, i.e. (1.1) with  $F = 0$ , is well posed in  $L^{s,p}(\mathbb{R}^n)$  for any  $s \in \mathbb{R}$ , (thus in the dispersive and hyperbolic cases one needs  $p = 2$ ). Then the problem reduces to determine the minimal  $s \in \mathbb{R}$  for which the IVP (1.1) is well posed in  $L^{s,p}(\mathbb{R}^n)$ .

We are interested in local well posedness where the properties of the solutions can be written as a function of the size of the data, for example, the time of existence depends on the size of the data. Thus, one may expect that this type of local well posedness together with the existence of conservation laws may imply global well posedness for a class of arbitrarily large data. Also in the cases where the existence of a global solution is uncertain these results may provide a description of the solution near the possible blow up time. These results are useful for other qualitative analysis of the problem, for example, in the study of stability of special solutions as traveling waves.

In the case of homogeneous spaces these local results may provide global ones for small data, which in many cases are optimal. These global small data results can be used to obtain further qualitative properties of the solutions, for example, to establish small nonlinear scattering.

In this regard a simple computation will be the starting point of our analysis. From our assumptions one finds that

$$(1.5) \quad \|\lambda^\theta u_0(\lambda x)\|_{\dot{L}^{s,p}} \equiv \|D^s(\lambda^\theta u_0(\lambda x))\|_p = \lambda^{s+\theta-n/p} \|D^s u_0\|_p.$$

Thus for  $\theta = n/p - s$  the norm of  $u_{0,\lambda}$  is independent of  $\lambda$ .

It should be pointed out that the scaling argument in (1.5) may be too general, and does not always recognize the type of equation considered.

Next we make precise our definition of well posedness.

**Definition 1.1 (local well posedness, sub-critical case).**

*The IVP (1.1) is locally well posed, in a sub-critical manner, in the function space  $\mathcal{X}$  if the following properties are satisfied :*

*(i)- Given  $u_0 \in \mathcal{X}$  there exists  $T = T(\|u_0\|_{\mathcal{X}}) > 0$ , with  $T(\delta) \rightarrow \infty$  when  $\delta \rightarrow 0$ , and a unique solution  $u$  of the IVP (1.1) with*

$$(1.6) \quad u \in X^T \equiv C([0, T] : \mathcal{X}) \cap \dots\dots\dots$$

*(ii)- There exists  $r = r(\|u_0\|_{\mathcal{X}}) > 0$  and a continuous nondecreasing function  $\mathcal{F}(\cdot) = \mathcal{F}(\cdot, \|u_0\|_{\mathcal{X}})$ , with  $\mathcal{F}(0) = 0$ , such that the map that takes the data to its*

solution  $\tilde{u}_0 \rightarrow \tilde{u}(t)$  from  $\{\tilde{u}_0 \in \mathcal{X} : \|\tilde{u}_0 - u_0\|_{\mathcal{X}} < r\}$  into  $X^T$  satisfies

$$(1.7) \quad \|u(t) - \tilde{u}(t)\|_{X^T} \leq \mathcal{F}(\|\tilde{u}_0 - u_0\|_{\mathcal{X}}).$$

(iii)- If  $u_0 \in \mathcal{Y} \hookrightarrow \mathcal{X}$ , then the above results hold in  $Y^T \equiv C([0, T] : \mathcal{Y}) \cap \dots$

### Remarks

(a) Part (i) only assumes the existence and uniqueness of a weak solution. However, from part (ii)-(iii) it follows that this weak solution is in fact a strong one, i.e. it can be achieved as a limit in  $X^T$  of classical solutions. The definition above was stated in this manner due to our interest in non-uniqueness results for weak solutions of (1.1) in  $C([0, T] : \mathcal{X})$ .

(b) The solution  $u \in X^T$  must guarantee that the nonlinearity  $F(u(t))$  makes sense, or else one might need a smaller class where uniqueness can be established, see [Kt7].

(c) We are not assuming that the time of existence  $T$  is the largest possible.

(d) Definition 1.1 requires that the time of existence of the solution, and the modulus of continuity of the solution upon the data depend only on the size of  $\|u_0\|_{\mathcal{X}}$ .

(e) In general, when the contraction principle is used to prove the local well posedness of the IVP (1.1) in its associated integral form

$$(1.8) \quad u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-t')A} F(u)(t') dt',$$

one has that  $\mathcal{F}(\rho) = M(\|u_0\|_{\mathcal{X}})\rho$  in (1.7), i.e. the map that takes data to its solution is locally Lipschitz. In this case, if in addition the nonlinearity  $F$  is an analytic function of its arguments then the map is analytic, see [Be], [Zh]. This follows as a consequence of the proof of the contraction principle and the implicit function theorem. In general, continuous dependence upon the data can be established by using the arguments in [BoSm] or [Kt1].

**Definition 1.2 (local well posedness, critical homogeneous case).**

The IVP (1.1) is locally well posed, in a critical manner, in the function space  $\mathcal{X}$  if the following properties are satisfied :

(i)-Definition 1.1, part (i)-(ii) hold with  $T$ ,  $r$  and  $\mathcal{F}$  depending on  $u_0$  itself.

(ii)-There exists  $\eta > 0$  such that if

$$(1.9) \quad \|u_0\|_{\mathcal{X}} < \eta,$$

then the above local results extend, uniformly, to the time interval  $[0, \infty)$ .

As a consequence of the scaling argument in (1.5), and the form of Definitions 1.1-1.2 we have the following general statements.

**Statement I.**

Let  $1 \leq p < \infty$ , and  $s > s_p = n/p - \theta$ , with  $\theta$  defined in (1.3). Then the IVP (1.1) is locally well posed in  $L^{s,p}(\mathbb{R}^n)$  in a sub-critical manner, as stated in Definition 1.1.

**Statement II.**

Let  $1 \leq p < \infty$ , and  $s_p = n/p - \theta$ , with  $\theta$  defined in (1.3). Then the IVP (1.1) is locally well posed in the homogeneous Sobolev space  $\dot{L}^{s_p,p}(\mathbb{R}^n)$  in a critical manner, as stated in Definition 1.2.

**Statement III.**

Let  $1 \leq p < \infty$ . If  $s < s_p$ , (resp.  $s \leq s_p$ ), then some part of Statement I, (resp. Statement II) fails.

Remarks

(a) Statement II still holds in the inhomogeneous Sobolev spaces  $L^{s_p,p}(\mathbb{R}^n)$ . However, the use of the homogeneous spaces,  $\dot{L}^{s_p,p}(\mathbb{R}^n)$ , yields the existence of global solutions for small data which in many cases is the best possible global result.

(b) In general to establish persistence properties in the homogeneous Sobolev spaces with negative indices, i.e.  $\dot{L}^{s,p}(\mathbb{R}^n)$ , with  $s < 0$ , one may need an equation in divergence form. In this case, the total mass of the data,  $\int u_0(x)dx$ , which may be required to be zero, is preserved.

Our goal will be to discuss the validity of Statements I-III in particular cases of the IVP (1.1). In particular, for fixed values of the parameters  $n, p, \theta$  one finds the following three possibilities:

- (A1) the results in Statement I suggested by the scaling argument in (1.5) hold,
- (A2) the results in Statement I suggested by the scaling argument in (1.5) cannot be achieved or they can be improved,
- (A3) the results in Statement I suggested by the scaling argument in (1.5) are unknown.

For Statement III we shall see examples in which the notion of well posedness fails due to the lack of at least one the following properties existence, uniqueness, persistence or the fact that the time of existence and the continuous dependence can be expressed as a function on the size of the data.

The rest of this paper is organized as follows. In section 2 we discuss the validity of Statements I-III for some parabolic models. The case of dispersive equations will be treated in section 3. Finally, section 4 is concerned with the hyperbolic case.

## §2. Parabolic Equations.

### (i) The semi-linear heat equation.

Consider the IVP associated to the semi-linear heat equation

$$(2.1) \quad \begin{cases} \partial_t u - \Delta u = a|u|^{\alpha-1}u, & t > 0, x \in \mathbb{R}^n, a = \pm 1, \alpha > 1 \\ u(x, 0) = u_0(x). \end{cases}$$

If  $u = u(x, t)$  is a solution of the IVP (2.1) then

$$(2.2) \quad u_\lambda(x, t) = \lambda^{2/(\alpha-1)}u(\lambda x, \lambda^2 t), \quad \lambda > 0,$$

satisfies the same equation with initial data

$$(2.3) \quad u_\lambda(x, 0) = \lambda^{2/(\alpha-1)} u_0(\lambda x).$$

Hence,

$$(2.4) \quad \|u_\lambda(x, 0)\|_{L^{s,p}} = \|D^s u_\lambda(x, 0)\|_{L^p} = c_{u_0} \lambda^{2/(\alpha-1)+s-n/p},$$

and

$$(2.5) \quad \theta = \frac{2}{\alpha-1}, \quad s_p = \frac{n}{p} - \frac{2}{\alpha-1}.$$

The following theorem is concerned with the existence of solutions for the IVP (2.1) with rough data.

**Theorem 2.1** ([BeFr]).

*If  $a = -1$ , and  $u_0(x) = \delta(x)$ , then the IVP (2.1) has a (weak) solution if and only if*

$$(2.6) \quad 0 < \alpha < \frac{(n+2)}{n}.$$

Let us see that in particular Theorem 2.1 implies Statement III for appropriate values of the parameters  $n$ ,  $\alpha$ ,  $s$ ,  $p$ . More precisely, it shows non local well posedness can hold in  $L^{s,p}(\mathbb{R}^n)$  with  $s_p > s > -n(p-1)/p$ .

Since  $\delta \in L^{s,p}(\mathbb{R}^n)$  for any  $s = -n/p' - \epsilon$ ,  $\epsilon > 0$ , and  $1/p + 1/p' = 1$ , in view of Statement III, local well posedness for the IVP (2.1) should fail for  $L^{s,p}(\mathbb{R}^n)$  where

$$(2.7) \quad s_p = \frac{n}{p} - \frac{2}{\alpha-1} > s = -\frac{n}{p'} - \epsilon = -n + \frac{n}{p} - \epsilon, \text{ i.e., } n(\alpha-1) > 2,$$

which agrees with (2.6).

The proof of Theorem 2.1 combines ideas on removable singularities for the equation (2.1) with previous results for the associated stationary elliptic problem.

In the same vain one has the following non-uniqueness result for the IVP (2.1).

**Theorem 2.2 ([HW]).**

Let  $a = +1$ , and  $1 \leq p < n(\alpha - 1)/2 < \alpha + 1$ . Then the IVP (2.1) with  $u_0 \equiv 0$  has a nontrivial positive solution in  $C([0, T] : L^p(\mathbb{R}^n))$ .

In addition, if  $1 \leq p < n(\alpha - 1)/(\alpha + 1)$ , (resp.  $1 \leq p < n(\alpha - 1)/2\alpha$ ) this nontrivial solution belongs to  $C([0, T] : L^{1,p}(\mathbb{R}^n))$ , (resp.  $C([0, T] : L^{2,p}(\mathbb{R}^n))$ ).

We observe that the conditions  $1 \leq p < n(\alpha - 1)/(\alpha + 1)$ , and  $1 \leq p < n(\alpha - 1)/2\alpha$  imply that  $p < 2$ , and  $p < 1 + 1/\alpha$ , respectively.

Also, since  $s_p = n/p - 2/(\alpha - 1)$ , Theorem 2.2 implies Statement III for  $s_p > 0$  and  $s = 0$ .

The proof of Theorem 2.2 is based on special behavior of self-similar solutions of the IVP (2.1), i.e.  $u(x, t) = t^{-1/(\alpha-1)}w(x/\sqrt{t})$ . Assuming that  $w$  is radial, i.e.  $w(y) = v(|y|)$ , the main step in the proof is the study of the asymptotic behavior of solutions to the IVP for the following differential equation

$$(2.8) \quad \begin{cases} v'' + \left(\frac{n-1}{x} + \frac{x}{2}\right)v' + \frac{v(x)}{\alpha-1} + |v|^{\alpha-1}v = 0, & x > 0, \\ v(0) = v_0, \quad v'(0) = 0, \end{cases}$$

respect to the parameter  $v_0$ .

Combining Theorem 2.2 and the Sobolev embedding theorem we obtain,

**Corollary 2.3.**

Let  $a = +1$ ,  $1 \leq p < n(\alpha - 1)/2 < \alpha + 1$ ,  $q(2 + s(\alpha - 1)) < n(\alpha - 1)$ , and  $p < q$ . Then in the following cases

$$(2.9) \quad \begin{aligned} & (i) \quad s < 0, \\ & (ii) \quad 1 \leq p < n \frac{\alpha - 1}{\alpha + 1}, \quad \text{and} \quad 0 < s < 1, \\ & (iii) \quad 1 \leq p < n \frac{\alpha - 1}{2\alpha}, \quad \text{and} \quad 1 < s < 2, \end{aligned}$$

the IVP (2.1) with  $u_0 \equiv 0$  has a nontrivial positive solution in  $C([0, T] : L^{s,q}(\mathbb{R}^n))$ .

For the case of Statement I we gather the results in [Gi], [KoYa], [W] as follows.

**Theorem 2.4,** ([Gi], [KoYa], [W]).

For  $a = \pm 1$ , and  $s \leq [\alpha - 1]$  if  $\alpha$  is not an odd integer, the IVP (2.1) satisfies Statement I.

The sharpness of the above theorem can be deduced in part from the following blow up result (see [F]), if  $a = +1$  there exists data  $u_0 \in S(\mathbb{R}^n)$ , and  $p \in [1, \infty)$  such that the corresponding solution  $u = u(x, t)$  of the IVP (2.1) provided by Theorem 2.4 blows up in the  $L^p$ -norm in a finite time  $T^*$ . Defining

$$(2.10) \quad u_\lambda(x, t) = \lambda^{2/(\alpha-1)} u(\lambda x, \lambda^2 t),$$

the solution of (2.1) corresponding to the data  $u_\lambda(x, 0) = \lambda^{2/(\alpha-1)} u_0(\lambda x)$ , one has that

$$(2.11) \quad \|u_\lambda(x, 0)\|_{L^p} \equiv \|u_0\|_{L^p}, \quad \lambda > 0,$$

and  $u_\lambda$  has a life span  $T_\lambda$  given by  $T_\lambda = T^*/\lambda^2$ . This proves that Theorem 2.4 is the best possible for  $a = +1$ ,  $s \geq 0$ .

In the case  $a = -1$  one has that local solutions of (2.1) satisfy

$$(2.12) \quad \|u(t)\|_{L^p} \leq \|u_0\|_{L^p}, \quad t \in [0, T), \quad p \geq 2.$$

This *a priori* estimate combined with Theorem 2.4 gives,

**Theorem 2.5.**

For  $a = -1$ ,  $s \leq 0$ ,  $s > s_p$ , and  $p \geq 2$  local solutions of the IVP (2.1) provided by Theorem 2.4 extend globally in  $L^{s,p}(\mathbb{R}^n)$ .

Concerning Statement II for  $s = 0$  we have the following global small data results in  $L^p(\mathbb{R}^n)$ .

**Theorem 2.6,** ([F], [W]).

The IVP (2.1) with  $a = \pm 1$ , and small data in  $L^p(\mathbb{R}^n)$  has a unique global solution in  $C([0, \infty) : L^p(\mathbb{R}^n))$  if

$$(2.13) \quad p = \frac{n}{2}(\alpha - 1) > 1.$$

For  $a = +1$  the condition (2.13) is also necessary.

Theorem 2.6 affirms that Statement II holds for  $a = +1$ ,  $s_p = 0$  if and only if  $p > 1$ . Thus,  $p = 1$  appears as an exceptional case.

The above results depend on the values of the parameters  $\alpha, n, s, p$ , and  $a$ . To illustrate them we consider the case  $\alpha = 3$ ,  $p = 2$ , i.e.

$$(2.14) \quad \begin{cases} \partial_t u - \Delta u = au^3, & t > 0, x \in \mathbb{R}^n, a = \pm 1, \\ u(x, 0) = u_0(x) \in H^s(\mathbb{R}^n). \end{cases}$$

Hence,  $s_2 = (n - 2)/2$ , and the previous results tell us that,

- If  $a = -1$ , then for  $s < -n/2$  the IVP (2.14) has no solution.
- If  $a = +1$ , then the IVP (2.14) with  $u_0 \equiv 0$  has at least two solutions in  $C([0, T] : H^s(\mathbb{R}^n))$ , with  $s < 0$  if  $n = 2$ , and  $s < 1/2$  if  $n = 3$ .
- If  $a = \pm 1$ , the IVP (2.14) is locally well posed in  $H^s(\mathbb{R}^n)$ , for  $s \geq (n - 2)/2$ . Solutions corresponding to large data and the case  $a = +1$  may blow up in finite time. For  $a = -1$ ,  $s \geq 0$ ,  $s > (n - 2)/2$ , these local solutions extend globally in time in the same class.
- If  $a = \pm 1$ , the IVP (2.14) has a unique global solution in  $L^2(\mathbb{R}^2)$  for any small data in  $L^2(\mathbb{R}^2)$ .
- If  $a = +1$ , the IVP (2.14) has local solutions corresponding to arbitrarily small data in  $L^2(\mathbb{R})$ , which blow up in finite time.

Thus the ill posedness of the IVP (2.14) in  $H^s(\mathbb{R}^n)$  with  $a = -1$  and  $s \in (-n/2, (n - 2)/2)$ , or  $a = +1$ , and  $s < (n - 2)/2$ ,  $n = 1, 4, 5, \dots$  remains open.

Finally, one can say that the IVP (2.1) belongs to the class (A.1) defined at the end of the introduction.

## (ii) The generalized Burgers equation.

Consider the IVP

$$(2.15) \quad \begin{cases} \partial_t u - \partial_x^2 u = -\partial_x(u^{k+1}), & t > 0, x \in \mathbb{R}, k = 1, 2, \dots \\ u(x, 0) = u_0(x). \end{cases}$$

In this case we have

$$(2.16) \quad \theta = \frac{1}{k}, \quad s_p = \frac{1}{p} - \frac{1}{k}.$$

The following result is concerned with the well posedness of the IVP for Burgers' equation, i.e.  $k = 1$  in (2.15), and for data in  $H^s(\mathbb{R})$ .

**Theorem 2.7** ([D]).

-If  $s > -1/2$ , then the IVP (2.14) with  $k = 1$  is locally well posed in  $H^s(\mathbb{R})$ .

-If  $s < -1/2$ , then the IVP (2.14) with  $k = 1$  and  $u_0 \equiv 0$  has infinitely many solutions in  $C([0, T] : H^s(\mathbb{R}))$ .

From (2.16) with  $k = 1$  we have that  $s_2 = -1/2$ , therefore Theorem 2.7 tells us that for  $p = 2$ , and  $k = 1$  Statements I, III (uniqueness fails) hold.

The proof in [D] is based on the Hopf-Cole transformation, which affirms that if  $v = v(x, t)$  is a solution of the one dimensional heat equation then  $u = -2\partial_x \ln(v)$  solves Burgers' equation. Applying this transformation to explicit solutions of the heat equation,  $v_a(x, t) = 1 + \sqrt{a/t} \exp(-x^2/4t)$ ,  $a > 0$  corresponding to data  $v_a(x, 0) = 1 + c_a \delta$  one obtains the non-uniqueness result.

In [Be] the well posedness result in Theorem 2.7 was extended to all  $1 < p < \infty$ , and  $k = 1, 2, \dots$ , including the critical case  $s_p = 1/p - 1/k$ . In other words, Statement I and II for the IVP (2.15) were proven in [Be]. As it was remarked there Statement III remains open except for the case  $n = 1$ ,  $k = 1$ , and  $p \geq 2$ .

We observe that solutions of the IVP (2.15) satisfy

$$(2.17) \quad \|u(t)\|_{L^p} \leq \|u_0\|_{L^p}, \quad \text{for any } t > 0,$$

where the case  $p = \infty$  corresponds to the maximum principle. Hence, in the case  $s > 0 > s_p$  the  $L^{s,p}$ -local solutions extend globally.

Related to the IVP (2.15) we consider the problem

$$(2.18) \quad \begin{cases} \partial_t u - \Delta u = \vec{a} \cdot \nabla(|u|^{\alpha-1}u), & t > 0, x \in \mathbb{R}^n, \vec{a} \in \mathbb{R}^n - \{0\}, \alpha > 1 \\ u(x, 0) = u_0(x). \end{cases}$$

In this case  $s_p = n/p - 1/(\alpha - 1)$ . In particular, for  $p = 1$  the scaling argument suggests that the IVP (2.18) is well posed in  $L^1(\mathbb{R}^n)$  when  $n(\alpha - 1) \leq 1$ .

The following result provides an example where the conditions of Statement I suggested by the scaling can be improved.

**Theorem 2.8** ([EsZu]).

*For any  $n \geq 1$ , and any  $\alpha > 1$  the IVP (2.18) is globally well posed in  $L^1(\mathbb{R}^n)$ . Moreover, the results in Statement I hold with*

$$(2.19) \quad X^T = C([0, \infty) : L^1(\mathbb{R}^n)) \cap C((0, \infty) : L^{2,p}(\mathbb{R}^n)) \cap C^1((0, \infty) : L^p(\mathbb{R}^n)),$$

for any  $p \in (1, \infty)$ .

The main idea in the proof of Theorem 2.8 is the fact that solutions  $u(t), v(t)$  of (2.18) with initial data  $u_0, v_0$ , in addition to (2.17), satisfy the inequality

$$(2.20) \quad \|u(t) - v(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}.$$

Thus the IVP (2.18) with  $p = 1$  is of type (A.2).

**(iii) The Navier-Stokes equation.**

Consider the initial value problem for Navier-Stokes equation

$$(2.21) \quad \begin{cases} \partial_t u - \mathcal{A}u + P\nabla \cdot (u \otimes u) = 0, & x \in \mathbb{R}^n, n \geq 2, t \geq 0 \\ u(x, 0) = Pu_0(x), \end{cases}$$

where  $u = u(t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the velocity field,  $\mathcal{A} = P\Delta$ ,  $P$  denotes the projection operator onto divergence free vectors along gradients, an homogeneous operator of order zero,  $u \otimes u$  is the tensor with  $jk$ -components  $u_j u_k$  and  $\nabla \cdot (u \otimes u)$  is the vector with  $j$ -component  $\partial_k(u_j u_k)$ .

As in (2.16) with  $k = 1$ , if  $u = u(x, t)$  is a solution of the IVP (2.21) then

$$(2.22) \quad u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad \lambda > 0$$

satisfies the same system with initial data

$$(2.23) \quad u_\lambda(x, 0) = \lambda u_0(\lambda x).$$

Therefore  $s_p = n/p - 1$ .

In [KtPo] it was shown that the IVP (2.21) is well posed in all  $L^{s,p}(\mathbb{R}^n)$  spaces dictated by the dimensional analysis in (2.22)-(2.23).

**Theorem 2.9** ([KtPo]).

*Statements I-II with  $1 < p < \infty$  hold for the IVP (2.21).*

In particular, the IVP (2.21) is of type (A.1).

The cases  $p = 2$  and  $p = n$  were previously proven in [KtF], [Kt3] respectively. Similar results in spaces of Besov type and Morrey spaces have been obtained in [Ca], [KoYa],[GiMi], [Kt5], [Ta].

In [KtPo], two proofs of Theorem 2.9 were given. The first one covering all values  $1 < p < \infty$  is based on the weighted time norm method introduced in [KtF]. This method basically provides the proof of all positive results discussed above. The second one uses the so called  $L^r L^{s,q}$ -(time-space) estimates [Gi], and is restricted to  $p \in [n, n+2]$ . The homogeneous version of these  $L^r \dot{L}^{s,q}$ -(time-space) estimates can be written as

$$(2.24) \quad \left( \int_0^\infty \|D_x^s e^{t\Delta} u_0\|_{L^{q''}^r}^r dt \right)^{1/r} \leq c \|u_0\|_{L^{q'}},$$

with

$$(2.25) \quad s \in [0, 2), \quad 0 < 1/r = (n/q' - n/q'')/2 + s/2 \leq 1/q'.$$

Let us briefly consider the problem of global existence of strong solutions for the Navier-Stokes system. In any dimension, strong solutions of the IVP (2.21) satisfy the identity

$$(2.26) \quad \|u(t)\|_{L^2} + \int_0^t \|\nabla_x u(t')\|_{L^2} dt' = \|u_0\|_{L^2}, \quad t > 0.$$

In dimension 2-D ( $n = 2$ ) one has that the vorticity of the velocity field  $\omega(t) = \nabla \times u(t)$  is a scalar, and satisfies the equation

$$(2.27) \quad \partial_t \omega - \Delta \omega + (u \cdot \nabla) \omega = 0.$$

From the equation (2.27) and the Biot-Savart law,  $u = \nabla \times (-\Delta)^{-1} \omega$ , one has that strong solutions of the 2-D Navier-Stokes equations satisfy the *a priori* estimate

$$(2.28) \quad \|\nabla_x u(t)\|_{L^p} \leq c_p \|\nabla_x u_0\|_{L^p}, \quad \text{for } p \in (1, \infty).$$

Thus in 2-D the *a priori* estimates in (2.26), (2.28) and the local results guarantee the existence of a global strong solution for sufficiently regular data.

The situation in 3-D is quite different. In this case the vorticity is a vector field and its evolution is described by a system, for which the estimates in (2.28) do not hold. In fact, the existence of a global strong solution in 3-D for sufficiently regular data is unknown and remains an outstanding open problem.

In this direction one has the following remark found in [Ca] as a consequence of Theorem 2.9 for the IVP (2.20). Since

$$(2.29) \quad \dot{L}^{r,p}(\mathbb{R}^n) \hookrightarrow \dot{L}^{r',p'}(\mathbb{R}^n), \quad \text{if } r' - n/p' = r - n/p \quad \text{and } r' < r,$$

the global small data result in  $\dot{L}^{n/p-1,p}(\mathbb{R}^3)$  implies the existence of arbitrarily large data in  $PH^1(\mathbb{R}^3)$ ,  $PL^3(\mathbb{R}^3)$ , and in any  $P\dot{L}^{3/p-1,p}(\mathbb{R}^3)$ , where  $1 < p < \infty$ , for which the existence of global 3-D strong solutions can be established.

## §2. Dispersive Equations.

For this type of equation we shall restrict to the case  $p = 2$ .

### (i) The semi-linear Schrödinger equation.

Consider the IVP for the semilinear Schrödinger equation

$$(3.1) \quad \begin{cases} i\partial_t u + \Delta u = F(u, \bar{u}), & t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\ v(x, 0) = v_0(x), \end{cases}$$

where the nonlinearity  $F(\cdot)$  satisfies

$$(3.2) \quad F(\lambda z, \lambda \bar{z}) = \lambda^\alpha F(z, \bar{z}), \quad \alpha > 1.$$

Thus the scaling argument in (2.2)-(2.5) gives  $s_2 = n/2 - 2/(\alpha - 1)$ . The following theorem tells us for which values of the parameters  $n, s, \alpha$  Statements I and II are known.

**Theorem 3.1** ([CzW1],[CzW2],[Ts1],[Ts2],[Cz] ).

(i)-For  $\alpha \geq (n + 4)/n$  and  $s > s_2 = n/2 - 2/(\alpha - 1)$ , with  $[s] < j$  if  $F(\cdot) \in C^j$ ,  $j \in \mathbb{Z}^+$ , the results in Statement I holds in  $H^s(\mathbb{R}^n)$ .

(ii)-For  $\alpha \geq (n+4)/n$ , the results in Statement II hold in  $\dot{H}^{s_2}(\mathbb{R}^n) = \dot{L}^{s_2, 2}(\mathbb{R}^n)$ .

(iii)-For  $\alpha < (n + 4)/n$ , the IVP (3.1) is well posed, in a subcritical manner, in  $L^2(\mathbb{R}^n)$ .

(iv)-For  $\alpha < (n + 2)/(n - 2)$ , if  $n > 2$  the IVP (3.1) is well posed, in a subcritical manner, in  $H^1(\mathbb{R}^n)$ .

The hypothesis  $\alpha \geq (n + 4)/n$  guarantees that  $s_2 = n/2 - 2/(\alpha - 1) \geq 0$ . In other words, the results in Theorem 3.1 require nonnegative Sobolev exponents. Also for a more general statement of Theorem 3.1, (iii)-(iv), with  $s \geq 0$ , we refer to [CzW2].

The proof of Theorem 3.1 is based on the version of the Strichartz estimates [Sc] for the free Schrödinger group  $\{e^{it\Delta}\}_{-\infty}^{\infty}$  found in [GnV13], i.e.

$$(3.3) \quad \left( \int_{-\infty}^{\infty} \|e^{it\Delta} u_0\|_{L^p}^q dt \right)^{1/q} \leq c \|u_0\|_{L^2},$$

with  $2/q = n/2 - n/p$ , and  $2 \leq p < 2n/(n - 2)$ , if  $n \geq 2$ ,  $2 \leq p \leq \infty$  if  $n = 1$ .

In fact, since the estimate (3.3) holds for  $\mathcal{L} = \partial_{x_1}^2 + \dots + \partial_{x_j}^2 - \partial_{x_{j+1}}^2 - \dots - \partial_{x_n}^2$  Theorem 3.1 holds for the IVP (3.1) with  $\mathcal{L}$  instead of the laplacian  $\Delta$ .

Well-posedness results for the IVP (3.1) in Sobolev spaces with negative index seem to depend not only on the order of the nonlinearity  $\alpha$  considered, but also on the structure of the nonlinearity  $F(\cdot)$ . In this regard one has the following results.

**Theorem 3.2** ([KePoVe5],[KePoVe6],[Sa]).

(i)-If  $n = 1$ , and  $F = N_1(u, \bar{u}) = uu$ , or  $F = N_2(u, \bar{u}) = \bar{u}\bar{u}$ , then the IVP (3.1) is locally well posed in  $H^s(\mathbb{R})$  for  $s > -3/4$ .

(ii)-If  $n = 1$ , and  $F = N_3(u, \bar{u}) = u\bar{u}$ , then the IVP (3.1) is locally well posed in  $H^s(\mathbb{R})$  for  $s > -1/4$ .

(iii)-If  $n = 1$ , and  $F = F(u, \bar{u}) = \bar{u}\bar{u}$ , then the IVP (3.1) is locally well posed in  $H^s(\mathbb{R})$  for  $s > -5/12$ .

(iv)-If  $n = 2$ , and  $F = F(u, \bar{u}) = \bar{u}\bar{u}$ , then the IVP (3.1) is locally well posed in  $H^s(\mathbb{R}^2)$  for  $s > -1/2$ .

Following our classification at the end of the introduction we have that the IVP (3.1) with  $\alpha \leq (n+4)/n$  is (A.1), while for  $\alpha < (n+4)/n$  it is (A.3).

More precisely, for  $s_2 = n/2 - 2/(\alpha - 1) < 0$  there is a gap between the result in Theorem 3.2 and that suggested by the scaling. In particular, for the case of the integrable model, i.e. the 1-D cubic Schrödinger equation,  $F = \pm|u|^2u$  (see [ZaSb]), the arguments in [KePoVe5], [KePoVe6] seem to indicate that the  $L^2(\mathbb{R})$  result provided by Theorem 3.1 is the best possible. A proof of it will provide an example where the value suggested for the scaling can not be achieved.

The proof of Theorem 3.2 is based on the relationship between bi-linear and tri-linear forms, and two parameter families of function spaces  $X_{s,b}$  introduced in [Bu]. For  $s, b \in \mathbb{R}$ ,  $X_{s,b}$  denotes the completion of the Schwartz space  $S(\mathbb{R}^{n+1})$  with respect to the norm

$$(3.4) \quad \|F\|_{X_{s,b}} = \left( \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} (1 + |\tau - |\xi|^2|)^{2b} (1 + |\xi|)^{2s} |\widehat{F}(\xi, \tau)|^2 d\xi d\tau \right)^{1/2}.$$

The definition of  $X_{s,b}$  uses the symbol of the operator of the associated linear problem. In particular, since the Fourier transform in space and time of  $e^{it\Delta}u_0$  is supported in the parabola  $\tau - |\xi|^2 = 0$  one has that for  $s \in \mathbb{R}$ ,  $b > 1/2$ ,

$$(3.5) \quad \|\psi(t)e^{it\Delta}u_0\|_{X_{s,b}} \leq c\|u_0\|_{H^s}.$$

In the cases  $N_1, N_2$  in Theorem 3.2 (i) the use of the space  $X_{s,b}$  guarantees that the nonlinear term makes sense, and does not follow from the smoothing effects which only provides a “gain” of  $1/2$  derivative, see [CoSa], [Sj], [Ve]. The work [KePoVe5] was motivated by [KIMa1] on the nonlinear wave equation, see the comments after (4.14).

Next we restrict our attention to a special form of the IVP (3.1)

$$(3.6) \quad \begin{cases} i\partial_t u + \Delta u = \mu|u|^{\alpha-1}u, & t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}, \quad \alpha > 1 \\ u(x, 0) = u_0(x). \end{cases}$$

Solutions of the IVP (3.6) satisfy, at least, two conservation laws

$$(3.7) \quad \|u(\cdot, t)\|_{L^2} = \|u_0\|_{L^2},$$

and

$$(3.8) \quad \int_{\mathbb{R}^n} (|\nabla_x u|^2 + \frac{2\mu}{\alpha+1}|u|^{\alpha+1})(x, t)dx = \|\nabla u_0\|_{L^2}^2 + \frac{2\mu}{\alpha+1}\|u_0\|_{L^{\alpha+1}}^{\alpha+1}.$$

These conservations laws combined with the result in Theorem 3.1 give the following global results

**Theorem 3.3** ([Cz],[GnV11],[Ts1]).

- If  $\alpha < (n+4)/n$ , then the result in Theorem 3.1 (iii) extends globally in time.
- If  $\mu > 0$ , then the result in Theorem 3.1 (iv) extends globally in time.

This global theorem is complemented with the following blow up result.

**Theorem 3.4** ([Gl]).

If  $\mu < 0$ , and  $\alpha \geq 1 + 4/n$ , then there exist  $u_0 \in H^1(\mathbb{R}^n)$  for which the corresponding local solution of the IVP (3.5) provided by Theorem 3.1 blows up in finite time  $T^*$ , i.e.

$$(3.9) \quad \lim_{t \uparrow T^*} \|\nabla u(t)\|_{L^2} = \infty.$$

The IVP (3.5) has the following version of Statement III.

**Theorem 3.5** ([BiKePoSvVe]).

-Statement I fails for the IVP (3.5) with  $s = s_2 = n/2 - 2/(\alpha - 1) \geq 0$  and  $\mu < 0$ .

-Statement II fails for the IVP (3.5) for any  $T > 0$ ,  $\mu < 0$  and  $\eta > C(\mu, \alpha, n)$ .

The proof of Theorem 3.5 for  $s \leq 1$  uses the following blow up result in (3.9). There exists a data  $u_0 \in H^1(\mathbb{R}^n)$  such that the corresponding solution  $u(x, t)$  of the IVP (3.5) blows up in finite time  $T^*$ , defining

$$(3.10) \quad u_\lambda(x, t) = \lambda^{2/(\alpha-1)} u(\lambda x, \lambda^2 t),$$

the solution corresponding to the initial data  $u_\lambda(x, 0) = \lambda^{2/(\alpha-1)} u_0(\lambda x)$ , one has that

$$(3.11) \quad \lim_{\lambda \rightarrow \infty} \|u_\lambda(x, 0)\|_{H^{n/2-2/(\alpha-1)}} = c_0 \neq 0,$$

and  $u_\lambda(x, t)$  has a life span  $T_\lambda$  given by  $T_\lambda = T^*/\lambda^2$ .

The proof of Theorem 3.5 for the case  $0 \leq s \leq 1$  uses the form of the solitary wave solution ( $n = 1$ ), and the ground state solutions ( $n > 1$ ) (see [BrLi] and [Sr]) to contradict the continuous dependence.

Finally, we comment on the case in  $H^1(\mathbb{R}^n)$ ,  $\alpha = (n + 2)/(n - 2)$ ,  $n > 2$  with  $\mu > 0$ . From Theorem 3.1 (ii) and the conservation laws (3.6)-(3.7) we have that the IVP (3.5) has a unique local solution for any  $u_0 \in H^1(\mathbb{R}^n)$ , which extends globally in the case of small data. In fact, in the later case, small  $H^1$  data, if  $u_0 \in H^s(\mathbb{R}^n)$ ,  $s > 1$ ,  $n = 3, 4$ , then the corresponding solution is a global  $H^s$ -solution. A similar result for large  $H^1$  data is unknown.

**(ii) The generalized Benjamin-Ono equation.**

Consider the IVP

$$(3.12) \quad \begin{cases} \partial_t u - \partial_x D_x u = -\partial_x(u^{k+1}), & t, x \in \mathbb{R}, k \in \mathbb{Z}^+ \\ u(x, 0) = u_0(x). \end{cases}$$

The case  $k = 1$  for equation (3.12) was deduced in [Bn] and [O] as a model in internal-wave theory and later was proven to be completely integrable.

For the equation in (3.12) the scaling is similar to that deduced in (2.13) for the case of the generalized Burgers equation, i.e.  $\theta = 1/k$ , and  $s_2 = 1/2 - 1/k$ . However, we may remark that although the scaling argument does not differentiate between the parabolic equation in (2.15) and the dispersive one in (3.12), the arguments and consequently the results for the former one are more complete than those of the later.

The best well posedness results for the IVP (3.12) can be gathered in the following Theorem.

**Theorem 3.6** ([I],[KePoVe3],[Po]).

- (i)-For  $k = 1$  the IVP (3.11) is globally well posed in  $H^s(\mathbb{R})$  with  $s \geq 3/2$ .
- (ii)-For  $k > 2$  the IVP (3.11) is locally well posed in  $H^s(\mathbb{R})$  with  $s > 3/2$ .
- (iii)-For  $s > 1$  if  $k = 2$ ,  $s > 5/6$  if  $k = 3$ , and  $s \geq 3/4$  if  $k \geq 4$  there exists  $\delta = \delta(k) > 0$  such that for any  $u_0 \in H^s(\mathbb{R})$  with  $\|u_0\|_{s,2} \leq \delta$  the IVP (3.11) is locally well posed in  $H^s(\mathbb{R})$ .

Part (i) of Theorem 3.6 uses the form of the conservation laws for the BO equation. They provide an *a priori* estimate of the solution in the  $H^{k/2}$ -norm for  $k \in \mathbb{Z}^+$ .

The result in Theorem 3.6 (ii) has a hyperbolic character in the sense that its proof does not use the dispersive part of the equation in (3.12). In fact, it is based on the same energy method which gives a similar result for the Burgers equation without viscosity, i.e. the equation in (3.12) without the second order term.

On the other hand, the result in Theorem 3.6 (iii) requires a smallness assumption on the data. This is due to the application of the inhomogeneous local smoothing effect in the associated integral problem (3.12). This inhomogeneous smoothing effect can be written as

$$(3.13) \quad \|D_x \int_0^t e^{(t-t')\partial_x D_x} F(\cdot, t') dt' \|_{L_x^\infty L_t^2} \leq c \|F\|_{L_x^1 L_t^2}.$$

In [KePoVe1] (3.13) was used to overcome the loss of one derivative introduced by the nonlinear term when problem (3.12) is written in the corresponding integral form. The estimate (3.13) has to be complemented with those for the maximal function (see [KeRu]), i.e.  $\|e^{t\partial_x D_x} v_0\|_{L_x^2 L_t^\infty([0,T])}$  which can not be made small by taking  $T$  tend to zero. This is the reason for the condition on the size of the data.

There are large gaps between the results in Theorem 3.6 and those suggested by the scaling argument. Thus for all powers  $k$  the IVP (3.12) is (A.3). Thus, for  $s \in ((k-2)/2k, 3/2)$ , with  $k \in \mathbb{Z}^+$ , no well posedness result for large data in  $H^s(\mathbb{R})$  is known. For the case  $s < (k-2)/2k$  one expects that the argument in [BiKePoSvVe] would give an ill posedness result (Statement III). The stability and instability of traveling wave solutions of (3.11) is deduced by using the conservation law

$$(3.14) \quad \Phi(u) = \int_{-\infty}^{\infty} ((D_x^{1/2}u)^2 - c_k u^{k+2})(x, t) dx.$$

However, a complete analysis requires that an *a priori* bound of the  $H^{1/2}$  norm allows control of the strong solution, but this requires a local well posedness result in  $H^s(\mathbb{R})$ , with  $s \leq 1/2$ , which as we remark it is unknown.

**(iii) The generalized Korteweg-de Vries equation.**

Consider the IVP

$$(3.15) \quad \begin{cases} \partial_t u + \partial_x^3 u = -\partial_x(u^{k+1}), & t, x \in \mathbb{R}, \quad k \in \mathbb{Z}^+ \\ u(x, 0) = u_0(x). \end{cases}$$

In this case the scaling argument tells us that  $s_2 = 1/2 - 2/k$ .

In the direction of Statement I-III one finds the following well posedness results.

**Theorem 3.7** ([KePoVe2], [KePoVe4]).

(i)-For  $k \geq 4$  the IVP (3.15) satisfies Statements I, II with  $p = 2$ .

(ii)- For  $s > -3/4$  if  $k = 1$ ,  $s > 1/4$  if  $k = 2$ , and  $s > 1/12$  if  $k = 3$  the IVP (3.15) is locally well posed in  $H^s(\mathbb{R})$ .

Therefore, for  $k = 1, 2, 3$  the IVP (3.15) is of type (A.3), and for  $k = 4, 5, \dots$  is of type (A.1).

**Theorem 3.8** ([BiKePoSvVe]).

(i)-For the IVP (3.15) Statement I fails for  $k \geq 4$ , and  $s = 1/2 - 2/k$ .

(ii)-For the IVP (3.15) with  $k \geq 4$ , and  $T > 0$  Statement II fails for  $\delta_k \geq a_k$ , where  $a_k$  is defined in (3.19).

First we observe that the results in Theorem 3.6 for the IVP (3.14) with  $k = 1, 2$  are consistent with the Miura transformation [Mu] which affirms that if  $v(\cdot)$  solves the modified KdV, i.e.  $k = 2$  in (3.15), then

$$(3.16) \quad u = c_0 \partial_x v + v^2$$

for an appropriate value of the complex constant  $c_0$  solves the KdV equation,  $k = 1$  in (3.15).

The proof of Theorem 3.8 uses the form of the traveling waves solutions of (3.15)

$$(3.17) \quad u_{k,c}(x, t) = \phi_{k,c}(x - ct), \quad c > 0,$$

where

$$(3.18) \quad \phi_{k,c}(x) = \left\{ \frac{(k+2)c}{2} \operatorname{sech}^2 \left( \frac{k}{2} \sqrt{c} x \right) \right\}^{1/k}.$$

By the scaling argument

$$(3.19) \quad \|D_x^{1/2-2/k} \phi_{k,c}\|_2^2 = a_k^2, \quad \text{for any } c > 0.$$

A simple computation shows that

$$(3.20) \quad \lim_{n \rightarrow \infty} \|\phi_{k,n} - \phi_{k,n+1}\|_{H^{s_k}} = 0, \quad s_k = 1/2 - 2/k,$$

and that for any  $t > 0$

$$(3.21) \quad \lim_{n \rightarrow \infty} \|(u_{k,n} - u_{k,n+1})(t)\|_{H^{s_k}} = \sqrt{2} a_k,$$

which contradicts the continuous dependence.

The following result shows that for  $k = 1$  one can not expect to have persistence properties in the critical homogeneous case.

**Theorem 3.9.**

For the IVP (3.15) with  $k = 1$  Statement II fails even locally.

*Proof.*

We shall show that for any  $u_0 \in \dot{H}^{-3/2}(\mathbb{R}) \cap S(\mathbb{R})$ ,  $u_0 \neq 0$  the corresponding solution  $u(\cdot)$  of the IVP (3.15) with  $k = 1$  verifies that

$$(3.22) \quad u(t) \notin \dot{H}^{-3/2}(\mathbb{R}), \quad \text{for any } t \neq 0.$$

First, we observe that if  $f \in \dot{H}^{-3/2}(\mathbb{R}) \cap S(\mathbb{R})$  then  $\widehat{f}(0) = \partial_\xi \widehat{f}(0) = 0$ .

From the results in [Kt2] it follows that for any  $u_0 \in \dot{H}^{-3/2}(\mathbb{R}) \cap S(\mathbb{R})$  the IVP (3.14) with  $k = 1$  has a unique global solution  $u \in C(\mathbb{R} : S(\mathbb{R}))$ .

Now multiplying the equation in (3.15) with  $k = 1$  by  $x$ , and integrating in the space variable we get

$$(3.23) \quad \frac{d}{dt} \int_{-\infty}^{\infty} xu(x, t) dx = \int_{-\infty}^{\infty} u^2(x, t) dx = \|u(t)\|_2 \neq 0.$$

Thus, our solution satisfies

$$(3.24) \quad \int_{-\infty}^{\infty} xu(x, t) dx = c \partial_\xi \widehat{u}(0, t) \neq 0, \quad \text{for any } t \neq 0,$$

which yields the result.

Solutions of (3.15) satisfy, at least, three conservation laws

$$(3.25) \quad I_1(u) = \int_{-\infty}^{\infty} u(x, t) dx, \quad I_2(u) = \int_{-\infty}^{\infty} u^2(x, t) dx,$$

and

$$(3.26) \quad I_3(u) = \int_{-\infty}^{\infty} ((\partial_x u)^2 - c_k u^{k+2})(x, t) dx.$$

From the conservation laws (3.25)-(3.26), and the local results it follows that for  $k = 2, 3$  and any real valued  $u_0 \in H^s(\mathbb{R}) = (1 - \Delta)^{-s/2} L^2(\mathbb{R})$  with  $s \geq 1$  the

IVP (3.15) possesses a unique global solution. For  $k = 1$ ,  $s \geq 0$  suffices, see [Bu]. A similar result for the higher nonlinearities  $k \geq 4$  is only known under smallness assumptions on the data  $u_0$ . In other words, the existence of global strong solution for the IVP (3.15) with  $k \geq 4$  for any  $u_0 \in H^1(\mathbb{R})$  remains open.

The case  $k = 4$  illustrates the difference between the time of existence as a function of the size of the data or as a function of the data itself. Theorem 3.7 (i) guarantees a local solution  $u \in C([0, T] : L^2(\mathbb{R})) \cap \dots$ , with  $T = T(u_0)$ . On the other hand, the conservation law tells us that  $\|u(t)\|_2 = \|u_0\|_2$ ,  $t \in [0, T]$ .

Following our classification at the end of the introduction we have that for  $k = 1, 2, 3$  the IVP (3.15) is (A.3), and for  $k \geq 4$  is (A.1).

Finally, we remark that the known local well posedness results for the IVP (3.15) in [KePoVe2], [KePoVe4] do not reach the values of the Sobolev exponent suggested by the scaling when  $k = 1, 2, 3$ . Although, they are better, for all  $k \in \mathbb{Z}^+$  except for the case  $k = 2$ , than those deduced in [Be], which agrees with the scaling, for  $p = 2$  for the parabolic IVP (2.15).

### §3. Hyperbolic Equations.

As in the case of dispersive equations we shall restrict to the case  $p = 2$ .

#### (i) Nonlinear wave equation.

Consider the IVP of the form

$$(4.1) \quad \begin{cases} \partial_t^2 u - \Delta u = F(u, \nabla_x u, \partial_t u), & t \in \mathbb{R}, x \in \mathbb{R}^n, \\ u(x, 0) = f(x), \\ \partial_t u(x, 0) = g(x), \end{cases}$$

where  $F : \mathbb{R}^{n+2} \rightarrow \mathbb{R}$  satisfies (1.3).

To simplify the exposition we shall restrict ourselves to the case  $n = 3$ , and first consider IVP's of the form

$$(4.2) \quad \begin{cases} \partial_t^2 u - \Delta u = au^k(\nabla_x u, \partial_t u)^\alpha, & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ u(x, 0) = f(x) \in H^s(\mathbb{R}^3), \\ \partial_t u(x, 0) = g(x) \in H^{s-1}(\mathbb{R}^3), \end{cases}$$

where  $a \in \mathbb{R} - \{0\}$ ,  $k \in \mathbb{Z}^+$ ,  $\alpha \in (\mathbb{Z}^+)^4$ .

For this problem the energy method establishes local well posedness for  $s > 5/2$  if  $j \geq 2$ , and for  $s > 3/2$  if  $j = 0, 1$ .

The scaling argument suggests that (4.2) is locally well posed for

$$(4.3) \quad s > s(k; j) = \frac{3k + 5j - 7}{2(k + j - 1)}, \quad \text{with } |\alpha| = j.$$

In this regard we collect the following results concerning local well posedness for the IVP (4.2).

**Theorem 4.1** ([LnSo], [PoSi]).

(i)-If  $j = 0$ , and  $k = 2$ , then the IVP (4.2) is locally well posed in  $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$  for  $s > 0$ .

(ii)-If  $j = 0$ , and  $k \geq 3$ , then the IVP (4.2) is locally well posed in a sub-critical manner, i.e. for  $j = 0$ , and  $k \geq 3$  Statement I holds with  $p = 2$ .

(iii)-If  $j = 0$ , and  $k \geq 4$ , then Statement II holds with  $p = 2$  for the IVP (4.2).

(iv)-If  $j = 1$ , and  $k = 1$ , then the IVP (4.2) is locally well posed in  $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$  for  $s > 1$ .

(v)-If  $j = 2$ , and  $k \in \mathbb{Z}^+$ , then the IVP (4.2) is locally well posed in  $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$  for  $s > 2$ .

(vi)-If  $j \geq 3$ , and  $k = 0$ , then the IVP (4.2) is locally well posed in a sub-critical manner, i.e. for  $j \geq 3$ , and  $k = 0$  Statement I holds with  $p = 2$ .

We observe that in the cases  $(j, k) = (0, 2)$ ,  $(1, 1)$ , and  $j = 2$ ,  $k \in \mathbb{Z}^+$ , Theorem 4.1 (part (i), (iv), (v)) does not reach the values suggested in (4.3) by the scaling argument. The following result shows that this is not a failure. In fact, it shows that in general Theorem 4.1, (i), (iv)-(v) is sharp, except for the limiting cases.

**Theorem 4.2** ([Ln1]).

For  $F_1 = u^2$ ,  $F_2 = u\partial_t u$ ,  $F_3 = (\partial_t u)^2$ , the IVP (4.3) is in general ill posed in  $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$  for  $s < 0$ ,  $s < 1$ ,  $s < 2$  respectively.

The nonlinearities  $F_1, F_2, F_3$  are particular cases of  $(j, k) = (0, 2), (1, 1), (2, 0)$  for which the scaling argument in (4.3) suggests local well posedness for  $s > -1/2, s > 1/2, s > 3/2$  respectively. Thus, in those cases the IVP (4.3) is (A.2).

The ill posedness is due to the fact that the time of existence can not be expressed as a function on the size of the data in  $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ .

Similarly, the sharpness of Theorem 4.1, (ii)-(iv) follows by using the blow up result in [J], [Si] as in (3.10)-(3.11). This also shows that the time of existence can not be expressed as a function of the size of the data in  $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ .

For some values of  $j, k$  one can strengthen this result. For example, it was shown in [Ln2], see also [ShTa], that the IVP

$$(4.4) \quad \begin{cases} \partial_t^2 u - \Delta u = u^3, & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ u(x, 0) = 0, \quad \partial_t u(x, 0) = 0 \end{cases}$$

has a nontrivial solution of the form

$$(4.5) \quad u(x, t) = 2 \frac{H(t - |x|)}{t}, \quad \text{where } H(x) = 1, x > 0, H(x) = 0, x \leq 0.$$

We observe that

$$(4.6) \quad u \in C([0, T] : H^s(\mathbb{R}^3)) \cap L^2([0, T] : L^\infty(\mathbb{R}^3)), \quad \text{for } s < 1/2,$$

with

$$(4.7) \quad \lim_{t \downarrow 0} \|u(t)\|_{H^s} = 0, \quad \text{for } s < 1/2.$$

Since  $j = 0, k = 3$ , the scaling (4.3) tells us that  $s(3; 0) = 1/2$ .

The following examples provide a stronger version of Theorem 4.2 which also shows that, in general, for the IVP (4.2) the well posedness result suggested by scaling cannot always be achieved.

**Theorem 4.3** ([Ln2]).

(i)-For any  $\epsilon > 0$  there exists  $(f, g) \in L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3)$  with

$$(4.8) \quad \|f\|_2 + \|g\|_{\dot{H}^{-1}} < \epsilon$$

for which the IVP (4.2) with  $F = u^2$  does not have distributional solution in  $u \in L^2([0, T] \times \mathbb{R}^3)$ .

(ii)-There exists  $(f, g) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$  for which the IVP (4.2) with  $F = (\mathcal{D}u)^2$ ,  $\mathcal{D} = \partial_{x_1} - \partial_t$ , does not have a weak solution or the uniqueness fails.

The idea in Theorems 4.2-4.3 is to construct explicit solutions that concentrate in one direction. Following [LnSo] we explain the situation in the n-dimensional semilinear case

$$(4.9) \quad \partial_t^2 u - \Delta u = |u|^k, \quad t \in \mathbb{R}, x \in \mathbb{R}^n, k = 2, 3, \dots, R^n.$$

First we look for solutions of (4.9) independent of  $x$ . Solving the o.d.e. one has that

$$(4.10) \quad u(x, t) = \frac{c_k}{t^{2/(k-1)}}$$

for  $t > 0$ . A combination of the Lorentz transformation and scaling provides the family of solutions

$$(4.11) \quad u_{a,\beta}(x, t) = \frac{c_k (1 - \beta^2)^{1/(k-1)}}{(a - (t - \beta x_1))^{2/(k-1)}}, \quad a, \beta \in \mathbb{R}^+,$$

which blows up at  $t - \beta x_1 = a$ . One can cut off the initial data outside  $|x| \geq a$  in a way that  $u_{a,\beta}(x, 0) = f_{a,\beta}(x)$ ,  $\partial_t u_{a,\beta}(x, 0) = g_{a,\beta}(x)$  satisfies

$$(4.12) \quad \|f_{a,\beta}\|_{\dot{H}^s} + \|g_{a,\beta}\|_{\dot{H}^{s-1}} \leq c_k \frac{(1 - \beta)^{(n+1)/4 - 1/(k-1) - s}}{a^{s + 2/(k-1) - n/2}}.$$

Hence, for  $s < (n+1)/4 - 1/(k-1)$  there are sequences  $a_m \rightarrow 0$ ,  $\beta_m \rightarrow 1$ , for which the solutions  $u_{a_m, \beta_m}$  have a lifespan bounded by  $a_m$  and data whose norm tends to zero. Therefore the IVP associated to the equation (4.9) with  $F(u) = |u|^k$  is ill posed for  $s < (n+1)/4 - 1/(k-1)$ . Also from the scaling argument and the blow up results in [J], [Si] one also obtains ill posedness for the same equation for  $s < n/2 - 2/(k-1)$ .

Well posedness results of the kind described in Statements I-II for most of the values  $k$ , and  $s \geq \max\{(n+1)/4 - 1/(k-1) : n/2 - 2/(k-1)\}$  has been established in [LnSo], see also [Ka].

The main ingredient in the proof of the positive results commented on above are the different versions of the Strichartz estimates for the linear wave equation found in [Pe], [Ka].

To complete our study of semilinear wave equations we consider the IVP (4.2) with  $k = 5$ ,  $j = |\alpha| = 0$ . Thus from (4.3)  $s(5, 0) = 1$ , and from Theorem 4.1 part (iii) one yields critical local well posedness for  $(f, g) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ , which extends globally if the data is small enough. On the other hand, for the case  $a = -1$ ,  $k = 5$ ,  $j = |\alpha| = 0$  in (4.2), the local strong solutions satisfy the identity

$$(4.13) \quad E(u(t)) = \int_{\mathbb{R}^3} \left( \frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6} \right) (x, t) dx = E(u(0)),$$

for  $t \in [0, T]$  which allows the construction of global weak solutions.

Thus it does not follow from the previous results that for data with further regularity, for example,  $(f, g) \in C^3(\mathbb{R}^3) \times C^2(\mathbb{R}^3)$ , the corresponding local solution given in Theorem 4.1 part (iii) extends globally or that there exists a weak global solution in  $C^2(\mathbb{R}^3 \times [0, \infty))$ , which will imply that this is unique in this class. This result was established in [St], for radial data, and in [Gr] for the general case, see also [ShSt]. As it was commented at the end of section 3 (i) a similar result for the IVP (3.5) with  $\alpha = (n+2)/(n-2)$ ,  $n > 2$  with  $\mu > 0$  is unknown.

In view of Theorems 4.1-4.3 the following question presents itself : for what kind of nonlinearities does the value of the Sobolev exponent suggested in (4.3) is achieved?

For the case  $j = |\alpha| = 2$ , it was proven in [KlMa1], [KlMa2] that for a special form of the nonlinearity this can be accomplished. More precisely, in [KlMa1] the authors considered IVP's of the form

$$(4.14) \quad \begin{cases} \partial_t^2 u^I - \Delta u^I = N^I(u, \nabla_x u, \partial_t) & t \in \mathbb{R}, \quad x \in \mathbb{R}^3 \\ u^I(x, 0) = f^I(x) \in H^s(\mathbb{R}^3) \\ \partial_t u^I(x, 0) = g^I(x) \in H^{s-1}(\mathbb{R}^3), \end{cases}$$

where  $u = u^I$  is a vector valued function, and the nonlinear terms  $N^I$  have the form

$$(4.15) \quad N^I(u, \nabla u, \partial_t u) = \sum_{j,k} \Gamma_{j,k}^I(u) B_{j,k}^I(\nabla u^j, \partial_t u^j, \nabla u^k, \partial_t u^k),$$

where  $B_{j,k}^I$  is any of the “null forms”:

$$(4.16) \quad \begin{aligned} Q_0(\nabla u^j, \partial_t u^j, \nabla u^k, \partial_t u^k) &= \sum_{i=1}^3 \partial_{x_i} u^j \partial_{x_i} u^k - \partial_t u^j \partial_t u^k \\ Q_{\alpha,\beta}(\nabla u^j, \partial_t u^j, \nabla u^k, \partial_t u^k) &= \partial_{x_\alpha} u^j \partial_{x_\beta} u^k - \partial_{x_\beta} u^j \partial_{x_\alpha} u^k, \end{aligned}$$

with  $0 \leq \alpha \leq \beta \leq 3$ ,  $\partial_{x_0} = \partial_t$ , and  $\Gamma_{j,k}^I(u)$  is a polynomial in  $u$ .

Such equations arise in the study of “wave maps” (for  $Q_0$ ) and Yang-Mills systems in a Coulomb gauge form for the general case.

In [KlMa1] it was shown that the IVP (4.14) is locally well posed for  $s \geq 2$ . This was done by studying the bilinear operators  $Q_0, Q_{\alpha,\beta}$  in (4.16) evaluated at  $(\nabla \phi, \partial_t \phi, \nabla \psi, \partial_t \psi)$  where  $\phi, \psi$  are homogeneous solutions to the linear wave equation, and then extended to the inhomogeneous wave equation via Duhamel’s formula. This result motivated those in [KePoVe5], [KePoVe6], [Ln1], [Ln2], [LnSo], [PoSi]. In this context, the idea of bilinear estimates, also appeared, simultaneous in [Bu].

In this regard one finds the following result.

**Theorem 4.4** ([KlMa2],[Kl]).

*The IVP (4.14) is local well posedness for  $s > 3/2$ , the value suggested by the scaling argument (4.3) with  $j = 2$ .*

The proof of Theorem 4.4 uses some of the ideas in [KePoVe4], to obtain better estimates for solutions of the wave equation with inhomogeneous terms  $Q_0, Q_{\alpha,\beta}$ . They use function spaces somewhat similar to those described in (3.3), but adapted to the wave operator, and allowed to obtain the theorem above.

Let us see that Theorem 4.4 is optimal. Consider the following IPV for a scalar wave equation

$$(4.17) \quad \begin{cases} \partial_t^2 u - \Delta_x u = \sum_{j=1}^3 (\partial_{x_j} u)^2 - (\partial_t u)^2, & t > 0, x \in \mathbb{R}^3 \\ u(x, 0) = 0, \\ \partial_t u(x, 0) = g(x). \end{cases}$$

Since  $w(x, t) = e^{u(x, t)} - 1$  solves the linear wave equation we can write the solution of (4.17) as

$$(4.18) \quad u(x, t) = \log(1 + w(x, t)) = \log\left(1 + \frac{t}{4\pi} \int_{|\omega|=1} g(x + t\omega) dS_\omega\right).$$

A simple calculation shows that the expression in (4.18) makes sense for initial data  $g \in H^s(\mathbb{R}^3)$ , with  $s > 3/2$ , which proves the sharpness of Theorem 4.4.

**ACKNOWLEDGMENT.** *The author takes pleasure in thanking T. Kato, C. E. Kenig, and L. Vega for helpful conversations and suggestions. This work was supported by NSF grant.*

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# TWO-PARAMETER NONLINEAR EIGENVALUE PROBLEMS

TETSUTARO SHIBATA

**Abstract:** We consider the following nonlinear two-parameter eigenvalue problem

$$\begin{aligned}u''(x) + \mu f(u(x)) &= \lambda g(u(x)), \quad x \in I = (0, 1), \\u(0) &= u(1) = 0,\end{aligned}$$

where  $\mu, \lambda \in R$  are parameters. We shall establish asymptotic formulas of the  $n$ -th variational eigencurves  $\lambda = \lambda_n(\mu, \alpha)$  obtained by Ljusternik-Schnirelman theory on general level set.

## 1. Introduction

We consider the following nonlinear two-parameter eigenvalue problem

$$(1.1) \quad \begin{aligned}u''(x) + \mu f(u(x)) &= \lambda g(u(x)), \quad x \in I = (0, 1), \\u(0) &= u(1) = 0,\end{aligned}$$

where  $\mu, \lambda \in R$  are parameters. The purpose of this paper is to study asymptotic behavior of the  $n$ -th variational eigencurves  $\lambda = \lambda_n(\mu, \alpha)$  obtained by Ljusternik-Schnirelman theory on the general level set

$$(1.2) \quad N_{\mu, \alpha} := \left\{ u \in W_0^{1,2}(I) : \int_0^1 \left( \frac{1}{2} u'(x)^2 - \mu F(u(x)) \right) dx = -\alpha \right\},$$

where  $F(u) := \int_0^u f(s) ds$  and  $\alpha > 0$  is a normalizing parameter. The typical examples of the nonlinearities  $f$  and  $g$  are as follows:

### Examples

- (1)  $f(u) = u, \quad g(u) = u + |u|^{p-1}u \quad (p > 1),$
- (2)  $f(u) = u, \quad g(u) = |u|^{p-1}u \quad (p > 1),$

- (3)  $f(u) = |u|^{p-1}u$ ,  $g(u) = |u|^{q-1}u$  ( $1 < q < p < q + 2$ ),  
(4)  $f(u) = |u|^{p-1}u + |u|^{r-1}u$ ,  $g(u) = |u|^{q-1}u + |u|^{s-1}u$  ( $1 < q < p < q + 2, q < s < p < r < s + 2$ ).

The motivation of the problem (1.1) is the following two-parameter linear equation

$$(1.3) \quad u''(x) + \mu r(x)u(x) = \lambda s(x)u(x), \quad x \in I.$$

Many authors have been studying the properties of  $n$ -th eigenvalue  $\lambda = \lambda_n(\mu)$  of (1.3) for given  $\mu \in R$ . One of the main problems there is to investigate asymptotic properties of  $\lambda_n(\mu)$  as  $\mu \rightarrow \infty$ . Our problem (1.1) is a nonlinear version of (1.3). However, there are some differences between (1.1) and (1.3) and one of the most important facts is that, since (1.1) is nonlinear, we need an additional parameter  $\alpha$  for  $\lambda$ , that is,  $\lambda = \lambda_n(\mu, \alpha)$ . We apply the variational method on general level set  $N_{\mu, \alpha}$  due to Zeidler [7] to (1.1) and establish asymptotic formulas of  $\lambda_n(\mu, \alpha)$  as  $\mu \rightarrow \infty, (n\pi)^2, 0, \alpha \rightarrow \infty$ .

## 2. Main Results for $f(u) = u$

We explain notations before stating our results. Let  $X := W_0^{1,2}(I)$  be the usual real Sobolev space. Let

$$\|u\|_X^2 = \int_I |u'(x)|^2 dx, \quad \|u\|_s := \int_I |u(x)|^s dx,$$

$$G(u) := \int_0^u g(s) ds, \quad \Psi(u) := \int_0^1 G(u(x)) dx.$$

For  $n \in N$  and  $\mu, \alpha > 0$ ,  $\lambda_n(\mu, \alpha)$  is called  $n$ -th variational eigenvalue of (1.1) if there exists  $u_n(\mu, \alpha, x) \in X$  satisfying the following conditions (2.1)-(2.3):

$$(2.1) \quad (u_n(\mu, \alpha, x), \mu, \lambda_n(\mu, \alpha)) \in N_{\mu, \alpha} \times R_+ \times R_+ \quad \text{satisfies (1.1).}$$

$$(2.2) \quad \left. \frac{d}{dx} u_n(\mu, \alpha, x) \right|_{x=0} > 0.$$

$$(2.3) \quad \Psi(u_n(\mu, \alpha, x)) = \beta_n(\mu, \alpha) := \inf_{K \in \Lambda_n} \sup_{u \in K} \Psi(u),$$

where

$$\Lambda_n := \{K \subset N_{\mu, \alpha} : K \text{ is compact, } u \in K \text{ implies } -u \in K, 0 \notin K, \gamma(K) \geq n\},$$

$$\gamma(K) := \inf\{k \in N : \text{there exists } h : K \rightarrow R^k \setminus \{0\}, h : \text{continuous and odd}\}.$$

We first introduce the results for the case  $f(u) = u$ ,  $g(u) = u + |u|^{p-1}u$  ( $p > 1$ ). The existence result of the  $n$ -th variational eigenvalue is due to Zeidler [7].

**Theorem 2.1** ([2, Theorem 2.1]). *Let  $\alpha > 0$  be fixed. Then  $\lambda_n(\mu, \alpha)$  is a continuous function of  $\mu > (n\pi)^2$ . Furthermore,*

$$(2.4) \quad \lim_{\mu \rightarrow \infty} \frac{\lambda_n(\mu, \alpha)}{\mu} = 1.$$

Moreover, as  $\mu \rightarrow (n\pi)^2$

$$(2.5) \quad \lambda_n(\mu, \alpha) = C_1(\mu - (n\pi)^2)^{\frac{p+1}{2}} + o((\mu - (n\pi)^2)^{\frac{p+1}{2}}),$$

where  $C_1 := \Gamma\left(\frac{p+3}{2}\right) / \left\{ \sqrt{\pi} 2^p \alpha^{\frac{p-1}{2}} \Gamma\left(\frac{p+2}{2}\right) \right\}$ .

**Theorem 2.2** ([4, Theorem]). *Let  $\mu > (n\pi)^2$  be fixed. Then the following asymptotic formula holds as  $\alpha \rightarrow \infty$ :*

$$(2.6) \quad \lambda_n(\mu, \alpha) = C_\mu \alpha^{\frac{(1-p)}{2}} + o\left(\alpha^{\frac{(1-p)}{2}}\right),$$

where

$$C_\mu = 2 \left(\frac{\mu}{2}\right)^{\frac{(p+1)}{2}} + O\left(\mu^{\frac{p}{2}}\right) \quad \text{as } \mu \rightarrow \infty.$$

Secondly, we introduce the results for the case  $f(u) = u$ ,  $g(u) = |u|^{p-1}u$  ( $p > 1$ ). Since the asymptotic behavior of  $\lambda_n(\mu, \alpha)$  as  $\mu \rightarrow (n\pi)^2$  is the same as that mentioned in Theorem 2.1, we restrict our attention to the case  $\mu \rightarrow \infty$ :

**Theorem 2.3** ([3, Theorem]). *Let  $\alpha > 0$  be fixed. Then the following formula holds as  $\mu \rightarrow \infty$ :*

$$(2.7) \quad \lambda_n(\mu, \alpha) = \frac{\mu^{\frac{p+1}{2}} - O\left(\mu^{\frac{p}{2}}\right)}{(2\alpha)^{\frac{p-1}{2}}}.$$

### 3. Main Results for General Nonlinearities

We begin with the existence result. We assume the following conditions:

(A.1)  $f, g$  are real-valued, odd, increasing function on  $R$ . Furthermore,  $g$  is locally Hölder continuous and  $f$  is locally Lipschitz continuous.

(A.2) Let  $f_1(s) := \frac{f(s)}{s}$ . Then  $f_1(s)$  is a increasing function of  $s \geq 0$  and  $f_1(s) \rightarrow \infty$  as  $s \rightarrow \infty$ .

(A.3)  $\{u \in N_\mu : \Psi(u) < C\} \subset X$  is bounded for any  $C > 0$ .

**Theorem 3.1** ([5, Theorem 2.1]). *Assume (A.1)-(A.3). Then there exists  $\lambda_n(\mu, \alpha)$  for  $\mu, \alpha > 0$  and  $n \in N$ .*

Theorem 3.1 is proved by an application of the existence results of Zeidler [7].

Next, we consider two special cases. At first, let  $f(u) = |u|^{p-1}u$ ,  $g(u) = |u|^{q-1}u$ , where  $1 < q < p < q + 2$ . We consider positive solutions of (1.1). We write  $\lambda(\mu) = \lambda_1(\mu, \alpha)$  and  $u_\mu = u(\mu, \alpha, x)$  for simplicity. Then we have

$$(3.1) \quad \lambda(\mu) = \frac{2\alpha + \frac{p-1}{p+1} \mu \|u_\mu\|_{p+1}^{p+1}}{\|u_\mu\|_{q+1}^{q+1}}.$$

(3.1) is obtained as follows. Multiplying (1.1) by  $u_\mu$  and integration by parts we obtain

$$(3.2) \quad -\|u_\mu\|_X^2 + \mu \|u_\mu\|_{p+1}^{p+1} = \lambda(\mu) \|u_\mu\|_{q+1}^{q+1};$$

this along with the fact that  $u_\mu \in N_{\mu, \alpha}$  implies (3.1).

**Theorem 3.2** ([6, Theorem 1.1]). *There uniquely exists a variational eigenvalue for fixed  $\mu, \alpha > 0$ , that is, if  $(\mu, \lambda_1(\mu), u_{\mu,1})$  and  $(\mu, \lambda_2(\mu), u_{\mu,2})$  satisfy (2.1) - (2.3) for the same  $\mu, \alpha > 0$ , then  $\lambda_1(\mu) = \lambda_2(\mu)$ . Furthermore,  $\lambda(\mu)$  is continuous in  $\mu > 0$  for a fixed  $\alpha > 0$ .*

**Theorem 3.3** ([5, Theorem 2.2]). *Let  $\alpha > 0$  ( $\mu > 0$ ) be fixed. Then the following asymptotic formula holds as  $\mu \rightarrow \infty$  ( $\alpha \rightarrow \infty$ ):*

$$(3.3) \quad \lambda(\mu) = C_2 \frac{2(p-q)}{p+3} \alpha \frac{2(p-q)}{p+3} \mu^{\frac{q+3}{p+3}} + o\left(\mu^{\frac{q+3}{p+3}}\right),$$

where

$$(3.4) \quad C_2 = \left(\frac{q+1}{p+1}\right)^{\frac{q+3}{2(p-q)}} \frac{(p+3)(q+1)(p-q)}{2(2q-p+3)} \sqrt{\frac{2}{\pi(q+1)}} \frac{\Gamma\left(\frac{p+3}{2(p-q)}\right)}{\Gamma\left(\frac{q+3}{2(p-q)}\right)}.$$

**Theorem 3.4** ([6, Theorem 1.2]). *Let  $\alpha > 0$  be fixed. Then as  $\mu \rightarrow 0$ , the following asymptotic formula holds:*

$$(3.5) \quad \lambda(\mu) = C_3 \mu^{(q-1)/(p-1)} + o\left(\mu^{(q-1)/(p-1)}\right),$$

where  $C_3 = \frac{(p-1)\|v_\infty\|_{p+1}^{p+1}}{(p+1)\|v_\infty\|_{q+1}^{q+1}}$  and  $v_\infty$  is a unique positive solution of the minimizing problem

$$(3.6) \quad \text{Minimize } \frac{1}{q+1} \|w\|_{q+1}^{q+1} \text{ under the constraint}$$

$$(3.7) \quad w \in V_0 := \left\{ w \in X : \frac{1}{2} \|w\|_X^2 = \frac{1}{p+1} \|w\|_{p+1}^{p+1}, w \neq 0 \right\}.$$

Finally, we consider Example (4) in Section 1.

**Theorem 3.5.** *Let  $f(u) = |u|^{p-1}u + |u|^{r-1}u$ ,  $g(u) = |u|^{q-1}u + |u|^{s-1}u$  ( $1 < q < p < q+2, q < s < p < r < s+2$ ). Then as  $\mu \rightarrow \infty$  the asymptotic formula (3.3) holds for a fixed  $\alpha > 0$ .*

#### 4. Sketch of the proof of Theorem 3.3 and 3.5

Theorem 3.3 and Theorem 3.5 can be proved by using the same arguments. Hence, we only treat Theorem 3.3 for simplicity. In what follows, we fix  $\alpha > 0$ . Let  $v_\mu = \mu^{\frac{1}{p-1}} u_\mu, \nu_\mu = \lambda \mu^{-\frac{q-1}{p-1}}$ . Then  $v_\mu$  satisfies

$$(4.1) \quad \begin{aligned} -v_\mu''(x) &= v_\mu(x)^p - \nu_\mu v_\mu(x)^q, & x \in I, \\ v_\mu(x) &> 0, & x \in I, \\ v_\mu(0) &= v_\mu(1) = 0. \end{aligned}$$

Furthermore, we put  $\xi_\mu = \nu_\mu^{\frac{p-1}{2(p-q)}}$ ,  $t = \xi_\mu(x - \frac{1}{2})$ ,  $y_\mu(t) = \nu_\mu^{-\frac{1}{p-q}} v_\mu(x)$ . Then it follows from (4.1) that  $y_\mu(t)$  satisfies

$$(4.2) \quad \begin{aligned} -y_\mu''(t) &= y_\mu(t)^p - y_\mu(t)^q, \quad t \in I_\mu := \left(-\frac{1}{2}\xi_\mu, \frac{1}{2}\xi_\mu\right), \\ y_\mu(t) &> 0, \quad t \in I_\mu, \\ y_\mu\left(-\frac{1}{2}\xi_\mu\right) &= y_\mu\left(\frac{1}{2}\xi_\mu\right) = 0. \end{aligned}$$

In connection with (4.2), we consider the nonlinear scalar field equation:

$$(4.3) \quad \begin{aligned} y''(t) + y(t)^p &= y(t)^q, \quad t \in R, \\ y(t) &> 0, \quad t \in R, \\ \lim_{t \rightarrow \pm\infty} y(t) &= 0. \end{aligned}$$

Since  $1 < q < p$ , we can apply Berestycki and Lions [1] to (4.3) and obtain that there uniquely exists a solution  $y$  of (4.3), which is called the *ground state* of (4.3). The key lemmas to prove our theorems are as follows:

**Lemma 4.1.**  $y_\mu \rightarrow y$  in  $L^{q+1}(R)$  as  $\mu \rightarrow \infty$ .

**Lemma 4.2** ([5, Lemma 4.6]). *Let  $y$  be the ground state of (4.3). Then*

$$(4.4) \quad \int_{-\infty}^{\infty} y(x)^{q+1} dx = \frac{2}{p-q} \sqrt{\frac{\pi(q+1)}{2}} \zeta_\infty^{\frac{q+3}{2}} \frac{\Gamma\left(\frac{q+3}{2(p-q)}\right)}{\Gamma\left(\frac{p+3}{2(p-q)}\right)}.$$

**Proof.** We see from [1] that for  $x > 0$

$$(4.5) \quad y'(x) = -y(x) \sqrt{\frac{2}{q+1} y(x)^{q-1} - \frac{2}{p+1} y(x)^{p-1}}.$$

Put  $s = y(x)$ ,  $t = \zeta_\infty^{-1} s$  and  $\sin \theta = t^{\frac{p-q}{2}}$  to obtain

$$(4.6) \quad \begin{aligned} \int_0^\infty y(x)^{q+1} dx &= \int_0^\infty y(x)^q \frac{-y'(x)}{\sqrt{\frac{2}{q+1} y(x)^{q-1} - \frac{2}{p+1} y(x)^{p-1}}} dx \\ &= \sqrt{\frac{q+1}{2}} \int_0^{\zeta_\infty} \frac{s^{\frac{q+1}{2}}}{\sqrt{1 - \frac{q+1}{p+1} s^{p-q}}} ds = \sqrt{\frac{q+1}{2}} \zeta_\infty^{\frac{q+3}{2}} \int_0^1 \frac{t^{\frac{q+1}{2}}}{\sqrt{1 - t^{p-q}}} dt \\ &= \frac{2}{p-q} \sqrt{\frac{q+1}{2}} \zeta_\infty^{\frac{q+3}{2}} \int_0^{\frac{\pi}{2}} \sin^{\frac{2q-p+3}{p-q}} \theta d\theta \\ &= \frac{1}{p-q} \sqrt{\frac{\pi(q+1)}{2}} \zeta_\infty^{\frac{q+3}{2}} \frac{\Gamma\left(\frac{q+3}{2(p-q)}\right)}{\Gamma\left(\frac{p+3}{2(p-q)}\right)}. \end{aligned}$$

□

By combining (3.1) with these two lemmas, we can obtain Theorem 3.3 and Theorem 3.5.

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**3rd step.** Using a solution  $\mathbf{w}$  of (2.3) and putting  $\mathbf{v}(t, x) = \mathbf{w}(x) + \mathbf{z}(t, x)$ , then (2.1) is reduced to the problem of finding  $\mathbf{z}(t, x)$  satisfying the following equation:

$$(2.4) \quad \begin{aligned} \partial_t \mathbf{z} - \Delta \mathbf{z} + (\mathbf{u}_\infty \cdot \nabla) \mathbf{z} + \mathcal{L}[\mathbf{w}] \mathbf{z} + \mathcal{N}[\mathbf{z}] + \nabla p &= \mathbf{0}, & \nabla \cdot \mathbf{z} &= 0 & \text{in } \Omega_\infty, \\ \mathbf{z} &= \mathbf{0} & & & \text{on } \partial \Omega_\infty, \\ \mathbf{z}(0, x) &= \mathbf{b}(x) & & & \text{in } \Omega, \end{aligned}$$

where  $\mathcal{L}[\mathbf{w}] \mathbf{z} = (\mathbf{w} \cdot \nabla) \mathbf{z} + (\mathbf{z} \cdot \nabla) \mathbf{w}$ ,  $\mathcal{N}[\mathbf{z}] = (\mathbf{z} \cdot \nabla) \mathbf{z}$  and  $\mathbf{b}(x) = \mathbf{a}(x) - \mathbf{u}_\infty - \mathbf{w}(x)$ . Instead of (2.4), employing the idea due to Kato [16] I solved the corresponding integral equation:

$$(2.5) \quad \mathbf{z}(t) = T_{\mathbf{u}_\infty}(t) \mathbf{b} - \int_0^t T_{\mathbf{u}_\infty}(t-s) \mathbb{P}[\mathcal{L}[\mathbf{w}] \mathbf{z}(s) + \mathcal{N}[\mathbf{z}(s)]] ds, \quad \forall t > 0.$$

And then, the following theorem is my main result obtained in Shibata [22] concerning a unique existence of solutions to (1.1) or (2.1) globally in time in the  $L_3$  framework.

**Theorem 3.** *Let  $3 < p < \infty$ ,  $0 < \beta < 1$  and  $0 < \kappa < \beta/3$  be given numbers. Let  $\mathbf{a}(x)$  and  $\mathbf{b}(x)$  be the same as in (1.1) and (2.4), respectively. Then, there exists a constant  $\epsilon > 0$  depending on  $p$ ,  $\beta$  and  $\kappa$  such that if  $0 < |\mathbf{u}_\infty| \leq \epsilon$ ,  $\mathbf{b} \in \mathbb{J}_3(\Omega)$  and  $\|\mathbf{a} - \mathbf{u}_\infty\|_3 \leq |\mathbf{u}_\infty|^\beta$ , then (2.5) admits a unique solution  $\mathbf{z}(t, x) \in \mathcal{B}([0, \infty); \mathbb{J}_3(\Omega)) \cap C^0((0, \infty); L_p(\Omega) \cap \dot{W}_3^1(\Omega))$  possessing the following properties:*

$$\begin{aligned} [\mathbf{z}]_{3,0,t} + [\mathbf{z}]_{p,\mu(p)/2,t} + [\nabla \mathbf{z}]_{3,1/2,t} &\leq |\mathbf{u}_\infty|^{\beta-2\kappa}, \quad \mu(p) = 1 - 3/p, \\ \lim_{t \rightarrow 0^+} (\|\mathbf{z}(t, \cdot) - \mathbf{b}\|_3 + [\mathbf{z}]_{p,\mu(p)/2,t} + [\nabla \mathbf{z}]_{3,1/2,t}) &= 0. \end{aligned}$$

Here and hereafter, we put  $[v]_{p,\rho,t} = \sup_{0 < s < t} s^\rho \|v(s, \cdot)\|_p$ .

### 3. Comments on proofs

**3.1 A proof of Theorem 3.** In order to prove Theorem 3, I used Theorems 1 and 2. The most important point is the estimation of the linear term:

$$L[\mathbf{z}] = \int_0^t T_{\mathbf{u}_\infty}(t-s) \mathcal{L}[\mathbf{w}] \mathbf{z}(s, \cdot) ds = \int_0^t T_{\mathbf{u}_\infty}(t-s) \mathbb{P}[(\mathbf{w} \cdot \nabla) \mathbf{z}(s, \cdot) + (\mathbf{z}(s, \cdot) \cdot \nabla) \mathbf{w}] ds.$$

To handle with the term:  $(\mathbf{z}(s, \cdot) \cdot \nabla) \mathbf{w}$  I used the following generalized Poincaré's inequality.

**Lemma 1.** *Let  $0 \leq \alpha < 1/3$  and put  $d_\alpha(x) = s(x)^\alpha |x|^{1-\alpha} \log |x|$ . Then, there exists a constant  $C_\alpha$  such that*

$$\|v/d_\alpha\|_3 \leq C_\alpha \|\nabla v\|_3 \quad \forall v \in \dot{W}_3^1(\Omega).$$

# AN EXTERIOR INITIAL BOUNDARY VALUE PROBLEM FOR THE NAVIER-STOKES EQUATION

YOSHIHIRO SHIBATA

**Abstract.** In this note, we study a unique existence of strong solutions globally in time to the Navier-Stokes equation in a three dimensional exterior domain with non-zero constant speed of fluid at the far field. The asymptotic behaviour of solutions is also addressed.

## 1. Introduction

The motion of nonstationary flow of an incompressible viscous fluid past an isolated rigid body is formulated by the following initial boundary value problem of the Navier-Stokes equation:

$$\begin{aligned}
 (1.1) \quad & \partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \mathfrak{p} = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = 0 && \text{in } \Omega_T, \\
 & \mathbf{u} = \mathbf{0} && \text{on } \partial\Omega_T, \\
 & \mathbf{u}(0, x) = \mathbf{a}(x) && \text{in } \Omega, \\
 & \lim_{|x| \rightarrow \infty} \mathbf{u}(t, x) = \mathbf{u}_\infty && \forall t \in (0, T),
 \end{aligned}$$

where  $\mathbf{0} = {}^T(0, 0, 0)$ ,  ${}^T M$  means the transposed  $M$  and for simplicity, the mass density and the viscosity coefficient are assumed to be 1. Throughout the paper  $\Omega$  represents a spatial region filled with fluid and is taken to be a domain of  $\mathbb{R}^3$  having the  $C^\infty$  boundary  $\partial\Omega$ , with a bounded complement  $\mathcal{O} = \mathbb{R}^3 - \Omega$  which is corresponding to the obstacle. Spatial points are denoted by  $x = (x_1, x_2, x_3)$  and the time variable by  $t$ . Space-time cylinder  $(0, T) \times \Omega$  is denoted by  $\Omega_T$ , and also  $(0, T) \times \partial\Omega = \partial\Omega_T$ . My problem is to find a three dimensional row vector of functions  $\mathbf{u}(t, x) = {}^T(u_1(t, x), u_2(t, x), u_3(t, x))$  which represents the velocity vector of the fluid and a scalar function  $\mathfrak{p}(t, x)$  which represents

the pressure.  $\mathbf{u}_\infty = T(u_{\infty 1}, u_{\infty 2}, u_{\infty 3})$  is a given constant vector which represents a velocity of the fluid at infinity.

After works due to Finn [6–10] ( cf. Galdi [12] and references therein ) concerning the asymptotic behaviour of the stationary problem of the Navier–Stokes equation, J. G. Heywood [13, 14] treated the stability globally in time of Finn’s physically reasonable solutions in the  $L_2$  framework (further works in the same line as in Hoywood [13, 14] were done by Masuda [19] and references therein). Concerning the unique existence of strong solutions to the Navier–Stokes equation and their asymptotic behaviour in  $L_3$  framework, the case when  $\mathbf{u}_\infty = \mathbf{0}$  has been investigated very well for small solutions. In fact, Kato [16] proved the unique existence of small strong solutions of (1.1) globally in time in the  $L_n$  framework when  $\Omega = \mathbb{R}^n$  and their properties of time decay. He employed various  $L_p$  norms and  $L_p$ – $L_q$  estimates for the evolutions of the Stokes operator. Iwashita [15] extended Kato’s work [16] to the case where  $\mathcal{O}$  is non-empty, that is he showed the  $L_p$ – $L_q$  estimates for the semigroup of the Stokes operator in  $\Omega$  with Dirichlet zero condition and proved the unique existence of small strong solutions of (1.1) globally in time in the  $L_n$  framework ( $n \geq 3$ ). Moreover, Borchers and Miyakawa [3] and Kozono and Yamazaki [18] proved the stability of small stationary solutions in the strong sense. But, the case when  $\mathbf{u}_\infty \neq \mathbf{0}$  has not studied well compared with the case when  $\mathbf{u}_\infty = \mathbf{0}$ , although the importance of the mathematical investigation of so called wake region is recognized very well. An unique existence of even small strong solutions of (1.1) globally in time in the  $L_3$  framework has been remained open for long time.

In the early of 1995, Shibata [22] proved that the problem (1.1) admits a unique strong solution globally in time when  $\mathbf{u}_\infty \neq \mathbf{0}$  but  $|\mathbf{u}_\infty|$  is very small,  $\nabla \cdot \mathbf{a} = 0$  in  $\Omega$  and the  $L_3$  norm of  $\mathbf{a} - \mathbf{u}_\infty$  is also very small. But, the smallness assumption on  $\mathbf{a} - \mathbf{u}_\infty$  depends on  $\mathbf{u}_\infty$ . More precisely if  $\mathbf{u}_\infty$  tends to zero, then the admissible initial datum is only zero vector, although Iwashita [15] proved the global existence theorem for small  $L_n$  initial data. It is the reason that the  $L_p$ – $L_q$  estimate of solutions to the linear Oseen equation obtained in [17] was not uniform with respect to  $\mathbf{u}_\infty$ . In the final section of this note, I shall give a lemma concerning the uniform estimate with respect to  $\mathbf{u}_\infty$  of the fundamental solutions of the Oseen equation and this lemma will enable me to improve theorems obtained in [17] regarding the uniformity with respect to  $\mathbf{u}_\infty$ , and hence the smallness assumption of  $\mathbf{a} - \mathbf{u}_\infty$  will be independent of  $\mathbf{u}_\infty$ . Nevertheless, I think that the results obtained in [17] and [22] made great progress in the study of asymptotic behaviour of solutions to the 3 dimensional Navier–Stokes equation, so that I believe that it is worth while explaining several results in [17] and [22] below, although the estimation is lacking in the uniformity with respect to  $\mathbf{u}_\infty$ .

Before going into detail, I would like to explain the notation used throughout this note. Three dimensional row vector-valued functions are denoted with bold-faced letter, for example,  $\mathbf{u} = T(u_1, u_2, u_3)$ .  $(a_{ij})$  means the  $3 \times 3$  matrix whose  $i^{\text{th}}$  column and  $j^{\text{th}}$  row component is  $a_{ij}$ . As usual, we put  $\partial_t = \partial/\partial t$ ,  $\partial_j = \partial/\partial x_j$ ,  $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ ,  $\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ ,  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  and  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ . For three dimensional row vector-valued functions  $\mathbf{u} = T(u_1, u_2, u_3)$ ,  $\mathbf{v} = T(v_1, v_2, v_3)$  and a scalar-valued function  $u$  we put  $\partial_x^m u = (\partial_x^\alpha u, |\alpha| = m)$ ,  $\bar{\partial}_x^m u = (\partial_x^\alpha u, |\alpha| \leq m)$ ,  $\partial_t^j \partial_x^\alpha \mathbf{u} = T(\partial_t^j \partial_x^\alpha u_1, \partial_t^j \partial_x^\alpha u_2, \partial_t^j \partial_x^\alpha u_3)$ ,  $\partial_x^m \mathbf{u} = (\partial_x^\alpha \mathbf{u}, |\alpha| = m)$ ,  $\bar{\partial}_x^m \mathbf{u} = (\partial_x^\alpha \mathbf{u}, |\alpha| \leq m)$ ,  $\nabla \mathbf{u} = (\partial_j u_i)$ ,

$\Delta \mathbf{u} = {}^T(\Delta u_1, \Delta u_2, \Delta u_3)$ ,  $(\mathbf{u} \cdot \nabla) \mathbf{v} = {}^T(\sum_{j=1}^3 u_j \partial_j v_1, \sum_{j=1}^3 u_j \partial_j v_2, \sum_{j=1}^3 u_j \partial_j v_3)$ ,  $\nabla \cdot \mathbf{u} = \sum_{j=1}^3 \partial_j u_j$ ,  $\nabla u = {}^T(\partial_1 u, \partial_2 u, \partial_3 u)$  and  $\nabla u : \nabla \mathbf{v} = {}^T(\nabla u \cdot \nabla v_1, \nabla u \cdot \nabla v_2, \nabla u \cdot \nabla v_3)$ . Put  $B_b = \{x \in \mathbb{R}^3 \mid |x| \leq b\}$  and  $\Omega_b = \Omega \cap B_b$ . Sobolev spaces of vector-valued functions are used, as well as of scalar-valued functions. If  $D$  is any domain in  $\mathbb{R}^3$ ,  $L_p(D)$  denotes the usual  $L_p$  space of scalar-functions on  $D$  and  $\|\cdot\|_{p,D}$  its usual norm. Moreover, we put  $\|u\|_{p,m,D} = \|\bar{\partial}_x^m u\|_{p,D}$ . For vector valued functions, we shall use the same symbols. For simplicity, we shall use the following abbreviation:  $(\cdot, \cdot) = (\cdot, \cdot)_\Omega$ ,  $\|\cdot\|_p = \|\cdot\|_{p,\Omega}$ ,  $\|\cdot\|_{p,m} = \|\cdot\|_{p,m,\Omega}$ ,  $|\cdot|_p = \|\cdot\|_{p,\mathbb{R}^3}$ ,  $|\cdot|_{p,m} = \|\cdot\|_{p,m,\mathbb{R}^3}$ .  $\mathcal{D}'$  denotes the set of all distributions on  $\mathbb{R}^3$ ,  $\mathcal{S}'$  the set of all tempered distributions on  $\mathbb{R}^3$  and  $C_0^\infty(D)$  the set of all functions of  $C^\infty(\mathbb{R}^3)$  whose support is contained in  $D$ . Moreover, we put  $L_{p,b}(D) = \{u \in L_p(D) \mid u(x) = 0 \ \forall x \notin B_b\}$ ,  $W_{p,loc}^m(\mathbb{R}^3) = \{u \in \mathcal{S}' \mid \partial_x^\alpha u \in L_p(B_b) \ \forall \alpha : |\alpha| \leq m \text{ and } \forall b > 0\}$ ,  $W_{p,loc}^m(D) = \{u \mid \exists U \in W_{p,loc}^m(\mathbb{R}^3) \text{ such that } u = U \text{ on } D\}$ ,  $L_{p,loc}(D) = W_{p,loc}^0(D)$ ,  $W_p^m(D) = \{u \in W_{p,loc}^m(D) \mid \|u\|_{p,m,D} < \infty\}$ ,  $\dot{W}_p^m(D) = \text{the completion of } C_0^\infty(D) \text{ with respect to } \|\cdot\|_{p,m,D}$ ,  $\dot{W}_{p,a}^m(D) = \{u \in \dot{W}_p^m(D) \mid \int_D u(x) dx = 0\}$  and  $\hat{W}_p^m(D) = \{u \in W_{p,loc}^m(D) \mid \|\partial_x^m u\|_{p,D} < \infty\}$ . To denote function spaces of vector-valued functions, we use the blackboard bold letters. For example,  $\mathbb{L}_q(D) = \{\mathbf{u} = {}^T(u_1, u_2, u_3) \mid u_j \in L_q(D), j = 1, 2, 3\}$ . Likewise for  $\mathbb{C}_0^\infty(D)$ ,  $\mathbb{L}_{p,b}(D)$ ,  $\mathbb{W}_{p,loc}^m(D)$ ,  $\mathbb{L}_{p,loc}(D)$ ,  $\mathbb{W}_p^m(D)$ ,  $\dot{\mathbb{W}}_p^m(D)$  and  $\hat{\mathbb{W}}_p^m(D)$ . Moreover, we put  $\mathbb{J}_p(D) = \text{the completion in } \mathbb{L}_p(D) \text{ of the set } \{\mathbf{u} \in \mathbb{C}_0^\infty(D) \mid \nabla \cdot \mathbf{u} = 0 \text{ in } D\}$  and  $\mathbb{G}_p(D) = \{\nabla p \mid p \in \hat{W}_p^1(D)\}$ . According to Fujiwara and Morimoto [11] and Miyakawa [20] (cf. also [12]), the Banach space  $\mathbb{L}_p(D)$  admits the Helmholtz decomposition:  $\mathbb{L}_p(D) = \mathbb{J}_p(D) \oplus \mathbb{G}_p(D)$ , where  $\oplus$  denotes the direct sum. Let  $\mathbb{P}_D$  be a continuous projection from  $\mathbb{L}_p(D)$  onto  $\mathbb{J}_p(D)$ . The Stokes operator  $\mathbb{A}_D$  and the Oseen operator  $\mathbb{O}_D(\mathbf{u}_\infty)$  are defined by the relations:  $\mathbb{A}_D = -\mathbb{P}_D \Delta$  and  $\mathbb{O}_D(\mathbf{u}_\infty) = \mathbb{A}_D + \mathbb{P}_D(\mathbf{u}_\infty \cdot \nabla)$  with the same domain:  $\mathcal{D}_p(\mathbb{A}_D) = \mathcal{D}_p(\mathbb{O}_D(\mathbf{u}_\infty)) = \mathbb{J}_p(D) \cap \dot{W}_p^1(D) \cap W_p^2(D)$ . For simplicity, we write:  $\mathbb{P} = \mathbb{P}_\Omega$ ,  $\mathbb{A} = \mathbb{A}_\Omega$  and  $\mathbb{O}(\mathbf{u}_\infty) = \mathbb{O}_\Omega(\mathbf{u}_\infty)$ . To denote various constants we use the same letter  $C$  and  $C_{A,B,\dots}$  denotes the constant depending on the quantities  $A, B, \dots$ .  $C$  and  $C_{A,B,\dots}$  will change from line to line. For two Banach spaces  $X$  and  $Y$ ,  $\mathcal{L}(X, Y)$  denotes the set of all bounded linear operators from  $X$  into  $Y$  with norm  $\|\cdot\|_{\mathcal{L}(X, Y)}$ ,  $\mathcal{B}(I; X)$  the set of all  $X$ -valued bounded continuous functions on  $I$  and  $\mathcal{C}(I; X)$  the set of all  $X$ -valued continuous functions on  $I$ .

## 2. Results obtained in Kobayashi and Shibata [17] and Shibata [22]

To solve (1.1) I put  $\mathbf{u} = \mathbf{u}_\infty + \mathbf{v}$  and then the equation for  $\mathbf{v}$  is the following Oseen equation:

$$\begin{aligned}
 (2.1) \quad & \partial_t \mathbf{v} - \Delta \mathbf{v} + (\mathbf{u}_\infty \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = \mathbf{0}, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_T, \\
 & \mathbf{v} = -\mathbf{u}_\infty \quad \text{on } \partial\Omega_T, \\
 & \mathbf{v}(0, x) = \mathbf{a}(x) - \mathbf{u}_\infty, \quad \text{in } \Omega,
 \end{aligned}$$

One interprets that the last condition in (1.1) is satisfied by finding solutions to (2.1) in the suitable Sobolev spaces, so that below instead of (1.1), I will treat (1.2).

**1st step.** First, I shall consider the following linear Oseen equation with zero boundary condition:

$$(2.2) \quad \begin{aligned} \partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{0}, & \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega^\infty, \\ \mathbf{u} &= \mathbf{0} & & & \text{in } \partial\Omega^\infty, \\ \mathbf{u}(0, x) &= \mathbf{a}(x) & & & \text{in } \Omega. \end{aligned}$$

Miyakawa [20] proved that  $\mathcal{O}(\mathbf{u}_\infty)$  generates an analytic semigroup  $T_{\mathbf{u}_\infty}(t)$  on  $\mathbb{J}_q(\Omega)$  for any  $1 < q < \infty$ . Kobayashi and Shibata [17] proved the following  $L_p$ - $L_q$  estimate for  $T_{\mathbf{u}_\infty}(t)$ .

**Theorem 1.** (1) Let  $1 < p \leq q < \infty$  and let  $\kappa > 0$  be any small number. Then, there exists a constant  $\sigma_0 : 0 < \sigma_0 \leq 1$  depending on  $p$  but independent of  $\kappa$ ,  $\mathbf{u}_\infty$  and  $q$  such that

$$\|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_q \leq C_{p,q,\kappa} |\mathbf{u}_\infty|^{-\kappa} t^{-\nu} \|\mathbf{a}\|_p, \quad \forall t > 0, \quad \forall \mathbf{a} \in \mathbb{J}_p(\Omega), \quad \nu = \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right),$$

provided that  $0 < |\mathbf{u}_\infty| \leq \sigma_0$ .

(2) In addition, we assume that  $1 < p \leq q \leq 3$ . Then,

$$\|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_q \leq C_{p,q,\kappa} |\mathbf{u}_\infty|^{-\kappa} t^{-(\nu+1/2)} \|\mathbf{a}\|_p, \quad \forall t > 0, \quad \forall \mathbf{a} \in \mathbb{J}_p(\Omega),$$

provided that  $0 < |\mathbf{u}_\infty| \leq \sigma_0$ .

**2nd step.** In order to treat the boundary data  $-\mathbf{u}_\infty$  in (2.1), I solved the following stationary problem:

$$(2.3) \quad -\Delta \mathbf{w} + (\mathbf{u}_\infty \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{w} + \nabla p = \mathbf{0}, \quad \nabla \cdot \mathbf{w} = 0 \text{ in } \Omega \text{ and } \mathbf{w} = -\mathbf{u}_\infty \text{ on } \partial\Omega.$$

Concerning the solvability and estimation of solutions, Shibata [22] proved the following theorem.

**Theorem 2.** Let  $3 < p < \infty$  and let  $\delta$  and  $\beta$  be any numbers such that  $0 < \delta < 1/4$  and  $0 < \delta < \beta < 1 - \delta$ . Then, there exists a constant  $\sigma_1 : 0 < \sigma_1 \leq 1$  depending on  $p$ ,  $\delta$  and  $\beta$  but independent of  $\mathbf{u}_\infty$  such that if  $0 < |\mathbf{u}_\infty| \leq \sigma_1$ , then (2.3) admits solutions  $\mathbf{w} \in \mathbb{W}_p^2(\Omega)$  and  $p \in W_p^1(\Omega)$  possessing the estimate:

$$\|\mathbf{w}\|_{p,2} + \|\mathbf{w}\|_\delta + \|p\|_{p,1} \leq |\mathbf{u}_\infty|^\beta.$$

Here and hereafter, we put  $s(\mathbf{u}_\infty)(x) = |x| - \mathbf{u}_\infty \cdot x / |\mathbf{u}_\infty|$  and

$$\begin{aligned} \|\mathbf{w}\|_\delta &= \sup_{x \in \Omega} (1 + |x|)(1 + s(\mathbf{u}_\infty)(x))^\delta |\mathbf{w}(x)| \\ &\quad + \sup_{x \in \Omega} (1 + |x|)^{3/2} (1 + s(\mathbf{u}_\infty)(x))^{1/2+\delta} |\nabla \mathbf{w}(x)|. \end{aligned}$$

In fact, since by Theorem 2

$$\begin{aligned} |\mathbf{w}(x)| &\leq (1 + |x|)^{-1} (1 + s(\mathbf{u}_\infty)(x))^{-\delta} |\mathbf{u}_\infty|^\beta, \\ |\nabla \mathbf{w}(x)| &\leq (1 + |x|)^{-3/2} (1 + s(\mathbf{u}_\infty)(x))^{-(1/2+\delta)} |\mathbf{u}_\infty|^\beta, \end{aligned}$$

putting  $\gamma = 3\delta/4$  and  $\epsilon = 1/(1 + \gamma)$ , we have

$$\begin{aligned} \|(\mathbf{w} \cdot \nabla) \mathbf{z}(s, \cdot)\|_{3/(2+\gamma)} &\leq \|\mathbf{w}\|_{3/(1+\gamma)} \|\nabla \mathbf{z}\|_3 \\ &\leq |\mathbf{u}_\infty|^\beta \|\nabla \mathbf{z}(s, \cdot)\|_3 \left( \int_\Omega [(1 + |x|)^{-1} s(\mathbf{u}_\infty)(x)^{-\delta}]^{3\epsilon} dx \right)^{1/3\epsilon} \leq C |\mathbf{u}_\infty|^\beta \|\nabla \mathbf{z}(s, \cdot)\|_3, \\ \|(\mathbf{z}(s, \cdot) \cdot \nabla) \mathbf{w}\|_{3/(2+\gamma)} &\leq \|\mathbf{z}(s, \cdot)/d_\alpha\|_3 \|d_\alpha \nabla \mathbf{w}\|_{3/(1+\gamma)} \leq C |\mathbf{u}_\infty|^\beta \|\nabla \mathbf{z}(s, \cdot)\|_3. \end{aligned}$$

Therefore, we have  $[L[\mathbf{z}]]_t \leq C |\mathbf{u}_\infty|^\beta [\mathbf{z}]_t$ , where  $[\mathbf{z}]_t = [\mathbf{z}]_{3,0,t} + [\nabla \mathbf{z}]_{3,1/2,t} + [\mathbf{z}]_{p,\mu(p)/2,t}$ . Since the nonlinear term is quadratic, by Theorem 1 we have  $[\mathbf{z}]_t \leq C |\mathbf{u}_\infty|^{-\kappa} (\|\mathbf{b}\|_3 + |\mathbf{u}_\infty|^\beta [\mathbf{z}]_t + [\mathbf{z}]_t^2)$ . If we choose  $|\mathbf{u}_\infty|$  so small that  $C |\mathbf{u}_\infty|^\beta \leq 1/2$ , we have  $[\mathbf{z}]_t \leq 2C |\mathbf{u}_\infty|^{-\kappa} (\|\mathbf{b}\|_3 + [\mathbf{z}]_t^2)$ , which immediately implies Theorem 3.

**Remark 1.** The estimate obtained by Theorem 1 is not uniform with respect to  $|\mathbf{u}_\infty|$ , so that I have to choose  $\|\mathbf{b}\|_3 |\mathbf{u}_\infty|^\kappa$  so small. This is the reason why the smallness assumption on  $\mathbf{a} - \mathbf{u}_\infty$  depends on  $\mathbf{u}_\infty$ . If the estimate in Theorem 1 is uniform with respect to  $\mathbf{u}_\infty$  (that is we can take  $\kappa = 0$  in Theorem 1), then the smallness assumption on  $\|\mathbf{b}\|_3$  is independent of  $\mathbf{u}_\infty$ , and hence so is that on  $\|\mathbf{a} - \mathbf{u}_\infty\|_3$ .

**3.2 A proof of Theorem 2.** In order to solve the exterior problem, in [17, 22] we considered it as the perturbation from the whole space. Namely, in principle the solution  $\mathbf{u}_{ext}$  to the exterior problem is represented in the form:

$$\mathbf{u}_{ext} = \varphi \mathbf{u}_{\mathbb{R}^3} + (1 - \varphi) \mathbf{u}_{int} + \mathbf{v},$$

where  $\varphi$  is a smooth function which is identically equal to 1 for  $|x| \geq R$  with sufficient large  $R$  and 0 in the neighborhood of  $\mathcal{O}$ ,  $\mathbf{u}_{\mathbb{R}^3}$  and  $\mathbf{u}_{int}$  are solutions to the corresponding problem in  $\mathbb{R}^3$  and in some bounded domain containing  $\mathcal{O}$ , respectively, and moreover  $\mathbf{v}$  is a compensating function to keep the divergence free condition:  $\nabla \cdot \mathbf{u}_{ext} = 0$ . The existence of such  $\mathbf{v}$  is guaranteed by the following theorem due to Bogovskii [1, 2] (cf. also Galdi [12]).

**Bogovskii's lemma.** *Let  $D$  be a bounded domain with smooth boundary. Let  $1 < p < \infty$  and let  $m$  be an integer  $\geq 0$ . Then, there exists a linear bounded operator  $\mathbf{B} : \dot{W}_{p,a}^m(D) \rightarrow \dot{W}_p^{m+1}(D)$  such that*

$$\nabla \cdot \mathbf{B}[f] = f \text{ in } D \text{ and } \|\mathbf{B}[f]\|_{p,m+1,D} \leq C_{p,m,D} \|f\|_{p,m,D}.$$

By using the symbol  $\mathbf{B}$ , the  $\mathbf{v}$  is defined by the formula :

$$\mathbf{v} = -\mathbf{B}[\nabla \varphi \cdot \mathbf{u}_{\mathbb{R}^3}] - \mathbf{B}[\nabla \varphi \cdot \mathbf{u}_{int}].$$

Since we may assume that  $\text{supp } \nabla\varphi$  is compact and contained in  $\Omega$ , such compensation never changes the boundary condition, so that in the case of the Navier-Stokes equation we can use the usual cut-off technique. From this point of view, the main step of the proof of Theorem 2 is the estimation of the convolution of the fundamental solutions of the Oseen equation and the right members. Let  $\chi(\mathbf{u}_\infty)(x)$  and  $\mathbf{p}$  be a system of the fundamental solutions to the following linear stationary Oseen equation:

$$-\Delta \mathbf{w} + (\mathbf{u}_\infty \cdot \nabla) \mathbf{w} + \nabla \mathbf{p} = \mathbf{f}, \quad \nabla \cdot \mathbf{w} = 0 \quad \text{in } \mathbb{R}^3.$$

Then, they are given by the following formula:

$$\begin{aligned} \chi_{jk}(\mathbf{u}_\infty)(x) &= (\delta_{jk} \Delta - \partial_j \partial_k) \Xi(\sigma)(x), \quad \Xi(\sigma)(x) = \int_0^{\sigma s(\mathbf{u}_\infty)(x)} \frac{1 - e^{-\alpha}}{8\pi\sigma\alpha} d\alpha, \quad \sigma = |\mathbf{u}_\infty|/2, \\ \pi_j(x) &= \frac{x_j}{4\pi|x|^3}, \quad \mathbf{p} = {}^T(\pi_1, \pi_2, \pi_3), \\ \chi(\mathbf{u}_\infty)(x) &= (\chi_{jk}(\mathbf{u}_\infty)(x)) = 3 \times 3 \text{ matrix whose } i, j \text{ component is } \chi_{jk}(\mathbf{u}_\infty)(x), \end{aligned}$$

The following lemma is concerned with the estimations of the  $\chi(\mathbf{u}_\infty)(x)$ .

**Lemma 2.** *Assume that  $\mathbf{u}_\infty \neq \mathbf{0}$ . Then, for any  $\delta: 0 \leq \delta \leq 1$  there exists a constant  $C_\delta > 0$  independent of  $\mathbf{u}_\infty$  such that*

$$|\chi(\mathbf{u}_\infty)(x)| \leq \frac{C_\delta}{(\sigma s(x))^\delta |x|}, \quad |\nabla \chi(\mathbf{u}_\infty)(x)| \leq \frac{C_\delta}{(\sigma s(x))^\delta s(x)^{1/2} |x|^{3/2}}.$$

The following is the one of the main step of the estimation.

**Lemma 3.** *Let  $0 < \delta < 1/4$  and put*

$$\langle \mathbf{g} \rangle_{2\delta} = \sup_{x \in \mathbb{R}^3} (1 + |x|)^{5/2+2\delta} |\mathbf{g}(x)|.$$

*If  $\mathbf{g} \in L_{1,loc}(\mathbb{R}^3)$  and  $\langle \mathbf{g} \rangle_{2\delta} < \infty$ , then*

$$\begin{aligned} |\chi(\mathbf{u}_\infty) * \mathbf{g}(x)| &\leq \frac{C_\delta \langle \mathbf{g} \rangle_{2\delta}}{|\mathbf{u}_\infty|^\delta |x| (1 + s(x))^\delta} && \text{for } |x| \geq 1, \\ |\nabla \chi(\mathbf{u}_\infty) * \mathbf{g}(x)| &\leq \frac{C_\delta \langle \mathbf{g} \rangle_{2\delta}}{|\mathbf{u}_\infty|^\delta |x|^{3/2} (1 + s(x))^{1/2+\delta}} && \text{for } |x| \geq 1. \end{aligned}$$

Farwig [4, 5] proved Lemma 3 essentially by refining the argument due to Finn [6-10]. Shibata [22] gave another proof of Lemma 3 based on the integration by parts with respect to the angular variables. By Lemma 3 and the compact perturbation method, Shibata [22] proved the following existence theorem.

**Lemma 4.** *Let us consider the exterior boundary value problem for the linear Oseen operator:*

$$(3.1) \quad -\Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla) \mathbf{u} + \nabla \mathbf{p} = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega.$$

Let  $3 < p < \infty$  and  $0 < \delta < 1/4$ . Put

$$\ll \mathbf{f} \gg_{2\delta} = \sup_{x \in \Omega} (1 + |x|)^{5/2} (1 + s(\mathbf{u}_\infty)(x))^{1/2+2\delta} |\mathbf{f}(x)|.$$

Assume that  $0 < |\mathbf{u}_\infty| \leq \sigma_0$ . Then, there exists a constant  $\sigma_1 > 0$  depending essentially only on  $p, \sigma_0$  and  $\delta$  such that if  $\mathbf{f} \in L_{1,loc}(\Omega)$  and  $\ll \mathbf{f} \gg_{2\delta} < \infty$ , then (3.1) admits a unique solution  $\mathbf{u} \in \mathbb{W}_p^2(\Omega)$  and  $\mathbf{p} \in W_p^1(\Omega)$  such that

$$\|\mathbf{u}\|_{p,2} + \|\mathbf{p}\|_{p,1} + \|\mathbf{u}\|_\delta \leq C_{p,\delta} |\mathbf{u}_\infty|^{-\delta} \ll \mathbf{f} \gg_{2\delta}.$$

Let us define the linear operator  $G$  by the formula:  $\mathbf{u} = G\mathbf{f}$ , where  $\mathbf{u}$  is a solution to (3.1). By using the Bogovskii's lemma, we can easily construct a vector  $\mathbf{v}(x)$  of functions in  $C^\infty(\mathbb{R}^3)$  such that  $\mathbf{v}(x) = -\mathbf{u}_\infty$  on  $\partial\Omega$ ,  $\text{supp } \mathbf{v}$  is compact and

$$|\partial_x^\alpha \mathbf{v}(x)| \leq C_\alpha |\mathbf{u}_\infty|, \quad \forall \alpha, \quad \forall x \in \mathbb{R}^3.$$

If we put  $\mathbf{w} = \mathbf{v} + \mathbf{y}$  to solve (2.3), by using the operator  $G$  we can rewrite (2.3) in the form:

$$\mathbf{y} = -G\mathbb{P}\{(\mathbf{z} \cdot \nabla)\mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \Delta \mathbf{v} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{v}\}.$$

Let  $\delta$  and  $\beta$  be given constants such that  $0 < \delta < 1/2$  and  $\delta < \beta < 1 - \delta$ . Let  $p$  be a fixed number such that  $3 < p < \infty$ . And then, as an invariant space we take

$$\mathcal{I} = \{\mathbf{y} \in \mathcal{D}_p(\mathcal{O}(\mathbf{u}_\infty)) \mid \|\mathbf{y}\|_{p,2} + \|\mathbf{y}\|_\delta \leq |\mathbf{u}_\infty|^\beta / 2\}.$$

If we choose  $|\mathbf{u}_\infty|$  small enough, then Theorem 2 can be proved by the usual contraction mapping principle in view of Lemma 4.

**3.3 A proof of Theorem 1.** As stated in the previous paragraph, in order to prove Theorem 1 we also used the cut-off technique. Therefore, we need the estimation for the semigroup  $E_{\mathbf{u}_\infty}(t)$  generated by  $\mathcal{O}_{\mathbb{R}^3}(\mathbf{u}_\infty)$  and the local energy estimation of  $T_{\mathbf{u}_\infty}(t)$ . Since  $E_{\mathbf{u}_\infty}(t)$  is given by the formula:

$$E_{\mathbf{u}_\infty}(t)\mathbf{a} = \left(\frac{1}{4\pi t}\right)^{3/2} \int_{\mathbb{R}^3} e^{-|x-t\mathbf{u}_\infty-y|^2/4t} \mathbf{a}(y) dy, \quad \forall \mathbf{a} \in \mathbb{J}_p(\mathbb{R}^3),$$

by using the classical Young's inequality we have the following lemma immediately.

**Lemma 5.** *Let  $\sigma_0 > 0$ ,  $1 \leq p \leq q \leq \infty$  and  $|\mathbf{u}_\infty| \leq \sigma_0$ . Then,*

$$|\partial_t^j \partial_x^\alpha E_{\mathbf{u}_\infty}(t) \mathbf{a}|_q \leq C_{j,\alpha,p,q,\sigma_0} t^{-(\nu+(j+|\alpha|)/2)} |\mathbf{a}|_p, \quad \nu = \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right),$$

for any  $\mathbf{a} \in \mathbb{J}_p(\mathbb{R}^3)$  and  $t > 0$ , where the constant  $C_{j,\alpha,p,q,\sigma_0}$  is independent of  $\mathbf{u}_\infty$  whenever  $|\mathbf{u}_\infty| \leq \sigma_0$ .

Moreover,

$$|\partial_t^j \partial_x^\alpha E_{\mathbf{u}_\infty}(t) \mathbf{a}|_p \leq C_{j,\alpha,p,\sigma_0} |\mathbf{a}|_{p,2j+|\alpha|}, \quad \forall \mathbf{a} \in \mathbb{J}_p(\mathbb{R}^3) \cap \mathbb{W}_p^{2j+|\alpha|}(\mathbb{R}^3), \quad \forall t \geq 0,$$

where the constant  $C_{j,\alpha,p,\sigma_0}$  is also independent of  $\mathbf{u}_\infty$  whenever  $|\mathbf{u}_\infty| \leq \sigma_0$ .

The following local energy decay estimate was obtained in [17].

**Lemma 6.** (*local energy decay*) *Let  $1 < p < \infty$  and let  $b_0$  be a fixed number such that  $B_{b_0} \supset \bar{\mathcal{O}}$ . Let  $\sigma_0$  be any positive number. Assume that  $0 < |\mathbf{u}_\infty| \leq \sigma_0$ . Then, for any  $b > b_0$ , integer  $M \geq 0$  and  $\kappa: 0 < \kappa \leq 1$ , there exists a constant  $C_{M,p,b,\kappa,\sigma_0} > 0$  independent of  $\mathbf{u}_\infty$  such that*

$$\|\partial_t^M T_{\mathbf{u}_\infty}(t) \mathbf{a}\|_{p,2,\Omega_b} \leq C_{M,p,b,\kappa,\sigma_0} |\mathbf{u}_\infty|^{-\kappa} t^{-(M+3/2)} \|\mathbf{a}\|_p, \quad \forall t > 0, \quad \forall \mathbf{a} \in \mathbb{J}_{p,b}(\Omega).$$

In order to prove Lemma 6, we use the usual representation formula of the semigroup in terms of the inverse operator  $(\lambda I + \mathbb{O}(\mathbf{u}_\infty))^{-1}$  given by the following formula:

$$(3.2) \quad T_{\mathbf{u}_\infty}(t) \mathbf{f} = -\frac{1}{2\pi i t} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{d}{d\lambda} (\lambda I + \mathbb{O}(\mathbf{u}_\infty))^{-1} \mathbf{f} d\lambda.$$

To estimate the right-hand side of the above formula, first we have to discuss the resolvent set  $\rho(\mathbb{O}(\mathbf{u}_\infty))$  of  $\mathbb{O}(\mathbf{u}_\infty)$ .

**Lemma 7.** (*resolvent set of  $\mathbb{O}(\mathbf{u}_\infty)$* ) *We have the formula:*

$$\rho(\mathbb{O}(\mathbf{u}_\infty)) \supset -\Sigma_{\mathbf{u}_\infty} = \{\lambda \in \mathbb{C} \mid -\lambda \in \Sigma_{\mathbf{u}_\infty}\},$$

where  $\Sigma_{\mathbf{u}_\infty} = \{\lambda \in \mathbb{C} \mid |\mathbf{u}_\infty|^2 \operatorname{Re} \lambda + (\operatorname{Im} \lambda)^2 > 0\}$ .

Moreover, for any  $p: 1 < p < \infty$ ,  $\lambda_0 > 0$  and  $\sigma_0 > 0$  there exists a constant  $C_{p,\lambda_0,\sigma_0}$  such that

$$|\lambda| \|(\mathbb{O}(\mathbf{u}_\infty) + \lambda I)^{-1} \mathbf{f}\|_p + \|(\mathbb{O}(\mathbf{u}_\infty) + \lambda I)^{-1} \mathbf{f}\|_{2,p} \leq C_{p,\lambda_0,\sigma_0} \|\mathbf{f}\|_p, \quad \forall \mathbf{f} \in L_p(\Omega),$$

provided that  $\operatorname{Re} \lambda \geq 0$ ,  $|\lambda| \geq \lambda_0$  and  $|\mathbf{u}_\infty| \leq \sigma_0$ .

In order to prove Lemma 6, we can use Lemma 7 to estimate  $(\mathbb{O}(\mathbf{u}_\infty) + \lambda I)^{-1}$  for large  $|\lambda|$ , so that the main step is estimation of  $(\mathbb{O}(\mathbf{u}_\infty) + \lambda I)^{-1}$  near  $\lambda = 0$ . To do this, the following lemma is the key.

**Lemma 8.** Let  $1 < p < \infty$ ,  $0 < \kappa \leq 1$ ,  $\sigma_0 > 0$ ,  $0 < |\mathbf{u}_\infty| \leq \sigma_0$  and  $b > b_0 + 4$  where  $b_0$  is a fixed number such that  $B_{b_0} \supset \overline{\mathcal{O}}$ . Put

$$\mathcal{H} = \mathcal{L}(\mathbb{J}_{p,b}(\Omega), \mathcal{D}_p(\mathcal{O}(\mathbf{u}_\infty))), \quad I_\gamma = \{\lambda = i\mu \mid \mu \in \mathbb{R}, 0 < |\mu| < \min(\gamma, 1)\},$$

$$D_\gamma = \{\lambda \in \mathbb{C} \mid 0 < \operatorname{Re} \lambda < \min(\gamma, 1), |\operatorname{Im} \lambda| < \min(\gamma, 1)\}.$$

Then, there exists a  $\gamma > 0$  depending on  $b, p$  and  $\sigma_0$  but independent of  $\mathbf{u}_\infty$  such that there exists an operator  $R_{\mathbf{u}_\infty}(\lambda) \in \mathcal{A}(D_\gamma, \mathcal{H}) \cup C^\infty(I_\gamma, \mathcal{H})$  such that

$$R_{\mathbf{u}_\infty}(\lambda)\mathbf{f} = (\mathcal{O}(\mathbf{u}_\infty) + \lambda I)^{-1}\mathbf{f}, \quad \|R_{\mathbf{u}_\infty}(\lambda)\mathbf{f}\|_{p,2,\Omega_b} \leq C_{p,b,\sigma_0}\|\mathbf{f}\|_p,$$

$$\|(\partial/\partial\lambda)^m R_{\mathbf{u}_\infty}(\lambda)\mathbf{f}\|_{p,2,\Omega_b} \leq \frac{C_{p,b,\sigma_0,m,\kappa}}{|\mathbf{u}_\infty|^\kappa |\operatorname{Im} \lambda|^{m-1/2}} \|\mathbf{f}\|_p, \quad \forall m \geq 1$$

for any  $\lambda \in D_\gamma \cup I_\gamma$  and  $\mathbf{f} \in \mathbb{J}_{p,b}(\Omega)$ .

In fact, combining Lemma 8 and the following lemma concerning the decay rate of the Fourier image of the function which are regular up to some fractional order, we can treat the low frequency part (that is near  $\lambda = 0$  in (3.2)), and hence we can prove Lemma 6.

**Lemma 9.** Let  $\mathcal{H}$  be a Banach space with norm  $|\cdot|_{\mathcal{H}}$ . Let  $f(\tau)$  be a function of  $C^\infty(\mathbb{R} - \{0\}, \mathcal{H})$  such that  $f(\tau) = 0$  for  $|\tau| \geq a$  with some  $a > 0$ . Assume that there exists a constant  $C_f$  depending on  $f$  such that

$$\left| (d/d\tau)^j f(\tau) \right| \leq C_f |\tau|^{-(j+1/2)}, \quad \forall \tau: 0 < |\tau| \leq a, \quad j = 0, 1.$$

If we put  $g(t) = (1/2\pi) \int_{-\infty}^{\infty} f(\tau) e^{it\tau} d\tau$ , then

$$|g(t)|_{\mathcal{H}} \leq C(1+t)^{-1/2} C_f$$

for some constant  $C > 0$  independent of  $f$ .

As stated in the previous paragraph, to prove Lemma 8, the main step is the estimation of solutions in  $\mathbb{R}^3$ , which is achieved by estimating the low frequency part of the fundamental solution  $E_{jk}^0(\mathbf{u}_\infty, \lambda)(x)$  which is defined by the following formula:

$$E_{jk}^0(\mathbf{u}_\infty, \lambda)(x) = \mathcal{F}^{-1} \left[ \frac{(\delta_{jk} - \xi_j \xi_k |\xi|^{-2}) \psi(\xi)}{|\xi|^2 + i\mathbf{u}_\infty \cdot \xi + \lambda} \right] (x),$$

where  $\psi$  is a smooth function whose support is contained in  $\{\xi \in \mathbb{R}^3 \mid |\xi| \leq 2\}$ ,  $i = \sqrt{-1}$ ,  $\delta_{jk} = 1$  for  $j = k$  and  $\delta_{jk} = 0$  for  $j \neq k$ , and  $\mathcal{F}^{-1}$  denotes the Fourier inverse transform. Note that if we put

$$E_{jk}(\mathbf{u}_\infty, \lambda)(x) = \mathcal{F}^{-1} \left[ \frac{\delta_{jk} - \xi_j \xi_k |\xi|^{-2}}{|\xi|^2 + i\mathbf{u}_\infty \cdot \xi + \lambda} \right] (x), \quad \pi_j(x) = \mathcal{F}^{-1} \left[ \frac{\xi_j}{i|\xi|^2} \right] (x),$$

$$E(\mathbf{u}_\infty, \lambda) = (E_{jk}(\mathbf{u}_\infty, \lambda)), \quad \mathbf{p} = {}^T(\pi_1, \pi_2, \pi_3),$$

then  $E(\mathbf{u}_\infty, \lambda)$  and  $\mathbf{p}$  give a system of fundamental solutions of the following stationary Oseen equation in  $\mathbb{R}^3$  with a complex parameter  $\lambda$ :

$$(\lambda - \Delta + (\mathbf{u}_\infty \cdot \nabla))\mathbf{u} + \nabla \mathbf{p} = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3.$$

Note that

$$|\xi|^2 + i\mathbf{u}_\infty \cdot \xi + \lambda \neq 0 \quad \text{for } \xi \in \mathbb{R}^3 \text{ and } \lambda \in \Sigma_{\mathbf{u}_\infty},$$

which plays an important role in order to prove Lemma 7. When  $\text{Re } \lambda \geq 0$  and  $\mathbf{u}_\infty \neq \mathbf{0}$ , by calculating the Fourier transform directly we have

$$E_{jk}(\mathbf{u}_\infty, \lambda)(x) = -\frac{1}{8\pi\sigma} (\delta_{jk}\Delta - \partial_j\partial_k) \int_0^\infty \frac{1 - e^{-F(\lambda, t, y)}}{\ell(t, y)} e^{-\lambda t/2\sigma} dt,$$

where

$$\sigma = |\mathbf{u}_\infty|/2, \quad \partial_j = \partial/\partial x_j, \quad \Delta = \sum_{j=1}^3 \partial^2/\partial x_j^2,$$

$$y = Sx, \quad S \text{ is the orthogonal matrix such that } S\mathbf{u}_\infty = |\mathbf{u}_\infty|\mathbf{e}_1, \quad (\mathbf{e}_1 = {}^T(1, 0, 0)),$$

$$F(\lambda, t, y) = \sqrt{\sigma^2 + \lambda} \ell(t, y) - \sigma(y_1 - t), \quad \text{the analytic branch of } \sqrt{\sigma^2 + \lambda} \text{ is chosen so that } \text{Re } \sqrt{\sigma^2 + \lambda} > 0,$$

$$\ell(t, y) = \sqrt{(y_1 - t)^2 + |y'|^2}, \quad y' = (y_2, y_3), \quad |y'| = \sqrt{y_2^2 + y_3^2}.$$

Using this formula, we have the following lemma.

**Lemma 10.** *Let  $\sigma_0 > 0$ ,  $\text{Re } \lambda \geq 0$ ,  $|\text{Im } \lambda| \leq 1$  and  $|\mathbf{u}_\infty| \leq \sigma_0$ . Then, for any integer  $n \geq 0$ ,  $\kappa \in \mathbb{R}: 0 < \kappa \leq 1$ ,  $\mu \in \mathbb{R}: 0 \leq \mu < 1/2$  and  $x \in \mathbb{R}^3$  we have the following estimations:*

$$(3.3) \quad \begin{aligned} & \left| \bar{\partial}_x^n E_{jk}^0(\mathbf{u}_\infty, \lambda)(x) \right| \leq C_n; \\ & \left| \bar{\partial}_x^n (\partial/\partial \lambda)^m E_{jk}^0(\mathbf{u}_\infty, \lambda)(x) \right| \leq \frac{C_{m,n,\kappa,\sigma_0} (1 + |x|)^{m+1}}{|\mathbf{u}_\infty|^\kappa |\text{Im } \lambda|^{m-1/2}} \end{aligned}$$

for any  $m \geq 1$ ,  $\text{Im } \lambda \neq 0$  and  $\mathbf{u}_\infty \neq \mathbf{0}$ .

**Remark 2.** To prove Lemma 10 is crucial in order to prove Theorem 1. Since the estimation (3.3) is not uniform with respect to  $\mathbf{u}_\infty$ , the estimation in Theorem 1 is also not uniform with respect to  $\mathbf{u}_\infty$ . If (3.3) was improved regarding the uniformity with respect to  $\mathbf{u}_\infty$ , the estimation in Theorem 1 was uniform with respect to  $\mathbf{u}_\infty$ . In the next section, to get such an improvement I would like to discuss a little bit.

#### 4. An essay of the uniform $L_p$ - $L_q$ estimate with respect to $\mathbf{u}_\infty$

In order to give a lemma concerning the decay rate of the Fourier image of functions which is regular up to a fractional order, we introduce the following space.

**Definition 1.** Let  $\mathcal{H}$  be a Banach space with norm  $|\cdot|_{\mathcal{H}}$ . Let  $N$  be a non-negative integer and  $0 < \sigma < 1$ . Put  $\mathcal{C}^{N+\sigma}(\mathbb{R}; \mathcal{H}) = \{u \in C^N(\mathbb{R}; \mathcal{H}) \mid \ll u \gg_{N+\sigma, \mathcal{H}} < \infty\}$  where

$$\begin{aligned} \ll u \gg_{N+\sigma, \mathcal{H}} = & \sum_{j=0}^N \int_{-\infty}^{\infty} |(d/d\tau)^j u(\tau)|_{\mathcal{H}} d\tau \\ & + \sup_{h \neq 0} |h|^{-\sigma} \int_{-\infty}^{\infty} |(d/d\tau)^N u(\tau+h) - (d/d\tau)^N u(\tau)|_{\mathcal{H}} d\tau. \end{aligned}$$

The following lemma is concerning the relationship between the regularity of the function and the decay rate of its Fourier image (cf. Shibata [21]).

**Lemma 11.** Let  $\mathcal{H}$  be a Banach space with norm  $|\cdot|_{\mathcal{H}}$ . Let  $N$  be a non-negative integer and  $0 < \sigma < 1$ . Assume that  $f \in \mathcal{C}^{N+\sigma}(\mathbb{R}; \mathcal{H})$ . Put  $g(t) = (1/2\pi) \int_{-\infty}^{\infty} f(\tau) e^{it\tau} d\tau$ . Then,

$$|g(t)|_{\mathcal{H}} \leq C(1+t)^{-(N+\sigma)} \ll f \gg_{N+\sigma, \mathcal{H}}.$$

While the assumption of Lemma 9 is pointwise, that of Lemma 11 is in the  $L_1$  sense, which is a key of the improvement of our estimate in [17] regarding the uniformity with respect to  $\mathbf{u}_{\infty}$ . The following lemma is corresponding to Lemma 10.

**Lemma 12.** Put

$$F_0(\mathbf{u}_{\infty}, s)(x) = E_{jk}^0(\mathbf{u}_{\infty}, is)(x), \quad F_1(\mathbf{u}_{\infty}, s)(x) = (\partial/\partial s)E_{jk}^0(\mathbf{u}_{\infty}, is)(x).$$

Then,

$$(4.1) \quad \sup_{s \in \mathbb{R}} |F_0(\mathbf{u}_{\infty}, s)|_{\infty} \leq C,$$

$$(4.2) \quad \sup_{s \in \mathbb{R}} |F_0(\mathbf{u}_{\infty}, s+h) - F_0(\mathbf{u}_{\infty}, s)|_{\infty} \leq C\sqrt{|h|},$$

$$(4.3) \quad \int_{-\infty}^{\infty} |F_1(\mathbf{u}_{\infty}, s)|_{\infty} ds \leq C,$$

$$(4.4) \quad \int_{-\infty}^{\infty} |F_1(\mathbf{u}_{\infty}, s+h) - F_1(\mathbf{u}_{\infty}, s)|_{\infty} ds \leq C\sqrt{|h|},$$

where  $C$  is an independent constant of  $\mathbf{u}_{\infty}$ .

*Proof.* Recall the definition of  $E_{jk}^0(\mathbf{u}_{\infty}, is)$  (cf. after Lemma 9), we have

$$\begin{aligned} |F_0(\mathbf{u}_{\infty}, s)|_{\infty} & \leq \int_{|\xi| \leq 2} \frac{d\xi}{\left| |\xi|^2 + i(\mathbf{u}_{\infty} \cdot \xi + s) \right|} \leq \int_{|\xi| \leq 2} \frac{d\xi}{|\xi|^2} < \infty; \\ \int_{-\infty}^{\infty} |F_1(\mathbf{u}_{\infty}, s)|_{\infty} ds & \leq \int_{|\xi| \leq 2} \int_{-\infty}^{\infty} \frac{ds d\xi}{\left| |\xi|^2 + i(\mathbf{u}_{\infty} \cdot \xi + s) \right|^2} = \{ \mathbf{u}_{\infty} \cdot \xi + s = t \} \\ & \leq \int_{|\xi| \leq 2} \int_{-\infty}^{\infty} \frac{dt d\xi}{|\xi|^4 + t^2} = \int_{|\xi| \leq 2} \frac{d\xi}{|\xi|^2} \int_{-\infty}^{\infty} \frac{dt}{t^2 + 1}, \end{aligned}$$

which shows (4.1) and (4.3).

When  $|h| \geq 1$ , (4.1) and (4.3) immediately implies (4.2) and (4.4), respectively. Therefore, we may assume that  $0 < |h| \leq 1$ . Observe that

$$(4.5) \quad \begin{aligned} & |F_0(\mathbf{u}_\infty, s+h) - F_0(\mathbf{u}_\infty, s)|_\infty \\ & \leq \int_{|\xi| \leq 2} \frac{|h|d\xi}{\left| |\xi|^2 + i(\mathbf{u}_\infty \cdot \xi + s+h) \right| \left| |\xi|^2 + i(\mathbf{u}_\infty \cdot \xi + s) \right|}. \end{aligned}$$

Since  $|\mathbf{u}_\infty \cdot \xi + s+h| \geq |h| - |\mathbf{u}_\infty \cdot \xi + s| \geq |h|/2$  when  $|\mathbf{u}_\infty \cdot \xi + s| \leq |h|/2$ , studying the case when  $|\mathbf{u}_\infty \cdot \xi + s| \leq |h|/2$  and the case when  $|\mathbf{u}_\infty \cdot \xi + s| \geq |h|/2$ , we see that

$$\left| |\xi|^2 + i(\mathbf{u}_\infty \cdot \xi + s+h) \right| \left| |\xi|^2 + i(\mathbf{u}_\infty \cdot \xi + s) \right| \geq \left( |\xi|^2 \sqrt{|\xi|^4 + h^2} \right) / 2,$$

which together with (4.5) implies (4.2). Observe that

$$\begin{aligned} & \int_{-\infty}^{\infty} |F_1(\mathbf{u}_\infty, s+h) - F_1(\mathbf{u}_\infty, s)|_\infty ds \\ & \leq \int_{|\xi| \leq 2} \int_{-\infty}^{\infty} \left| \frac{1}{\left| |\xi|^2 + i(\mathbf{u}_\infty \cdot \xi + s+h) \right|^2} - \frac{1}{\left| |\xi|^2 + i(\mathbf{u}_\infty \cdot \xi + s) \right|^2} \right| ds d\xi \\ & \leq \sum_{j=1}^5 \int_{\omega_j} K(\xi, s, h) d\xi ds = \sum_{j=1}^5 I_j \end{aligned}$$

where

$$\begin{aligned} K(\xi, s, h) &= \frac{2|\xi|^2|h| + 2|s||h| + |h|^2}{(|\xi|^4 + (s+h)^2)(|\xi|^4 + s^2)}, \quad \omega_1 = \{|\xi| \leq \sqrt{|h|}, |s| \leq |h|/2\}, \\ \omega_2 &= \{|\xi| \leq \sqrt{|h|}, |h|/2 \leq |s| \leq 2|h|\}, \quad \omega_3 = \{|\xi| \leq \sqrt{|h|}, |s| \geq 2|h|\}, \\ \omega_4 &= \{|\xi| \geq \sqrt{|h|}, |s| \leq 2|h|\}, \quad \omega_5 = \{|\xi| \geq \sqrt{|h|}, |s| \geq 2|h|\}. \end{aligned}$$

Since  $|s+h| \geq |h| - |s| \geq |h|/2$  and  $2|\xi|^2|h| + 2|s||h| + |h|^2 \leq C|h|^2$  when  $(\xi, s) \in \omega_1$ , we have

$$\begin{aligned} I_1 & \leq \int_{\omega_1} \frac{C|h|^2 d\xi ds}{(|\xi|^4 + s^2)(|\xi|^4 + h^2)} \leq C \int_0^{|h|/2} ds \int_0^{\sqrt{|h|}} \frac{r^2 dr}{r^4 + s^2} \\ & = \left\{ \begin{array}{l} r = \sqrt{|s|}t, \quad r^2 dr = |s|^{3/2} dt \\ r^4 + s^2 = s^2(t^4 + 1), \end{array} \right\} = C \int_0^{|h|/2} \frac{ds}{\sqrt{s}} \int_0^\infty \frac{t^2 dt}{t^4 + 1} \leq C\sqrt{|h|}. \end{aligned}$$

Since  $|\xi|^4 + s^2 \geq |\xi|^4 + (|h|/2)^2$ ,  $2|\xi|^2|h| + 2|s||h| + |h|^2 \leq C|h|^2$  and  $|s+h| \leq 3|h|$  when  $(\xi, s) \in \omega_2$ , we have

$$I_2 \leq \int_{\omega_2} \frac{C|h|^2 d\xi ds}{(|\xi|^4 + (|h|/2)^2)(|\xi|^4 + (s+h)^2)} \leq C \int_0^{3|h|} ds \int_0^{\sqrt{|h|}} \frac{r^2 dr}{r^4 + s^2} \leq C\sqrt{|h|}.$$

Since  $|s + h| \geq |s| - |h| \geq |s|/2$  and  $2|\xi|^2|h| + 2|s||h| + |h|^2 \leq C|s||h|$  when  $(\xi, s) \in \omega_3$ , we have

$$\begin{aligned} I_3 &\leq \int_{\omega_3} \frac{C|s||h|d\xi ds}{(|\xi|^4 + s^2)^2} = C|h| \int_{2|h|}^{\infty} s ds \int_0^{\sqrt{|h|}} \frac{r^2 dr}{(r^4 + s^2)^2} \\ &= C \int_{2|h|}^{\infty} \frac{|h|ds}{s^{3/2}} \int_0^{\infty} \frac{t^2 dt}{(t^4 + 1)^4} \leq C\sqrt{|h|}. \end{aligned}$$

When  $(\xi, s) \in \omega_4$ ,  $2|\xi|^2|h| + 2|s||h| + |h|^2 \leq C|\xi|^2|h|$ , because  $|h| \leq |\xi|^2$  and  $|s| \leq 2|h| \leq 2|\xi|^2$ . Therefore, we have

$$\begin{aligned} I_4 &\leq \int_{\omega_4} \frac{C|\xi|^2|h|d\xi ds}{(|\xi|^4 + s^2)(|\xi|^4 + (s+h)^2)} \leq C|h| \int_{|\xi| \geq \sqrt{|h|}} \frac{d\xi}{|\xi|^2} \int_0^{2|h|} \frac{ds}{|\xi|^4 + s^2} \\ &= C|h| \int_{|\xi| \geq \sqrt{|h|}} \frac{d\xi}{|\xi|^4} \int_0^{\infty} \frac{dt}{t^2 + 1} \leq C|h| \int_{\sqrt{|h|}}^{\infty} \frac{dr}{r^2} \leq C\sqrt{|h|}. \end{aligned}$$

When  $(\xi, s) \in \omega_5$ ,  $2|\xi|^2|h| + 2|s||h| + |h|^2 \leq C(|\xi|^2 + |s|)|h|$  and  $|s + h| \geq |s| - |h| \geq |s|/2$ , because  $2|h| \leq |s|$  and  $|\xi|^2 \geq |h|$ . Therefore, we have

$$\begin{aligned} I_5 &\leq C|h| \int_{|s| \geq 2|h|} \int_{|\xi| \geq \sqrt{|h|}} \frac{|\xi|^2 + |s|}{(|\xi|^4 + s^2)^2} d\xi ds \\ &\leq C|h| \int_{|\xi| \geq \sqrt{|h|}} \frac{d\xi}{|\xi|^4} \int_0^{\infty} \frac{1 + |t|}{(t^2 + 1)^2} dt \leq C\sqrt{|h|}. \end{aligned}$$

Combining these five estimations, we have the second part of (4.2), which completes the proof of the lemma.

**Remark 3.** According to the method due to Kobayashi and Shibata [17], I will be able to get Theorem 1 with  $\kappa = 0$  (that is the uniform estimate with respect to  $\mathbf{u}_\infty$ ) by using Lemma 12 instead of Lemma 10. Therefore, in Theorem 3 instead of assuming that  $\|\mathbf{a} - \mathbf{u}_\infty\|_3 \leq |\mathbf{u}_\infty|^\beta$ , we may assume that  $\|\mathbf{a} - \mathbf{u}_\infty\|_3 \leq \epsilon$ , where  $\epsilon$  is independent of  $\mathbf{u}_\infty$ . This means that the smallness assumption on  $\mathbf{a}$  is independent of  $\mathbf{u}_\infty$ . Therefore, we can prove the continuity of solutions to (1.1) with respect to  $\mathbf{u}_\infty$  near  $\mathbf{u}_\infty = \mathbf{0}$ . The detail will be published elsewhere.

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# AN INITIAL BOUNDARY VALUE PROBLEM FOR SOME HYPERBOLIC-PARABOLIC COUPLED SYSTEM

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**Abstract.** This note is concerned with a unique existence of small solutions globally in time of the initial boundary value problem for some coupled system of nonlinear hyperbolic and parabolic equations in a bounded domain. The exponential decay of the energy is also addressed. The model equation of our theory is a nonlinear thermoviscoelastic equation and a nonlinear viscoelastic equation describing the elastic motion with thermal effect and viscosity.

## 1. Introduction

We usually think that the classical solutions of the initial boundary value problem for nonlinear wave equations in a bounded domain develop the singularity within a finite time, because there is no dissipation (cf. Klainerman and Majda [5]). On the other hand, from the mechanical experiments the motion of the elastic body usually decays very quickly by the thermal effects, the viscosity effect, the friction effect and so on. A thermoviscoelastic equation is one of good model of explaining such decaying phenomena of the motion of the elastic body (cf. Dafermos [1]). Recently, Shibata [7] proved an unique existence of small solutions globally in time of nonlinear thermoviscoelastic equations in a bounded domain with zero Dirichlet boundary condition and the exponential decay of the solutions. In this note I will give more general results including Shibata [7] and also Kawashima and Shibata [4].

## 2. Statement of main results

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . Let  $x = (x_1, \dots, x_n)$  denote points of  $\mathbb{R}^n$  and  $t$  time. We consider the following initial boundary value

problem:

$$\begin{aligned}
 \mathbf{u}_{tt}^H &= \sum_{i,j=1}^n \{ A_{ij}^H(\Phi) \partial_i \partial_j \mathbf{u}^H + B_{ij}^H(\Phi) \partial_i \partial_j \mathbf{u}_t^H \\
 &\quad + P_{ij}^H(\Phi) \partial_i \partial_j \mathbf{u}^P \} + \sum_{j=1}^n P_j^H(\Phi) \partial_j \mathbf{u}^P \quad \text{in } \Omega_T, \\
 \text{(HP)} \quad A^P(\Phi') \mathbf{u}_t^P - \sum_{j=1}^n Q_j^P(\Phi) \partial_j \mathbf{u}_t^H &= \sum_{i,j=1}^n \{ A_{ij}^P(\Phi) \partial_i \partial_j \mathbf{u}^P \\
 &\quad + Q_{ij}^P(\Phi) \partial_i \partial_j \mathbf{u}^H + R_{ij}^P(\Phi) \partial_i \partial_j \mathbf{u}_t^H \} \quad \text{in } \Omega_T, \\
 \mathbf{u}^H &= 0, \quad \mathbf{u}^P = 0 \quad \text{on } \partial\Omega_T, \\
 \mathbf{u}^H(0, x) &= \mathbf{u}_0^H(x), \quad \mathbf{u}_t^H(0, x) = \mathbf{u}_1^H(x), \quad \mathbf{u}^P(0, x) = \mathbf{u}_0^P(x) \quad \text{in } \Omega,
 \end{aligned}$$

where  $\Omega_T = (0, T) \times \Omega$ ,  $\partial\Omega_T = (0, T) \times \partial\Omega$ ,  $\partial_j = \partial/\partial x_j$ , the subscript  $t$  denotes the partial derivative with respect to  $t$ ,  $\mathbf{u}^L = {}^T(u_1, \dots, u_{N_L})$  are  $N_L$  vectors of unknown functions for  $L = H$  and  $P$  ( ${}^T M$  means the transposed  $M$  and the superscripts  $H$  and  $P$  mean the hyperbolic part and the parabolic part, respectively),  $A_{ij}^H(\Phi)$  and  $B_{ij}^H(\Phi)$  are  $N_H \times N_H$  matrices  $P_{ij}^H(\Phi)$  and  $P_j^H(\Phi)$  are  $N_P \times N_H$  matrices,  $A^P(\Phi')$  and  $A_{ij}^P(\Phi)$  are  $N_P \times N_P$  matrices,  $Q_j^P(\Phi)$ ,  $Q_{ij}^P(\Phi)$  and  $R_{ij}^P(\Phi)$  are  $N_H \times N_P$  matrices, all the matrices are smoothly depending on  $\Phi$  and  $\Phi'$ , and  $\Phi = (\nabla \mathbf{u}^H, \nabla \mathbf{u}_t^H, \mathbf{u}^P, \nabla \mathbf{u}^P)$  and  $\Phi' = (\nabla \mathbf{u}^H, \mathbf{u}^P)$  ( $\nabla = (\partial_1, \dots, \partial_n)$ ).

Now, we introduce the assumptions:

$$\text{(A.1)} \quad {}^T A_{ij}^H = A_{ji}^H, \quad {}^T B_{ij}^H = B_{ji}^H, \quad {}^T A^P = A^P, \quad {}^T A_{ij}^P = A_{ji}^P;$$

$$\begin{aligned}
 \text{(A.2)} \quad &\exists \delta > 0 \text{ such that} \\
 &\sum_{i,j=1}^n A_{ij}^L(0) \xi_i \xi_j \geq \delta |\xi|^2 I_{N_L}, \quad L = H \text{ and } P, \\
 &\sum_{i,j=1}^n B_{ij}^H(0) \xi_i \xi_j \geq \delta |\xi|^2 I_{N_H}, \quad A^P(0) \geq \delta I_{N_P}
 \end{aligned}$$

for all  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$  where  $I_{N_L}$  is the  $N_L \times N_L$  identity matrices;

$$\text{(A.3)} \quad {}^T P_j^H(0) = A^P(0)^{-1} Q_j^P(0), \quad P_{ij}^H(0) = Q_{ij}^P(0) = R_{ij}^P(0) = 0.$$

The first formula in (A.3) is a generalization of a condition from the usual constitutive relation of the thermoelastic material. Since we consider only small solutions, all the assumptions may be imposed only for  $\Phi = 0$  and  $\Phi' = 0$ . Let  $H^s$  denote the usual

Sobolev space of order  $s$  on  $\Omega$  in the usual  $L^2$  sense equipped with norm  $\|\cdot\|_s$ . For a time interval  $I$  and an integer  $N \geq 2$ , let us put

$$X^N(I) = \left\{ (\mathbf{u}^H, \mathbf{u}^P) \mid (\mathbf{u}^H, \mathbf{u}_t^H, \mathbf{u}^P) \in \bigcap_{j=0}^{N-2} C^j(I; H^{N-j}), \right. \\ \left. \mathbf{u}^H \in C^N(I; L^2), \partial_t^N \mathbf{u}^H \in L^2(I; H^1), \mathbf{u}^P \in C^{N-1}(I; L^2), \partial_t^{N-1} \mathbf{u}^P \in L^2(I; H^1) \right\}, \\ \|\!(\mathbf{u}^H, \mathbf{u}^P)(t)\!\|_N^2 = \sum_{j=0}^{N-2} \|(\mathbf{u}^H, \mathbf{u}_t^H, \mathbf{u}^P)(t, \cdot)\|_{N-j}^2 + \|\partial_t^N \mathbf{u}^H(t, \cdot)\|_0^2 + \|\partial_t^{N-1} \mathbf{u}^P(t, \cdot)\|_0^2, \\ [(\mathbf{u}^H, \mathbf{u}^P)(t)]_N^2 = \|\!(\mathbf{u}^H, \mathbf{u}^P)(t)\!\|_N^2 + \|\nabla \partial_t^N \mathbf{u}^H(t, \cdot)\|_0^2 + \|\nabla \partial_t^{N-1} \mathbf{u}^P(t, \cdot)\|_0^2,$$

where  $\partial_t = \partial/\partial t$ .

To explain the compatibility conditions which initial data should satisfy, for a moment we assume that (HP) admits a solution  $(\mathbf{u}^H, \mathbf{u}^P) \in X^N([0, T])$ . Put  $\mathbf{u}_j^H(x) = \partial_t^j \mathbf{u}^H(0, x)$  and  $\mathbf{u}_j^P(x) = \partial_t^j \mathbf{u}^P(0, x)$ , and then  $\mathbf{u}_j^H, j \geq 2$ , and  $\mathbf{u}_j^P, j \geq 1$ , are computed by recurrence from (HP). To get a solution  $(\mathbf{u}^H, \mathbf{u}^P) \in X^N([0, T])$ , it is necessary to assume that

$$(A.4) \quad \mathbf{u}_0^H(x) \in H^N \cap H_0^1; \quad \mathbf{u}_{j+1}^H(x) \in H^{N-j} \cap H_0^1, j = 0, 1, \dots, N-2; \quad \mathbf{u}_N^H(x) \in L^2; \\ \mathbf{u}_j^P(x) \in H^{N-j} \cap H_0^1, j = 0, 1, \dots, N-2; \quad \mathbf{u}_{N-1}^P(x) \in L^2,$$

where  $H_0^1 = \{u \in H^1 \mid u = 0 \text{ on } \partial\Omega\}$ . Given initial data  $\mathbf{u}_0^H, \mathbf{u}_1^H$  and  $\mathbf{u}_0^P$ , we put

$$D(N, \mathbf{u}_0^H, \mathbf{u}_1^H, \mathbf{u}_0^P) = \sum_{j=0}^N \|\mathbf{u}_j^H\|_{N-j} + \sum_{j=0}^{N-1} \|\mathbf{u}_j^P\|_0.$$

Then, our main result is the following.

**Main Theorem.** Put  $N_0 = [n/2] + 3$ . Let  $N$  be an integer  $\geq N_0$  and  $\nu \geq 0$ . Assume that (A.1)–(A.3) hold. Then, there exists an  $\epsilon > 0$  depending on  $n$  such that if initial data  $\mathbf{u}_0^H, \mathbf{u}_1^H$  and  $\mathbf{u}_0^P$  satisfy (A.4) and  $D(N_0, \mathbf{u}_0^H, \mathbf{u}_1^H, \mathbf{u}_0^P) \leq \epsilon$ , then the problem (HP) admits a unique solution  $(\mathbf{u}^H, \mathbf{u}^P) \in X^N([0, \infty))$  for any  $T > 0$ .

Moreover, there exists a  $\gamma > 0$  such that for any  $L : N_0 \leq L \leq N$  there exists a constant  $\Gamma_L$  depending on  $L, \gamma, n$  and  $D(L, \mathbf{u}_0^H, \mathbf{u}_1^H, \mathbf{u}_0^P)$  such that

$$e^{2\gamma t} \|\!(\mathbf{u}^H, \mathbf{u}^P)(t)\!\|_L^2 + \int_0^t e^{2\gamma s} [(\mathbf{u}^H, \mathbf{u}^P)(s)]_L^2 ds \leq \Gamma_L.$$

A local existence theorem will be proved in the standard manner (cf. T. Kato [3], T. Kobayashi, H. Pecher and Y. Shibata [6] and also W. Dan [2] where the more complicated boundary condition is handled with). Thanks to the term :  $\sum_{i,j=1}^n B_{ij}^H(0) \partial_i \partial_j \mathbf{u}_t^H$  (corresponding to the viscosity) and  $A^P(0) \mathbf{u}_t^H - \sum_{i,j=1}^n A_{ij}^P(0) \partial_i \partial_j \mathbf{u}^P$  (corresponding to the thermal effect), by using the usual energy method, one can show the exponential

decay of the first energy. The method due to Kawashima and Shibata [4] and Shibata [7] can be applied to get the estimate of the higher derivatives. And then, using the sharp estimations of the nonlinear functions and composite functions, one can prove Main Theorem in the usual manner by regarding the nonlinear terms as a small perturbation from the linear part.

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# STABILITY, INSTABILITY AND REGULARITY OF NONLINEAR WAVES

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**Abstract:** In these expository lectures I discuss stability and regularity theory within the context of particular examples. The topics covered are: (1) stability theory with brief illustrations of a scalar wave equation, the Boltzmann equation and the KdV equation; (2) instability of BGK equilibria in the theory of plasmas; (3) regularity of solutions of the Vlasov-Poisson and Vlasov-Maxwell systems; (4) breathers, instability of wave maps, and their relationship; (5) gain of regularity of dispersive waves, particularly those of KdV type.

## 1. Stability of Non Linear Waves.

Consider

$$(1) \quad \frac{du}{dt} = A(u) \quad , \quad u(0) = u_0$$

where  $A$  is a nonlinear operator. We are concerned with equations with a "constant energy". (For instance, the energy could be  $\|u(t)\|^2$ .) A key concept of interest to scientists is *stability*.

**Definition.** A solution of  $A(\varphi) = 0$  is called an *equilibrium*. An equilibrium is called *stable* (or *nonlinearly stable*) if:  $\forall \epsilon > 0 \exists \delta > 0$  such that if  $\|u_0 - \varphi\|_1 < \delta$ , then there exists a unique solution  $u(\cdot)$  of (1), defined for all  $0 \leq t < +\infty$ , with values in some space  $X$  such that

$$(2) \quad \sup_{0 \leq t < \infty} \|u(t) - \varphi\|_2 < \epsilon.$$

Examples in the plane: a center (stable), a node (stable or unstable), a saddle point (unstable). A stable node is asymptotically stable.

**Remarks.** (i) This definition is not precise: it requires specification of the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  and of the space  $X$ .

- (ii) It must be modified for traveling waves, standing waves, etc.
- (iii) If an equilibrium is stable, is it asymptotically stable?
- (iv) If unstable, what happens to the unstable orbits? Do they converge to a different equilibrium? Do they blow up in a finite time?
- (v) Can an equilibrium be stable in one set of norms and unstable in another?

Linearization.

It is commonly assumed that stability is governed by the linearized equation

$$(3) \quad \frac{dv}{dt} = Lv \quad , \quad \text{where } L = A'(\varphi).$$

One looks at whether the solutions  $v(t)$  of (3) grow or decay as  $t \rightarrow +\infty$ . This question is equivalent to studying the spectrum of  $L$ . If the spectrum meets the open right half-plane  $\{Re\lambda > 0\}$ , then  $\varphi$  is called linearly unstable. The left half-plane is associated with stability. The imaginary axis is associated with marginal stability.

Linearization can be dangerous, however, as the following example (shown to me by J. Goodman) illustrates. Consider the P.D.E.  $u_t - xu_x = u^2$  and its equilibrium  $\varphi \equiv 0$ . Because solutions blow up at  $x=0$ ,  $\varphi$  must be unstable. Nevertheless, the linearized equation is  $v_t - xv_x = 0$ , whose solutions decay:

$$\int_{-\infty}^{\infty} v^2 dx = ce^{-t}.$$

What is wrong with this example? It is that the  $L^2$  norm and the behavior at the single point  $x=0$  are incompatible. (In fact, the  $L^2$  norm of  $v(t)$  decays but the  $H^1$  norm grows exponentially.)

In general, the passage from the linear to the nonlinear theory is not trivial, and may even be false. This is especially true for nonlinear wave equations, where the spectrum is typically marginal and continuous.

A nonlinear hyperbolic wave equation.

$$(4) \quad u_{tt} - \Delta u + f(u) = 0,$$

$x \in \mathbf{R}^n$ . The equilibrium equation is

$$(5) \quad -\Delta \varphi + f(\varphi) = 0.$$

For our example we choose  $f(u) = u - |u|^{p-1}u$  with  $p > 1$ . Then there exists an equilibrium  $\varphi(x) > 0$ , with  $\varphi(\infty) \rightarrow 0$  as  $|x| \rightarrow \infty$ , provided  $p < 1 + \frac{4}{n-2}$  (with no restriction if  $n = 1$  or  $2$ .) Furthermore we use the energy norm:  $u \in H^1, u_t \in L^2$ .

**Theorem 1.** *This equilibrium  $\varphi$  is always unstable (in the energy norm).*

One first proves that  $\varphi$  is linearly unstable. I will only outline this part of the proof. The linearized equation is

$$(6) \quad v_{tt} + Mv = 0 \quad , \quad M = -\Delta + 1 - p\varphi^{p-1}.$$

Notice that from (5) we have  $\int (|\nabla\varphi|^2 + \varphi^2)dx = \int \varphi^{p+1}dx$ . Hence

$$\begin{aligned}(M\varphi, \varphi) &= \int (|\nabla\varphi|^2 + \varphi^2 - p\varphi^{p+1})dx \\ &= (1-p) \int \varphi^{p+1}dx < 0.\end{aligned}$$

Now the solution of (6) is a linear combination of  $e^{\pm t\sqrt{-M}}$ . The operator  $M$  is self-adjoint and its continuous spectrum is the interval  $[1, \infty)$ . Because of the inequality just proved,  $M$  must have at least one negative eigenvalue. Call it  $\lambda$ . Then either  $\sqrt{-M}$  or  $-\sqrt{-M}$  has the eigenvalue  $+\sqrt{-\lambda} > 0$ . Furthermore  $\pm\sqrt{-M}$  has its continuous spectrum entirely on the imaginary axis. Therefore (6) has at least one solution that grows exponentially in time.

Another example is the standing wave  $\varphi(x)e^{i\omega t}$  for the complex-valued version of (4). It turns out to be stable for certain  $p$  if  $\varphi$  is the ground state and  $\omega$  is not too small. Thus we can say that *rotation may stabilize motion*.

Yet another example is (4) with  $f(u) = u + |u|^{p-1}u$  (with a + sign). This equation has no equilibria vanishing at  $\infty$ . For certain  $p$  and  $n$ , all its solutions approach linear waves as  $t \rightarrow \pm\infty$ . This phenomenon is called *scattering*.

#### Boltzmann equation for a gas.

Using non-standard notation, we let  $u(t, x, v)$  = the density of particles in phase space  $(x, v)$ ,  $x$  = the position in  $\mathbf{R}^3$ , and  $v$  = the velocity in  $\mathbf{R}^3$ . The Boltzmann equation is

$$\partial_t u + v \cdot \nabla_x u = Q(u)$$

where the collision term  $Q$  is defined by

$$Q(u)(t, x, v) = \int [u(t, x, v')u(t, x, w') - u(t, x, v)u(t, x, w)]dw d\Omega,$$

and  $\Omega$  is the scattering angle. The rules of interaction of a pair of particles are their conservation of momentum and energy:

$$v + w = v' + w', \quad |v|^2 + |w|^2 = |v'|^2 + |w'|^2.$$

A continuous distribution of particles also inherits the laws of conservation of mass, momentum and energy. These invariants are the integrals over phase space of  $f$ ,  $vf$  and  $|v|^2 f$ . Nevertheless, the entropy increases:

$$\frac{d}{dt} \iint -u \log u \, dv \, dx \leq 0.$$

This equation possesses few equilibria. Essentially the only one is the maxwellian  $\varphi(v) = e^{-|v|^2}$ . Using the entropy, one proves ([U] *et al.*)

**Theorem 2.** *The maxwellian  $\varphi$  is asymptotically stable (in several different norms).*

### Korteweg-deVries equation.

The generalized KdV equation is

$$u_t + u_{xxx} + (u^p)_x = 0.$$

The solitary wave  $\varphi(x - ct)$  satisfies an O.D.E. and is given by an explicit formula. There is exactly one that is positive, radial and of lowest energy.

**Theorem 3.** *The solitary wave is stable if  $1 < p < 5$ .*

This theorem is due to [B] et al. In the classical cases  $p = 2$  and  $3$ , the solitary waves enjoy amazing stability properties. This is the celebrated *soliton*. Pego and Weinstein [PW] recently proved its asymptotic stability in a one-sided weighted norm. For larger  $p$ , on the other hand, the solitary wave is unstable [SoS].

**Theorem 4.** *The solitary wave is unstable if  $p \geq 5$ .*

An open problem is to prove blow up if  $p \geq 5$ .

### Some Other Examples.

$$\begin{aligned} \text{NLS:} \quad & iu_t - \Delta u + f(|u|) \frac{u}{|u|} = 0 \\ & \varphi(x)e^{i\omega t} \end{aligned}$$

$$\begin{aligned} \text{BBM:} \quad & u_t - u_{xxt} - f(u)_x = 0 \\ & \varphi(x - ct) \end{aligned}$$

$$\begin{aligned} \text{Boussinesq:} \quad & u_{tt} + u_{xxxx} - f(u)_{xx} = 0 \\ & \varphi(x - ct) \end{aligned}$$

$$\begin{aligned} \text{Incompressible Euler:} \quad & u_t + (u \cdot \nabla)u = \nabla p \quad , \quad \nabla \cdot u = 0 \\ & \varphi(x) \end{aligned}$$

See [L] and [FSV] for the last two examples.

## **2. Instability of BGK Equilibria.**

A plasma is a collection of charged particles. Examples: fluorescent bulb, fusion reactor, solar wind, comet tail, particle accelerator. We assume that collisions between the particles are negligible, that there is no magnetic field, that the space dimension is 1, and that both the mass and the charge of an individual particle are 1. We consider a collection of electrons and ions. Let

$$f_{\pm}(t, x, v) = \text{density of ions(+)} \text{ and electrons(-)}; \quad E(t, x) = \text{electric field.}$$

(All quantities are scalars, because the problem is one-dimensional.) The system governing such a continuous distribution of particles is the Vlasov-Poisson system

$$(VP) \quad \begin{cases} \partial_t f_{\pm} + v \partial_x f_{\pm} \pm E \partial_v f_{\pm} = 0 \\ \partial_x E = \int_{\mathbf{R}} (f_+ - f_-) dv. \end{cases}$$

A similar system describes a continuous distribution of particles under gravity (e.g. stars in a galaxy).

Any equilibrium must satisfy  $(v\partial_x \pm E\partial_v)f_{\pm} = 0$ . Thus

$$(1) \quad \begin{cases} f_{\pm} = \mu_{\pm}(\frac{1}{2}v^2 \mp \beta(x)) \\ E = \beta'(x) \end{cases}$$

satisfies the Vlasov equation for arbitrary  $\mu_+, \mu_-, \beta$ . In order to satisfy the Poisson equation, we require

$$(2) \quad \frac{d^2\beta}{dx^2} = \int_{-\infty}^{\infty} [\mu_+(\frac{1}{2}v^2 - \beta(x)) - \mu_-(\frac{1}{2}v^2 + \beta(x))] dv$$

Such solutions are called *BGK equilibria* after the authors of the original paper [BGK]. There are hundreds of physics papers about them. Here we assume neutrality:

$$(3) \quad \int_{-\infty}^{\infty} [\mu_+(\frac{1}{2}v^2) - \mu_-(\frac{1}{2}v^2)] dv = 0.$$

The homogeneous case  $\beta \equiv 0$ .

This case can be studied easily by linearization. The linearized system is

$$(4) \quad \begin{cases} (\partial_t + v \partial_x) g_{\pm} = \mp (\partial_v \mu_{\pm}) E \\ \partial_x E = \int (g_+ - g_-) dv. \end{cases}$$

We look for exponential solutions of (4):

$$E = e^{i(\xi x - \omega t)}, \quad g_{\mp} = e^{i(\xi x - \omega t)} \tilde{g}_{\mp}(v),$$

from which we easily get the dispersion relation

$$(5) \quad \xi^2 = \int_{-\infty}^{\infty} \frac{\partial_v [\mu_+ + \mu_-]}{v - z} dv = F(z),$$

where  $z = \omega/\xi$ . We look for real  $\xi$  and complex  $\omega$ . Since  $|e^{-i\omega t}| = e^{t\Im\omega}$ , we ask whether there exists  $\Im\omega > 0$  or not. Thus the homogeneous equilibrium is linearly unstable if and only if the image of  $\{\Im z > 0\}$  under  $F$  meets the positive real axis. Penrose [P] found a nice necessary and sufficient condition on  $\mu_{\pm}$  for this to be true. In particular, if  $\mu_+ + \mu_-$  is a decreasing function of  $|v|$ , the equilibrium is linearly stable, but if  $\mu_+ + \mu_-$  deviates sufficiently from monotonicity, it is linearly unstable. The following statement is a special case.

$$(6) \quad \text{If } \int_{-\infty}^{\infty} \frac{\partial_v [\mu_+ + \mu_-]}{v} > 0, \text{ then it is linearly unstable.}$$

Here is an important open problem. If there is a little bump in the graph of  $\mu_+ + \mu_-$  (a small deviation from monotonicity), is the equilibrium *nonlinearly* stable or unstable?

The inhomogeneous case  $\beta \neq 0$ .

We still must solve equation (2). Now we assume condition (6) as well as (3). Then (2) takes the form  $d^2\beta/dx^2 + H'(\beta) = 0$ , where  $H'(0) = 0$  and  $H''(0) > 0$ . Thus the origin is a center. We choose  $\beta(x)$  to be one of the periodic solutions near the origin, of period  $P_\beta$ , say.

**Theorem.** *These BGK equilibria are unstable (both linearly and nonlinearly) with respect to the norm*

$$(5') \quad \|f_\pm\|_1 = \int_0^{2P_\beta} \int_{-\infty}^{\infty} (|f_+| + |f_-|) dv dx + \sup_x \left| \int_{-\infty}^{\infty} (f_+ - f_-) dv \right|$$

under perturbations of period  $2P_\beta$ .

More precisely, assume (3), (6) and  $|\mu''_\pm(s)| = O(|s|^{-\gamma})$  for some  $\gamma > 2$ . Assume  $\beta(\cdot)$  as above. Then there exist  $\epsilon_0 > 0$  and solutions  $\{f_\pm^\delta(t, x, v) : \delta > 0\}$  of (VP) of period  $2P_\beta$  in  $x$  such that

$$(6') \quad \sum_{\pm} \|f_\pm^\delta(0) - \mu_\pm(\frac{1}{2}v^2 \mp \beta(x))\|_{W^{1,1}} < \delta$$

but

$$(7) \quad \sup_{0 \leq t < \infty} \|f_\pm(t) - \mu_\pm\|_1 \geq \epsilon_0.$$

For this theorem see [GuS]. We do not know whether the period-doubling is required.

Proof of linear instability.

The linearized system around the inhomogeneous equilibrium is

$$(VP)_{lin} \quad \begin{cases} [\partial_t + v\partial_x \pm \beta_x\partial_v] g_\pm = -E \partial_v \mu_\pm(\frac{1}{2}v^2 \mp \beta(x)) \\ \partial_x E = \int_{-\infty}^{\infty} (g_+ - g_-) dv. \end{cases}$$

Because of the inhomogeneity  $\beta(x)$ , we cannot expect there to be a solution that is an exponential in  $x$ . We want to perturb from the known case  $\beta \equiv 0$ , but the perturbation is not lower-order. Nevertheless, we invert the operator  $\partial_t + v\partial_x \pm \beta_x\partial_v$  by integrating along its characteristic curves

$$\dot{x} = v, \quad \dot{v} = \pm \beta_x(x).$$

This system has a phase plane portrait just like that of a pendulum. The periodic orbits correspond to the trapped particles. We denote by  $(X^\pm(t; 0, x', v'), V^\pm(t; 0, x', v'))$  the path that passes through the point  $(x', v')$  at  $t = 0$ . Thus from  $(VP)_{lin}$  we get an equation like

$$(8) \quad g_\pm = \int_{characteristics} E \partial_v \mu_\pm.$$

Then we look for solutions

$$\begin{cases} E = e^{-i\omega t} \tilde{E}(x) \\ g_{\pm} = e^{-i\omega t} \tilde{g}_{\pm}(x, v) \end{cases}$$

and plug them into the Poisson equation. We obtain

$$(9) \quad \partial_x \tilde{E}(x) = \int_{-\infty}^{\infty} k(x, x', \omega) \tilde{E}(x') dx',$$

where  $k = k^+ - k^-$  and

$$(10) \quad k^+(x, x', \omega) = - \int_0^{\infty} \int_{-\infty}^{\infty} \delta(x - X^+(t; 0, x', v')) \partial_{v'} \mu_+(\frac{1}{2}v'^2 - \beta(x')) e^{+i\omega t} dv' dt.$$

Integrating (9) from  $-\infty$  to  $x$ , we obtain an equation of the form

$$(11) \quad \tilde{E} = \mathcal{C}(\omega, \beta) \tilde{E}.$$

In case  $\beta \equiv 0$  we know there exists a solution  $\tilde{E} \neq 0$  with  $\Im\omega > 0$ . We want to perturb this solution. We prove three statements:

$$\begin{aligned} \omega &\longrightarrow \mathcal{C}(\omega, \beta) \text{ is analytic in } \{\Im\omega > 0\}. \\ \beta &\longrightarrow \mathcal{C}(\omega, \beta) \text{ is continuous in a certain sense near } \beta = 0. \\ \tilde{E} &\longrightarrow \mathcal{C}(\omega, \beta)\tilde{E} \text{ is a compact operator in } L^1. \end{aligned}$$

Under these conditions it is known that the poles of  $(I - \mathcal{C}(\omega, \beta))^{-1}$  vary continuously as a function of  $\beta$ . (Steinberg's Theorem). Therefore we deduce that (11) has a non-trivial solution with  $\Im\omega > 0$ .

#### Proof of nonlinear instability.

Let us drop the notation  $\pm$ , and write  $(VP)_{lin}$  as

$$(\partial_t + L) g = 0.$$

We have just shown there exists a solution  $g = e^{\lambda t} R(x, v)$  (where  $\lambda = -i\omega$ ). The full nonlinear system (VP) is

$$(12) \quad (\partial_t + L)(f - \mu) = (E - \beta') \cdot \partial_v(f - \mu)$$

where we have simplified the notation. We choose

$$f(0) = \mu + \delta R$$

where  $\delta$  is a small parameter and  $R e^{\lambda t}$  is the solution of  $(VP)_{lin}$  with the maximum  $\Re\lambda$ . Next we write (12) in the integral form

$$(13) \quad f(t) - \mu = \delta R e^{\lambda t} + \int_0^t e^{-L(t-\tau)} (E - \beta') \partial_v(f - \mu) d\tau.$$

which we estimate crudely as

$$(14) \quad \|f(t) - \mu - \delta R e^{\lambda t}\|_{L^1} \leq \int_0^t e^{\Re\lambda(t-\tau)} \|E - \beta'\|_{L^\infty} \|\partial_v(f - \mu)\|_{L^1} d\tau.$$

We treat the dangerous factor involving a derivative of  $f - \mu$  by the

**Lemma.** *If  $\|f - \mu\|_{L^1} = O(e^{\alpha t})$  and if certain norms of  $f - \mu$  are bounded, then*

$$\|\partial_v(f - \mu)\|_{L^1} = O(e^{\alpha t}).$$

Then the instability follows from (14), with  $\alpha = \Re\lambda$ .

### 3. Regularity of Plasma Waves in Kinetic Theory.

#### Vlasov-Poisson system.

The Vlasov-Poisson system in 3 dimensions is

$$(V) \quad \partial_t f + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0 \quad (\gamma = \pm 1)$$

$$(P) \quad E = \nabla \varphi \quad , \quad \Delta \varphi = \int f dv$$

Here  $f(t, x, v)$  is the density of particles,  $E(t, x)$  is the electric field,  $x$  is the position in  $\mathbf{R}^3$  and  $v$  is the velocity in  $\mathbf{R}^3$ . The plasma case is  $\gamma = 1$  and the gravity case is  $\gamma = -1$ .

**Theorem 1.** *If  $0 \leq f_0 \in C_c^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$ , then there exists a unique solution of (VP) in  $C^\infty$  with  $f|_{t=0} = f_0$ , that satisfies certain conditions at  $\infty$ .*

This theorem was proved independently by [Pf] and [LP]. For contrast, in the case  $\gamma = -1$  with  $v$  replaced by  $v/\sqrt{1+v^2}$ , some solutions “blow up in a finite time” [GS1].

For the proof of Theorem 1, Pfaffelmoser estimates the finite support of  $f(t, x, v)$  in the  $v$ -variable. Lions and Perthame estimate the moments in  $v$ , obtaining the *a priori* bound

$$(1) \quad \sup_{0 \leq t \leq T} \iint |v|^m f dv dx < \infty \quad \forall m, \forall T.$$

We follow the latter method. There are two easy *a priori* bounds, the  $L^q$  norm and the energy:

$$\iint |f|^q dv dx < \infty \quad \forall q \quad \text{and} \quad \iint |v|^2 f dv dx + \int |E|^2 dx < \infty.$$

**Lemma 1.** *Let  $M_m(f) = \iint |v|^m f dv dx$ . Then*

$$(2) \quad \left\| \int |v|^\ell f dv \right\|_{L_x^{\frac{m+\ell}{\ell+3}}} \leq c \|f\|_{L^\infty}^{\frac{m-\ell}{m+3}} M_m(f)^{\frac{\ell+3}{m+3}}$$

for  $0 \leq \ell \leq m$ .

Proof. Split

$$\int |v|^\ell f dv = \int_{|v| < A} + \int_{|v| > A} \leq c \|f\|_\infty A^{3+\ell} + c A^{-m+\ell} \int |v|^m f dv$$

and then optimize  $A$ .

**Lemma 2.** *If  $\sup_{[0, T]} M_m(f)$  is bounded  $\forall m \forall T$ , then  $E(t, x)$  is bounded and Theorem 1 follows.*

Proof. Let  $\rho = \int f dv$ . Then by the Poisson equation

$$E(t, x) = c \int_{\mathbf{R}^3} \frac{x-y}{|x-y|^3} \rho(y) dy = \int_{|x-y| > 1} + \int_{|x-y| < 1}.$$

Hence for all  $q > 3$

$$\begin{aligned} |E(t, x)| &\leq c \|\rho\|_{L^1} + c \|\rho\|_{L^q} \\ &\leq c \|f\|_{L^1} + c \|f\|_{L^\infty}^\alpha M_m(f)^\beta \end{aligned}$$

by Lemma 1, choosing  $q = \frac{m+3}{3}$ ,  $\ell = 0$  and  $m > 6$ . Thus  $E$  is bounded.

**Lemma 3.**

$$(3) \quad \frac{dM_m(f)}{dt} \leq c \|E\|_{L^{m+3}} [M_m(f)]^{\frac{m+2}{m+3}}$$

Proof. By the Vlasov equation,  $dM_m/dt = -\gamma \int E (\int |v|^m \nabla_v f dv) dx$ . Integrating by parts and using the Hölder inequality,

$$\begin{aligned} \frac{dM_m}{dt} &\leq m \|E\|_{L^{m+3}} \left\| \int |v|^{m-1} f dv \right\|_{L^{\frac{m+3}{m+2}}} \\ &\leq m \|E\|_{L^{m+3}} \|f\|_{L^\infty}^{\frac{1}{m+3}} M_m^{\frac{m+2}{m+3}} \end{aligned}$$

by Lemma 1. By Gronwall, it suffices to prove

$$(4) \quad \|E\|_{L^{m+3}} \leq c_1 + c_2 M_m^{\frac{1}{m+3}}.$$

**Lemma 4.**

$$(5) \quad \|E(t)\|_{L_x^{m+3}} \leq c + c \int_0^t (t-\tau) \left\| \int_{\mathbf{R}^3} (Ef)(\tau, x - (t-\tau)v, v) dv \right\|_{L_x^{m+3}} d\tau.$$

Proof. Write (V) as  $(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot [-Ef]$ . Integrate in  $v$ , and use the straight line characteristics. Let  $\rho = \int f dv$  and  $\rho_0 = \int f_0(x - tv, v) dv$ . Thus

$$\rho = \rho_0 + c \nabla_x \cdot \int_0^t (t-\tau) \int_{\mathbf{R}^3} (Ef)(\tau, x - (t-\tau)v, v) dv d\tau.$$

Next use the Poisson equation  $E = \nabla \varphi$  and  $\Delta \varphi = \rho$  in order to estimate  $E$ .

Proof of Theorem 1. We estimate the right side of Lemma 4. The proof is tricky. An observation of Bouchut permits the use of the Marcinkiewicz space  $M^p(\mathbf{R}^3)$ . For fixed  $t, x$  we estimate

$$(6) \quad \left| \int (Ef)(\tau, x - (t-\tau)v, v) dv \right| \leq \|E\|_{M_v^{3/2}} \|f\|_{L_v^1}^{1/3} \|f\|_{L_v^\infty}^{2/3}.$$

Next,

$$\|E\|_{M_v^{3/2}} = (t-\tau)^{-2} \|E\|_{M_x^{3/2}} \leq c(t-\tau)^{-2} \|\rho\|_{L_x^1}$$

by the  $L^1$  theory of the Laplacian. The other factor in (6) is estimated in  $L_x^{m+3}$  as

$$\begin{aligned} \left\| \|f(\tau, x - (t-\tau)v, v)\|_{L_v^1}^{1/3} \right\|_{L_x^{m+3}} &= \left\| \int f(\tau, x - (t-\tau)v, v) dv \right\|_{L_x^{1+m/3}} \\ &\leq c M_m(f)^{\frac{1}{m+3}} \quad \text{by Lemma 1.} \end{aligned}$$

Thus (5) together with (6) takes the form

$$(7) \quad \|E(t)\|_{L^{m+3}} \leq c + c \int_0^t \frac{1}{t-\tau} [M_m(f)(\tau)]^{\frac{1}{m+3}} d\tau.$$

We insert (7) into (3). This fails to be sufficient because of the non-integrable kernel  $(t-\tau)^{-1}$ . So the estimate must be modified near  $\tau = t$  in order to make it work. We omit this final part of the proof.

Plasma with magnetic field.

The *Relativistic Vlasov-Maxwell System (RVM)* is

$$(V) \quad \partial_t f_\alpha + \hat{v}_\alpha \cdot \nabla_x f_\alpha + e_\alpha (E + \hat{v}_\alpha \wedge B) \cdot \nabla_v f_\alpha = 0$$

$$(M) \quad \begin{aligned} \partial_t E &= \nabla \wedge B - j & \nabla \cdot E &= \rho \\ \partial_t B &= -\nabla \wedge E & \nabla \cdot B &= 0 \end{aligned}$$

$$\rho = \sum_\alpha \int e_\alpha f_\alpha dv \quad , \quad j = \sum_\alpha \int e_\alpha \hat{v}_\alpha f_\alpha dv$$

where position  $x \in \mathbf{R}^3$ , momentum  $v \in \mathbf{R}^3$ , velocity  $\hat{v}_\alpha = v/\sqrt{|v|^2 + m_\alpha^2}$ , mass of one particle =  $m_\alpha$ , charge of one particle =  $e_\alpha$ , speed of light = 1, density of  $\alpha$ -species of particle =  $f_\alpha(t, x, v)$  (for  $1 \leq \alpha \leq N$ ), electric field =  $E$ , magnetic field =  $B$ , charge =  $\rho$  and current =  $j$ . In general, regularity is an open problem for (RVM). But it is known in some cases.

**Theorem 2.** *Let initial data be given in  $C_c^\infty$  that satisfies the appropriate constraints. If there is an a priori bound on the support, namely,*

$$\sup\{|v| : f_\alpha(t, x, v) \neq 0 \text{ for some } t \in [0, T], x \in \mathbf{R}^3, 1 \leq \alpha \leq N\} < \infty$$

for all  $T < \infty$ , and the bound is uniform in a certain approximation, then there is a unique  $C^\infty$  solution for all time (vanishing at spatial infinity).

Thus a singularity (if there is any) must come from the particles moving arbitrarily near the speed of light. This theorem [GS1] can be used to deduce the regularity of a plasma that is “almost neutral” [GS2]. For the proof of the theorem, let us omit the subscript  $\alpha$  and  $m_\alpha$  and  $e_\alpha$ . Thus we abbreviate (RVM) as

$$(V) \quad \partial_t f + \hat{v} \cdot \nabla_x f + (E + \hat{v} \wedge B) \cdot \nabla_v f = 0$$

$$\rho = \int f dv \quad , \quad j = \int \hat{v} f dv \quad , \quad \hat{v} = v/\sqrt{1 + |v|^2}$$

together with the Maxwell equations (M). Our main goal is to prove a  $C^1$  estimate.

**Lemma 1.** *Let  $D = \partial/\partial x_k$ . Then*

$$(8) \quad |Df(t)|_\infty \leq c + c \int_0^t (|DE(\tau)|_\infty + |DB(\tau)|_\infty) |\nabla_v f(\tau)|_\infty d\tau.$$

A similar estimate holds for  $D = \partial/\partial v_k$ .

Proof. Differentiate the equation and integrate along the characteristics.

**Lemma 2.** *The field  $(E, B)$  and its first derivatives  $(DE, DB)$  can be represented as integrals of  $f$  itself on the backward characteristic cone  $K$ .*

Proof. From (M) we have  $(\partial_t^2 - \Delta)E = -\nabla_x \rho - \partial_t j$  and a similar equation for  $B$ . Hence

$$E(t, x) = E^{(0)} + \int_K \int_{\mathbf{R}^3} (\dots) Df \, dv \, dK$$

where  $(\dots)$  is an explicit kernel depending only on the independent variables, and  $E^{(0)}$  depends on the initial data. We must get rid of the derivative  $Df$ . Now  $Df$  is a linear combination of time and space derivatives of  $f$ . Split it as a linear combination of  $Sf, T_1f, T_2f, T_3f$ , where

$$S = \partial_t + \sum_k \hat{v}_k \partial_{x_k} \quad , \quad T_j = -\omega_j \partial_t + \partial_{x_j}$$

( $j = 1, 2, 3$ ) are the characteristic derivatives and  $\omega_j$  is an angular variable on  $K$ . Since  $|\hat{v}| < 1$ , they are linearly independent. The operators  $T_1, T_2$  and  $T_3$  are tangential along the cone  $K$ , so an integration by parts makes them act only on the kernel. By (V), we may substitute  $Sf = -\nabla_v \cdot [(E + \hat{v} \wedge B)f]$  and then we integrate by parts in  $v$ . We end up with the representation

$$(9) \quad E = E^{(0)} + \iint (\dots) f \, dv \, dK + \iint (\dots) (E + \hat{v} \wedge B) f \, dv \, dK.$$

The new kernels, first derivatives of the old ones, are harmless because  $v$  runs over the support of  $f$ , which is assumed to be bounded. Doing the splitting twice, we obtain a similar representation for derivatives, of the form

$$(10) \quad DE = DE^{(0)} + \iint \{(\dots)f + (\dots)Ef + (\dots)E^2f + (\dots)(DE)f\} \, dv \, dK$$

(plus similar terms with  $B$  instead of  $E$ ).

Proof of Theorem 2. From Lemma 2, we estimate

$$|DE|_\infty + |DB|_\infty \leq c + c \int_0^t \{|DE|_\infty + |DB|_\infty + \log^+ |Df|_\infty\} \, d\tau.$$

The last term arises because one of the final kernels leads to a singular integral. Combining (1) and (6) leads to the estimate

$$|Df(t)|_\infty \leq c + c \int_0^t (\log^+ |Df(\tau)|_\infty) |Df(\tau)|_\infty \, d\tau.$$

This estimate implies that  $Df$  is bounded (for bounded  $t$ ), just as in Gronwall's inequality. Thus we have the desired  $C^1$  estimate.

#### 4. Breathers and Wave Maps.

##### Breathers.

The sine-Gordon equation

$$(SG) \quad \alpha_{tt} - \alpha_{xx} + \sin \alpha = 0$$

is completely integrable. It has traveling wave solutions called “kinks” that are asymptotic to constants at  $\pm\infty$ . A special combination of kinks, traveling in opposite directions, is periodic in  $t$  and localized in  $x$ , but is not a standing wave. It is the “breather” and is given by the explicit formula

$$(1) \quad \alpha(x, t) = 4 \arctan \left\{ \frac{b \sin(at)}{a \cosh(bx)} \right\}, \quad a^2 + b^2 = 1.$$

For years people searched for breather solutions of other equations, without success. In recent years it has been proved that the classical breather (1) is unique among various classes of equations ([BMW], [Ky], [D]).

In the following discussion, we will switch the roles of  $t$  and  $x$ , so that the breather will be periodic in  $x$  and vanish as  $t \rightarrow \pm\infty$ .

##### Wave Maps.

Consider a map between two manifolds  $M \xrightarrow{u} N$  that satisfies the Euler-Lagrange equation of the Lagrangian

$$\mathcal{L}(u) = \int_M \|du\|^2.$$

Such a map is called a *harmonic map*. [The simplest example is the classical Dirichlet integral  $\mathcal{L}(u) = \int |\nabla u|^2 dx$ , where  $u$  is a classical harmonic function.] However, we shall choose  $M = \text{Minkowski space} = \mathbf{R}_t \times \mathbf{R}_x^n$ , in which case our map is called a *wave map* by some authors. We shall also choose the target manifold  $N$  to be a Riemannian manifold and consider  $N$  to be embedded in some  $\mathbf{R}^m$ . The Euler-Lagrange equation asserts that at each point  $u_{tt} - \Delta u$  is a normal vector to  $N$ . In fact,

$$(WM) \quad u_{tt}^i - \Delta u^i + \Gamma_{jk}^i(u) (u_t^j u_t^k - \nabla u^j \cdot \nabla u^k) = 0$$

where the  $\Gamma_{jk}^i$  are the Christoffel symbols of  $N$ . (WM) is a simpler field theory than Yang-Mills and General Relativity but possesses some of the same difficulties. In fact, certain special solutions of these other theories are themselves wave maps.

In this lecture I will make the following choices:

$$M = M_p^2 = \mathbf{R}_t^1 \times S_x^1 \quad (\text{periodic } 1 + 1 \text{ dim. Minkowski space})$$

$$N = S^2 \subset \mathbf{R}^3 \quad (\text{ordinary 2-sphere}).$$

Then  $u_{tt} - \Delta u = \lambda u$  where  $\lambda$  is a scalar function. Hence

$$(WM) \quad \begin{cases} u_{tt} - u_{xx} + (|u_t|^2 - |u_x|^2) u = 0 \\ |u(t, x)| = 1. \end{cases}$$

##### Relationship between (WM) and (SG).

If  $u(t, x)$  is a wave map, define its angle function  $\alpha(t, x)$  as the angle between the characteristic derivatives  $u_t - u_x$  and  $u_t + u_x$ . Let  $h = \frac{1}{2}|u_t - u_x|$  and  $k = \frac{1}{2}|u_t + u_x|$ . The first result is well-known to geometers.

**Theorem 1.** *If  $u$  satisfies (WM), then  $\alpha$  satisfies the generalized sine-Gordon equation*

$$(GSG) \quad \alpha_{tt} - \alpha_{xx} - h(x-t)k(x+t) \sin \alpha = 0.$$

**Theorem 2.** *Conversely, let  $\alpha(t, x)$  be a breather solution of*

$$(GSG) \quad \alpha_{tt} - \alpha_{xx} - (\ell^2 - \omega^2) \sin \alpha = 0,$$

*for some positive integer  $\ell$  and some real  $\omega$ , with  $\omega^2 < \ell^2$ . Then there exists a family of wave maps  $u : M_p^2 \rightarrow S^2$  where  $h = \frac{1}{2}(\ell - \omega)$ ,  $k = \frac{1}{2}(\ell + \omega)$  and  $\alpha$  is the angle function of  $u$ .*

The wave maps in Theorem 2 are called the *breather wave maps*. They are given by explicit formulas. We summarize these two theorems as follows. “Each wave map leads to a solution of (GSG), and each breather comes from a family of wave maps.”

#### Relationship with Stability Theory.

Consider the traveling wave map

$$M_p^2 \ni (t, x) \longrightarrow \varphi(t, x) = \begin{bmatrix} \cos(\ell x - \omega t) \\ \sin(\ell x - \omega t) \\ 0 \end{bmatrix} \in S^2 \subset \mathbf{R}^3.$$

**Theorem 3.**  *$\alpha$  is stable if  $\omega^2 > \ell^2$ , and unstable if  $\omega^2 < \ell^2$ , in the energy norm.*

For instance, if  $\omega = 0$  and  $\ell$  is a positive integer, there exist exactly  $\ell$  unstable modes. That is,  $\ell$  is the dimension of the unstable manifold of the linearized problem.

**Theorem 4.** *If  $\omega = 0$  and  $\ell$  is a positive integer, then each unstable mode corresponds to a wave map  $u(t, x)$  such that*

$$\lim_{t \rightarrow -\infty} \|u(t, x) - \varphi(x)\|_{energy} = 0.$$

*Among them,  $(\ell - 1)$  are breather wave maps (with different choices of the parameters), and the remaining one satisfies a simple O.D.E.*

The breather wave maps also satisfy

$$\lim_{t \rightarrow +\infty} \|u(t, x) - \varphi(x)\|_{energy} = 0.$$

Therefore the map  $t \mapsto u(t, \cdot)$  may be called *homoclinic*. Theorem 3 may be found in [GSS] and Theorems 1,2 and 4 in [SS].

#### Non-round target manifold.

We define a *generalized breather* as a homoclinic wave map from  $M_p^2$  into a compact Riemannian manifold  $N^2$ .

**Theorem 5.** *There exist generalized breathers whose target manifolds  $N^2$  are 2-spheres with non-standard metrics.*

Proof. We only consider the map on the top half cylinder  $\{t < 0\}$  because on the bottom half  $\{t > 0\}$  it is symmetric. The wave map  $u : (M_p^2, \eta) \mapsto (S^2, h)$  is factored as  $u = f_2 \circ f_1$ , thus

$$u : (M_p^2, \eta) \xrightarrow{f_1} (N, g) \xrightarrow{f_2} (S^2, h)$$

by defining  $N = M_p^2$  as sets, and the metric  $g$  of  $N$  as the pulled-back metric of  $S^2$ . Thus  $N$  has a singular metric  $\geq 0$  and  $f_2$  is an isometric immersion. This can be done so that  $f_1$  is still a wave map.

Furthermore,  $N = N_1 \cup N_3 \cup N_{sing}$ , where  $f_2$  is a 1-1 mapping on  $N_1$ ,  $f_2$  is a 3-1 mapping on  $N_3$ , and  $N_{sing}$  is the curve where the metric  $g$  is singular. We perturb the metric  $g$  within  $N_1$  to obtain a new metric  $\tilde{g}$  in such a way that  $\tilde{f}_1$  is still a wave map. We push  $\tilde{g}$  forward to get a new (non-round) metric  $\tilde{h}$  on the sphere. Thus

$$\tilde{u} : (M, \eta) \xrightarrow{\tilde{f}_1} (N, \tilde{g}) \xrightarrow{\tilde{f}_2} (S, \tilde{h})$$

is the same as the original map but considered with the new metrics. Since  $\tilde{f}_1$  is a wave map and  $\tilde{f}_2$  is an isometric immersion,  $\tilde{u}$  is a wave map from  $(M_p^2, \eta)$  into  $(S, \tilde{h})$ . It is still homoclinic.

## 5. Gain of Regularity of Dispersive Waves.

Previously (for Vlasov systems) we discussed the regularity problem: do regular initial data lead to a regular solution? Now we shall discuss the gain-of-regularity problem: do *non-regular* initial data lead to a regular solution for  $t > 0$ ?

Example 1. The free Schrödinger equation  $i\partial u/\partial t = \Delta u$  ( $x \in \mathbf{R}^n$ ). Let  $u(0, x) = \varphi(x)$  be the initial data. Then  $\varphi \mapsto u(t)$  is unitary on  $L^2(\mathbf{R}^n)$  so that the solution at a fixed time is no more regular than  $\varphi$ . Nevertheless, it is true that

$$(1) \quad \sup_R \int_{|x| < R} \int_{-\infty}^{\infty} |D_x^{1/2} u(t)|^2 dt dx \leq C \|\varphi\|_{L^2(\mathbf{R}^n)}.$$

This property is called “local smoothing” ([Sj],[V],[CS]). If we assume  $\varphi \in L^1(\mathbf{R}^n)$  with compact support, then the solution is analytic for  $t \neq 0$ , as we see immediately from the classical formula

$$(2) \quad u(t) = c_n t^{-n/2} e^{i|x|^2/4t} * \varphi.$$

Thus rapid decay of  $\varphi(x)$  as  $|x| \rightarrow \infty$  leads to great smoothness for  $t \neq 0$ .

Example 2. The KdV equation  $u_t + u_{xxx} + u u_x = 0$ . This equation, although nonlinear, enjoys a similar property to the previous example, as proved in [K]. In fact, if  $\int_{-\infty}^{\infty} |\varphi(x)|^2 (1 + e^x) dx < \infty$ , then  $u \in C^\infty$  for  $t > 0$ .

The intuition in both examples is as follows. Singularities travel along bicharacteristic rays and at infinite speed. If no initial singularities come from spatial infinity, then all the singularities disappear immediately.

Generalized bicharacteristics.

Suppose that, after linearization, the equation takes the form

$$(3) \quad i \frac{\partial u}{\partial t} = a(x, D)u \quad (D_j = \frac{1}{i} \frac{\partial}{\partial x_j})$$

with a real symbol  $a(x, \xi)$ . Then the bicharacteristics are the solutions of the O.D.E.

$$(4) \quad \dot{x} = \frac{\partial a}{\partial \xi} \quad , \quad \dot{\xi} = -\frac{\partial a}{\partial x}.$$

For the 1-dimensional Schrödinger equation,  $a(x, \xi) = -\xi^2$  and  $\dot{\xi} = -2\xi$ . Thus for  $\xi$  positive the bicharacteristic rays move to the left, while for  $\xi$  negative they move to the right. So the analysis will require splitting into  $\xi < 0$  and  $\xi > 0$ , and therefore the use of pseudo-differential operators.

For the Airy equation (linearized KdV),  $u_t + u_{xxx} = 0$ , we have  $a(x, \xi) = -\xi^3$  and  $\dot{x} = -3\xi^2$ . Therefore all singularities move to the left. This is the reason for Kato's assumption that  $\varphi(x) \rightarrow 0$  rapidly as  $x \rightarrow +\infty$ . (The solitons move to the right.)

**Theorem 1.** Consider the linear equation (3). Assume

- (i) it is dispersive:  $|\partial a / \partial \xi| \rightarrow \infty$  uniformly in  $x$  as  $|\xi| \rightarrow \infty$ .
- (ii) no rays are trapped: If  $x(t)$  is a solution of (4), then  $|x(t)| \rightarrow \infty$  uniformly for  $|x(0)|$  and  $|\xi(0)|^{-1}$  bounded.
- (iii) it is flat at  $\infty$ :

$$|\partial_x^\alpha \partial_\xi^\beta [a(x, \xi) - a(\infty, \xi)]| \leq C_{\alpha\beta} \langle \partial a / \partial \xi \rangle \langle \xi \rangle^{1-|\beta|} \langle x \rangle^{-1-\delta-|\alpha|}$$

for all  $\alpha, \beta$  and for some  $\delta > 0$ .

(iv)  $\int_{\mathbf{R}^n} |\varphi(x)|^2 |x|^k dx < \infty, \quad \forall k \geq 0.$

Then  $u$  is a  $C^\infty$  function for  $t \neq 0$ .

Nonlinear Dispersive Waves.

If the equation is nonlinear, but the dispersive term dominates, then the gain of regularity is still true. KdV-type equations are particularly simple because the singularities all move in one direction so that microlocal analysis can be avoided. Here is a generalization of Kato's Theorem.

**Theorem 2.** Consider the equation

$$(5) \quad u_t + f(u_{xxx}, u_{xx}, u_x, u) = 0$$

where  $f \in C^\infty$  and

$$(6) \quad \partial f / \partial (u_{xxx}) \geq c > 0 \quad , \quad \partial f / \partial (u_{xx}) \leq 0.$$

If  $u(t, x)$  is a solution in any time interval  $(t_1, t_2) \times \mathbf{R}_x$  such that

$$(7) \quad \sup_{t_1 < t < t_2} \int_{-\infty}^{\infty} \{|u_{xxxxx}|^2 + |u|^2\} (1 + x_+)^K dx < \infty$$

$\forall K \geq 0$ , then  $u \in C^\infty$  in the same time-interval  $(t_1, t_2) \times \mathbf{R}_x$ .

Remarks. (a) There is a local-in-time existence and uniqueness theorem in the space defined by (7). Thus Theorem 2 says there is regularity up to the (possible) blow-up time.

(b) A more precise version of Theorem 2 assumes (7) for a fixed  $K$  and concludes a finite gain of regularity. (Similarly for Theorem 1.)

(c) We require 5 derivatives in  $L^2$  because of the strong nonlinearity.

(d) Both Theorem 1 and Theorem 2 are proved by a generalization of the energy method ([CKS1] and [CKS2]).

Sketch of Proof of Theorem 2.

For simplicity, consider the equation  $u_t + f(u_{xxx}) = 0$  where  $f' \geq c > 0$ . We write  $\partial = \partial/\partial x$  and  $u_N = \partial^N u$ . We apply the operator  $\partial^N$  to the equation and then multiply it by  $pu_N$  where  $p = p(t, x)$  is a weight function to be chosen later. Thus

$$(8) \quad \frac{\partial u_N}{\partial t} \cdot pu_N + \partial^N [f(u_3)] \cdot pu_N = 0.$$

The first term in (8) equals  $\frac{\partial}{\partial t}(\frac{1}{2}p u_N^2) - \frac{1}{2} \frac{\partial p}{\partial t} u_N^2$ . The second term in (8) has the *leading part*

$$\begin{aligned} p f'(u_3) u_{N+3} u_N &= -p f'(u_3) u_{N+2} u_{N+1} - \partial[p f'(u_3)] u_{N+2} u_N + \partial\{\dots\} \\ &= \frac{3}{2} \partial[p f'(u_3)] u_{N+1}^2 + \partial^2[p f'(u_3)] u_{N+1} u_N + \partial\{\dots\} \\ &= \frac{3}{2} \partial[p f'(u_3)] u_{N+1}^2 - \frac{1}{2} \partial^3[p f'(u_3)] u_N^2 + \partial\{\dots\}. \end{aligned}$$

The *next part* is

$$\begin{aligned} N f''(u_3) u_4 u_{N+2} \cdot pu_N &= -N p f''(u_3) u_4 u_{N+1}^2 - N \partial[p f''(u_3) u_4] u_{N+1} u_N + \partial\{\dots\} \\ &= -N p \partial[f'(u_3)] u_{N+1}^2 + \frac{N}{2} \partial^2[p f''(u_3) u_4] u_N^2 + \partial\{\dots\} \end{aligned}$$

The succeeding terms do not yield  $u_{N+1}^2$  but only  $u_N^2$ ,  $u_{N-1}^2$ , etc. Therefore (8) yields an identity of the form

$$(9) \quad \frac{\partial}{\partial t} [p u_N^2] + q u_{N+1}^2 + r u_N^2 + \dots = 0$$

where  $q = 3f'(u_3) \partial p + (3 - 2N) \partial[f'(u_3)] p$  and the dots represent lower-order terms. We choose

$$0 < q(t, x) \sim \begin{cases} e^{-\sigma|x|} \\ x^k \end{cases} \quad \text{and} \quad 0 < p(t, x) \sim \begin{cases} e^{-\sigma|x|} & \text{as } x \rightarrow -\infty \\ x^{k+1} & \text{as } x \rightarrow +\infty \end{cases}$$

and  $p(0, x) = 0$ . When (9) is integrated, it provides the gain of one derivative in  $L^2$ , from the  $N^{th}$  to the  $(N + 1)^{st}$ . At each induction step, the weight function loses one power of  $x$  (from  $x_+^k$  to  $x_+^{k-1}$ ) and gains one power of  $t$  (from  $t^k$  to  $t^{k+1}$ ).

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**Acknowledgements.** This research is supported in part by NSF Grant 93-22146 and ARO Grant DAAH 04-93-G0198.

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$$-\Delta u = \lambda \exp\left(\frac{\alpha u}{\alpha + u}\right) \text{ v.s. } -\Delta u = \lambda e^u$$

TAKASHI SUZUKI

**Abstract.** The problem

$$-\Delta u = \lambda \exp\left(\frac{\alpha u}{\alpha + u}\right) \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega$$

arises in the theory of gas combustion, where  $\Omega \subset \mathcal{R}^n$  is a bounded domain with smooth boundary  $\partial\Omega$ , and  $\lambda > 0$  and  $\alpha > 0$  stand for parameters. Several works have been devoted to its bifurcation diagram, but not so much is known for general domains. Our purpose is to present a theoretical study for S-shaped bifurcation and mushroom.

## 1. Introduction

Given a chemical constant  $\alpha > 0$ , the modified Gel'fand problem

$$-\Delta u = \lambda \exp\left(\frac{\alpha u}{\alpha + u}\right) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (1)$$

describes the steady state of gas combustion subject to the Arrhenius law (c.f. [13]). Here,  $\Omega \subset \mathcal{R}^n$  denotes a bounded domain with sufficiently smooth boundary  $\partial\Omega$ , and  $\lambda > 0$  a physical parameter. We are concerned with the total set of solutions

$$\mathcal{S} = \{(\lambda, u(x)) \mid \text{classical solutions of (1)}\} \subset \mathcal{R}_+ \times C(\bar{\Omega}),$$

or more precisely, the effect of domain shape to its connected components. We have the following facts on uniqueness and nonuniqueness ([1], [2], [10], [14], [15]).

**Proposition 1** *If  $\alpha \gg 1$ , there exists a nonempty bounded open interval  $\Lambda \subset (0, +\infty)$  such that (1) admits an ordered triple of solutions.*

**Proposition 2** *Given  $\alpha > 0$ , the solution  $u(x)$  of (1) is unique and stable for  $0 < \lambda \ll 1$  and  $\lambda \gg 1$ .*

The nonlinearity  $f(u) = \exp\left(\frac{\alpha u}{\alpha + u}\right)$  is uniformly bounded and hence Schauder's fixed point theorem assures the existence of a solution  $u(x)$ , and furthermore, a priori bounds of  $\|u\|_{L^\infty}$  for each  $\lambda > 0$ . Combined with Proposition 2, those facts imply the following.

**Theorem 3** *Let  $\mathcal{S}_0 \subset \mathcal{R}_+ \times C(\bar{\Omega})$  be the connected component of  $\mathcal{S}$  containing the trivial solution  $(\lambda, u(x)) = (0, 0)$  on its boundary. Then, any connected component  $\mathcal{S}_1 \neq \mathcal{S}_0$  of  $\mathcal{S}$ , if it exists, must be bounded.*

If such  $\mathcal{S}_1$  is a continuum, we call it a *mushroom* ([7]). Furthermore, any subset of  $\mathcal{S}$  homeomorphic to  $\mathcal{R}$  we call a *branch*.

Proposition 2 has been proven by ordered Banach space or the Green function, but another way based on Hardy's inequality is proposed ([6]). Furthermore, Proposition 1 can be refined to control the component  $\mathcal{S}_0$ . Namely, it forms branches for  $0 < \lambda \ll 1$  and  $\lambda \gg 1$ , bending in the "generalized sense" unless  $\mathcal{S}$  itself makes up a unified branch parametrized by  $\lambda$ .

## 2. S-shaped bifurcation

Proposition 1 indicates two more bendings of  $\mathcal{S}_0$ , the S-shaped bifurcation, while the following theorem is valid for the general nonlinearity satisfying  $f(u) \geq 0$  for  $u \geq 0$  ([5]).

**Theorem 4** *Let  $\Omega \subset \mathcal{R}^2$  be symmetric with respect to  $x_1$  and  $x_2$  axes, where  $x = (x_1, x_2) \in \Omega$ . Furthermore, let it be convex with respect to both axes, which means that any segment parallel to an axis is contained in  $\Omega$  if its end points are so. Then  $\mathcal{S}$  is a branch.*

Take the case that  $\Omega$  has still two axile symmetries, but is convex with respect to only one direction, say  $x_2$ . Such  $\Omega$  does not satisfy the assumptions of the above theorem, but can express a dumbbell-like region. Let  $u(x)$  be a solution symmetric with respect to both axes. The following lemma provides some information concerning the secondary bending or the asymmetric bifurcation.

**Lemma 5** *Under those circumstances, the second eigenfunction  $\psi_2(x)$  of the linearized operator can be taken in the following forms whenever the second eigenvalue is nonnegative.*

1.  $\psi_2(x)$  is symmetric and anti-symmetric with respect to  $x_1$  and  $x_2$  axes, respectively. Its nodal domains are  $\Omega \cap \{\pm x_1 > 0\}$ .

2.  $\psi_2(x)$  is symmetric with respect to both axes and has a nodal domain with its closure contained in  $\Omega$ .

### 3. Gel'fand equation

S-shaped bifurcation suggested by Proposition 1 is supposed to produce mushrooms if parameters change significantly more. Our viewpoint is to take into account of the domain perturbation. First, we deform a ball dumbbell-likely, preserving the symmetries on each axis. Then, breaking one of them will cause mushrooms.

We have not got any rigorous proof, but the consideration developed below seems to support the procedure. This is performed for two dimensional case  $n = 2$  through the study on  $\alpha = +\infty$ , that is, the Gel'fand problem

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (2)$$

Let  $\tilde{\mathcal{S}}$  be the total set of solutions  $\{(\lambda, u(x))\}$  of (2). Justifying the above idea, we note the following theorem of [3].

**Theorem 6** *Any bounded branch of  $\tilde{\mathcal{S}}$  has a homeomorphic copy in  $\mathcal{S}$  if  $\alpha \gg 1$ .*

As for (2) we have the following ([8], [12], [11]).

**Proposition 7** *Let  $\{(\lambda, u(x))\}$  be any family of solutions with  $\lambda \downarrow 0$  and set*

$$\Sigma = \int_{\Omega} \lambda e^u dx.$$

*Then,  $\{\Sigma\}$  accumulates to  $8\pi m$  with some  $m = 0, 1, 2, \dots, +\infty$ . Passing to a subsequence,  $\{u(x)\}$  behaves as follows.*

1. *If  $m = 0$ ,  $\|u\|_{L^\infty} \rightarrow 0$ .*
2. *If  $m = +\infty$ ,  $u(x) \rightarrow +\infty$  for any  $x \in \Omega$ .*
3. *If  $0 < m < +\infty$ , there is a set  $\mathcal{B} = \{x_1^*, x_2^*, \dots, x_m^*\} \subset \Omega$  of  $m$  points so that*

$$u|_{\mathcal{B}} \rightarrow +\infty, \quad \|u\|_{L^\infty(K)} = O(1), \quad u(x) \rightarrow 8\pi \sum_{j=1}^m G(x, x_j^*), \quad (3)$$

*where  $K \subset\subset \bar{\Omega} \setminus \mathcal{B}$  and the convergence is locally uniform in  $\bar{\Omega} \setminus \mathcal{B}$ .*

Here and henceforth,  $G(x, y)$  denotes the Green function for  $-\Delta$  in  $\Omega$  with  $\cdot|_{\partial\Omega} = 0$  and  $\{x_j^*\}_{j=1}^m$  are so related as

$$\frac{1}{2}\nabla R(x_j^*) + \sum_{l \neq j} \nabla_x G(x_l^*, x_j^*) = 0 \quad (1 \leq j \leq m), \quad (4)$$

where

$$R(x) = \left[ G(x, Y) + \frac{1}{2\pi} \log|x - y| \right]_{y=x}.$$

In the case of  $m = 1$ , (4) indicates that  $x_1^*$  is a critical point of  $R(x)$ . Such a point is called a *core*. If it is nondegenerate as a critical point of  $R(x)$ , we call it a nondegenerate core.

**Proposition 8** *Let  $\Omega \subset \mathcal{R}^2$  be simply connected and  $x_1^*$  a nondegenerate core. Then, there exists a unique continuous family of solutions of (2), denoted by*

$$\tilde{\mathcal{S}}^* = \{(\lambda, u(x))\}_{0 < \lambda \ll 1} \subset \mathcal{R}_+ \times C(\bar{\Omega}),$$

satisfying (3) as  $\lambda \downarrow 0$ .

The following are known. Let  $\Omega \subset \mathcal{R}^2$  be simply connected, a core  $x_1^* \in \Omega$  be given, and take a conformal mapping  $g : B \equiv \{|z| < 1\} \subset \mathcal{C} \rightarrow \Omega$  satisfying  $g(0) = x_1^*$ . Then we have  $g''(0) = 0$ , and the nondegeneracy of  $x_1^*$  is expressed as  $\sigma = |g'''(0)/g'(0)| \neq 2$ . Any domain admits a core. It is unique if  $\Omega$  is convex, and furthermore, then  $\sigma < 2$  so that is nondegenerate.

**Proposition 9** *The connected component  $\tilde{\mathcal{S}}_0$  of  $\tilde{\mathcal{S}}$ , containing  $(0, 0)$  on the boundary, forms a branch bending just once, provided that*

$$-|a_1|^2 + \sum_{k=3}^{\infty} \frac{k^2}{k-2} |a_k|^2 < 0,$$

where  $g(z) = \sum_{k=0}^{\infty} a_k z^k$ .

Recently, we have found the following.

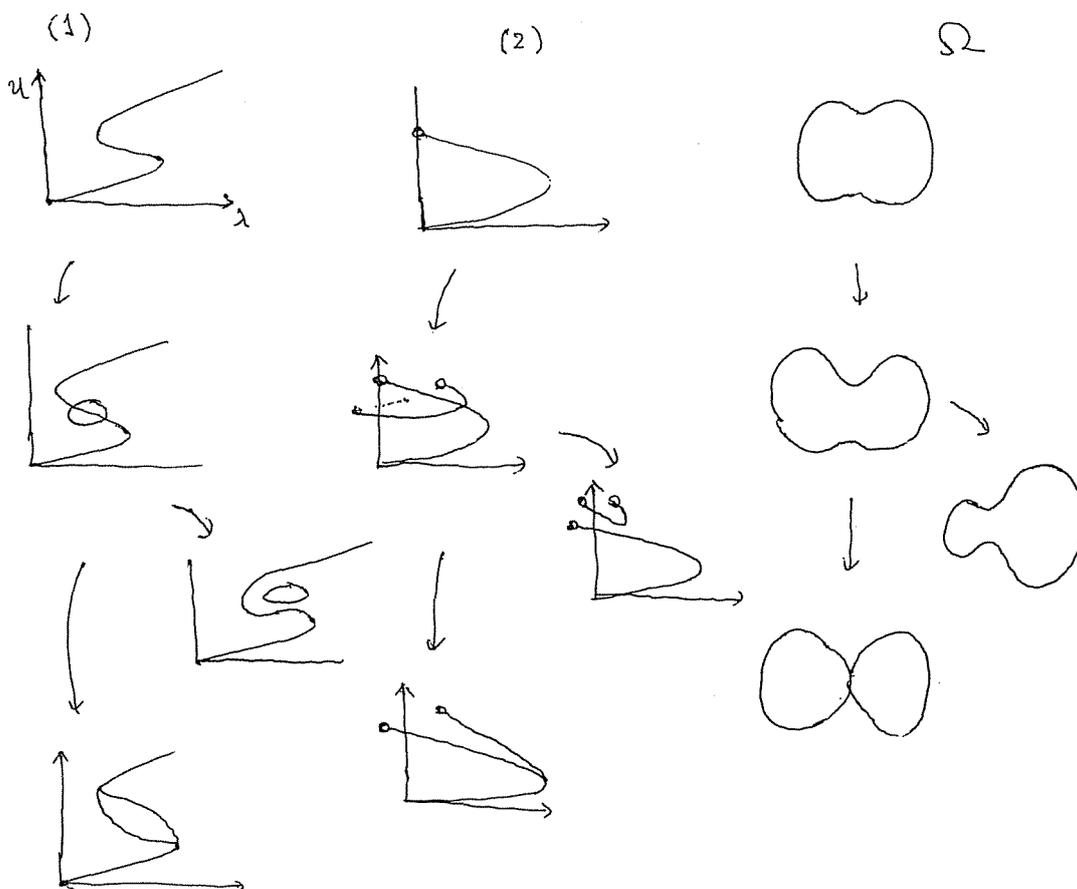
1. A rough estimate for the number of blowup points exists. In particular it is finite for any simply connected domain.
2. Morse indices for solutions created by Proposition 8 are between 1 and 3.

Those are utilized effectively to control the global bifurcation diagram for (2) ([6]).

#### 4. Mushroom ?

The theorem [9] is described as follows. Let  $\Omega \subset \mathcal{R}^2$  be a domain with two axile symmetries, and break its convexity to one axis. Then, the pitchfork bifurcation of nondegenerate cores will arise eventually.

Under those circumstances, Proposition 8 guarantees three families of solutions for (2). The argument [6] is described as follows. First, some more implicit, but reasonable assumptions for  $\Omega$ , bring their connectivity via the theory of topological degree. Then, naturally is expected the imperfect bifurcation as the domain perturbs asymmetrically for  $x_2$  axis ([4], e.g.). Finally, thanks to Theorem 6, those components are to be imbedded into  $\mathcal{S}$ , and we can suggest the generation of a mushroom for (1).



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# WEAK SOLUTIONS FOR THE EVOLUTION PROBLEMS OF HARMONIC MAPS ON NONDECREASING DOMAINS

ATSUSHI TACHIKAWA

**Abstract.** In this paper we construct a weak solution of an initial and boundary value problem on a nondecreasing domain  $\Omega_t$  for the heat flow for harmonic maps into a sphere. To construct the weak solution we employ the method that is given by combining Rothe's time-discretization method and the direct method of calculus of variations.

## 1 Introduction

Let  $M = (M^m, g)$  and  $N = (N^\ell, h)$  be Riemannian  $m$ - and  $\ell$ -manifolds ( $m, \ell \geq 2$ ) respectively. Let  $x = (x^1, \dots, x^m)$  and  $u = (u^1, \dots, u^\ell)$  be local coordinates on  $M$  and  $N$  respectively. We shall write  $(g_{\alpha\beta}(x))$  and  $(h_{ij}(u))$  for the metric tensors with respect to the local coordinates on  $M$  and  $N$  respectively.

For a map  $u \in C^1(M, N)$  and a bounded domain  $\Omega \subset M$ , we define the *energy* of  $u$  on  $\Omega$  as

$$\mathcal{E}(u; \Omega) = \int_{\Omega} e(u) d\mu,$$

with the *energy density*

$$e(u)(x) = \frac{1}{2} g^{\alpha\beta}(x) D_{\alpha} u^i(x) D_{\beta} u^j(x) h_{ij}(u(x))$$

and the volume element  $d\mu = \sqrt{g(x)} dx$ ,  $g(x) = \det(g_{\alpha\beta}(x))$ . Here and in the sequel,  $D_{\alpha}$  denotes  $\partial/\partial x^{\alpha}$ . A map  $U : M \rightarrow N$  is said to be *harmonic* if it is of class  $C^2$  and is

a critical point of the energy functional. The Euler-Lagrange equation for the energy functional is given by

$$(1.1) \quad (\tau(u))^i := \Delta_M u^i(x) + g^{\alpha\beta}(x) \Gamma_{jk}^i(u(x)) D_\alpha u^j(x) D_\beta u^k(x) = 0 \quad \text{for } 1 \leq i \leq \ell,$$

where  $\Gamma_{jk}^i$  denote the Christoffel symbols on  $N$  and  $\Delta_M$  denotes the Laplace-Beltrami operator on  $M$ , i.e.

$$(1.2) \quad \Delta_M = \frac{1}{\sqrt{g}} D_\beta \left( \sqrt{g} g^{\alpha\beta} D_\alpha \right) = g^{\alpha\beta}(x) D_\alpha D_\beta - g^{\alpha\beta}(x) \Gamma_{\alpha\beta}^\gamma(x) D_\gamma.$$

The basic existence problem for harmonic maps is to find a harmonic map in the given homotopy class. A natural approach to this problem is to study the evolution equation

$$(1.3) \quad \frac{\partial u}{\partial t} - \tau(u) = 0, \quad \text{in } M \times \mathbf{R}_+.$$

For the case that  $\partial M = \emptyset$ , the global existence of the solution to the Cauchy problem for the equation (1.3) was shown by Eells-Sampson [7] assuming that the sectional curvatures of the target manifold  $N$  are nonpositive. In case  $N = S^\ell$ , the global existence of weak solution was shown by Chen [3]. Moreover, for general target manifolds, the global existence and partial regularity result was given by Chen-Struwe [4].

For the case that  $\partial M \neq \emptyset$ , one can consider the initial-boundary value for (1.3). Hamilton [10] has shown the global existence of smooth solution to the problem (1.3) for the case that the sectional curvatures of the target manifold  $N$  are nonpositive. For general target manifolds, the global existence and partial regularity result was obtained by Chen-Lin [5].

In this paper we consider the initial-boundary value problem on a nondecreasing domain for evolution of harmonic maps into a sphere. Let  $\Omega$  be a bounded domain of a smooth Riemannian  $m$ -manifold  $M$  and  $S^\ell$  a sphere  $\{u : u \in \mathbf{R}^{\ell+1}, |u| = 1\} \subset \mathbf{R}^{\ell+1}$ . For  $C^1$ -map  $u = (u^1(x), \dots, u^{\ell+1}(x)) : \Omega \rightarrow S^\ell \subset \mathbf{R}^{\ell+1}$  the energy of  $u$  on  $\Omega$  is given by

$$(1.4) \quad \mathcal{E}(u; \Omega) = \int_\Omega \frac{1}{2} \|Du(x)\|^2 d\mu.$$

Here and in the sequel,  $\|\xi\| = \left( \sum_{i=1}^{\ell+1} g^{\alpha\beta} \xi_\alpha^i \xi_\beta^i \right)^{1/2}$  for  $\xi = (\xi_\alpha^i) \in \mathbf{R}^{m(\ell+1)}$ , and  $|\cdot|$  denote the standard Euclidean norms. The Euler-Lagrange equation of the energy functional  $\mathcal{E}$  is given by

$$(1.5) \quad \Delta_M u + u \|Du\|^2 = 0 \quad \text{in } \Omega .$$

Let  $\Omega_t$  be a one-parameter family of bounded domains of  $M$  with Lipschitz boundaries  $\partial\Omega_t$ . Let us define  $V_{\sigma,\tau}$  and  $\Sigma_{\sigma,\tau}$  in  $M \times \mathbf{R}$  as follows:

$$V_{\sigma,\tau} = \{(x, t) \mid \sigma < t < \tau, x \in \Omega_t\},$$

$$\Sigma_{\sigma,\tau} = \partial V_{\sigma,\tau} \setminus \{(\Omega_\sigma \times \{\sigma\}) \cup (\Omega_\tau \times \{\tau\})\}.$$

We construct a weak solution of the following evolution problem of harmonic maps into the sphere  $S^\ell$  on a nondecreasing domain  $\Omega_t$ .

$$(1.6) \quad \frac{\partial u}{\partial t} - \Delta_M u - u \|Du\|^2 = 0, \quad \text{for } (x, t) \in V_{0, \infty},$$

$$(1.7) \quad u(x, t) = w(x), \quad \text{for } (x, t) \in \Sigma_{0, \infty},$$

$$(1.8) \quad u(x, 0) = u_0(x), \quad \text{for } x \in \Omega_0.$$

Now we can state our main result.

**Theorem 1.** *Let  $M$  be a smooth Riemannian  $m$ -manifold and  $\{\Omega_t\}$  ( $0 \leq t < +\infty$ ) a one-parameter family of bounded domains of  $M$  with Lipschitz boundaries. Assume that  $\{\Omega_t\}$  is monotone nondecreasing i.e.*

$$(1.9) \quad \Omega_s \subset \Omega_t \quad \text{if } 0 \leq s < t.$$

*Then for any  $u_0(x) \in H^{1,2}(\Omega_0, S^\ell)$  and  $w \in H_{\text{loc}}^{1,2}(M, S^\ell)$  with  $u_0 = w$  on  $\partial\Omega_0$  there exists a weak solution of the initial-boundary problem (1.6), (1.7) and (1.8).*

In order to prove the above theorem, we use Rothe's time-discretization method ([16]) and the direct method of calculus of variations.

Rothe's time-discretization method has been used to construct solutions of parabolic and hyperbolic equations. Moreover, in 1971, Rektorys [15] combined the time-discretization method and the direct method of calculus of variations to construct solutions of parabolic equations. Roughly speaking, their method is summarized as follows: For the equation

$$(1.10) \quad \frac{\partial u}{\partial t} - \left( \text{the Euler-Lagrange equation of } \int_{\Omega} F(x, u, Du) d\mu \right) = 0,$$

they consider the auxiliary variational functionals

$$(1.11) \quad G_n(u) = \int_{\Omega} \left\{ \frac{|u - u_{n-1}|^2}{h} + F(x, u, Du) \right\} d\mu,$$

and define  $u_n$  successively as the minimizer of  $G_n(u)$ . Using the sequence  $\{u_n\}$ , they construct approximate solutions and prove that the approximate solutions converge to a solution of (1.10) as  $h \rightarrow 0$ . In [15] existence of weak solutions of linear parabolic equations was proved.

Recently, this method was rediscovered by Kikuchi [11]. In [11] parabolic systems associated to the variational functionals of harmonic map type are studied. Moreover, several authors subsequently use the above method. For instance, Bethuel-Coron-Ghidaglia-Soyeur [2] showed the existence of the weak solutions of evolution problems for harmonic maps into  $S^\ell$  in the same procedure. Our proof is based on the method of [2].

Similar ideas to one above have been applied to the other type of equations. (See, for example, [12, 13, 14, 15, 17].) Especially, in [1] and [8], we can find remarkable use of them combined with the geometric measure theory. Moreover, these ideas are closely related to the notion of *minimizing movement* which is introduced by De Giorgi [6].

## 2 The outline of the proof of Theorem 1

To construct a weak solution of (1.6), we proceed as in [14]. For fixed  $h > 0$  and given  $\hat{u}_{n-1} \in H_{\text{loc}}^{1,2}(M, \mathbf{R}^{\ell+1})$ ,  $n \geq 1$ , we consider the following functional for  $u \in H^{1,2}(\Omega_{(n-1)h}, \mathbf{R}^{\ell+1})$ :

$$(2.1) \quad \mathcal{F}_n(u) = \int_{\Omega_{(n-1)h}} \left\{ \frac{1}{2} \frac{|u - \hat{u}_{n-1}|^2}{h} + \frac{1}{2} \|Du\|^2 \right\} d\mu.$$

The general theories of calculus of variations guarantee the existence of a minimizer of  $\mathcal{F}_n$  in the class

$$(2.2) \quad H_w^{1,2}(\Omega_{(n-1)h}) = \left\{ u \in H^{1,2}(\Omega_{(n-1)h}, S^\ell) : u = w \text{ on } \partial\Omega_{(n-1)h} \right\}.$$

(See, for example, [9].) Let  $u_n$  be a minimizer of  $\mathcal{F}_n$  in  $H_w^{1,2}(\Omega_{(n-1)h})$ , then  $u_n$  satisfies

$$(2.3) \quad \int_{\Omega_{(n-1)h}} \left\{ \frac{u - \hat{u}_{n-1}}{h} \cdot \varphi - \frac{1}{2h} |u - \hat{u}_{n-1}|^2 u \cdot \varphi + g^{\alpha\beta} D_\alpha u \cdot D_\beta \varphi - \|Du\|^2 u \cdot \varphi \right\} d\mu = 0$$

for all  $\varphi \in H_0^{1,2}(\Omega_{(n-1)h}, \mathbf{R}^{\ell+1})$ . Here and in the sequel, “ $\cdot$ ” denotes the standard Euclidean scalar product in  $\mathbf{R}^{\ell+1}$ . We define  $\hat{u}_n \in H_{\text{loc}}^{1,2}(M, S^\ell)$  by

$$(2.4) \quad \hat{u}(x) = \begin{cases} u_n(x) & \text{for } x \in \Omega_{(n-1)h} \\ w(x) & \text{for } x \in M \setminus \Omega_{(n-1)h}. \end{cases}$$

Since  $\mathcal{F}_n(\hat{u}_n) = \mathcal{F}_n(u_n) \leq \mathcal{F}_n(\hat{u}_{n-1})$ , denoting  $D_n = \Omega_{nh} \setminus \Omega_{(n-1)h}$ , we see that  $\{\hat{u}_n\}$  satisfies

$$(2.5) \quad \int_{\Omega_{(n-1)h}} \frac{1}{2} \frac{|\hat{u}_n - \hat{u}_{n-1}|^2}{h} d\mu + \mathcal{E}(\hat{u}_n; \Omega_{(n-1)h}) \leq \mathcal{E}(\hat{u}_{n-1}; \Omega_{(n-1)h}) = \mathcal{E}(\hat{u}_{n-1}; \Omega_{(n-2)h}) + \mathcal{E}(\hat{u}_{n-1}; D_{n-1})$$

Summing up the estimate (2.5) from  $n = 1$  to  $n = N$ , we get

$$(2.6) \quad \sum_{n=1}^N \int_{\Omega_{(n-1)h}} \frac{1}{2} \frac{|\hat{u}_n - \hat{u}_{n-1}|^2}{h} d\mu + \mathcal{E}(\hat{u}_N; \Omega_{(N-1)h}) \leq \mathcal{E}(u_0; \Omega_0) + \mathcal{E}(w; \Omega_{(N-1)h}).$$

Now, let us define

$$\begin{cases} \bar{u}_h(x, t) = \begin{cases} \hat{u}_0(x) & \text{for } t = 0, \\ \hat{u}_n(x) & \text{for } (n-1)h < t \leq nh, \quad n \geq 1, \end{cases} \\ u_h(x, t) = \frac{t - (n-1)h}{h} \hat{u}_n(x) + \frac{nh-t}{h} \hat{u}_{n-1}(x) \quad \text{for } (n-1)h < t \leq nh, \quad n \geq 1. \end{cases}$$

Then, from (2.6), we have the following lemma.

**Lemma 2.1.** *For any  $T \geq 0$ ,  $\bar{u}_h$  and  $u_h$  satisfy*

$$(2.7) \quad \int_0^T \int_{\Omega_T} \frac{1}{2} |D_t u_h|^2 d\mu dt + \sup_{0 \leq t \leq T} \mathcal{E}(u_h; \Omega_T) \leq \mathcal{E}(u_0; \Omega_0) + \mathcal{E}(w; \Omega_T),$$

$$(2.8) \quad \sup_{0 \leq t \leq T} \mathcal{E}(\bar{u}_h; \Omega_T) \leq \mathcal{E}(u_0; \Omega_0) + \mathcal{E}(w; \Omega_T).$$

As in [2], Lemma 2.1 implies that

$$\begin{aligned} u_h &\rightharpoonup u && H^{1,2}(\Omega_T \times (0, T); \mathbf{R}^{\ell+1}), \\ u_h, \bar{u}_h &\rightarrow u && L^2(\Omega_T \times (0, T); \mathbf{R}^{\ell+1}), \\ D_\alpha \bar{u}_h &\rightharpoonup D_\alpha u && L^2(\Omega_T \times (0, T); \mathbf{R}^{\ell+1}), \end{aligned}$$

for some  $u \in H^{1,2}(\Omega_T \times (0, T); \mathbf{R}^{\ell+1})$ .

Now, let define  $\Omega_t$  as  $\Omega_t = \Omega_0$  for  $t < 0$ , and let

$$V'_{h,\sigma,\tau} = \{(x, t) | \sigma < t < \tau, x \in \Omega_{t-h}\}.$$

Then, (2.3) implies that  $u_h$  and  $\bar{u}_h$  satisfy the following equation

$$(2.9) \quad D_t u_h - \Delta_M \bar{u}_h - \left( \frac{h}{2} |D_t u_h|^2 + \|D \bar{u}_h\|^2 \right) \bar{u}_h = 0 \quad \text{in } V'_{h,0,T}.$$

Taking wedge product of (2.9) with  $\bar{u}_h$ , we can see that

$$(2.10) \quad D_t u_h \wedge \bar{u}_h - \frac{1}{\sqrt{g}} D_\beta (\sqrt{g} g^{\alpha\beta} D_\alpha \bar{u}_h \wedge \bar{u}_h) = 0 \quad \text{in } V'_{h,0,T},$$

in weak sense.

Thus, letting  $h \rightarrow 0$ , we can see that  $u$  satisfies

$$(2.11) \quad D_t u \wedge u - \frac{1}{\sqrt{g}} D_\beta (\sqrt{g} g^{\alpha\beta} D_\alpha u \wedge u) = 0 \quad \text{in } V_{0,T},$$

weakly. This means that  $u(x, t)$  is a weak solution of (1.6) (see [3]). On the other hand, since  $u_h(x, 0) = u_0(x)$  in  $\Omega_0$ ,  $u_h(x, t) = w(x)$  on  $\partial V_{0,T}$  and  $u_h \rightharpoonup u$  in  $H^{1,2}(\Omega_T \times (0, T), \mathbf{R}^{\ell+1})$ , we can see that  $u$  satisfies (1.7) and (1.8) also.

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# NONEXISTENCE OF GLOBAL SOLUTIONS TO SEMILINEAR WAVE EQUATIONS

HIROYUKI TAKAMURA

**Abstract:** In this talk, I would like to outline the nonexistence of global solutions to initial value problems for semilinear wave equations with positive power nonlinearities. The initial data is assumed to be sufficiently small. If the support of the data is non-compact, there are blowing-up solutions even for global existence power in compactly supported case because of “bad” spatial decay of the data. Moreover, a critical decay is conjectured.

## 1. Introduction.

Main result stated here is appeared in H. Takamura [21]. We shall study classical solutions to the following initial value problem for semilinear wave equations;

$$(1.1) \quad \square u = |u|^p \quad \text{in } \mathbb{R}^n \times [0, \infty), \text{ or}$$

$$(1.2) \quad \square u = |u_t|^p \quad \text{in } \mathbb{R}^n \times [0, \infty),$$

where  $u$  is a scalar unknown function,  $\square \equiv \partial^2/\partial t^2 - \Delta_x$  is d'Alembertian and  $p > 1$ . The initial condition :

$$(1.3) \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}^n$$

is considered for smooth functions  $f, g$ . Throughout this talk, we assume  $n \geq 2$ .

Positive nonlinearities in (1.1) and (1.2) almost cause blow-up solutions provided the initial data is large in some sense. For instance, see R.T. Glassey [5]. Hence, we may formulate the question as follows : what is the critical value of  $p$ , say  $p_0(n)$ , depending on  $n$ , with the property that (1.1) and (1.2) admit a global solution for all “small”  $f, g$  if  $p > p_0(n)$ , and that there are blowing-up solutions if  $1 < p \leq p_0(n)$ .

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If the initial data has compact support, it has been conjectured that

$$(1.4) \quad p_0(n) = \begin{cases} \frac{n+1 + \sqrt{n^2 + 10n - 7}}{2(n-1)} & \text{for (1.1)} \\ \frac{n+1}{n-1} & \text{for (1.2)}. \end{cases}$$

Both numbers are related with some integrability. This conjecture was partially verified in several space dimensions.

For the equation (1.1), the case  $n = 3$  was done by F. John [9] except for  $p = p_0(3)$  and J. Schaeffer [19] for  $p = p_0(3)$ . The case  $n = 2$  was done by R.T. Glassey [6] [7] except for  $p = p_0(2)$  and J. Schaeffer [19] for  $p = p_0(2)$ . In higher space dimensions, T.C. Sideris [20] showed its blow-up part  $1 < p < p_0(n)$ . Recently, Y. Zhou [26] proved global existence part  $p > p_0(n)$  in the case  $n = 4$ . During this conference, I was informed of papers by V. Georgiev [4] and H. Lindblad & C.D. Sogge [14]. They solved independently global existence part for  $n \geq 4$  by completely different methods.

For the equation (1.2), the conjecture was verified by F. John [10] for the blow-up part and T.C. Sideris [19] for the global existence part in the case  $n = 3$ , and both by J. Schaeffer [16] in the case  $n = 5$ . The blow-up part in the case  $n = 2$  was shown by J. Schaeffer [18] for  $p = p_0(2)$  and by R. Agemi [1] for  $1 < p < p_0(2)$ . In other space dimensions, M.A. Rammaha [15] proved the blow-up in the case  $n \geq 4$  with  $p = p_0(n)$  for odd  $n$  and  $1 < p < p_0(n)$  for even  $n$ . Y. Zhou [25] studied the precise upper bound of the existence time in low space dimensions. His methods may solve the blow-up part in all space dimensions.

Remarkable results on (1.1) and (1.2) is that, if the support of the initial data is noncompact, one can get blowing-up solutions even for the existence  $p > p_0(n)$  because of the slow spatial decay of the data. Hence we may formulate the main question as follows : what is the critical decay, say  $\kappa_0$ , with the property that (1.1) and (1.2) admits a global solution for all "small"  $f, g$  satisfying that

$$(1.5) \quad \nabla_x^\alpha f(x), \nabla_x^\beta g(x) = O\left(\frac{1}{|x|^\kappa}\right) \quad \text{as } |x| \rightarrow \infty$$

for suitable multi-indices  $\alpha, \beta$  and  $\kappa \geq \kappa_0, \quad p > p_0(n),$

and that there are blowing-up solutions if the initial data satisfies some positivity condition. For example,

$$(1.6) \quad f(x) \equiv 0, \quad g(x) \geq \frac{M}{(1+|x|)^\kappa} \quad \text{with } M > 0, \quad 0 < \kappa < \kappa_0.$$

We note that  $\kappa_0$  does not depend on space dimensions  $n$ . More precisely, it has been conjectured that

$$(1.7) \quad \kappa_0 = \begin{cases} \frac{p+1}{p-1} & \text{for (1.1)} \\ \frac{1}{p-1} & \text{for (1.2)}. \end{cases}$$

For the equation (1.1), this conjecture was verified by F. Asakura [3] except for  $\kappa = \kappa_0$  and by K. Kubota [13], or K. Tsutaya [24] for  $\kappa = \kappa_0$  in the case  $n = 3$ . The case  $n = 2$  was done by R. Agemi & H. Takamura [2] for the blow-up part, K. Kubota [13] for the global existence part, or both by K. Tsutaya [22] [23]. These all results are independently obtained. The global existence under (1.5) will be proved by H. Kubo & K. Kubota [12] in the spherically symmetric case for all odd  $n$ .

For the equation (1.2), the blow-up under (1.6) was proved by H. Kubo [11] in the case  $n = 2, 3$ . The global existence under (1.5) will be proved by H. Kubo & K. Kubota [12] in the spherically symmetric case for all odd  $n$ . Independently, both parts in the case  $n = 3$  will appear in K. Hidano [8].

The aim of this talk is to show that the blow-up of (1.1) and (1.2) under (1.6) is valid for all  $n \geq 2$ . The main difficulty in high space dimensions  $n \geq 4$  lies in the fact that the fundamental solution of  $\square$  contains many time derivatives. Avoiding this situation, we consider the spherically symmetric case. But, then, fundamental solution is no longer positive for full space. Previous two blow-up results on the compactly supported case by T.C. Sideris [19] for (1.1) and M.A. Rammaha [15] for (1.2) required the full space integral of a solution and reduced to the blow-up theory for ordinary differential inequalities. Such a method is not applicable to the noncompactly supported case. Some new ideas are required. Our success is essentially due to making  $u$ -closed (which means with respect to only  $u$ ) integral inequalities for both problems, (1.1) and (1.2). It is remarkable especially for (1.2). We note that, in low space dimensions  $n = 2, 3$ , H. Kubo [11] used  $u_t$ -closed integral inequality for (1.2) which follows from a integral equation equivalent to (1.2) by differentiation in  $t$ . But, such a differentiation yields no positivity in higher space dimensions. Iterating pointwise estimates of  $u$  in  $u$ -closed integral inequalities, we will get the required blow-up.

**Theorem 1.1.** *Let  $n \geq 2$ . Assume (1.6) and (1.7) in the spherically symmetric case  $g = g(|x|)$ . Then, classical solutions to each initial value problem (1.1) and (1.3), (1.2) and (1.3) blow-up in finite time.*

## 2. Preliminaries.

We shall start from the representation formula of solutions to the linear problem. Let  $u^0$  be a solution to the following initial value problem :

$$(2.1) \quad \begin{aligned} \square u^0 &= 0 && \text{in } \mathbb{R}^n \times [0, \infty), \\ u^0|_{t=0} &= 0, \quad u_t^0|_{t=0} = g(|x|), && x \in \mathbb{R}^n. \end{aligned}$$

The following two representations are due to M.A. Rammaha [15]. See (6a) and (6b) in [15]. But the even dimensional case is slightly modified here.

**Lemma 2.2 (The representation formula in odd space dimensions).** *Let  $n = 2m + 1$ ,  $m \in \mathbb{N}$  and  $u^0$  be a solution to (2.1). Then, for  $r \equiv |x|$ , we have*

$$(2.2) \quad u^0(r, t) = \frac{1}{2r^m} \int_{|r-t|}^{r+t} \lambda^m g(\lambda) P_{m-1} \left( \frac{\lambda^2 + r^2 - t^2}{2r\lambda} \right) d\lambda,$$

where  $P_k$  denotes Legendre polynomials of degree  $k$ .

**Lemma 2.3 (The representation formula in even space dimensions).** *Let  $n = 2m$ ,  $m \in \mathbb{N}$  and  $u^0$  be a solution to (2.1). Then, for  $r \equiv |x|$ , we have*

$$(2.3) \quad u^0(r, t) = \frac{2}{\pi r^{m-1}} \int_0^t \frac{\rho d\rho}{\sqrt{t^2 - \rho^2}} \times \\ \times \int_{|r-\rho|}^{r+\rho} \frac{\lambda^m g(\lambda)}{\sqrt{\lambda^2 - (r-\rho)^2} \sqrt{(r+\rho)^2 - \lambda^2}} T_{m-1} \left( \frac{\lambda^2 + r^2 - \rho^2}{2r\lambda} \right) d\lambda$$

where  $T_k$  denotes Tschebyscheff polynomials of degree  $k$ .

*Remark 2.4.* We note that  $(\lambda^2 + r^2 - t^2)/(2r\lambda)$  and  $(\lambda^2 + r^2 - \rho^2)/(2r\lambda)$  travel from  $-1$  to  $1$  in the above two representation formulas. As is well-known,  $P_k(z)$  and  $T_k(z)$  have  $k$  zero points in the region  $-1 < z < 1$ . Hence, we need the following lemma to gain the positivity.

**Lemma 2.5.** *There exists a positive constant  $\delta_m$ , depending only on  $m \in \mathbb{N}$ , such that*

$$(2.4) \quad P_{m-1}(z), T_{m-1}(z) \geq \frac{1}{2} \quad \text{for} \quad 1 \geq z \geq \frac{1}{1 + \delta_m}.$$

This lemma follows from well-known facts. Now, we state the key lemma.

**Lemma 2.6.** *Let  $n = 2m + 1$  or  $n = 2m$ ,  $m \in \mathbb{N}$  and  $u$  be a solution to (1.1), or (1.2), and (1.3) with  $f \equiv 0$ ,  $g = g(r) \geq 0$  for  $r \equiv |x|$ ,  $x \in \mathbb{R}^n$ . Then, we have*

$$(2.5) \quad u(r, t) \geq u^0(r, t) + \frac{1}{8r^m} \int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} \lambda^m H(\lambda, \tau) d\lambda, \\ u^0(r, t) \geq \frac{1}{8r^m} \int_{r-t}^{r+t} \lambda^m g(\lambda) d\lambda,$$

where  $H \equiv |u|^p$ , or  $|u_t|^p$  provided

$$(2.6) \quad r - t \geq \frac{2}{\delta_m} t > 0.$$

Here  $\delta_m$  is the one in Lemma 2.5.

*Sketch of proof.* By standart Duhamel's principle, we have to show only the second line of (2.5). It is easy to see that

$$(2.7) \quad \frac{\lambda^2 + r^2 - t^2}{2r\lambda} \geq \frac{(r-t)^2 + r^2 - t^2}{2r(r+t)} = \frac{r-t}{r+t}$$

for  $|r-t| \leq \lambda \leq r+t$ ,  $(r, t) \in (0, \infty)^2$ . The condition (2.6) is equivalent to

$$(2.8) \quad \frac{r-t}{r+t} \geq \frac{1}{1 + \delta_m}$$

Hence, by Lemma 2.2 and Lemma 2.5, we get Lemma 2.6 for odd dimensional case. For even dimensional cases, more technical calculations are required.

**3. Sketch of proof of Theorem 1.1.**

We shall follow basically F. John [9]’s iterating argument. Let  $u(r, t)$  be a global solution to our problems, where  $r = |x|$ . If  $t$  is larger than some constant, we will get a contradiction. For small fixed  $\delta > 0$ , we define the “blow-up set” ;

$$(3.1) \quad \Sigma = \left\{ (r, t) \in (0, \infty)^2 ; r - t \geq \max\left\{ \frac{2}{\delta_m} t, \delta \right\} \right\},$$

and let  $n = 2m + 1$ , or  $n = 2m$  for  $m \in \mathbb{N}$ .

Using the assumption (1.6) with  $g = g(r)$  and Lemma 2.6, we get an estimate

$$(3.2) \quad u(r, t) \geq u_0(r, t) \geq \frac{C_0 t^{m+1}}{r^m (r + t)^\kappa} \quad \text{for} \quad (r, t) \in \Sigma,$$

where  $C_0$  is a positive constant.

For the nonlinearity  $|u|^p$ , iterating (3.2) infinitely many times in the first line of (2.5), we will find  $(r_0, t_0) \in \Sigma$  such that  $u(r_0, t_0) = \infty$  provided  $0 < \kappa < \kappa_0$ . This is the desired contradiction.

For  $|u_t|^p$ , the same argument as above gives us a requirement of only  $u$ -closed integral inequality. It follows from (1.6) with  $g = g(r)$  and Lemma 2.6 that, for  $(r, t) \in \Sigma$ ,

$$(3.3) \quad u(r, t) \geq \frac{1}{8r^m} \int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} \lambda^m |u_t(\lambda, \tau)|^p d\lambda.$$

Inverting the order of  $(\lambda, \tau)$ -integral, we find that, for  $(r, t) \in \Sigma$ ,

$$(3.4) \quad \begin{aligned} u(r, t) &\geq \frac{1}{8r^m} \left( \int_{r-t}^r d\lambda \int_0^{\lambda-(r-t)} d\tau + \int_r^{r+t} d\lambda \int_0^{r+t-\lambda} d\tau \right) \lambda^m |u_t(\lambda, \tau)|^p \\ &\geq \frac{1}{8r^m} \int_r^{r+t} \lambda^m d\lambda \int_0^{r+t-\lambda} |u_t(\lambda, \tau)|^p d\tau. \end{aligned}$$

Hölder’s inequality yields that

$$(3.5) \quad \left| \int_0^{r+t-\lambda} u_t(\lambda, \tau) d\tau \right|^p \leq (r + t - \lambda)^{p-1} \int_0^{r+t-\lambda} |u_t(\lambda, \tau)|^p d\tau.$$

Since  $u(\lambda, 0) = f(\lambda) \equiv 0$  by (1.6), we obtain from (3.4) and (3.5) that, for  $(r, t) \in \Sigma$ ,

$$(3.6) \quad u(r, t) \geq \frac{1}{8r^m} \int_r^{r+t} \lambda^m (r + t - \lambda)^{1-p} |u(\lambda, r + t - \lambda)|^p d\lambda.$$

This is the required  $u$ -closed integral inequality.

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# GEVREY REGULARIZING EFFECT FOR A NONLINEAR SCHRÖDINGER EQUATION

K. TANIGUCHI

**Abstract.** Let  $u(t, x)$  be a solution of a nonlinear Schrödinger equation  $i\partial u/\partial t + \Delta u = f(t, x, u)$ ,  $u(0, x) = \phi$ . In this paper, we study the global Gevrey property for the solution  $u(t, x)$  with respect to a dilation operator  $P = 2\partial/\partial t + x \cdot \nabla_x$ , and the local Gevrey property with respect to the space variables.

## 1. Introduction.

This is a joint work with Professor Keiichi Kato. We consider the following Cauchy problem of a nonlinear Schrödinger equation in  $n$  space dimensions,

$$(1) \quad \begin{cases} Lu \equiv i\partial_t u + \Delta u = f(t, x, u), \\ u(0, x) = \phi(x), \end{cases}$$

where  $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$  and  $f(t, x, u)$  is a complex valued function of Gevrey class in  $(t, x, u) \in \mathbf{R}_t \times \mathbf{R}^n \times \mathbf{C}$ . In Hayashi - Kato [2] and De Bouard - Hayashi - Kato [1], they proved the following theorems.

**Theorem 1.** *Suppose that  $f(t, x, u) \equiv F(u, \bar{u})$  is a polynomial of  $u$  and its conjugate  $\bar{u}$  with  $F(0, 0) = 0$ , and set  $m = [n/2] + 1$ . Assume*

$$(2) \quad \|(x \cdot \nabla_x)^\ell \phi\|_{H^m(\mathbf{R}_x^n)} \leq CA_1^\ell \ell!^s.$$

*Then, there exists a positive constant  $T$  such that the equation (1) has a unique solution  $u(t, x)$  in  $C^0([0, T]; H^m(\mathbf{R}^n)) \cap C^1([0, T]; H^{m-2}(\mathbf{R}^n))$  and it satisfies*

$$\sup_t \|P^\ell u(t, x)\|_{H^m(\mathbf{R}_x^n)} \leq CA_2^\ell \ell!^s \quad \text{for any } \ell,$$

where  $P = 2t\partial_t + x \cdot \nabla_x$  is a dilation operator.

**Theorem 2.** *In theorem 1 we assume, furthermore, that  $n = 1$ . Then, under the condition (2) the solution  $u(t, x)$  of (1) satisfies the following property: For any positive number  $R$  there exist constants  $C = C_R$  and  $A = A_R$  such that*

$$\|\partial_x^\alpha u(t, x)\|_{H^1(-R, R)} \leq CA^\alpha \alpha!^\sigma \quad \text{for } t \in [0, T] \text{ with } t \neq 0,$$

where  $\sigma = \max(1, s/2)$ .

In this paper, we extend their results to the case that nonlinear term  $f(t, x, u)$  is a general function. We put the following assumptions.

(A.1)  $f(t, x, u)$  is a complex-valued  $C^\infty$ -function defined in  $[0, T] \times \mathbb{R}_x^n \times \mathbb{C}$  which satisfies

$$\|P^\ell f(t, x, 0)\|_{H^m(\mathbb{R}_x^n)} \leq CA^\ell \ell!^s$$

with  $m = [n/2] + 1$  and a dilation operator  $P = 2t\partial_t + x \cdot \nabla_x$ .

(A.2) For any  $K > 0$  there exist constants  $C = C_K$  and  $A = A_K$  such that

$$\begin{aligned} |\partial_x^\gamma P^\ell \partial_u^k \partial_{\bar{u}}^{k'} f(t, x, u)| &\leq CA^{\ell+k+k'} \ell!^s k!^s k'!^s \\ &\text{for } x \in \mathbb{R}_x^n, |u| \leq K, |\gamma| \leq m, k + k' \geq 1, \end{aligned}$$

where  $\partial_{\bar{u}} = \partial/\partial \bar{u}$  is a differentiation with respect to the conjugate complex of  $u$ .

(A.3) Let  $\sigma$  satisfy  $\max(s/2, 1) \leq \sigma \leq s$ . Then, for any fixed  $R > 0$  there exist constants  $C = C_R$  and  $A = A_R$  such that

$$\begin{aligned} |\partial_t^\ell \partial_x^\alpha \partial_u^k \partial_{\bar{u}}^{k'} f| &\leq C^{1+\ell+|\alpha|+k+k'} \ell!^s \alpha!^\sigma k!^\sigma k'!^\sigma \\ &\text{for } |x| \leq R, |u| \leq K. \end{aligned}$$

Our main theorems are the following:

**Theorem 3.** *Assume (A.1) and (A.2). Then, the same result of Theorem 1 holds.*

**Theorem 4.** *Let  $\sigma$  satisfy  $\max(s/2, 1) \leq \sigma \leq s$  and assume (A.1)-(A.3). Then, under the condition (2) the solution  $u(t, x)$  of (1) satisfies the following property: For any positive number  $R$  there exist constants  $C = C_R$  and  $A = A_R$  such that*

$$(3) \quad \|\partial_x^\alpha u(t, x)\|_{H^m(B_R)} \leq CA^{|\alpha|} \alpha!^\sigma \quad \text{for } t \in [0, T] \text{ with } t \neq 0,$$

where  $B_R$  is a ball in  $\mathbb{R}^n$  with the radius  $R$ .

We give several examples of nonlinear terms which satisfy the assumptions (A.1), (A.2) and (A.3) and give a example of initial data which satisfies (2). In the following, we denote, by  $G^s(x \cdot \nabla; H^m)$ , the set of functions  $a(x)$  satisfying  $\|(x \cdot \nabla)^\ell a(x)\|_{H^m(\mathbb{R}_x^n)} \leq CM^\ell \ell!^s$ .

### Examples of nonlinear terms

1.  $f(t, x, u)$  is a polynomial  $F(u, \bar{u})$  of  $u$  and  $\bar{u}$  with  $F(0, 0) = 0$ .
2.  $f(t, x, u) = \frac{a(x)}{1 + |u|^4}$  where  $a(x) \in G^\sigma(x \cdot \nabla; H^m)$  and  $a(x)$  is locally in Gevrey class of order  $\sigma$ .
3.  $f(t, x, u) = \frac{F(u, \bar{u})}{1 + |u|^2}$ , where  $F(u, \bar{u})$  is a polynomial of  $u$  and  $\bar{u}$  with  $F(0, 0) = 0$ .

### Example of initial data

$|x|^a(1 + |x|^2)^b$  with  $2b - n/2 > a > m - n/2$  is in  $G^\sigma(x \cdot \nabla; H^m(\mathbf{R}^n))$ . If  $a$  is not even integer,  $|x|^a(1 + |x|^2)^b$  has a singularity at the origin.

## 2. Outline of the proofs of Theorem 3 and Theorem 4.

**Lemma 2.1.** *Let  $k \leq m$  and let multi-indices  $\gamma_1, \dots, \gamma_k$  satisfy  $|\gamma_1| + \dots + |\gamma_k| \leq m$ . Then, there exists a constant  $C$  such that for any  $u_j \in H^m$  ( $j = 1, \dots, k$ ) we have*

$$\|\partial_x^{\gamma_1} u_1 \partial_x^{\gamma_2} u_2 \dots \partial_x^{\gamma_k} u_k\|_{L^2(\mathbf{R}_x^n)} \leq C \prod_{j=1}^k \|u_j\|_m$$

Here and in what follows, we denote  $\|u\|_m = \|u\|_{H^m(\mathbf{R}_x^n)}$ .

**Lemma 2.2.** *Let  $g(x, u) \in C^\infty(\mathbf{R}_x^n \times \mathbf{C})$  satisfy for  $k + k' \leq m$  and  $|\gamma| \leq m$*

$$|\partial_u^k \partial_{\bar{u}}^{k'} \partial_x^\gamma g(x, u)| \leq M_K \quad \text{for } x \in \mathbf{R}_x^n, |u| \leq K.$$

*Then, there exists a constant  $\rho$  and for any  $K$  there exists a constant  $C_K$  such that for  $u, v \in H^m$  with  $\|u\|_m \leq K$  we have*

$$g(x, u(x))v(x) \in H^m$$

and

$$\|g(x, u(x))v(x)\|_m \leq C_K M_{\rho K} \|v\|_m.$$

We can prove this lemma by using Lemma 2.1.

**Lemma 2.3.** *Suppose that  $f(t, x, u)$  satisfies (A.1) and (A.2). Then, for any  $K$  there exist constants  $C$  and  $A_3$  such that for any  $u \in H^m$  with  $\|u\|_m \leq K$  an inequality*

$$(4) \quad \|(P^\ell f)(t, x, u(t, x))\|_m \leq C A_3^\ell \ell!^s$$

*holds for any  $\ell$ .*

*Proof.* Write

$$(5) \quad P^\ell f(t, x, u(t, x)) = P^\ell f(t, x, 0) + \int_0^1 \partial_u f(t, x, \theta u(t, x)) d\theta \cdot u(t, x) \\ + \int_0^1 \partial_{\bar{u}} f(t, x, \theta u(t, x)) d\theta \cdot \bar{u}(t, x).$$

Then, using Lemma 2.2 with  $g(x, u) = \int_0^1 \partial_u f(t, x, \theta u) d\theta$  (or  $g(x, u) = \int_0^1 \partial_{\bar{u}} f(t, x, \theta u) d\theta$ ) and  $M_K = CA^\ell \ell!^s$  for the second and third terms of the right hand side of (5), we get (4).

*Outline of the proof of Theorem 3.* Consider the linearized equation with respect to (1):

$$\begin{cases} i\partial_t u + \Delta u = f(t, x, v) \\ u(0, x) = \phi(x), \end{cases}$$

and denote the mapping which corresponds  $v$  to  $u$  by  $\mathcal{S}$ . Set  $C([0, T]; G_M^s(C_0))$  by

$$C([0, T]; G_M^s(C_0)) = \{u(t, x) \in C([0, T]; H^m); \|u; M\| \leq C_0\}$$

with

$$\|u; M\| = \|u\|_{H^m} + \sum_{\ell=1}^{\infty} \frac{M^{\ell-1}}{\ell!(\ell-1)!^{s-1}} \|P^\ell u(t, \cdot)\|_{H^m}.$$

Then, using Lemma 2.2, Lemma 2.3 and the differentiation of composite function, we can prove, for a small  $T$ , that the mapping  $\mathcal{S}$  maps  $C([0, T]; G_M^s(C_0))$  into  $C([0, T]; G_M^s(C_0))$  for an appropriate  $C_0$ , and  $\mathcal{S}$  is a contraction mapping. Hence, we can obtain the desired solution by the fixed point of  $\mathcal{S}$ .

*Outline of the proof of Theorem 4.* Take a positive constant  $R$  and take a  $C^\infty$ -function  $r(x)$  with the property

$$\begin{cases} r(x) = 1 & \text{for } |x| \leq R, \\ r(x) = 0 & \text{for } |x| \geq R + 1. \end{cases}$$

Let  $u(x)$  be a solution of (1). Then, since  $[L, P] = 2L$ , we have

$$LP^\ell u = (P + 2)^\ell \{f(t, x, u)\}$$

and from  $\partial_t = \frac{1}{2t}P - \frac{1}{2t}x \cdot \nabla_x$  we have

$$\begin{aligned} \Delta P^\ell u &= -i\partial_t P^\ell u + (P + 2)^\ell \{f(t, x, u)\} \\ &= -\frac{i}{2t} P^{\ell+1} u + \frac{i}{2t} x \cdot \nabla_x P^\ell u + (P + 2)^\ell \{f(t, x, u)\}. \end{aligned}$$

Using this equation we can prove first

$$\|r(x)^{|\alpha|} \partial_x^\alpha P^\ell u\|_m \leq C_1 A_4^\ell t^{-|\alpha|} \ell!^s \quad \text{for any } \ell$$

for  $|\alpha| \leq 2$  and next

$$\|r^{|\alpha|} \partial_x^\alpha P^\ell u\|_m \leq C_2 A_5^{|\alpha|+\ell-1} t^{-|\alpha|} (|\alpha| + \ell - 2)!^\sigma \ell!^{s-\sigma} \quad \text{for any } \ell$$

for  $|\alpha| \geq 3$  by the induction on  $|\alpha|$ . This proves (3).

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# ON THE HOMOGENIZATION OF NONLINEAR CONVECTION-DIFFUSION EQUATIONS WITH OSCILLATORY INITIAL AND FORCING DATA

TAMIR TASSA

## Abstract

We study the behavior of oscillatory solutions to convection-diffusion problems, subject to initial and forcing data with modulated multi-scale oscillations. We determine the weak  $W^{-1,\infty}$ -limit of the solutions when the small scales of the modulations tend to zero and quantify the weak convergence rate. Moreover, in case the solution operator of the equation is compact, this weak convergence is translated into a strong one. Examples include nonlinear conservation laws and equations with nonlinear degenerate diffusion.

## 1 Introduction

In this manuscript we present the main results of [1] (a joint work with Eitan Tadmor) and [2]. The subject of our study is the behavior of oscillatory weak entropy solutions for equations of the form

$$u_t = K(u, u_x)_x + h(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \quad (1.1)$$

where  $K = K(u, p)$  is nondecreasing in  $p := u_x$ . This large family includes equations which mix both types – hyperbolic equations dominated by purely convective terms ( $K_p \equiv 0$ ), or, parabolic equations dominated by possibly degenerate diffusive terms ( $K_p \geq 0$ ).

We are concerned with the initial value problem for (1.1) where the initial data,  $u_0^\varepsilon(x)$ , and the forcing data,  $h^\varepsilon(x, t)$ , are subject to modulated oscillations. Specifically, we are interested in the behavior of  $u^\varepsilon$ , the entropy solution of

$$u_t^\varepsilon = K(u^\varepsilon, u_x^\varepsilon)_x + h^\varepsilon(x, t), \quad u^\varepsilon(x, 0) = u_0^\varepsilon(x), \quad (1.2)$$

where the modulation of the initial and forcing data takes the form

$$u_0^\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right), \quad h^\varepsilon(x, t) = h\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, t\right), \quad \varepsilon_i \downarrow 0. \quad (1.3)$$

Here,  $u_0(x, y)$  and  $h(x, y, t)$  ( $t$  is a parameter) are functions in  $BV_x(\Omega \times T^n)$  – the space of bounded functions  $f = f(x, y)$ ,  $x \in \mathbb{R}_x$ ,  $y \in T^n$  (the  $n$ -dimensional unit torus), which are constant outside the bounded interval  $\Omega \subset \mathbb{R}_x$  and have a bounded variation with respect to  $x$ .

As a first step, one needs to study the behavior of oscillatory functions such as the initial condition and the forcing term in the above problem, when the small scales tend to zero. Namely, if  $f(x, y) = f(x, y_1, \dots, y_n) \in BV_x(\Omega \times T^n)$ , what is the limit, in some sense, of the oscillatory function  $f_\varepsilon(x) = f(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n})$  when  $\varepsilon_i \downarrow 0$ . In §2 we answer this question in terms of the  $W^{-1, \infty}$ -norm in  $\Omega = [a, b]$ ,

$$\|g(x)\|_{W^{-1, \infty}(\Omega)} = \left\| \int_a^x g(\xi) d\xi \right\|_{L^\infty(\Omega)} .$$

We first address the question in the simple one-scale case ( $n = 1$ ). We then turn to the more complex two-scale case and show that the weak limit depends on the relation between the two small scales. Finally, we address the question in the general multi-scale case ( $n \geq 2$ ).

Now, let  $\bar{u}_0(x)$  and  $\bar{h}(x, t)$  denote the  $W^{-1, \infty}$  weak limits of  $u_0^\varepsilon(x)$  and  $h^\varepsilon(x, t)$ , respectively. Then in §3 we show that the entropy solution,  $u^\varepsilon(x, t)$ , approaches the corresponding entropy solution of the homogenized problem

$$u_t = K(u, u_x)_x + \bar{h}(x, t), \quad u(x, 0) = \bar{u}_0(x) . \quad (1.4)$$

We *quantify* the convergence rate of  $u^\varepsilon$  towards  $u$  in the weak  $W^{-1, \infty}$ -topology. Furthermore, in case the solution operator is compact, we are able to translate this weak convergence into a strong one, with  $L^p$ -convergence rate estimates for every  $t > 0$ .

Finally, in §4 we give examples of equations with compact solution operators to which our analysis applies and provide a graphical demonstration of the convergence to the homogenized solution.

## 2 Weak limits of one- and multi-scale homogenization

### 2.1 One small scale

The following fundamental lemma plays a central role in our analysis. For the proof, see [1, Lemma 2.1].

**Lemma 2.1** *Assume that  $g(x, y) \in BV_x(\Omega \times T^1)$ ,  $\Omega$  being a possibly unbounded interval in  $\mathbb{R}_x$ , and let  $g_\varepsilon(x) := g(x, \frac{x}{\varepsilon})$  and  $\bar{g}(x) := \int_0^1 g(x, y) dy$ . Then*

$$\|g_\varepsilon(x) - \bar{g}(x)\|_{W^{-1, \infty}} \leq C\varepsilon, \quad C = \|g\|_{L^1(T^1; BV(\mathbb{R}_x))} . \quad (2.1)$$

## 2.2 Two small scales

The case of two small scales is by far more interesting and complex than the simple one small scale case. The answer, or, better yet, the array of answers which we reveal here is quite interesting, sometimes even surprising. T. Hou has discovered in [3] a part of the picture: he found that  $\bar{f}(x)$  depends on the limit ratio  $\alpha = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon_1}{\varepsilon_2}$  in the following unstable manner: If  $\alpha$  is 0 (or, equivalently, infinite) or an irrational number, the weak limit is the average of  $f(x, y)$  over the 2-dimensional torus,

$$\bar{f}(x) = \int_{T^2} f(x, y) dy ; \quad (2.2)$$

in case  $\alpha$  is a nonzero rational number,  $\frac{m}{n}$ , the weak limit is the average of  $f(x, y)$  over the projection of the straight line  $Span_{\mathbb{R}}\{(n, m)\}$  on  $T^2$ ,

$$\bar{f}(x) = \int_{T^1} f(x, ny_1, my_1) dy_1 . \quad (2.3)$$

The assumption under which these limits were obtained, was that  $r := \frac{\varepsilon_1}{\varepsilon_2} - \alpha$  tends to zero **faster** than  $\varepsilon_1$  and  $\varepsilon_2$ .

In the theorems which follow we complete the task and unveil the entire picture in the 2-scale case (for proofs, see [2, §2]). If  $\alpha$  is zero or irrational, we prove that the weak limit is as in (2.2), **regardless** of the rate in which  $r$  vanishes (Theorems 2.1 and 2.2). If, however,  $\alpha$  is a nonzero rational number, the weak limit depends on the value of  $\alpha$  and, in addition, on the **rate** in which  $\alpha$  is approached by  $\frac{\varepsilon_1}{\varepsilon_2}$ , namely – the order of magnitude of  $r$ . In Theorem 2.3 we show that (2.3) holds only when  $|r| \ll \mathcal{O}(\varepsilon_1, \varepsilon_2)$ ; if  $|r| = \mathcal{O}(\varepsilon_1, \varepsilon_2)$ ,  $\bar{f}(x)$  takes a similar form of an  $f$ -average over an affine curve on  $T^2$  which is parallel to the linear curve along which the integral in (2.3) is taken; however, if  $|r| \gg \mathcal{O}(\varepsilon_1, \varepsilon_2)$ , the weak limit switches unexpectedly from a one-dimensional integral to the double integral in (2.2).

We use below the notations  $BV(y_i)$ ,  $Lip(x)$  etc. for the spaces of functions which are uniformly  $BV$  or Lipschitz continuous in  $\Omega \times T^n$  with respect to the variable in brackets.

**Theorem 2.1** (*Case 1: Zero limit*). *Assume that  $\frac{\varepsilon_1}{\varepsilon_2} \rightarrow 0$  and that  $f \in Lip(y_2)$  or  $f \in Lip(x) \cap BV(y_2)$ . Then*

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1, \infty}(\Omega)} \leq Const \cdot \left( \varepsilon_2 + \frac{\varepsilon_1}{\varepsilon_2} \right) , \quad (2.4)$$

where

$$\bar{f}(x) = \int_{T^2} f(x, y) dy . \quad (2.5)$$

**Theorem 2.2** (*Case 2: Irrational limit*). *Assume that  $\frac{\varepsilon_1}{\varepsilon_2} \rightarrow \alpha \in \mathbb{R} \setminus \mathbb{Q}$  and that  $f \in L^\infty(\Omega, H^s(T^2))$ ,  $s > 1$ . Then*

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1, \infty}(\Omega)} \rightarrow 0 , \quad (2.6)$$

where

$$\bar{f}(x) = \int_{T^2} f(x, y) dy . \quad (2.7)$$

**Theorem 2.3** (Case 3: Nonzero rational limit). Assume that  $\frac{\varepsilon_1}{\varepsilon_2} \rightarrow \frac{m}{n}$  where  $m, n \in \mathbb{Z}^*$  and  $r = \frac{\varepsilon_1}{\varepsilon_2} - \frac{m}{n}$ .  
(1) If  $\frac{r}{\varepsilon_2} \rightarrow c$  then

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq \text{Const} \cdot \left( \varepsilon_2 + \left| \frac{r}{\varepsilon_2} - c \right| \right), \quad (2.8)$$

where

$$\bar{f}(x) = \int_{T^1} f(x, ny_1 - \frac{n^2 cx}{m^2}, my_1) dy_1, \quad (2.9)$$

provided that  $f \in \text{Lip}(y_1)$ .

(2) If  $\frac{\varepsilon_2}{r} \rightarrow 0$  then

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq \text{Const} \cdot \left( |r| + \frac{\varepsilon_2}{|r|} \right), \quad (2.10)$$

where

$$\bar{f}(x) = \int_{T^2} f(x, y) dy, \quad (2.11)$$

provided that  $f \in \text{Lip}(y_2)$  or  $f \in \text{Lip}(x) \cap BV(y_2)$ .

In Cases 1 and 3 of rational limits, the weak convergence results are accompanied by convergence rate estimates. In Case 2 of an irrational limit the task of obtaining convergence rate estimates is much more intricate. We refer the reader to [2, §3] for some results in this direction.

### 2.3 Multiple scales

Let  $f(x, y) \in BV_x(\Omega \times T^n)$ ,  $n \geq 2$ , and  $f_\varepsilon(x) = f(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n})$ ,  $\varepsilon_i \downarrow 0$ . For the sake of simplicity, we assume that all scales are proportional (the reader is referred to [2, §4] for a more general analysis). Hence, we aim at finding the  $W^{-1,\infty}$ -weak limit of

$$f_\varepsilon(x) = f\left(x, \frac{\alpha_1 x}{\varepsilon}, \dots, \frac{\alpha_n x}{\varepsilon}\right),$$

where  $\alpha_i > 0$ ,  $1 \leq i \leq n$  and  $\alpha_1 = 1$ .

Let  $a = (\alpha_1, \dots, \alpha_n)$ . We let  $\mathcal{M}(a)$  denote the  $\mathbb{Z}$ -module of vectors in  $\mathbb{Z}^n$  which are orthogonal to  $a$ ,  $\mathcal{M}_{\mathbb{R}}(a)$  denote the  $\mathbb{R}$ -subspace of  $\mathbb{R}^n$  spanned by the vectors of  $\mathcal{M}(a)$ , and, finally,  $\mathcal{M}_{\mathbb{R}}(a)^\perp$  be its orthogonal complement in  $\mathbb{R}^n$ . Our statement is as follows:

**Theorem 2.4** Under the above assumptions, if  $f \in L^\infty(\Omega, H^s(T^n))$ ,  $s > \frac{n}{2}$ , then

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \rightarrow 0, \quad (2.12)$$

where

$$\bar{f}(x) = \int_{\mathcal{P}\mathcal{M}_{\mathbb{R}}(a)^\perp} f(x, y) dy, \quad (2.13)$$

$\mathcal{P}$  being the modulo-1 projection of  $\mathbb{R}^n$  onto  $T^n$ .

### 3 Convergence to the homogenized solution

We return to the problem of determining the limit of  $u^\varepsilon(x, t)$ , the entropy solution of (1.2), (1.3), when  $\varepsilon_i \downarrow 0$ . Let  $\bar{u}_0(x)$  and  $\bar{h}(x, t)$  denote the  $W^{-1, \infty}$  weak limits of  $u_0^\varepsilon(x)$  and  $h^\varepsilon(x, t)$ , respectively, with the  $W^{-1, \infty}$ -error bounds

$$\|u_0^\varepsilon(x) - \bar{u}_0(x)\|_{W^{-1, \infty}} \leq \mathcal{E}(\varepsilon) \quad , \quad \|h^\varepsilon(x, t) - \bar{h}(x, t)\|_{W^{-1, \infty}} \leq \mathcal{F}(\varepsilon, t) . \quad (3.1)$$

Then the following holds (consult [1, §2]):

**Theorem 3.1** (*W<sup>-1,∞</sup>-Convergence*). *Let  $u^\varepsilon$  be the entropy solution of (1.2), (1.3) and let  $u$  be the entropy solution of the corresponding homogenized equation*

$$u_t = K(u, u_x)_x + \bar{h}(x, t), \quad u(x, 0) = \bar{u}_0(x) . \quad (3.2)$$

Then,

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{W^{-1, \infty}} \leq \mathcal{E}(\varepsilon) + \int_0^t \mathcal{F}(\varepsilon, \tau) d\tau := \mathcal{G}(\varepsilon, t) . \quad (3.3)$$

**Remarks.**

1. Note that in the homogeneous case (where  $h \equiv 0$ ) the error bound in (3.3),  $\mathcal{G}(\varepsilon, t)$ , is independent of  $t$ .

2. In the one-scale case,  $\mathcal{G}(\varepsilon, t) = \mathcal{O}(\varepsilon)$  in view of Lemma 2.1.

Next, we translate the weak  $W^{-1, \infty}$ -convergence rate estimate, (3.3), into strong  $L^p$ -convergence rate estimates. To this end we focus our attention on nonlinear equations for which the solution operator is compact. Specifically, we concentrate on solution operators,  $S(t) : u(\cdot, 0) \mapsto u(\cdot, t)$ , which map bounded sets in  $L^\infty$  into bounded sets in the regularity spaces,  $W_{loc}^{s, r}$ ,  $s > 0$ ,  $1 \leq r \leq \infty$ . This compactness is clearly of a nonlinear nature and it implies that the solution operator immediately cancels out oscillations which may have been present at  $t = 0$ .

**Theorem 3.2** *If equation (1.2) possesses a  $W^{s, r}$ -regularizing effect then  $u^\varepsilon$  converges to  $u$  – the solution of the homogenized equation (3.2), and the following error estimates hold*

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p(\Omega)} \leq C \cdot B_\varepsilon^{s, r}(t)^{1-\theta} \cdot \mathcal{G}(\varepsilon, t)^\theta \quad \forall p \in [1, (\frac{1}{r} - s)_+^{-1}] . \quad (3.4)$$

Here,  $\theta$ ,  $p_*$  and  $B_\varepsilon^{s, r}$  are given by

$$\theta = \frac{\frac{1}{p_*} - \frac{1}{r} + s}{1 - \frac{1}{r} + s} \in [0, 1], \quad p_* := \max\{p, r(s+1)\} , \quad (3.5)$$

$$B_\varepsilon^{s, r}(t) = \|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{W^{s, r}} , \quad (3.6)$$

and  $C$  is some constant which depends on  $p$ ,  $|\Omega|^{\frac{1}{p} - \frac{1}{p_*}}$  and  $t$ .

These strong  $L^p$ -convergence rate estimates are obtained by means of interpolation between  $W^{-1, \infty}$  (in which the error tends to zero) and  $W^{s, r}$  (in which the error is bounded), see [1, Theorem 3.1].

## 4 Examples

Here, we mention briefly several families of convection-diffusion equations, of both hyperbolic and parabolic type, which are equipped with a certain  $W^{s,r}$ -regularity. Please see [1, §4,§5] for details, proofs and references. For the sake of clarity, we state the  $L^p$ -convergence rate estimates only in the one small scale case ( $n = 1$ ), in which the  $W^{-1,\infty}$ -error bound,  $\mathcal{G}(\varepsilon, t)$ , is of order  $\mathcal{O}(\varepsilon)$ .

### 1. Hyperbolic conservation laws

Convex hyperbolic conservation laws,

$$u_t + f(u)_x = h, \quad f'' \geq \alpha > 0,$$

are equipped with  $BV$ -regularity, which we identify with  $W^{1,1}$ -regularity. Hence, error estimate (3.4) reads in this case

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p(\Omega)} \leq C \cdot B_\varepsilon(t)^{1-\frac{1}{p_*}} \cdot \varepsilon^{\frac{1}{p_*}} \quad \forall p \in [1, \infty) ; \quad p_* := \max\{p, 2\}. \quad (4.1)$$

Here,  $B_\varepsilon(t)$  abbreviates the  $BV$ -size of the difference,

$$B_\varepsilon(t) = B_\varepsilon^{1,1}(t) = \|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{BV}, \quad (4.2)$$

and the constant  $C$  depends on  $p$ ,  $|\Omega|^{\frac{1}{p} - \frac{1}{p_*}}$ , and (in the inhomogeneous case) also on  $t$ .

In the homogeneous case,  $B_\varepsilon(t)$  is bounded with respect to  $\varepsilon$  and we get that for any fixed  $t > 0$

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p} \leq Const \cdot \varepsilon^{\frac{1}{p_*}} \quad p_* = \max\{p, 2\} \quad \forall p \in [1, \infty). \quad (4.3)$$

In the inhomogeneous case, however,  $B_\varepsilon(t)$  grows like  $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$  and that implies the following error estimate:

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p} \leq Const \cdot \varepsilon^{\frac{3}{2p_*} - \frac{1}{2}} \quad p_* = \max\{p, 2\}. \quad (4.4)$$

This shows that the nonlinear regularizing effect outpaces the persisting generation of modulated oscillations due to the oscillatory forcing term, and still yields strong convergence, though of a slower rate than in the homogeneous case.

### 2. Convection-diffusion equations with nonlinear flux

Consider the viscous conservation law

$$u_t + f(u)_x = Q(u)_{xx}, \quad Q' \geq 0, \quad (4.5)$$

where the flux  $f$  is *nonlinear* in the sense that there exists  $k \geq 2$  such that  $f^{(k)}$  never vanishes. This nonlinearity assumption implies  $W^{s,1}$ -regularity with  $s = \frac{1}{2k-1}$  which yields the error estimate

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p} \leq Const. \begin{cases} \varepsilon^{\frac{s}{s+1}} & \forall p \in [1, s+1), \\ \varepsilon^{\frac{1-p(1-s)}{sp}} & \forall p \in [s+1, \frac{1}{1-s}). \end{cases} \quad (4.6)$$

### 3. The porous media equation

Next, we focus on the regularizing effect due to the nonlinearity of the degenerate diffusivity. The porous media equation,

$$u_t = (u^m)_{xx}, \quad u \geq 0, \quad m > 1, \quad (4.7)$$

serves as a prototype model example for parabolic, 'convection-free' equations with degenerate diffusion. In this context, we identify  $m = 2$  as a critical exponent: when  $m > 2$  the equation is known to possess  $W^{s,\infty}$ -regularity with  $s = \frac{1}{m-1} < 1$ ; however, when  $m \leq 2$ , the equation is in fact  $W^{2,1}$ -regular. These two regularity results, combined, imply that if  $u^\varepsilon$  and  $u$  are an oscillatory and the corresponding homogenized solutions of (4.7), then for any fixed  $t > 0$  it holds that

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^\infty(\Omega)} \leq \text{Const} \cdot \varepsilon^{\min\{\frac{1}{m}, \frac{1}{2}\}}. \quad (4.8)$$

### 4. Convection-diffusion equations with nonlinear diffusion

We revisit the viscous conservation law (4.5); this time the  $C^1$  flux  $f$  could be arbitrary and the nonlinearity of the equation is related to the possibly degenerate diffusion. We assume that the diffusion term,  $Q(u)$ , is *nonlinear* in the sense that

$$\exists \alpha \in (0, 1), \delta_0 > 0 : \text{meas}\{u : 0 \leq Q'(u) \leq \delta\} \leq \text{Const} \cdot \delta^\alpha, \quad \forall \delta \leq \delta_0. \quad (4.9)$$

If (4.9) holds then equation (4.5) is at least  $W^{s,1}$ -regular with  $s = \frac{2\alpha}{\alpha+4}$ . Hence,  $L^p$ -error estimate (4.6) holds with this value of  $s$ .

Finally, we give an illustrated example to demonstrate our convergence analysis. To this end, let

$$f(y_1, y_2) = \cos(2\pi y_1) \cos(2\pi y_2) \quad \text{and} \quad f_\varepsilon(x) = f\left(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}\right). \quad (4.10)$$

The first part of Theorem 2.3 implies that in case  $\varepsilon_1 = \varepsilon_2$ ,

$$f_\varepsilon(x) \rightharpoonup \int_0^1 f(y_1, y_1) dy_1 = \frac{1}{2}, \quad (4.11)$$

while if  $\varepsilon_1 = \varepsilon_2 + \varepsilon_2^2$ ,

$$f_\varepsilon(x) \rightharpoonup \int_0^1 f(y_1 - x, y_1) dy_1 = \frac{1}{2} \cos(2\pi x). \quad (4.12)$$

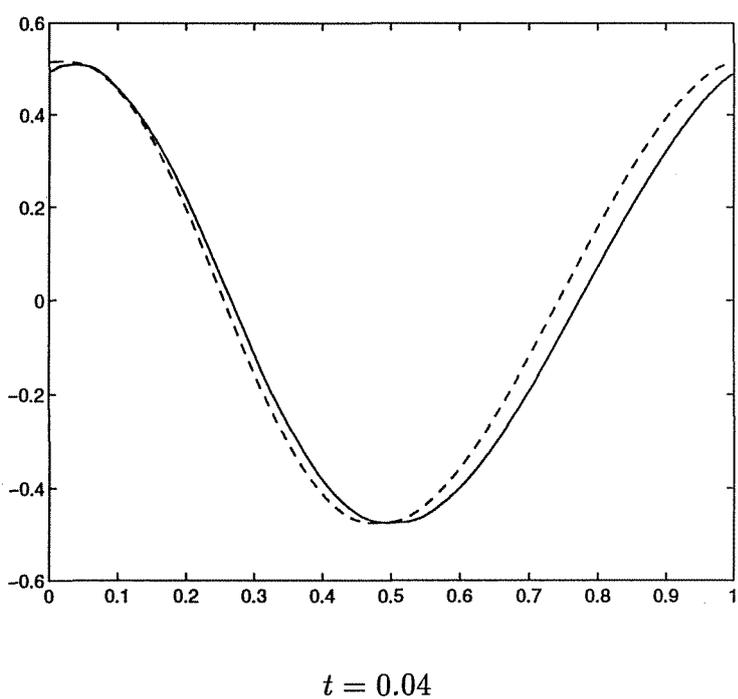
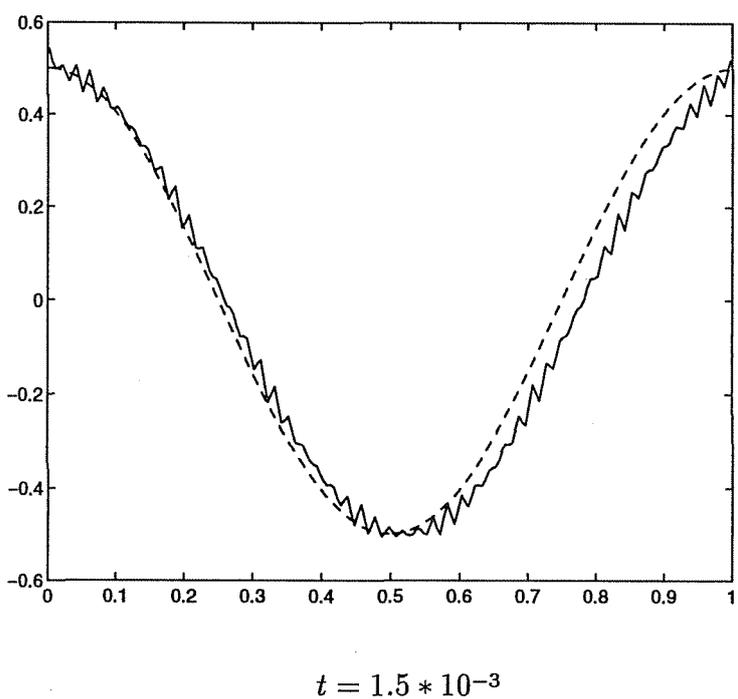
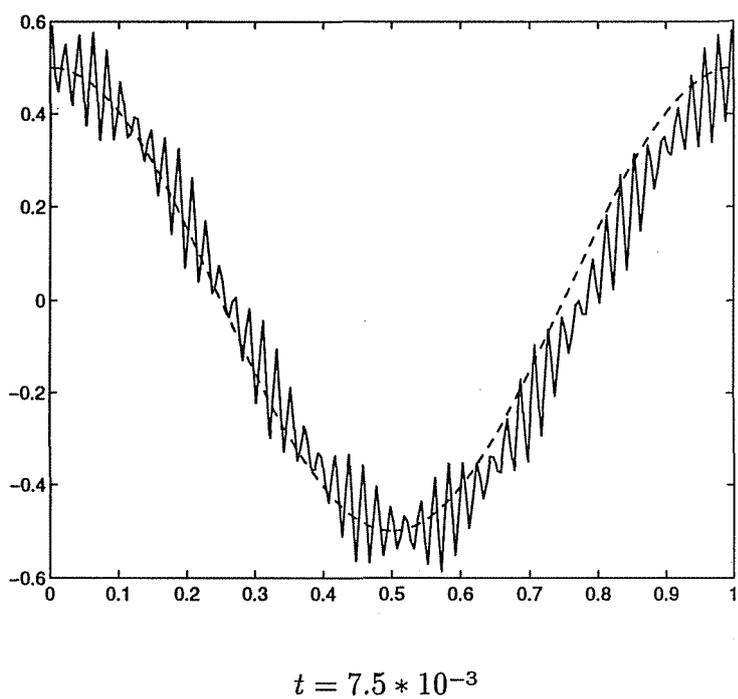
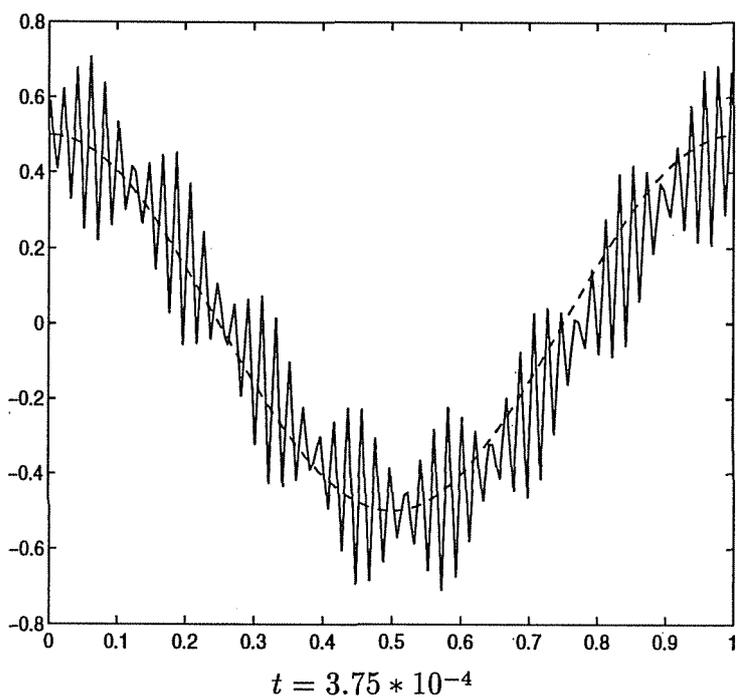
We now consider the following initial value problem for the Burgers' equation:

$$u_t^\varepsilon + \frac{1}{2}((u^\varepsilon)^2)_x = f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}\right) \quad ; \quad u^\varepsilon(x, 0) = f\left(\frac{x}{\varepsilon + \varepsilon^2}, \frac{x}{\varepsilon}\right), \quad (4.13)$$

where  $f(\cdot, \cdot)$  is given in (4.10). The weak limits of the forcing term and of the initial value are given, respectively, in (4.11) and (4.12). Hence, the entropy solution of (4.13),  $u^\varepsilon(\cdot, t)$ , tends weakly in  $W^{-1,\infty}$  to  $u(\cdot, t)$ , the entropy solution of the homogenized problem,

$$u_t + \frac{1}{2}(u^2)_x = \frac{1}{2} \quad ; \quad u(x, 0) = \frac{1}{2} \cos(2\pi x). \quad (4.14)$$

Moreover, apart from an initial layer of width  $\mathcal{O}(\varepsilon)$ ,  $u^\varepsilon(\cdot, t)$  converges strongly to  $u(\cdot, t)$ . In the figure next page we plot  $u^\varepsilon(\cdot, t)$ , with  $\varepsilon = 0.0408$ , versus  $u(\cdot, t)$  for four values of  $t$  in the initial layer ( $u^\varepsilon$  is described by the solid line and  $u$  by the dashed one). We see how the oscillations diminish in time and that they no longer exist at  $t = 0.04 \approx \varepsilon$ .



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# Global Existence of Small Amplitude Solutions for the Klein-Gordon-Zakharov Equations

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## Abstract

We study the Cauchy problem for the Klein-Gordon-Zakharov equations in three space dimensions. We show the existence of global solutions for small initial data using the invariant Sobolev space.

## 1 Introduction and Results

This note is based on the paper [15] and is intended to present a recent developments on the Klein-Gordon-Zakharov equations. We consider the Cauchy problem of the Klein-Gordon-Zakharov equations in three space dimensions :

$$\partial_t^2 u - \Delta u + u = -nu, \quad t > 0, \quad x \in \mathbf{R}^3, \quad (1)$$

$$\partial_t^2 n - \Delta n = \Delta |u|^2, \quad t > 0, \quad x \in \mathbf{R}^3, \quad (2)$$

$$u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad (3)$$

$$n(0, x) = n_0(x), \quad \partial_t n(0, x) = n_1(x),$$

where  $\partial_t = \partial/\partial t$ , and  $u(t, x)$  and  $n(t, x)$  are functions from  $\mathbf{R}_+ \times \mathbf{R}^3$  to  $\mathbf{C}^3$  and from  $\mathbf{R}_+ \times \mathbf{R}^3$  to  $\mathbf{R}$ , respectively. The system (1)–(2) describes the propagation of strong turbulence of the Langmuir wave in a high frequency plasma (see [14]).

Many results have been obtained concerning the global existence of small amplitude solutions for the coupled systems of the Klein-Gordon and wave equations with quadratic nonlinearity (see, e.g., [1–4, 6, 7, 9–11]). Two methods are known to be applicable to solve those systems. One is to use the invariant Sobolev space with respect to the generators of the Lorentz group. This was developed by Klainerman [7]. He also introduced the notion of the null condition to prove the existence of global solutions for the wave equations with quadratic nonlinearity. We note that the null condition technique is based on the Lorentz invariance of the equations. Recently, Bachelot [1] and Georgiev [3] improved the null condition technique to show the global existence result for the Dirac-Klein-Gordon equations and the Maxwell-Dirac equations, respectively, which are physically important (see also Georgiev [2]). Another method is based on the theory of normal forms introduced by Shatah [10], which is an extension of Poincaré’s theory of normal forms for the ordinary differential equations to the partial differential equations. The idea of this method is to transform the original system with quadratic nonlinearity into a new system with cubic nonlinearity. See also [12] and its references. Recently, applying the argument of normal forms to (1)–(2), Ozawa, Tsutaya and Tsutsumi [9] have proved the existence of global solutions to (1)–(3) for small initial data. However, in [9] one needs the high regularity assumptions on the data to ensure the global existence. In fact, the assumptions on the data in [9] are the following :

$$u_0 \in H^{52} \cap W^{29,6/(5+2\varepsilon)}, \quad u_1 \in H^{51} \cap W^{28,6/(5+2\varepsilon)},$$

$$n_0 \in H^{51} \cap W^{28,220/217} \cap \dot{H}^{-1}, \quad n_1 \in H^{50} \cap W^{27,220/217} \cap \dot{H}^{-2}, \quad 0 < \varepsilon \leq 10^{-2}.$$

Moreover, the global solution  $n$  of (1)–(3) constructed by [9] must belong to the homogeneous Sobolev space  $\dot{H}^{-1}$  of negative index. In this note we show that there exist the global solutions of (1)–(3) for small data using the invariant Sobolev space but without applying the null condition technique and improve the regularity requirements on the initial data. The reason why we do not need the null condition technique is due to the nonlinearity of (1)–(2). The nonlinear terms in (1) and (2) do not seem to satisfy the null

condition as in [1] or [3]. The main difference between the nonlinear terms in the system (1)–(2) and those in the Maxwell-Dirac system or the Dirac-Klein-Gordon system is the fact that the system (1)–(2) has the nonlinear term including the Laplacian  $\Delta$ . Then, we can estimate the  $L^2$ -norm of the solution  $n$  of (2) in the following procedure : (i) apply the generators of the Poincaré group to (2), (ii) rewrite the equation to the integral equation, (iii) integrate by parts, (iv) use the Hardy-Littlewood-Sobolev inequality. The point is that we do not need the  $L^2 - L^2$  estimate due to Klainerman [7] used in [1] and [3] to evaluate the solution  $n$  of  $\partial_t^2 n - \Delta n = \Delta|u|^2$  with zero data. Our proof seems simpler and shorter than that of [9].

Before we state the main results in the present note, we give several notations. We put  $\partial_j = \partial/\partial x_j$  for  $j = 1, 2, 3$ . Let  $\Gamma = (\Gamma_j; j = 1, \dots, 10)$  denote the generators of the Poincaré group  $(\partial_t, \partial_1, \partial_2, \partial_3, L_1, L_2, L_3, \Omega_{12}, \Omega_{23}, \Omega_{13})$ , where

$$\begin{aligned} L_j &= x_j \partial_t + t \partial_j, \quad j = 1, 2, 3, \\ \Omega_{ij} &= x_i \partial_j - x_j \partial_i, \quad 1 \leq i < j \leq 3, \end{aligned}$$

and we put

$$\partial = (\partial_t, \partial_1, \partial_2, \partial_3).$$

For a multi-index  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ , we put

$$\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}.$$

For a multi-index  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ , we put

$$\partial^\alpha = \partial_t^{\alpha_1} \partial_1^{\alpha_2} \partial_2^{\alpha_3} \partial_3^{\alpha_4}.$$

For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_{10})$ , we put

$$\Gamma^\alpha = \Gamma_1^{\alpha_1} \dots \Gamma_{10}^{\alpha_{10}}.$$

For  $1 \leq p \leq \infty$ , let  $L^p$  denote the standard  $L^p$  space on  $\mathbf{R}^3$ . For  $m \geq 0$  and  $s \geq 0$ , we define the weighted Sobolev space  $H^{m,s}$  on  $\mathbf{R}^3$  as follows:

$$H^{m,s} = \{v \in L^2; (1 + |x|^2)^{s/2} (1 - \Delta)^{m/2} v \in L^2\}$$

We put  $H^m \equiv H^{m,0}$  for  $m \geq 0$ . Let  $\omega = (1 - \Delta)^{1/2}$ .

We have the following theorem concerning the global existence of solutions to (1)-(3) for small initial data.

**Theorem 1** *Let  $0 < \varepsilon < 1/6$  and  $k \geq 4$ . Assume that  $u_0 \in H^{k+5,k+4}$ ,  $u_1 \in H^{k+4,k+4}$ ,  $n_0 \in H^{k+4,k+4}$  and  $n_1 \in H^{k+3,k+4}$ . Then, there exists a  $\delta > 0$  such that if*

$$\|u_0\|_{H^{k+5,k+4}} + \|u_1\|_{H^{k+4,k+4}} + \|n_0\|_{H^{k+4,k+4}} + \|n_1\|_{H^{k+3,k+4}} \leq \delta, \quad (4)$$

then (1)-(3) has the unique global solutions  $(u, n)$  satisfying

$$u \in \bigcap_{j=0}^{k+5} C^j([0, \infty); H^{k+5-j}), \quad (5)$$

$$n \in \bigcap_{j=0}^{k+4} C^j([0, \infty); H^{k+4-j}), \quad (6)$$

$$\begin{aligned} & \sum_{|\alpha|=k+4} \sup_{t \geq 0} (1+t)^{-\varepsilon} \{ \|\partial_t \Gamma^\alpha u(t)\|_{L^2} + \|\omega \Gamma^\alpha u(t)\|_{L^2} \} \\ & + \sum_{|\alpha| \leq k+4} \sup_{t \geq 0} (1+t)^{-\varepsilon} \|\Gamma^\alpha u(t)\|_{L^2} + \sum_{|\alpha| \leq k+4} \sup_{t \geq 0} \|\Gamma^\alpha n(t)\|_{L^2} \\ & + \sum_{|\alpha| \leq k} \sup_{\substack{t \geq 0 \\ x \in \mathbb{R}^3}} \{ |(1+t+|x|)^{3/2-2\varepsilon} \Gamma^\alpha u(t, x)| \\ & \quad + |(1+t+|x|) \Gamma^\alpha n(t, x)| \} < \infty. \end{aligned} \quad (7)$$

**Remark 1** (i) By (7), we see that the right hand sides of (1)–(2) are integrable in time and therefore we find that the solutions  $(u, n)$  of (1)–(3) constructed by Theorem 1 asymptotically approach the free solutions as  $t \rightarrow \infty$ . See [9].

(ii) As compared to [9], we have brought down the regularity assumptions on the data significantly. Instead, we need some spatial decay on the data, which is inevitable as far as the method depends on the invariant Sobolev norms.

(iii) The estimate (7) is close to the optimal one as regards the space-time behavior of solutions.

The following corollary follows easily from the proof of Theorem 1.

**Corollary 1**      *In addition to all the assumptions in Theorem 1, if  $u_0, u_1, n_0, n_1 \in \bigcap_{m \geq 1} H^m$ , then the solutions  $(u, n)$  given by Theorem 1 satisfy*

$$u(t, x), n(t, x) \in C^\infty([0, \infty) \times \mathbf{R}^3).$$

We can prove Theorem 1 by the contraction argument. The main tools in the proof are the decay estimate of the inhomogeneous linear Klein-Gordon equation due to Georgiev [4] and the Sobolev inequality in the Minkowski space by Klainerman [6,8] and Hörmander [5]. We can show the global existence result for (1)–(3) by using these two inequalities and adopting the weight function in the norm slightly different from [2,3] and [1].

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## 2. Assumptions and Results.

We suppose that all functions and solutions are real-valued. We set

$$W = C(R^+, V) \cap C^1(R^+, L^2(\Omega)) \cap C^2(R^+, V'),$$

where  $V = H_0^1$  and  $V'$  is its dual. When  $u \in W$ , the regularity of  $u$  is sufficient for proving our results.

We shall show the following theorem under some assumptions:

**Theorem 1.** *Let  $u$  be a global unique solution of (1) and  $u \in W$ . If  $u$  does not vanish identically, there exist some  $(x_1, t_1)$  and  $(x_2, t_2) \in \Omega \times [0, \infty)$  such that*

$$u(x_1, t_1)u(x_2, t_2) < 0.$$

We set some assumptions:

**Assumption 1.** *Let  $u_0$  and  $u_1$  be sufficiently regular. Suppose that  $u(t_0) = u_0$  and  $\partial_t u(t_0) = u_1$  are imposed on (1) at any fixed  $t = t_0$  as initial data. Then the problem (1) has a unique global solution  $u \in W$ .*

There are enormous literature about the results concerning Assumption 1 (cf. [2], [5] and the references of [2]).

**Assumption 2.** (i)  $G \in C^2$  on each variable and there exists a constant  $G_0$  such that  $0 < G_0 \leq G(x, t; u)$  for any  $(x, t) \in \Omega \times [0, \infty)$  and for any solution  $u$  of (1).

(ii)  $g(x, t, u)$  is nonnegative for any  $(x, t, u) \in \Omega \times [0, \infty) \times R$ .

We shall use the results of the eigenvalue problem:

$$(3) \quad \begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega. \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well-known that the eigenvalue problem (3) has eigenvalues  $\{\lambda_n\}$  such that  $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots \rightarrow \infty$  and eigenfunctions  $\phi_n$  corresponding to  $\lambda_n$  such that  $\phi_n \in C^\infty(\Omega) \cap V$  and especially  $\phi_1$  is positive in  $\Omega$ . For the proof of this fact we refer to [1] and [2].

Let us prove Theorem 1. Let  $u$  be the solution of (1), and  $\phi_1$  be the positive eigenfunction corresponding to  $\lambda_1$ . By applying the Green theorem, we have

$$\begin{aligned} \int_{\Omega} \{\Delta(G(x, t; u)u)\} \phi_1(x) dx &= \int_{\Omega} G(x, t; u)u \Delta\phi_1(x) dx \\ &= -\lambda_1 \int_{\Omega} G(x, t; u)u \phi_1(x) dx. \end{aligned}$$

We put

$$U(t) = \int_{\Omega} u(x, t) \phi_1(x) dx.$$

Integrating  $Lu\phi_1 = 0$  in  $\Omega$ , we have

$$(4) \quad (\alpha(t)U')' + \beta(t)U' + \lambda_1\gamma(t) \int_{\Omega} Gu\phi_1 dx + \int_{\Omega} gu\phi_1 dx = 0,$$

where  $U' = \frac{d}{dt}U$ .

Let  $u(x, t_0) > 0$  for any  $x \in \Omega$ . Then there is some time interval  $I$  with  $t_0$  as its left end point such that  $u(x, t) > 0$  for any  $(x, t) \in \Omega \times I$ . We shall show the length of  $I$  to be finite. It follows from Assumption 2 and the positivity of  $\phi_1$  that in  $I$

$$(5) \quad (\alpha(t)U')' + \beta(t)U' + \lambda_1 G_0 \gamma(t)U \leq 0.$$

We consider the ordinary differential equation:

$$(6) \quad (\alpha(t)v')' + \beta(t)v' + \gamma^*(t)v = 0.$$

**Lemma 1.** *Let  $\gamma^*(t) = \lambda_1 G_0 \gamma(t)$  and let  $v$  be a solution of (6) with an initial condition  $v(t_0) = U(t_0)$ ,  $v'(t_0) = U'(t_0)$ . If  $v(t) \geq 0$  for  $t \geq t_0$ , then  $U(t) \leq v(t)$  for  $t \geq t_0$ .*

*Proof.* From (5)  $\times v - (6) \times U$ , we have

$$\begin{aligned} & (\alpha U')'v - (\alpha v')'U + \beta(U'v - Uv') \\ &= \alpha(U'v - Uv')' + (\alpha' + \beta)(U'v - Uv') \leq 0. \end{aligned}$$

We can rewrite the above inequality as follows,

$$(7) \quad \left\{ \exp\left(\int_{t_0}^t \frac{\alpha' + \beta}{\alpha} ds\right)(U'v - Uv') \right\}' \leq 0.$$

From integrating (7) from  $t_0$  to  $t$  and by using the initial condition, we have

$$U'(t)v(t) \leq U(t)v'(t).$$

Integrating once more, we obtain the desired inequality

$$U(t) \leq v(t).$$

There are many results concerning the distribution of zeros of solutions of ordinary differential equations. We refer to [3] and [4]. We apply the following:

**Lemma 2.** (Leighton and Kreith). *Let  $\alpha \in C^1$ ,  $\beta, \gamma^* \in C$  and  $\alpha > 0$ . If for any real number  $h$ ,*

$$\int_h^\infty \frac{1}{\alpha(t)} dt = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \left\{ \frac{\beta(t)}{2\alpha(t)} + \int_h^t \left( \gamma^*(s) - \frac{\beta^2(s)}{4\alpha(s)} \right) ds \right\} = \infty.$$

*Then every nontrivial solution of (6) has an infinite number of zeros in every interval of the form  $[h, \infty)$ .*

For the proof we refer to [4].

We continue to show the proof of Theorem 1.

We suppose  $\alpha, \beta$  and  $\gamma^* = \lambda_1 G_0 \gamma$  to satisfy the assumptions of Lemma 2. Then it follows from Lemma 1 that there exists  $T (> t_0)$  such that  $U(T) = 0$ . Hence noting  $\phi_1(x) > 0$  in  $\Omega$ , we have  $u(x', t') < 0$  for some  $(x', t') \in \Omega \times N(T)$ , or  $u(x, t) \geq 0$  for any  $(x, t) \in \Omega \times N(T)$ , where  $N(T)$  is the neighborhood of  $T$ . The former case is included in the statement of Theorem 1. Let us consider the latter case. If  $u(x, t) \geq 0$  for any  $(x, t) \in \Omega \times N(T)$  and  $U(T) = 0$ , then we have  $u(x, T) = 0$  in  $\Omega$  and we can say that  $u(x, t)$  for any fixed  $x \in \Omega$  attains to the minimum value 0 at  $t = T$ . Hence  $\partial_t u(x, T) = 0$ . Thus it follows from the uniqueness of solutions (Assumption 1) that  $u(x, t) = 0$  identically in  $\Omega \times [0, \infty)$ . Thus the proof of Theorem 1 has completed.

Now we shall confine our equations to the Kirchhoff equation (2). For the solution  $u$  of (1) concernig the Kirchhoff equation we put

$$(8) \quad U(t) = \int_{\Omega} u(x, t) \phi_1(x) dx.$$

Then in a similar calculation of the integration by parts in Theorem 1 we have the identity

$$U''(t) + \lambda_1 c^2 (1 + \|\nabla u(t)\|^2) U(t) = 0.$$

Since the energy estimate of solutions, we can take the following constant  $M$  and  $m$  such that  $0 < m \leq \lambda_1 c^2 (1 + \|\nabla u(t)\|^2) \leq M$ . Then we set the following two ordinary differential equation with the initial values  $U(T) = 0$  and  $U'(T)$ ,

$$(9) \quad y'' + my = 0 \quad \text{with} \quad y(T) = 0, \quad y'(T) = U'(T),$$

and

$$(10) \quad z'' + Mz = 0 \quad \text{with} \quad z(T) = 0, \quad z'(T) = U'(T).$$

The solutions of (9) and (10) are respectively

$$y(t) = \frac{U'(T)}{\sqrt{m}} \sin \sqrt{m}(t - T) \quad \text{and} \quad z(t) = \frac{U'(T)}{\sqrt{M}} \sin \sqrt{M}(t - T),$$

and their zeros are for  $n = 1, 2, \dots$ , respectively

$$\left\{ T + \frac{n\pi}{\sqrt{m}} \right\} \quad \text{and} \quad \left\{ T + \frac{n\pi}{\sqrt{M}} \right\}.$$

Let  $U'(T) > 0$ . When  $U'(T) < 0$ , we have the same conclusion, too. In a similar calculation in Lemma 1 for any  $t \geq T$  we have

$$z(t) \leq U(t) \leq y(t).$$

Thus we have

**Theorem 2.** Let  $u$  be the solution of (1) for the Kirchhoff equation. Then  $U(t)$  defined in (8) has countably infinite zeros  $\{T_n\}$  such that

$$T + \frac{n\pi}{\sqrt{M}} \leq T_n \leq T + \frac{n\pi}{\sqrt{m}}.$$

And then there exist at least countably infinite  $(x_n, t_n), (x'_n, t'_n) \in \Omega \times I_n$  such that

$$u(x_n, t_n)u(x'_n, t'_n) < 0,$$

where  $I_n =$  the neighborhood of  $T_n$ .

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# ASYMPTOTIC BEHAVIOR FOR A QUASILINEAR WAVE EQUATION OF KIRCHHOFF TYPE

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**Abstract:** We prove the global solvability for a quasilinear hyperbolic wave equation of Kirchhoff type with small initial data in some function spaces. Then we construct the wave operators and the scattering operator in a neighborhood of the origin in the function spaces above, and show the continuity of the scattering operator with respect to suitable topologies.

## 1. Introduction.

In this paper we show the global solvability and the existence of the scattering operator for the equation

$$(QE) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2}(x, t) = m(\|\nabla_x u(\cdot, t)\|_{L^2}^2)^2 \Delta_x u(x, t) & \text{in } \mathbb{R}_x^n \times \mathbb{R}_t, \\ u(x, 0) = \phi_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = \psi_0(x) & \text{on } \mathbb{R}_x^n. \end{cases}$$

Throughout this paper we assume that  $m \in C^1([0, \infty))$  satisfies  $\inf_{\lambda \geq 0} m(\lambda) = m_0 > 0$ .

On the global existence for non-analytic initial data, we know only the results of Greenberg and Hu [4], Pokhozhaev [5] and D'Ancona and Spagnolo [1], [2], [3]. Greenberg and Hu [4] first showed the global existence for small initial data satisfying some decay conditions when  $n = 1$  and  $m(x)^2 = x + 1$ , and obtained some results on the asymptotic behavior. In particular, they proved that the function  $m(\|\nabla u(t)\|_{L^2}^2)$  converges to a number  $c_\infty$  as  $t \rightarrow \pm\infty$  for initial data as above. Pokhozhaev [5] showed the global existence for an arbitrary initial data belonging to  $H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$  in case  $m(x)^2 = (a_1 x + a_2)^{-2}$  with positive constants  $a_1$  and  $a_2$ . D'Ancona and Spagnolo [1], [3] improved and generalized the result on the global solvability of Greenberg and Hu [4] for hyperbolic type equations (of more general form in [1]) with general  $m$  as above

and general  $n$  for small initial data satisfying some decay conditions, and D'Ancona and Spagnolo [2] showed the global existence for (QE) with perturbation for initial data with compact support.

The purpose of this paper is to show the global solvability, to construct the wave operators and the scattering operator on a neighborhood of the origin in some suitable space included in  $\mathcal{H}^{J+1}(\mathbb{R}^n) \times H^J(\mathbb{R}^n)$  for  $J \geq 1/2$ , (see Notation 1 for the definition of  $\mathcal{H}^J$ ), and to show the continuity of the scattering operator with respect to a suitable topology. Conditions on functions data sufficient for belonging to this neighborhood are described in terms of the  $\dot{H}^r(\mathbb{R}^n)$  norm or the  $H^r(\mathbb{R}^n)$  norm with decay condition.

Details are given in [6].

## 2. Function spaces.

**Notation 1.** For  $s \in \mathbb{R}$ , let  $\mathcal{H}^s(\mathbb{R}^n) = \{f \in \mathcal{S}'/\mathbb{C}; \nabla f \in H^{s-1}(\mathbb{R}^n)\}$  with norm  $\|f\|_{\mathcal{H}^s} = \|\nabla f\|_{H^{s-1}}$ .

**Remark 1.** Suppose that  $s > 0$ . Then the Sobolev imbedding theorem implies  $\mathcal{H}^s(\mathbb{R}^n) \subset L^{2n/(n-2s)}(\mathbb{R}^n)$  for  $n > 2s$ , but  $\mathcal{H}^s(\mathbb{R}^n)$  is not contained in  $L^2(\mathbb{R}^n)$ .

**Notation 2.** Let  $p$  be a non-negative number. Let  $\nu$  be a positively homogeneous measurable function of order 0 on  $\mathbb{R}^n$  such that  $\{\nu(\xi); \xi \in \mathbb{R}^n\} \subset \{1, -1\}$ . For a function  $f \in L^1(\mathbb{R}^n)$ , we denote

$$\|f\|_{p,\nu} = \sup_{\tau \in \mathbb{R}} (1 + |\tau|)^p \left| \int_{\mathbb{R}^n} e^{i|\xi|\nu(\xi)\tau} \nu(\xi) f(\xi) d\xi \right|.$$

**Notation 3.** Let  $p$  and  $\varepsilon$  be positive numbers. Let  $\nu$  be a function in Notation 2. We define a set  $X_{p,\nu}$  and a subset  $Y_{p,\nu,\varepsilon}$  in  $X_{p,\nu}$  as follows:

$$\begin{aligned} X_{p,\nu} &= \{(\phi, \psi) \in \mathcal{H}^{3/2}(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n); \|(\phi, \psi)\|_{X_{p,\nu}}^2 \equiv \\ &\quad \|\phi\|_{\mathcal{H}^{3/2}}^2 + \|\psi\|_{H^{1/2}}^2 + \left\| |\xi|^3 |\hat{\phi}|^2 \right\|_{p,\nu} + \left\| |\xi| |\hat{\psi}|^2 \right\|_{p,\nu} + \left\| |\xi|^2 \nu(\xi) \Re(\hat{\phi} \bar{\hat{\psi}}) \right\|_{p,\nu} < \infty\}, \\ Y_{p,\nu,\varepsilon} &= \{(\phi, \psi) \in \mathcal{H}^{3/2}(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n); \|(\phi, \psi)\|_{X_{p,\nu}}^2 \leq \varepsilon\}. \end{aligned}$$

For  $J \geq 1/2$ , we put

$$X_{p,\nu}^J = X_{p,\nu} \cap (\mathcal{H}^{J+1}(\mathbb{R}^n) \times H^J(\mathbb{R}^n)), \quad Y_{p,\nu,\varepsilon}^J = Y_{p,\nu,\varepsilon} \cap (\mathcal{H}^{J+1}(\mathbb{R}^n) \times H^J(\mathbb{R}^n)).$$

We define the distance  $d_{p,\nu}^J$  on the set  $X_{p,\nu}^J$  as follows:

$$\begin{aligned} d_{p,\nu}^J((\phi_1, \psi_1), (\phi_2, \psi_2)) &= \|\phi_1 - \phi_2\|_{\mathcal{H}^{J+1}} + \|\psi_1 - \psi_2\|_{H^J} + \left\| |\xi|^3 (|\hat{\phi}_1|^2 - |\hat{\phi}_2|^2) \right\|_{p,\nu} \\ &\quad + \left\| |\xi| (|\hat{\psi}_1|^2 - |\hat{\psi}_2|^2) \right\|_{p,\nu} + \left\| |\xi|^2 \nu(\xi) \Re(\hat{\phi}_1 \bar{\hat{\psi}}_1 - \hat{\phi}_2 \bar{\hat{\psi}}_2) \right\|_{p,\nu}. \end{aligned}$$

**Remark 2.** We use  $Y_{p,\nu,\varepsilon}^J$  with  $p > 1$  for the global solvability and  $p > 2$  for the existence of the wave operator.

**Notation 4.** Let  $s_1, s_2 \in \mathbb{R}$ . We denote

$$\|f\|_{[s_1, s_2]}^2 = \int_{|\xi| \leq 1} \left( |\xi|^{s_1} |\hat{f}(\xi)| \right)^2 d\xi + \int_{|\xi| > 1} \left( |\xi|^{s_2} |\hat{f}(\xi)| \right)^2 d\xi.$$

**Remark 3.** It holds that

$$\{f \in \mathcal{S}' ; \|f\|_{[s_1, s_2]}^2 < \infty\} = \begin{cases} \dot{H}^{s_1} + \dot{H}^{s_2} & \text{if } s_1 \geq s_2, \\ \dot{H}^{s_1} \cap \dot{H}^{s_2} & \text{if } s_1 \leq s_2. \end{cases}$$

In the following, we give some estimates of  $\|(\phi, \psi)\|_{X_{p, \nu}}$ .

The following proposition is essentially the same as Lemma A of D'Ancona-Spagnolo [3], where they treated  $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$  with  $\nu \equiv 1$  and considered the  $H^{3/2}$  norm and the  $H^{1/2}$  norm in place of the  $\dot{H}^{3/2}$  norm and the  $\dot{H}^{1/2}$  norm respectively. But it is easy to see that their proof is also applicable to our case for every  $\nu$  in Notation 2, in view of the denseness of  $C_0^\infty(\mathbb{R}^n)$  in the function spaces below.

**Proposition A.** *Let  $k$  be an integer such that  $0 \leq k \leq n + 1$ . Then there exists a positive constant  $C = C_{k, n}$  such that the following inequality holds for every  $(\phi, \psi) \in \mathcal{H}^{3/2}(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n)$  and every  $\nu$  in Notation 2:*

$$(1) \quad \|(\phi, \psi)\|_{X_{k, \nu}}^2 \leq C \left\{ \|\nabla \phi\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \sum_{|\alpha| \leq k} \left( \|x^\alpha \phi\|_{\dot{H}^{3/2}}^2 + \|x^\alpha \psi\|_{\dot{H}^{1/2}}^2 \right) \right\}.$$

Here this inequality should be interpreted as follows: If the right-hand side is finite, then the left-hand side is finite and dominated by the right-hand side. The conclusions of the following three propositions should be interpreted in the same way.

In view of Remark 2, the above proposition covers the cases needed except for the scattering in the case  $n = 1$ . We give other conditions with no upper bound of  $k$ .

**Proposition 1.** *Let  $p$  be a positive number and  $k$  be an integer such that  $k \geq p$ . Then there exists a positive constant  $C = C_{p, k, n}$  such that the following inequality holds for every  $(\phi, \psi) \in \mathcal{H}^{3/2}(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n)$  and every  $\nu$  in Notation 2:*

$$\|(\phi, \psi)\|_{X_{p, \nu}}^2 \leq C \sum_{|\alpha| \leq k} \left( \|x^\alpha \phi\|_{[(3-p)/2 + |\alpha|, 3/2]}^2 + \|x^\alpha \psi\|_{[(1-p)/2 + |\alpha|, 1/2]}^2 \right).$$

In the case  $n = 1$ , we show another sufficient condition, in which we does not use an integer  $k \geq p$ :

**Proposition 2.** *Let  $n = 1$  and  $\nu(\xi) = \text{sgn}(\xi)$  ( $= \xi/|\xi|$ ). Let  $p$  be a positive number. Then there exists a positive constant  $C = C_p$  such that the following inequality holds for every  $(\phi, \psi) \in \mathcal{H}^2(\mathbb{R}) \times H^1(\mathbb{R})$ :*

$$(2) \quad \|(\phi, \psi)\|_{X_{p, \nu}}^2 \leq C \left\| (|x|^p + 1) \left( \left| \frac{\partial^2 \phi}{\partial x^2} \right| + \left| \frac{\partial \phi}{\partial x} \right| + \left| \frac{\partial \psi}{\partial x} \right| + |\psi| \right) \right\|_{L^2}^2.$$

By using the proof of the above propositions, we see that the distance  $d_{p, \nu}^J$  is also dominated by the norms which we are familiar with. We only state one sufficient condition corresponding to Proposition 2: Other sufficient conditions, which correspond to Proposition A and Proposition 1, can be described in the same manner.

**Proposition 3.** *Let  $n, p, \nu(\xi)$  and  $C = C_p$  be the same as in Proposition 2, and assume that  $J \geq 1$ . Then the following inequality holds for every  $(\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathcal{H}^{J+1}(\mathbb{R}) \times H^J(\mathbb{R})$ :*

$$\begin{aligned} d_{p,\nu}^J((\phi_1, \psi_1), (\phi_2, \psi_2)) &\leq \|\phi_1 - \phi_2\|_{\mathcal{H}^{J+1}} + \|\psi_1 - \psi_2\|_{H^J} \\ &+ C \left\| (|x|^p + 1) \sum_{j=1}^2 \left( \left| \frac{\partial^2 \phi_j}{\partial x^2} \right| + \left| \frac{\partial \phi_j}{\partial x} \right| + \left| \frac{\partial \psi_j}{\partial x} \right| + |\psi_j| \right) \right\|_{L^2} \\ &\times \left\| (|x|^p + 1) \left( \left| \frac{\partial^2(\phi_1 - \phi_2)}{\partial x^2} \right| + \left| \frac{\partial(\phi_1 - \phi_2)}{\partial x} \right| + \left| \frac{\partial(\psi_1 - \psi_2)}{\partial x} \right| + |\psi_1 - \psi_2| \right) \right\|_{L^2}. \end{aligned}$$

### 3. Global solvability.

D'Ancona-Spagnolo [3] showed the global solvability for smooth initial data  $(\phi_0, \psi_0) \in Y_{p,\nu,\varepsilon}^J$ , with small  $\varepsilon$ , in case  $\nu(\xi) = 1$ , although they did not introduce the set  $Y_{p,\nu,\varepsilon}^J$  explicitly. Applying their Lemma A with  $k = 2$ , they proved the global solvability of (QE) for smooth  $\phi$  and  $\psi$  with small  $\sum_{|\alpha| \leq 2, |\beta| \leq 1} (\|x^\alpha D^\beta \nabla \phi_0\|_{L^2}^2 + \|x^\alpha D^\beta \psi_0\|_{L^2}^2)$ .

By using the function  $\nu$ , we prove the global solvability of (QE) for initial data  $(\phi_0, \psi_0) \in Y_{p,\nu,\varepsilon}^J$  with small  $\varepsilon$  for every  $\nu$  in Notation 2. When  $n = 1$ , we obtain the global solvability for initial data which makes the right-hand side of the inequality (2) with  $p > 1$  sufficiently small, by choosing  $\nu(\xi) = \text{sgn } \xi$  and applying the following theorem and Proposition 2. This solvability does not follow from the result for  $\nu(\xi) = 1$ .

**Theorem 1 (Unique global solvability).** *Let  $J \geq 1/2$  and  $p > 1$ . Let  $\nu$  be a function in Notation 2. Then there exists a positive constant  $\varepsilon_0$  depending only on the number  $p$  and the function  $m$  such that, for every  $(\phi_0, \psi_0) \in Y_{p,\nu,\varepsilon_0}^J$ , the Cauchy problem (QE) has a unique global solution  $(u(t), (\partial u / \partial t)(t)) \in \bigcap_{i=0}^1 C^i(\mathbb{R}, \mathcal{H}^{J+1-i}(\mathbb{R}^n)) \times C^i(\mathbb{R}, H^{J-i}(\mathbb{R}^n))$ .*

**Remark 4.** In Theorem 1, assume furthermore that  $\phi \in L^2(\mathbb{R}^n)$ . Then the solution  $u(t)$  belongs to  $\bigcap_{i=0}^2 C^i(\mathbb{R}, H^{J+1-i}(\mathbb{R}^n))$ .

**Remark 5.** There is a positive constant  $K$  depending only on  $p$  such that if

$$M_1 \left( \frac{\varepsilon_0}{m_0} \left( M_0(\varepsilon_0) + \frac{1}{m_0} \right) \right) \left( M_0(\varepsilon_0) + \frac{1}{m_0} \right) \varepsilon_0 \leq K m_0^2,$$

then  $\varepsilon_0$  satisfies the assertion of Theorem 1, where  $M_j(R) = \sup_{0 \leq \lambda \leq R} \left| \frac{\partial^j}{\partial \lambda^j} m(\lambda) \right|$  for  $j = 0, 1$  and  $R \in \mathbb{R}$ .

### 4. Scattering.

Assuming decay order on initial data more than that needed for the global solvability, we consider the asymptotic behavior of the solution.

**Theorem 2 (Asymptotic behavior).** Let  $J \geq 1/2$ ,  $p > 2$ ,  $\nu$  be a function in Notation 2, and  $\varepsilon_0$  be a constant in Theorem 1. Then there exist positive constants  $\varepsilon_1 (\leq \varepsilon_0)$  and  $E_1$  such that the following holds. Assume that  $u(t)$  is the solution of (QE) corresponding to  $(\phi_0, \psi_0) \in Y_{p,\nu,\varepsilon_1}^J$ . Then we have the equalities  $\lim_{t \rightarrow \pm\infty} \|\nabla u(t)\|_{L^2}^2 = \lambda_\infty$  and  $\lim_{t \rightarrow \pm\infty} \|\frac{\partial u}{\partial t}(t)\|_{L^2}^2 = m(\lambda_\infty)\lambda_\infty$ , where  $\lambda_\infty$  is determined uniquely by the equality

$$\frac{1}{2}M(\lambda_\infty) + \frac{1}{2}m(\lambda_\infty)^2\lambda_\infty = \frac{1}{2}M(\|\nabla\phi_0\|_{L^2}^2) + \frac{1}{2}\|\psi_0\|_{L^2}^2$$

with  $M(x) = \int_0^x m(z)^2 dz$ . That is, the ratio of the potential energy  $M(\|\nabla u(t)\|_{L^2}^2)/2$  to the kinetic energy  $\|\partial u(t)/\partial t\|_{L^2}^2/2$  tends to  $M(\lambda_\infty)/m(\lambda_\infty)^2\lambda_\infty$ . Put  $c_\infty = m(\lambda_\infty)$ . Then

$$c_\infty - m(\|\nabla u(t)\|_{L^2}^2) = O(|t|^{1-p}) \text{ as } t \rightarrow \pm\infty,$$

and there exists a unique global solution  $(v(t), (\partial v/\partial t)(t)) = (v_\pm(t), (\partial v_\pm/\partial t)(t)) \in \bigcap_{i=0}^1 C^i(\mathbb{R}; \mathcal{H}^{J+1-i}(\mathbb{R}^n)) \times C^i(\mathbb{R}; H^{J-i}(\mathbb{R}^n))$  of

$$(LE, c_\infty) \quad \frac{\partial^2 v}{\partial t^2} = c_\infty^2 \Delta v \text{ in } \mathbb{R}_x^n \times \mathbb{R}_t,$$

such that

$$\lim_{t \rightarrow \pm\infty} \left( \|\nabla u(t) - \nabla v_\pm(t)\|_{L^2} + \left\| \frac{\partial u}{\partial t}(t) - \frac{\partial v_\pm}{\partial t}(t) \right\|_{L^2} \right) = 0.$$

Furthermore it holds that

$$\lim_{t \rightarrow \pm\infty} \left( \|\nabla u(t) - \nabla v_\pm(t)\|_{H^J} + \left\| \frac{\partial u}{\partial t}(t) - \frac{\partial v_\pm}{\partial t}(t) \right\|_{H^J} \right) = 0,$$

$$\left( \|\nabla u(t) - \nabla v_\pm(t)\|_{H^{J-1}} + \left\| \frac{\partial u}{\partial t}(t) - \frac{\partial v_\pm}{\partial t}(t) \right\|_{H^{J-1}} \right) = O(|t|^{2-p}) \text{ as } t \rightarrow \pm\infty.$$

The inverse of the wave operator  $W_\pm^{-1} : (\phi_0, \psi_0) \mapsto (\phi_\pm, \psi_\pm) = (v_\pm(0), (\partial v_\pm/\partial t)(0))$  maps  $Y_{p,\nu,\varepsilon}^J$  to  $Y_{p,\nu,E_1\varepsilon}^J$  for every  $\varepsilon$  such that  $0 < \varepsilon \leq \varepsilon_1$ , and the following formula holds:

$$c_\infty = m \left( \frac{\|\nabla\phi_\pm\|_{L^2}^2 + c_\infty^{-2}\|\psi_\pm\|_{L^2}^2}{2} \right).$$

**Theorem 3 (Existence and continuity of the scattering operator).** Assume that  $m \in C^2([0, \infty); [m_0, \infty))$ . Let  $J \geq 1/2$  and  $p > 2$ . Let  $\nu$  be a function in Notation 2. Let  $\varepsilon_1$  and  $E_1$  be constants in Theorem 2. Then there exist a positive constant  $K_0$  depending only on  $p$ , positive constants  $\varepsilon_2 (\leq \varepsilon_1)$ ,  $E_2$  and  $E_3$  such that the following holds for every  $\varepsilon$  such that  $0 < \varepsilon \leq \varepsilon_2$ :

(i) Let  $(\phi, \psi) \in Y_{p,\nu,\varepsilon}^J$ , and let  $c_\infty$  be the positive number determined by the equality

$$c_\infty = m \left( \frac{\|\nabla\phi\|_{L^2}^2 + c_\infty^{-2}\|\psi\|_{L^2}^2}{2} \right).$$

Let  $v$  be a solution of  $(LE, c_\infty)$  with  $v(0) = \phi$  and  $\frac{\partial v}{\partial t}(0) = \psi$ . Then there exists a unique global solution  $(u_\pm(t), (\partial u_\pm / \partial t)(t)) \in \bigcap_{i=0}^1 C^i(\mathbb{R}; \mathcal{H}^{J+1-i}(\mathbb{R}^n)) \times C^i(\mathbb{R}; H^{J-i}(\mathbb{R}^n))$  of  $(QE)$  such that

$$\lim_{t \rightarrow \pm\infty} \left( \|\nabla u_\pm(t) - \nabla v(t)\|_{L^2} + \left\| \frac{\partial u_\pm}{\partial t}(t) - \frac{\partial v}{\partial t}(t) \right\|_{L^2} \right) = 0,$$

and that

$$(3_\pm) \quad \limsup_{t \rightarrow \pm\infty} \left( (1 + |t|)^p \left| \frac{d}{dt} \log(m(\|\nabla u_\pm(t)\|_{L^2}^2)) \right| \right) \leq K_0 m_0.$$

It holds furthermore that

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \left( \|\nabla u_\pm(t) - \nabla v(t)\|_{H^J} + \left\| \frac{\partial u_\pm}{\partial t}(t) - \frac{\partial v}{\partial t}(t) \right\|_{H^J} \right) &= 0, \\ \left( \|\nabla u_\pm(t) - \nabla v(t)\|_{H^{J-1}} + \left\| \frac{\partial u_\pm}{\partial t}(t) - \frac{\partial v}{\partial t}(t) \right\|_{H^{J-1}} \right) &= O(|t|^{2-p}) \text{ as } t \rightarrow \pm\infty, \\ \sup_{t \in \mathbb{R}} \left( (1 + |t|)^p \left| \frac{d}{dt} \log(m(\|\nabla u_\pm(t)\|_{L^2}^2)) \right| \right) &\leq E_3 \varepsilon. \end{aligned}$$

The operator  $W_\pm$  maps  $Y_{p,\nu,\varepsilon}^J$  to  $Y_{p,\nu,E_2\varepsilon}^J$ .

(ii) The scattering operator  $S = W_+^{-1}W_-$  is continuous from  $Y_{p,\nu,\varepsilon}^J$  to  $Y_{p,\nu,E_1E_2\varepsilon}^J$  with respect to the topology from  $d_{p,\nu}^J$  of  $X_{p,\nu}^J$  to  $\mathcal{H}^{J+1}(\mathbb{R}^n) \times H^J(\mathbb{R}^n)$ .

**Remark 6.** The condition  $(3_\pm)$  can be replaced by  $(u_\pm(0), \frac{\partial u_\pm(0)}{\partial t}) \in Y_{p,\nu,E_2\varepsilon}^J$ .

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# Integrability of the long- and short- wave interaction equation

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**Abstract:** The integrability in the sense of Painlevé property is examined in the long- and short- wave interaction equation. The equation described in a coupled form of the NLS equation with the K-dV equation has only two parameters in the normalized form. When the equation is reduced to the ODE through the traveling wave transformation, it is shown to pass the Painlevé test for three cases of the parameters. On the other hand, for these parameters, when the test is directly applied to the original PDE, it is found that two cases except for one do not pass the test without any restrictions. However, the test is found not to be successful in the nearly integrable region. Furthermore, the possibility of ‘finite time integrability’ is discussed for a special case of the parameters.

## 1. Introduction.

In dispersive media, wave interactions play an important role in energy exchange among two or more different wave modes, if resonance conditions with respect to wave frequencies (and wave numbers) or wave velocities are satisfied in these wave modes. The long- and short-wave interaction is one of such interactions and can strongly occur under the resonance condition that a group velocity of the short wave is nearly equal to a phase velocity of the long wave. In this article, we deal with the following model equation to describe this interaction, which is expressed in a coupled form of the Nonlinear Schrödinger (NLS) equation with the Korteweg-de Vries (K-dV) equation: [1]

$$i S_t \pm S_{xx} = SL, \quad L_t + \alpha L L_x + \beta L_{xxx} = |S|_x^2, \quad (1)$$

where  $L$  and  $S$  denote, respectively, the real long wave and the complex amplitude of the envelope of the short wave, while  $x$  and  $t$  are spatial and temporal coordinates in a frame of reference moving with the phase velocity of the long wave or the group velocity of the short wave.

In the above equation, which is expressed in the normalized form with only two parameters  $\alpha$  and  $\beta$ , the parameters and the alternative of the  $\pm$  signs in front of  $S_{xx}$  depend upon the individual properties of the waves and the media concerned: [1] the gravity and capillary waves in a single layer fluid ( $\alpha, \beta \leq 0$  and  $+$  sign), the gravity waves in a two-layer fluid ( $\beta \leq 0$  and  $-$  sign), the ion acoustic and electron plasma waves ( $\alpha \geq 0, \beta \leq 0$  and  $+$  sign) and so on. However, since the case of  $-$  sign can be formally obtained if  $t, L$  and  $\beta$  in eq.(1) are replaced by  $-t, -L$  and  $-\beta$ , we will consider only the case of  $+$  sign in the followings.

Depending upon the parameters  $\alpha$  and  $\beta$ , physical meanings and mathematical properties of this equation can be said as follows: When both  $\alpha$  and  $\beta$  are equal to zero, eq.(1) represents the case when the magnitude of the long wave is much less than that of the short wave ( $|L| \ll |S|$ ). For this case, the equation is proved to be integrable or to have the n-soliton solution by means of the inverse scattering transform (IST) method. [2, 3] On the other hand, when both  $\alpha$  and  $\beta$  have finite values, the equation represents the case for which the magnitudes of the long and short waves are of the same order ( $|L| \sim |S|$ ). In this case, not only analytic solitary wave (one-soliton) solutions, but also a variety of numerical solitary wave solutions including ones with oscillatory damped tails are found. [4] It is expected, however, that the long time asymptotic wave behavior may become chaotic for general initial waves or soliton interactions, since the equation for  $\beta = 1$  is shown to be non-integrable through IST [5]. Additionally, in the Hirota bilinear form for  $\alpha = -6\beta$ , the n-soliton solution has not been found for  $\alpha, \beta \neq 0$ . [5, 6] Nevertheless, for the nearly integrable case in the vicinity of  $\alpha = \beta = 0$ , it is numerically shown that the wave behavior is regular or irregular depending upon initial conditions and values of the parameters. [4]

As is seen in the above, though eq.(1) is shown to be non-integrable for the particular  $\alpha$  and  $\beta$ , the integrability has not yet been analytically surveyed for all values of the parameters, in particular, in the nearly integrable region. Therefore, in this article, the integrability of eq.(1) is examined in the  $(\alpha, \beta)$  parameter space by means of the Painlevé test, which is known as one of the useful and practical techniques to test the integrability despite some drawbacks. [7, 8]

The organization of this article is as follows: In section 2, the results of the test are shown for the reduced ordinary differential equation (ODE) through a variable transformation (Painlevé ODE test). In section 3, for the cases which pass the ODE test, the original partial differential equation (PDE) is directly tested (Painlevé PDE test). And finally, in section 4, we remark the validity of the test in the nearly integrable region and the possibility of the 'finite time integrability'.

## 2. Painlevé ODE test.

For the Painlevé ODE test, we first reduce eq.(1) to the ODE through the following traveling wave transformation:

$$S = f(\zeta) \exp[i(\lambda/2)(x - Vt)], \quad L = g(\zeta), \quad (\zeta = x - \lambda t) \quad (2)$$

where  $\lambda$  and  $V$  are constants. Substituting (2) into eq.(1) and integrating  $g$  with respect to  $\zeta$ , we can easily obtain the reduced ODE

$$f_{\zeta\zeta} + (\lambda/2)(V - \lambda/2)f = fg, \quad \beta g_{\zeta\zeta} + (\alpha/2)g^2 - \lambda g = f^2 - C^2, \quad (3)$$

where we have imposed the boundary conditions:  $f \rightarrow C$  (const.),  $f_\zeta, f_{\zeta\zeta}, g, g_\zeta, g_{\zeta\zeta} \rightarrow 0$  as  $|\zeta| \rightarrow \infty$ , and  $\lambda = 2V$  for  $C \neq 0$ .

Making use of the following variable transformation into eq.(3)

$$g \rightarrow (2/\beta)^{1/2}g, \quad \zeta \rightarrow (\beta/2)^{1/4}\zeta, \quad (4)$$

we can show that our system has Hénon-Heiles Hamiltonian

$$H = (1/2)[(d f / d \zeta)^2 + (d g / d \zeta)^2] + I(f, g), \quad (5)$$

where

$$I = (\beta/2)^{1/2}(\lambda/4)(V - \lambda/2)f^2 - (2/\beta)^{1/2}(\lambda/4)g^2 - (f^2 - C^2)g/2 + \alpha g^3/(6\beta).$$

Since the Painlevé properties (P-properties) in the above system have been examined by Chang *et al.* [9] for  $\beta > 0$  and  $C = 0$ , it is expected that our ODE has similar singular structures. In fact, it is found that eq.(3) has similar P-properties. [4]

According to the procedure of the test by Ablowitz *et al.*, [10] the solutions of eq.(3) are expanded in the following Laurent series:

$$f = (\zeta - \zeta_0)^{-a} \sum_{j=0}^{\infty} f_j(\zeta - \zeta_0)^j, \quad g = (\zeta - \zeta_0)^{-b} \sum_{j=0}^{\infty} g_j(\zeta - \zeta_0)^j. \quad (6)$$

where  $\zeta_0$  denotes an arbitrary movable singularity depending upon initial conditions. Substituting the above expression into eq.(3) and equating coefficients of powers of  $\zeta$ , we can obtain the leading orders  $a$  and  $b$  for  $j = 0$ , and the recursion relations with respect to  $f_j$  and  $g_j$  for  $j \geq 1$ . From the recursion relations, we can see that the coefficients  $f_j$  or/and  $g_j$  become arbitrary for particular values of  $j = r$ , which is called resonances. The resonances for  $r = -1$  and  $0$  are, respectively, corresponding to the arbitrariness of  $\zeta_0$  and  $f_0$  (and/or  $g_0$ ), though negative resonances for  $r < -1$  are ignored. [12] For the P-property, these  $a$ ,  $b$  and  $r$  are required, at least, to be integers, which means that the solutions should be of the pole type or the single-valued. Then, Table I shows that the candidates for the P-property are limited to three significant cases of  $\alpha$  and  $\beta$ . It is found in this table that the case  $\alpha = \beta = 0$  has only general solution, while the other cases have both general and singular solutions in pairs. In these solutions, the general solution means that the equation has equal arbitrary parameters to the order of the equation, while the singular solution means that the solution has less arbitrariness than the order of the equation. However, in order for these three candidates to have the P-property, the self-consistency of the resonance must be checked in the recursion relations. Resulting from this, it is finally found that the Case I for  $\alpha = -\beta$  has the P-property under the restrictions that either  $V - \lambda/2 + 2/\beta = 0$  for  $C = 0$  or  $V = \lambda = 0$  for  $C \neq 0$ , while the other cases have P-property without any restrictions.

### 3. Painlevé PDE test.

It is known that the test in the reduced ODE gives only necessary conditions for the

Table 1: Painlevé ODE Test

$\alpha = \beta = 0$		$a = 1, b = 2, f_0^2 = -2\lambda, g_0 = 2$ $r = -1, 4$ (general solution) P-property
$\alpha = -6\beta$	Case I	$a = 2, b = 2, f_0^2 = -72\beta, g_0 = 6$ $r = -3, -1, 6, 8$ (singular solution) P-property
	Case II	$a = 1, b = 2, g_0 = 2, f_0$ : arbitrary $r = -1, 0, 3, 6$ (general solution) P-property
$\alpha = -\beta$	Case I	$a = 2, c = 2, f_0^2 = 18\beta, g_0 = 6$ $r = -1, 2, 3, 6$ (general solution) P-property for $V - \lambda/2 + 2/\beta = 0$ ( $C = 0$ ) or $V = \lambda = 0$ ( $C \neq 0$ )
	Case II	$a = -4, b = 2, g_0 = 12, f_0$ : arbitrary $r = -7, -1, 0, 6$ (singular solution) P-property

original PDE to be completely integrable. [10] In other words, a given PDE is not completely integrable when the ODE reduced from the PDE does not have the P-property. Therefore, in this section, the integrability of the original PDE is directly examined for the three cases that pass the ODE test in the preceding section.

Let us apply the Painlevé PDE test, whose direct procedure was introduced by Weiss *et al.* [11] In this test, a given partial differential equation is said to have the P- property if the solutions are single-valued in the neighborhood of the arbitrary and analytic (movable) singular manifold. Since the singular manifold for the ODE reduces to the singularity with respect to a single variable, the PDE test may be considered as a straightforward extension of the ODE test with similar procedure. For convenience, rewriting eq.(1) in the following form:

$$i u_t + u_{xx} = uw, \quad -i v_t + v_{xx} = vw, \quad w_t + \alpha w w_x + \beta w_{xxx} = (uv)_x, \quad (7)$$

the solutions are set as

$$u = \phi^{-a} \sum_{j=0}^{\infty} u_j \phi^j, \quad v = \phi^{-b} \sum_{j=0}^{\infty} v_j \phi^j, \quad w = \phi^{-c} \sum_{j=0}^{\infty} w_j \phi^j. \quad (8)$$

Making use of (8) into eq.(7), we can determine the leading order  $a$ ,  $b$  and  $c$  and the resonances  $r$  like in the ODE test, whose values are integers for the same three cases of  $\alpha$  and  $\beta$  as in Table I. The results of the PDE test are shown in Table II, where the case  $\alpha = \beta = 0$  have only general solution, while the other two cases have both singular and general solutions. [13] Checking the recursion relations for the self-consistency of the resonances, it is finally found that the case of  $\alpha = \beta = 0$  and the Case II for  $\alpha = -\beta$  hold the P-property without any restrictions. The latter case, however, is excluded in the present context, since the solutions  $u$  and  $v$  are regular to vanish closely near the singular manifold  $\phi = 0$ . Consequently, the significant solution is only  $w$  which is nothing but that

Table 2: Painlevé PDE Test

$\alpha = \beta = 0$		$a = b = 1, c = 2, u_0 v_0 = 2\phi_x \phi_t$ ( $u_0$ or $v_0$ : arbitrary), $w_0 = 2\phi_x^2$ $r = -1, 0, 2, 3, 4$ (general solution) P-property
$\alpha = -6\beta$	Case I	$a = b = 2, c = 2, u_0 v_0 = -72\beta\phi_x^4$ , ( $u_0$ or $v_0$ : arbitrary) $w_0 = 6\phi_x^2$ $r = -3, -1, 0, 4, 5, 6, 8$ (singular solution) Conditional P-property
	Case II	$a = b = 1, c = 2, w_0 = 2\phi_x^2$ , $u_0$ and $v_0$ : arbitrary $r = -1, 0, 0, 3, 3, 4, 6$ (general solution) Conditional P-property
$\alpha = -\beta$	Case I	$a = b = 2, c = 2, u_0 v_0 = 18\beta\phi_x^4$ ( $u_0$ or $v_0$ : arbitrary), $w_0 = 6\phi_x^2$ $r = -1, 0, 2, 3, 4, 5, 6$ (general solution) Conditional P-property
	Case II	$a = b = -4, c = 2, w_0 = 12\phi_x^2$ , $u_0$ and $v_0$ : arbitrary $r = -7, -7, -1, 0, 0, 4, 6$ (singular solution) P-property

of the K-dV equation, where the resonances occur for  $r = -1, 4, 6$ . On the other hand, the other cases have the P-property through the traveling wave transformation like  $\phi = x - ct$  ( $c$ :const.), that is to say, the P-property is conditional. Thus, only the case of  $\alpha = \beta = 0$  is completely integrable, which is consistent with the result of IST method. [2, 3]

#### 4. Concluding remarks.

We can see in Table II that the leading orders and some coefficients in the expansions are coincident or adjustable between the completely integrable case  $\alpha = \beta = 0$  and the case for  $\alpha = -6\beta$  (Case II). Although this suggests that these two cases are closely related to each other, the test is found not to be successful in the nearly integrable region  $\alpha, \beta \sim 0$  for  $\alpha = -6\beta$ , since the singular manifold expansions become non-uniformly valid when  $\beta$  tends to zero. This non-uniformity may be due to the small parameter  $\beta$  in the highest order derivative term in eq.(1). Additionally, since there exists one-soliton solutions which are uniformly valid for  $\alpha = -6\beta$  including  $\alpha = \beta = 0$ , [4] the usual singular manifold expansions (6) and (8) is not appropriate to examine the integrability in this region.

On the other hand, in the general solution for  $\alpha = -6\beta$  (Case II), we should remark that the compatibility condition that permits the P-property is found to be relaxed considerably for a finite time. That is to say, since the significant compatibility condition for the P- property is written as

$$\theta_t - \theta\theta_x = 0, \quad (9)$$

through  $\theta = \phi_t/\phi_x$ , the general solution of the above wave equation

$$\theta = \Theta(x + t\theta), \quad (10)$$

is analytic for a finite time depending upon initial conditions, where  $\Theta$  denotes an arbitrary function. Therefore, for a certain class of  $\phi$  which is given by (10) through  $\theta = \phi_t/\phi_x$ , the

compatibility condition (9) can be satisfied for a finite time during which the solution (10) is analytic and arbitrary. This means that the equation holds the P-property for the finite time and is expected to have the multi-soliton solution for the time. As a special case, it is easily seen that the condition (9) is identically satisfied for an infinitely long time under the traveling wave transformation  $\phi = x - ct$ , which is confirmed by the existence of one-soliton solution. [4] Thus, for  $\alpha = -6\beta$ , though one soliton state is valid for an infinitely long time, the soliton interactions due to multi-soliton state might be elastic for the finite time, that is to say, the possibility of 'finite time integrability' is expected.

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# FINITE-TIME BEHAVIOUR FOR NONLINEAR SCHRÖDINGER EQUATIONS

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**Abstract:** We study the initial-boundary value problems for a class of Schrödinger equations. By constructing so called modal integration, the finite-time behaviour including the local estimates and the finite-time blowup of solutions for the problems is shown. In addition some complete conclusions on linear equations are given.

## 1. Introduction

Consider nonlinear Schrödinger equations

$$iu_t + \Delta u = f(u). \quad (1)$$

So far, the most situation studied for (1) is concerning the classical situation  $f = ku|u|^p$  ( $p > 0, k \in \mathbb{R}$ ). For general nonlinearity  $f(u)$ , Klainerman and Ponce investigated the global solutions to the Cauchy problems, Hayashi and Tsutsumi etc. further investigated the Cauchy problems to the situation of polynomial. In this note, we consider the following general polynomial situation:

$$f(u) = c_0 + c_1u + c_2|u|^{1+p} \quad (p > 0, c_j \in C(j = 0, 1, 2)). \quad (2)$$

Let  $\Omega$  be a bounded domain with smooth boundary  $\partial\Omega$  in  $\mathbb{R}^n$  ( $n \geq 1$ ). We consider the initial boundary value problems of (1) with the conditions

$$u(0, x) = u_0(x), \quad x \in \Omega \subset \mathbb{R}^n \quad (3)$$

$$u(t, x) = 0, \quad t \geq 0, x \in \partial\Omega. \quad (4)$$

As is known, the study for the initial boundary value problems to (1) is also less. By constructing the modal integration, we may get the local estimates with wave-shape, then

get the finite-time blowup properties. At last applying the modal integration, we obtain some complete conclusions on linear equations.

## 2. Finite-time Behaviour

We introduce the linear eigenvalue problem of the Laplace operator on  $\Omega$ :

$$\Delta\phi(x) + \lambda\phi(x) = 0, \quad x \in \Omega, \quad (5)$$

$$\phi(x) = 0, \quad x \in \partial\Omega. \quad (6)$$

Then there exist the first eigenvalue  $\lambda > 0$  and the corresponding eigenfunction  $\phi(x) > 0$  satisfying the problem (5), (6). Furthermore we can take  $\phi(x)$  such that  $\int_{\Omega} \phi(x) dx = 1$ . Obviously  $\lambda$  and  $\phi(x)$  obtained by this method are uniquely determined by  $\Omega$ .

Let  $u = u(t, x)$  be a solution of the problems (1), (3), (4) under the condition (2), and

$$u \in C^1([0, T], L^1(\Omega) \cap C([0, T], W^{2,1} \cap L^{p+1}(\Omega))). \quad (7)$$

We construct the integral including parameter for  $u$ :

$$J(t) = e^{i(\lambda-\mu)t} \int_{\Omega} \phi u dx \quad (8)$$

Where  $\mu \in C$ . (8) is called the modal integration of the problems (1), (3), (4), which includes a complex parameter  $\mu$ .

For (8), by (1),(3),(4),(5),(6),(7), we have

$$J'(t) = ie^{i(\lambda-\mu)t} \left( -\mu \int_{\Omega} \phi u dx - \int_{\Omega} \phi f dx \right). \quad (9)$$

As  $c_j \in C(j = 0, 1, 2)$  in (2), it includes many cases. Here we only discuss a typical case:

$$f(u) = (1+i)|u|^{1+p} + u \quad (p > 0) \quad (10)$$

The other various cases can be dealt with in the similar way.

For (10), in (8), take  $\mu = -1$ , and let  $J_1(t) = ReJ(t)$ , it follows from (9) that

$$J_1'(t) = [\cos(\lambda+1)t + \sin(\lambda+1)t] \int_{\Omega} \phi |u|^{p+1} dx. \quad (11)$$

When  $T \leq \frac{3\pi}{4(\lambda+1)}$ , for  $t \in [0, T]$ , it is true that

$$\cos(\lambda+1)t + \sin(\lambda+1)t \geq 0. \quad (12)$$

From the Hölder's inequality and (8), we have

$$\int_{\Omega} \phi |u|^{p+1} dx \geq |J(t)|^{p+1} \geq |J_1(t)|^{p+1} \quad (13)$$

It follows from (11),(12),(13) that

$$J_1'(t) \geq [\cos(\lambda+1)t + \sin(\lambda+1)t] |J_1(t)|^{p+1}, \quad t \in [0, T]. \quad (14)$$

Let  $J_1(0) = \int_{\Omega} \phi(x) \operatorname{Re} u_0 dx > 0$ , then by (14), we may get

$$J_1(t) \geq J_1(0) \left[ 1 - \frac{p}{\lambda+1} (J_1(0))^p (1 + \sin(\lambda+1)t - \cos(\lambda+1)t) \right]^{-\frac{1}{p}} \quad (15)$$

Thus the following estimate theorem of the solution in finite time can be shown.

**Theorem 1.** Let  $f = (1+i)|u|^{p+1} + u, p > 0$ , and  $u_0 = u_0(x)$  satisfy  $\int_{\Omega} \phi(x) \operatorname{Re} u_0 dx = J_1(0) > 0$ . Then when  $T \leq \frac{3\pi}{4(\lambda+1)}$ , the solution of the problems (1),(3),(4) satisfying (7) has the following estimate:

$$\|u\|_{L^1(\Omega)} \geq c \left[ 1 - \frac{p}{\lambda+1} (J_1(0))^p (1 + \sin(\lambda+1)t - \cos(\lambda+1)t) \right]^{\frac{1}{p}}, \quad t \in [0, T] \quad (16)$$

where  $c$  is a positive constant.

For (10), in (8), take  $\mu = -1$ , and let  $J_2(t) = \operatorname{Im} J(t)$ , then the following estimate theorem can be shown

**Theorem 2.** Let  $f = (1+i)|u|^{p+1} + u, (p > 0), u_0 = u_0(x)$  satisfy  $\int_{\Omega} \phi(x) \operatorname{Im} u_0 dx = J_2(0) < 0$ . Then when  $T \leq \frac{\pi}{4(\lambda+1)}$ , the solution of the problems (1),(3),(4) satisfying (7) has the following estimate

$$\|u\|_{L^1(\Omega)} \geq c' \left[ 1 - \frac{p}{\lambda+1} |J_2(0)|^p (\cos(\lambda+1)t + \sin(\lambda+1)t - 1) \right]^{\frac{1}{p}}, \quad t \in [0, T] \quad (17)$$

where  $c'$  is a positive constant.

**Remark 1.** By (11) It may be seen that  $\operatorname{Re} J(t)$  acts a periodical oscillation with the period  $\frac{3\pi}{4(\lambda+1)}$ . From the some reason,  $\operatorname{Im} J(t)$  acts a periodical oscillation with the period  $\frac{\pi}{4(\lambda+1)}$ .

**Remark 2.** From (16), (17), it may be seen that the estimates are with wave-shape.

For (16), it can be seen that when  $J_1(0) \geq \left[ \frac{\lambda+1}{(1+\sqrt{2})^p} \right]^{\frac{1}{p}}$ , for

$$1 - \frac{p}{\lambda+1} (J_1(0))^p (1 + \sin(\lambda+1)t - \cos(\lambda+1)t), \quad (18)$$

there exists zeros in  $[0, \frac{3\pi}{4(\lambda+1)}]$ , Moreover when

$$\left[ \frac{\lambda+1}{(1+\sqrt{2})^p} \right]^{\frac{1}{p}} \leq J_1(0) \leq \left( \frac{\lambda+1}{2p} \right)^{\frac{1}{p}},$$

the zero of (18) is

$$T_0 = \frac{\pi}{2(\lambda+1)} - \frac{1}{2(\lambda+1)} \arcsin \left[ \frac{\lambda+1}{p(J_1(0))^p} \left( 2 - \frac{\lambda+1}{p(J_1(0))^p} \right) \right];$$

when

$$J_1(0) > \left( \frac{\lambda+1}{2p} \right)^{\frac{1}{p}},$$

the zero of (18) is

$$T'_0 = \frac{1}{2(\lambda+1)} \arcsin \left[ \frac{\lambda+1}{p(J_1(0))^p} \left( 2 - \frac{\lambda+1}{p(J_1(0))^p} \right) \right]$$

For (17), by the similar analyzing and computing, it can be obtained that when

$$J_2(0) \leq -\left[\frac{\lambda+1}{(\sqrt{2}-1)^p}\right]^{\frac{1}{p}},$$

in  $[0, \frac{\pi}{4(\lambda+1)}]$  for

$$1 - \frac{p}{\lambda+1}|J_2(0)|^p(\cos(\lambda+1)t + \sin(\lambda+1)t - 1)$$

there exists unique zero:

$$T_0 = \frac{1}{2(\lambda+1)} \arcsin\left[\frac{2(\lambda+1)}{p|J_2(0)|^p} + \frac{(\lambda+1)^2}{p^2|J_2(0)|^{2p}}\right]$$

Therefore from the estimates (16),(17),we can show the following blowingup theorem of the solution in finite time.

**Theorem 3.** Let  $f(u) = (1+i)|u|^{1+p} + u$ ,  $p > 0$ .

1. Let  $J_1(0) = \int_{\Omega} \phi(x) Reu_0 dx$ , then

1). when  $u_0$  satisfies

$$\left[\frac{\lambda+1}{(1+\sqrt{2})^p}\right]^{\frac{1}{p}} \leq J_1(0) \leq \left(\frac{\lambda+1}{2p}\right)^{\frac{1}{p}},$$

the solution of the problems (1),(3),(4) in (7) blows up in some finite time

$$T^* \leq T_0 = \frac{\pi}{2(\lambda+1)} - \frac{1}{2(\lambda+1)} \arcsin\left[\frac{\lambda+1}{p(J_1(0))^p} \left(2 - \frac{\lambda+1}{p(J_1(0))^p}\right)\right],$$

and the solution on  $[0, T^*)$  satisfies the estimate (16);

2). when  $u_0$  satisfies

$$J_1(0) > \left(\frac{\lambda+1}{2p}\right)^{\frac{1}{p}},$$

the solution of the problems (1), (3), (4) in (7) blows up in some finite time

$$T^{**} \leq T'_0 = \frac{1}{2(\lambda+1)} \arcsin\left[\frac{\lambda+1}{p(J_1(0))^p}\right].$$

and the solution on  $[0, T^{**})$  satisfies the estimate (16).

2. Let  $J_2(0) = \int_{\Omega} \phi(x) Imu_0 dx$ , then when  $u_0$  satisfies

$$J_2(0) \leq -\left[\frac{\lambda+1}{(\sqrt{2}-1)^p}\right]^{\frac{1}{p}}$$

the solution of the problems (1),(3),(4) in (7) blows up in some finite time

$$T^* \leq T_0 = \frac{1}{2(\lambda+1)} \arcsin\left[\frac{2(\lambda+1)}{p|J_2(0)|^p} + \frac{(\lambda+1)^2}{p^2|J_2(0)|^{2p}}\right].$$

and the solution on  $[0, T^*)$  satisfies the estimate (17).

**Remark 3.** From Theorem 3, it is seen that the less the initial value is, the longer the estimate of the lifespan of the solution is, and the bigger the initial value is, the more easily the blowingup occurs.

### 3. Conclusions on Linear Equations

Consider (1) is linear equations, that is

$$iu_t + \Delta u = c_1 u + c_2 \quad (19)$$

where  $c_1$  and  $c_2$  are complex constants.

Take  $\mu = c_1$ , then from (9) it follows that

$$J'(t) = -ic_2 e^{i(\lambda+c_1)t} \quad (20)$$

thus we can obtain the following results.

**Theorem 4.** Let  $f(u) = c_1 u + c_2$ , and  $c_1 \neq -\lambda$ , then for any  $T > 0$  the solution of the problems (1),(3),(4) satisfying (7) has

$$\int_{\Omega} \phi u dx = -\frac{c_2}{\lambda+1} + \left(\frac{c_2}{\lambda+c_1} + \int_{\Omega} \phi u_0 dx\right) e^{-i(\lambda+c_1)t} \quad t \in [0, T]. \quad (21)$$

**Theorem 5.** Let  $f(u) = -\lambda u + c_2$ , then for any  $T > 0$ , the solution of the problems (1),(3),(4) satisfying (7) has

$$\int_{\Omega} \phi u dx = \int_{\Omega} \phi u_0 dx - ic_2 t \quad t \in [0, T]. \quad (22)$$

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**Acknowledgement:** The author would like to express his gratitude to Professor Yoshio Tsutsumi for his advice and encouragement .

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