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**THE 21st SAPPORO SYMPOSIUM ON PARTIAL  
DIFFERENTIAL EQUATIONS**

**August 5 ~ 7, 1996**

THE 21st SAPPORO SYMPOSIUM ON PARTIAL  
DIFFERENTIAL EQUATIONS

August 5 ~ 7, 1996

Department of Mathematics  
Hokkaido University  
060 Sapporo, Japan

Chairman Rentaro Agemi  
(Hokkaido Univ.)

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## Rarefaction Waves in a Radiating Gas

Shuichi Kawashima  
Kyushu University

We discuss the asymptotic stability of rarefaction waves in a radiating gas.

Consider a model system of a radiating gas. This model system is a hyperbolic-elliptic system and consists of the inviscid Burgers equation for the velocity and a simple elliptic equation for the heat flux.

This model system admits three different nonlinear waves which are rarefaction waves, diffusion waves, and shock waves (traveling waves with shock profiles). The dissipation involved in the system is subtle and therefore any strong shock wave does contain an inner discontinuity (discontinuous shock wave) in the velocity.

The rarefaction wave of our system is defined in terms of the centered rarefaction wave of the inviscid Burgers equation. Similarly, the diffusion wave is defined in terms of the self-similar solution of the (viscous) Burgers equation. These two waves are not exact solutions of our system. On the other hand, the shock wave is an exact solution of the traveling wave form to the system.

In this talk, we only discuss the asymptotic stability of the rarefaction waves. Our main result is stated as follows.

Suppose that the strength of the rarefaction wave is small and that the initial velocity is close to the step function corresponding to the velocity component of the rarefaction wave. Then our system admits a unique global solution and this solution behaves like the rarefaction wave as time goes to infinity. The rate of the convergence toward the rarefaction wave is given explicitly.

In the proof, we use a smooth approximation of the rarefaction wave. Though the rarefaction wave is based on the inviscid Burgers equation, our approximation is defined in terms of the exact solution to the (viscous) Burgers equation with the corresponding step initial function. This new approximation combined with Ito's technique of deriving the rate of the convergence enables us to prove our main result.

# Monge–Kantorovich Mass Transfer Methods and PDE

by

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We have recently understood that several interesting nonlinear PDE can be interpreted as supporting “fast” mass transfer, governed by Monge–Kantorovich theory, and “slow” evolution of other parameters. A starting insight is that the dual problem (for the Monge–Kantorovich problem of optimally rearranging a measure  $\mu^+$  into  $\mu^-$ ) reads

$$(1) \quad \mu^+ - \mu^- \in \partial I[u],$$

where  $I[u] = 0$  if  $|Du| \leq 1$  a.e.,  $= +\infty$  otherwise. The idea is to study variants of (1) involving changes in time.

**Example 1: Sandpiles.** The evolution

$$(2) \quad \begin{cases} f - u_t \in \partial I[u] & (t > 0) \\ u = 0 & (t = 0) \end{cases}$$

is interpreted in work of L. Prigozhin [P], Aronsson–Evans–Wu [A-E-W] as modelling the growth of sandpiles, fed by the source  $f \geq 0$ . The physical stability condition  $|Du| \leq 1$  a.e. for the height function is precisely that from Monge–Kantorovich theory. Thus we can interpret (2) as saying the mass  $\mu^+ = f dx$  is instantly and optimally being rearranged to  $\mu^- = u_t dy$ , thereby determining the dynamics on the  $O(1)$ -scale.

A further model, due to Evans–Gariepy–Feldman [E-F-G], tracks the “collapse” of unstable sandpiles by the  $p \rightarrow \infty$  limit of

$$(3) \quad \begin{cases} u_t = \operatorname{div}(|Du|^{p-2} Du) & (t > 0) \\ u = g & (t = 0), \end{cases}$$

where  $\|Dg\|_{L^\infty} > 1$ . Invoking various rescalings we recast (2) (in the limit  $p \rightarrow \infty$ ) into a form to which Monge–Kantorovich “mass balance” relationships hold. These in turn imply  $O(1)$ -scale dynamics. For instance, if

$$g = L \operatorname{dist}_+(x, \Gamma_\tau)$$

where  $L > 1$ ,  $\tau = L^{-1}$ , and  $\Gamma_\tau$  is a smooth curve in  $\mathbb{R}^2$ , we derive a nonlocal geometric law of motion for the “base of the evolving sandpile”, namely

$$(4) \quad V = \frac{\gamma}{3t} \left( \frac{3 - 2\kappa\gamma}{2 - \kappa\gamma} \right) \quad (t \geq \tau),$$

where

$$(5) \quad \begin{cases} V = & \text{outward normal velocity} \\ \kappa = & \text{curvature} \\ \gamma = & \text{distance to the ridge.} \end{cases}$$

**Example 2: Compression molding.** In forthcoming work in collaboration with Aronsson we demonstrate that certain asymptotic compression molding problems give rise to related nonlocal geometric motions, again as a consequence of Monge–Kantorovich mass balance. The relevant geometric evolution in this case is

$$(6) \quad V = \gamma \left( 1 - \frac{\kappa\gamma}{2} \right) \quad (t \geq 0),$$

$V, \kappa, \gamma$  as in (5).

Feldman [F] has rigorously analyzed the motions (4), (6).

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# Existence of a moving boundary of the Hele-Shaw flow

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We consider the following 1-phase mathematical model for the Hele-Shaw flow. We assume that the moving boundary is a family of curves  $\{\Gamma(t)\}_{0 \leq t < T}$  in the  $x$ - $y$  plane  $\mathbf{R}^2 \cong \mathbf{C}$ , ( $z = x + iy$ ) parametrized by time  $t$  and  $\Gamma(t) = \{(x, y) \in \mathbf{R}^2; y = h(x, t), x \in \mathbf{R}\}$ . We set  $\Omega(t) = \{(x, y) \in \mathbf{R}^2; y > h(x, t), x \in \mathbf{R}\}$  which is occupied by viscous fluid. The outer unit normal vector and the outer normal velocity of  $\Gamma(t)$  from  $\Omega(t)$  are denoted by  $\nu$  and  $v$ . The curvature of  $\Gamma(t)$  is denoted by  $\kappa$ . (If  $\Omega$  is convex,  $\kappa \geq 0$ .) We suppose that there are some injection and/or suction points  $\{z_j\}_{j=1}^m \in \Omega(t)$ , and  $q_j \in \mathbf{R}$  denotes an inflow from  $z_j$  in unit time. If  $q_j < 0$ ,  $z_j$  is a suction point. Let  $\beta > 0$  be the surface tension coefficient on the moving boundary  $\Gamma(t)$ . We are interested in the next moving boundary problem for the 1-phase Hele-Shaw flow.

**Problem 1. (Hele-Shaw problem)** For given  $\Gamma_0 = \{(x, y); y = h_0(x)\}$ ,  $\{q_j\}_{j=1}^m \subset \mathbf{R}$ ,  $\{z_j\}_{j=1}^m \subset \{(x, y); y > h_0(x)\}$  and  $\beta > 0$ , find a moving boundary  $\{\Gamma(t)\}_{t \geq 0}$  satisfying the following conditions;

$$\begin{cases} v = -\frac{\partial P_{\Gamma(t)}}{\partial \nu} & \text{on } \Gamma(t), (t > 0), \\ \Gamma(0) = \Gamma_0 \end{cases},$$

where  $P_{\Gamma(t)}$  denotes the solution of the next Dirichlet problem in  $\Omega(t)$  for fixed  $\varepsilon > 0$ .

$$\begin{cases} -\Delta P_{\Gamma(t)}(z) = \sum_{j=1}^m q_j \delta(z - z_j), & z \in \Omega(t), \\ P_{\Gamma(t)}(z) = \beta \kappa, & z \in \Gamma(t), \\ \sup_{z \in \Omega(t), |z - z_j| > \varepsilon} |P_{\Gamma(t)}(z)| < \infty, \end{cases}$$

Let  $s$  be the length on  $\Gamma_0$  and let  $\varphi_0$  be the angle between the tangent vector and the  $x$ -axis. Namely,

$$\Gamma_0 = \{\gamma_0(s) \in \mathbf{C}; s \in \mathbf{R}\}, \quad \gamma_0'(s) = e^{i\varphi_0(s)}.$$

In the same manner, we set

$$\Gamma(t) = \{\gamma(s, t) \in \mathbf{C}; s \in \mathbf{R}\}, \quad \frac{\partial \gamma}{\partial s}(s, t) = e^{i\varphi(s, t)}.$$

To determine the parametric representation  $\gamma(s, t)$  uniquely, we assume

$$\gamma(s, 0) = \gamma_0(s), \quad \gamma_t(0, t) = v(0, t)\nu(0, t).$$

The following theorem is the main result of this research.

**Main Theorem.** For given  $\varphi_0 \in H^2(\mathbf{R})$ ,  $\|\varphi_0\|_{L^\infty(\mathbf{R})} < \pi/2$ ,  $\{q_j\}_{j=1}^m \subset \mathbf{R}$ ,  $\{z_j\}_{j=1}^m \subset \{(x, y); y > h_0(x)\}$  and  $\beta > 0$ , there is some  $T > 0$  and there exists unique solution of Problem 1  $\{\Gamma(t)\}_{0 \leq t \leq T}$  such that  $\varphi$  belongs to the function space

$$H^1((0, T); L^2(\mathbf{R})) \cap L^2((0, T); H^3(\mathbf{R})) \cap C^0([0, T]; H^2(\mathbf{R})).$$

In 1984, Duchon and Robert [5] proved similar theorem in the case that there is no injection or suction point ( $q_j = 0$ ,  $j = 1, \dots, m$ ). As the case  $q_j < 0$  is related to so-called “fingering” phenomena, the above result is important from this point of view. Moreover, the main theorem is still valid without the assumption  $\|\varphi_0\|_{L^\infty(\mathbf{R})} < \pi/2$ , namely,  $\Gamma(t)$  is not necessarily a graph. This generalization is significant when we study the fingering phenomena mathematically.

Some related results are found in the following references.

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## Existence of the singular ground state with maximal intensity

Tokushi Sato (Tohoku University)

In this talk, we consider singular ground states of the scalar field equation in  $\mathbb{R}^n$  with space dimension  $n \geq 2$ . For  $p > 1$  we call  $u$  a *ground state* of the scalar field equation if  $u \in C^2(\mathbb{R}^n)$  and  $u$  satisfies

$$(P)_0 \quad \begin{cases} -\Delta u + u = u^p, & u > 0 \quad \text{in } \mathbb{R}^n, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

It is known that  $(P)_0$  has a solution if and only if  $1 < p < (n+2)/(n-2)$ . (We agree that  $(n+2)/(n-2) = n/(n-2) = \infty$  for  $n = 2$ .) For simplicity, we assume that  $u$  attains its maximum at the origin for a solution  $u$  to  $(P)_0$ . Next we call  $u$  a *singular ground state* of the scalar field equation if  $u \in C^2(\mathbb{R}^n \setminus \{0\})$  and  $u$  satisfies

$$(P) \quad \begin{cases} -\Delta u + u = u^p, & u > 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\ u(x) \rightarrow \infty & \text{as } x \rightarrow 0, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{cases}$$

Note that any solution to  $(P)$  or  $(P)_0$  is radially symmetric. Concerning this problem, Ni–Serrin (1986) showed that  $(P)$  has no solution if  $n \geq 3$  and  $p \geq (n+2)/(n-2)$ . Recently, existence results of solutions to  $(P)$  for  $1 < p < (n+2)/(n-2)$  are proved by several authors.

In the following, we only consider the case where  $1 < p < n/(n-2)$ . Then the behavior of the singularity of any solution at the origin must be

$$u(x) \sim \kappa E(x) \quad \text{as } x \rightarrow 0$$

for some constant  $\kappa > 0$  depending on  $u$  which is called the *intensity* of the singularity. Here  $E$  is the fundamental solution for  $-\Delta$  on  $\mathbb{R}^n$ , i.e.

$$E(x) := \begin{cases} \frac{1}{(n-2)n\omega_n} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\ \frac{1}{2\pi} \log \frac{1}{|x|} & \text{if } n = 2 \end{cases}$$

( $\omega_n$  denotes the volume of a unit ball in  $\mathbb{R}^n$ ). Thus we consider the problem

$$(P)_\kappa \quad \begin{cases} -\Delta u + u = u^p, & u > 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\ u(x) \sim \kappa E(x) & \text{as } x \rightarrow 0, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \end{cases}$$

instead of  $(P)$ . Any solution  $u \in C^2(\mathbb{R}^n \setminus \{0\})$  to  $(P)_\kappa$  satisfies

$$-\Delta u + u = g(u) + \kappa \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \quad u \geq \kappa E_1 \quad \text{on } \mathbb{R}^n \setminus \{0\},$$

where  $E_1$  is the fundamental solution for  $-\Delta + 1$  on  $\mathbb{R}^n$ , i.e.

$$E_1(x) := \frac{1}{(2\pi)^{n/2}} \frac{1}{|x|^{(n-2)/2}} K_{(n-2)/2}(|x|)$$

(  $K_\nu$  denotes the modified Bessel function of order  $\nu$  ). Note that  $E_1$  satisfies

$$-\Delta E_1 + E_1 = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n)$$

and

$$E_1(x) \sim E(x) \quad \text{as } x \rightarrow 0, \quad E_1(x) \sim c_n \frac{e^{-|x|}}{|x|^{(n-1)/2}} \quad \text{as } |x| \rightarrow \infty.$$

Concerning this problem, we know the following fact.

**Fact.** (i) There exists  $\kappa^* > 0$  such that problem  $(P)_\kappa$  has a solution for  $0 < \kappa < \kappa^*$  and has no solution for  $\kappa > \kappa^*$ .

(ii) Problem  $(P)_\kappa$  has at least two solutions for  $0 < \kappa \ll 1$ .

Now we consider the existence of a solution to  $(P)_{\kappa^*}$  where

$$\kappa^* := \sup\{ \kappa > 0 \mid (P)_\kappa \text{ has a solution} \}$$

is called the *maximal intensity*. Our main result is the following.

**Theorem.** Let  $n \geq 2$  and  $1 < p < n/(n-2)$ .

(A) There exists a unique solution  $u_1 \in C^2(\mathbb{R}^n \setminus \{0\})$  to  $(P)_{\kappa^*}$ .

(B) Problem  $(P)_\kappa$  has at least two solutions for  $0 < \kappa^* - \kappa \ll 1$  near  $u_1$  in an appropriate sense.

In the following, we describe the outline of the proof of Theorem. From two propositions below, we have part (A) of Theorem.

**Proposition 1.** Let  $n \geq 2$  and  $1 < p < n/(n-2)$ . Assume that  $u_1 \in C^2(\mathbb{R}^n \setminus \{0\})$  is a solution to  $(P)_{\kappa_1}$  and the linearized problem

$$(L; u_1) \quad \begin{cases} -\Delta \varphi + \varphi = p u_1^{p-1} \varphi, & \varphi > 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\ \varphi(0) = 1, \quad \varphi(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

has a radial solution  $\varphi_1 \in C_*^2(\mathbb{R}^n \setminus \{0\})$ . Then  $\kappa_1 = \kappa^*$  and a solution to  $(P)_{\kappa_1}$  is unique. Here

$$C_*^2(\mathbb{R}^n \setminus \{0\}) := C^2(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n).$$

**Proposition 2.** Let  $n \geq 2$  and  $1 < p < n/(n-2)$ . Then there exists  $(u_1, \varphi_1; \kappa_1)$  which satisfies the assumption of Proposition 1.

In order to prove Proposition 1 we use the properties below. Let  $(u_1, \varphi_1; \kappa_1)$  be a solution in the sence of Proposition 1 and set

$$u_1 - \kappa_1 E_1 = \zeta^\nu v_1, \quad \varphi_1 = \zeta^\nu \psi_1,$$

where  $0 < \nu < 1$  and  $\zeta \in C^\infty(\mathbb{R}^n)$  is a radial function which is nonincreasing in  $\tau = |x|$  and satisfies

$$\zeta(x) = \begin{cases} 1 & \text{for } 0 \leq |x| \ll 1, \\ E_1(x) & \text{for } |x| \gg 1. \end{cases}$$

Then  $(v_1, \psi_1; \kappa_1)$  satisfies

$$v_1 = V[v_1; \kappa_1] \geq 0, \quad \psi_1 = \Psi[v_1; \kappa_1] \psi_1 > 0,$$

where

$$V[v; \kappa] := \zeta^{-\nu} \cdot E_1 * [(\zeta^\nu v + \kappa E_1)_+^p], \quad \Psi[v; \kappa] \psi := \zeta^{-\nu} \cdot E_1 * [p(\zeta^\nu v + \kappa E_1)_+^{p-1} \zeta^\nu \psi].$$

Note that  $\Psi[v_1; \kappa_1] : L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$  is a compact operator if  $p < q < n/(n-2)$ . Positivity of  $\psi_1$  deduces that

$$\text{Ker}(I - \Psi[v_1; \kappa_1]) = [\psi_1] \ (\subset L^q(\mathbb{R}^n)),$$

and we can see that

$$(I - \Psi[v_1; \kappa_1])(L^q(\mathbb{R}^n)) = [\psi_1^*]^\perp, \quad \psi_1^* := p u_1^{p-1} \varphi_1 \zeta^\nu$$

by using Fredholm's alternative. Proposition 1 follows from this fact and the convexity of the nonlinearity function.

In order to prove Proposition 2 we introduce a parameter  $\tau \in [0, 1]$  and consider

$$(P_\tau)_\kappa \quad \begin{cases} -\Delta u + u = u^p - (1 - \tau)(\kappa E_1)^p, & u \geq \kappa E_1 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\ u(x) \sim \kappa E(x) \quad \text{as } x \rightarrow 0, & u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{cases}$$

We set up the following problem.

**Definition.** For  $\tau \in [0, 1]$ ,  $(u, \varphi; \kappa)$  is a solution to  $(Q_\tau)$  if

- (i)  $\kappa > 0$ ,
- (ii)  $u \in C^2(\mathbb{R}^n \setminus \{0\})$  is a radial solution to  $(P_\tau)_\kappa$ ,
- (iii)  $\varphi \in C_*^2(\mathbb{R}^n \setminus \{0\})$  is a radial solution to  $(L; u)$ .

We set

$$T := \{ \tau \in [0, 1] \mid (Q_\tau) \text{ has a solution} \}.$$

We claim that  $T = [0, 1]$  which is equivalent to that  $T$  is nonempty, closed and open in  $[0, 1]$ . We divide the proof into three steps.

Step 1 [  $0 \in T$  ].

First note that  $u_0 = \kappa_0 E_1$  is a solution to  $(P_0)_{\kappa_0}$  for all  $\kappa_0 > 0$ . So we claim that  $(L; u_0)$  has a radial solution for some  $\kappa_0 > 0$ . To do this, we consider the minimizing problem

$$\inf \left\{ \frac{\|\nabla \varphi\|_2^2 + \|\varphi\|_2^2}{\|E_1^{(p-1)/2} \varphi\|_2^2} \mid \varphi \in W^{1,2}(\mathbb{R}^n) \setminus \{0\} \right\} (\equiv \bar{\lambda}).$$

By the standard argument we can see that  $\bar{\lambda} > 0$  and there exists a minimizer  $\varphi_0 \in W^{1,2}(\mathbb{R}^n) \setminus \{0\}$ . Furthermore, we see that  $\varphi_0 \in C_x^2(\mathbb{R}^n \setminus \{0\})$ ,  $\varphi_0$  is radial and satisfies

$$\begin{cases} -\Delta \varphi_0 + \varphi_0 = \bar{\lambda} E_1^{p-1} \varphi_0 & \text{in } \mathbb{R}^n \setminus \{0\}, \\ \varphi_0(0) = 1, \quad \varphi_0(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

(by a normalization). For  $\kappa_0 > 0$  such that  $\bar{\lambda} = p\kappa_0^{p-1}$ ,  $\varphi_0$  is a solution to  $(L; u_0)$  and  $(u_0, \varphi_0; \kappa_0)$  is a solution to  $(Q_0)$ .

Step 2 [ Closedness of  $T$  ].

For  $\tau \in T$  we denote a solution to  $(Q_\tau)$  by  $(u_\tau, \varphi_\tau; \kappa_\tau)$  and set

$$u_\tau - \kappa_\tau E_1 = w_\tau = z_\tau E_1, \quad \varphi_\tau = y_\tau E_1.$$

Then we see that  $z_\tau$  and  $y_\tau$  are increasing in  $r$ , while  $u_\tau$  and  $\varphi_\tau$  are decreasing in  $r$ . Moreover,  $\{\kappa_\tau\}_{\tau \in T}$  is decreasing in  $\tau$  and hence  $0 < \kappa_\tau \leq \kappa_0$ . For  $0 < \nu < 1$  we multiply  $\zeta^\nu$  both side of

$$-\Delta w_\tau + w_\tau = u_\tau^p - (1 - \tau)(\kappa_\tau E_1)^p \quad \text{in } \mathcal{D}'(\mathbb{R}^n)$$

and integrate on  $\mathbb{R}^n$ . Then we have

$$(*) \quad \int_{\mathbb{R}^n} u_\tau^p \zeta^\nu dx \leq M_\nu$$

for some  $M_\nu > 0$ , by making use of integration by part and Young's inequality. From the integral representation of solutions we see that  $\{z_\tau\}_{\tau \in T}$  and  $\{\varphi_\tau\}_{\tau \in T}$  are locally uniformly bounded and locally equi-continuous on  $\mathbb{R}^n$ .

Now we assume  $\{\tau_j\}_{j=1}^\infty \subset T$ ,  $\tau_j \rightarrow \tau$  as  $j \rightarrow \infty$ . By the Ascoli-Arzelà theorem there exist a subsequence  $\{j_i\}_{i=1}^\infty$ , radial functions  $z, \varphi \in C(\mathbb{R}^n)$  and  $\kappa \geq 0$  such that

$$z_{\tau_{j_i}} \rightarrow z, \quad \varphi_{\tau_{j_i}} \rightarrow \varphi \quad \text{locally uniformly on } \mathbb{R}^n, \quad \kappa_{\tau_{j_i}} \rightarrow \kappa \quad \text{as } i \rightarrow \infty.$$

We set  $u - \kappa_\tau E_1 = w = z E_1$  and  $\varphi = y E_1$ . Then we have

$$z(0) = 0, \quad \varphi(0) = 1, \quad u \geq \kappa E_1, \quad \varphi \geq 0 \quad \text{on } \mathbb{R}^n \setminus \{0\}.$$

Furthermore,  $z$  and  $y$  are nondecreasing in  $r$  and hence  $\varphi > 0$  on  $\mathbb{R}^n$ . Since  $u$  and  $\varphi$  are nonincreasing in  $r$ , there exist  $\gamma, \tilde{\gamma} \geq 0$  such that

$$u(x) \rightarrow \gamma, \quad \varphi(x) \rightarrow \tilde{\gamma} \quad \text{as } |x| \rightarrow \infty.$$

From  $(P_{\tau_j})_{\kappa_{\tau_j}}$ ,  $(L; u_{\tau_j})$  and  $(*)$  we have

$$-\Delta u + u = u^p - (1 - \tau)(\kappa E_1)^p, \quad -\Delta \varphi + \varphi = pu^{p-1}\varphi \quad \text{in } \mathbb{R}^n \setminus \{0\}$$

and  $u \not\equiv 0$ . As  $|x| \rightarrow \infty$ , we have  $\gamma = 0, 1$  and  $\tilde{\gamma} = 0$ .

If  $\gamma = 1$ , then  $pu^{p-1} - 1 \geq p - 1 > 0$  on  $\mathbb{R}^n$  and  $\varphi(x) = \varphi(|x|)$  satisfies

$$\varphi'' + \frac{n-1}{r}\varphi' + (pu^{p-1} - 1)\varphi = 0 \quad \text{for } r > 0$$

and hence  $\varphi$  is oscillating, which is a contradiction. Therefore,  $\gamma = 0$  holds true.

Finally we note that  $(L; u)$  has no nontrivial radial solution if  $u \in C^2(\mathbb{R}^n)$  is a solution to  $(P)_0$ . From this fact we can deduce  $\kappa > 0$  and  $(u, \varphi; \kappa)$  is a solution to  $(Q_\tau)$ . Therefore,  $T$  is closed.

**Step 3 [ Openness of  $T$  ].**

Assume  $\tau_0 \in T$  and set

$$\begin{cases} \Lambda_{\tau_0} := \left\{ \xi \in X(\mathbb{R}^n)_r \mid \int_{\mathbb{R}^n} pu_{\tau_0}^{p-1}\varphi_{\tau_0} \cdot \xi E_1 dx = 0 \right\}, \\ \Sigma_{\tau_0} := \left\{ \xi \in X(\mathbb{R}^n)_r \mid \int_{\mathbb{R}^n} p(p-1)u_{\tau_0}^{p-2}\varphi_{\tau_0}^2 \cdot \xi E_1 dx = 1 \right\}, \end{cases}$$

where

$$X(\mathbb{R}^n)_r := \{ \xi \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \mid \xi \text{ is radial and } \xi(0) = 0 \}.$$

We introduce a small parameter  $\varepsilon$  and solve  $(Q_\tau)$  for  $|\tau_0 - \tau| \ll 1$  in the form

$$(z, y; \kappa, \tau) = \begin{cases} (\varepsilon(y_0 + \varepsilon\xi), y_0 + \varepsilon\eta; \kappa_0 - \varepsilon\rho, \varepsilon^2\sigma) & \text{for } 0 < \varepsilon \ll 1 \quad \text{if } \tau_0 = 0, \\ (z_{\tau_0} + \varepsilon\xi, y_{\tau_0} + \varepsilon\eta; \kappa_{\tau_0} - \varepsilon\rho, \tau_0 + \varepsilon\sigma) & \text{for } 0 < |\varepsilon| \ll 1 \quad \text{if } \tau_0 > 0, \end{cases}$$

where  $(\xi, \eta; \rho, \sigma) \in \Lambda_0^2 \times \mathbb{R}^2$  if  $\tau_0 = 0$ ,  $(\xi, \eta; \rho, \sigma) \in (\Sigma_{\tau_0} \times \Lambda_{\tau_0}) \times \mathbb{R}^2$  if  $\tau_0 > 0$ , by using the contraction mapping principle repeatedly.

By the three steps above we can conclude  $T = [0, 1]$  and part (A) of Theorem is established. In order to prove part (B) we introduce a small parameter  $\varepsilon$  and solve  $(P)_\kappa$  for  $0 < \kappa_1 - \kappa \ll 1$  in the form

$$(z; \kappa) = (z_1 + \varepsilon(y_1 + \varepsilon\xi); \kappa_1 - \varepsilon^2\rho) \quad \text{for } 0 < |\varepsilon| \ll 1,$$

where  $(\xi; \rho) \in \Lambda_1 \times \mathbb{R}$ . This completes the proof of Theorem.

A NUMERICAL METHOD  
BASED ON THE DISCRETE MORSE SEMIFLOW

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1. Parabolic case

Let  $\Omega(\subset \mathbb{R}^n)$  be a bounded domain with smooth boundary ( $n \geq 1$ ). Firstly we consider a minimizing problem of a functional:

$$I(u) := \int_{\Omega} \left( |\nabla u|^2 + f(u) \right) dx, \quad \text{in } \mathcal{K} := \{u : \Omega \rightarrow \mathbb{R}^N; u \in \text{suitable space}\} \quad (1.1)$$

where  $N \geq 1$ ,  $\mathcal{K}$  is an admissible function space and  $f$  is a nonnegative function. In this paper,  $f$  are chosen  $f(u) := \frac{1}{8}(|u|^2 - 1)^2$ , in the Ginzburg-Landau ( $n = N = 2$ ), with  $\mathcal{K} := \{u \in W^{1,2}(\Omega; \mathbb{R}^N) \cap L^4(\Omega; \mathbb{R}^N); u = \varphi \text{ on } \partial\Omega\}$  and  $f(u) := \chi_{u>0}(x)$ , in the free boundary problem<sup>3)</sup> ( $n = 2, N = 1$ ), with  $\mathcal{K} = \{u \in L^1_{\text{loc}}(\Omega); \nabla u \in L^2(\Omega), u = u_0 \text{ on } \partial\Omega\}$ .

For (1.1), we introduce the notion of the discrete Morse semiflow which is based on so called Kikuchi's scheme<sup>2),3)</sup>. Let  $h$  be a positive number which tends to zero later. Consider the following type of time semidiscretized functional:

$$J_m(u) = \int_{\Omega} \frac{|u - u_{m-1}|^2}{h} dx + I(u), \quad (m = 1, 2, \dots) \quad (1.2)$$

and determine a sequence  $\{u_m\}$  of functions in  $\mathcal{K}$  inductively. Firstly, for an initial data  $u_0 \in \mathcal{K}$  with  $I(u_0) < \infty$ , we define  $u_1$ , as a minimizer of  $J_1$  in  $\mathcal{K}$ . The next function  $u_2 \in \mathcal{K}$  is a minimizer of  $J_2$  in  $\mathcal{K}$ , and so on.

The essential estimate on this flow is based on the following property:

$$J_m^h(u_m^h) \equiv \int_{\Omega} \frac{|u_m^h - u_{m-1}^h|^2}{h} dx + I(u_m^h) \leq J_m^h(u_{m-1}^h) \equiv I(u_{m-1}^h),$$

and therefore we have  $\int_{\Omega} |u_m^h - u_{m-1}^h|^2 / h dx \leq I(u_{m-1}^h) - I(u_m^h)$ . Summing up from  $m = 1$  to  $M$ , we have a estimate:

$$I(u_M^h) + \sum_{m=1}^M \int_{\Omega} \frac{|u_m^h - u_{m-1}^h|^2}{h} dx \leq I(u_0). \quad (1.3)$$

This estimate is a basic estimate of this flow, from which many properties are obtained.

Before showing a convergence theory, firstly, we define an approximate solution of a heat equation.

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

DEFINITION 1.1. We define functions  $\bar{u}^h$  and  $u^h$  on  $\Omega \times (0, \infty)$  by

$$\bar{u}^h(x, t) = u_m^h(x), \quad u^h(x, t) = \frac{t - (m-1)h}{h} u_m^h(x) + \frac{mh - t}{h} u_{m-1}^h(x),$$

for  $(x, t) \in \Omega \times ((m-1)h, mh]$ .

It is easy to see that if  $f$  is differentiable, approximate solutions satisfy

$$\int_{\Omega} u_0 \eta(x, 0) dx = \int_0^T \int_{\Omega} D_t u^h \eta dx dt + \int_0^T \int_{\Omega} D \bar{u}^h D \eta dx dt + \int_0^T \int_{\Omega} f'(\bar{u}^h) \eta dx dt.$$

Note that by (1.3) we have:

THEOREM 1.2. If  $f'$  is continuous, then a limit function  $u$  belongs to  $\mathring{V}_2((0, T) \times \Omega)$  and satisfies

$$\int_{\Omega} u_0 \eta(x, 0) dx = \int_0^T \int_{\Omega} D_t u \eta dx dt + \int_0^T \int_{\Omega} D u D \eta dx dt + \int_0^T \int_{\Omega} f'(u) \eta dx dt \quad (1.4)$$

for all  $\eta \in \mathring{W}_2^{1,1}((0, T) \times \Omega)$  with  $\eta(x, T) = 0$ , where  $\mathring{V}_2((0, T) \times \Omega) = \{u \in L^2(Q_T), u_x \in L^2(Q_T); |u|_{Q_T} = \text{ess sup}_{0 \leq t \leq T} \|u(x, t)\|_{L^2(\Omega)} + \|u_x\|_{L^2(Q_T)} < \infty\}$ . We call  $u$  a weak solution. Note that continuity of  $f'$  is one of a sufficient condition.

From (1.3), we easily have

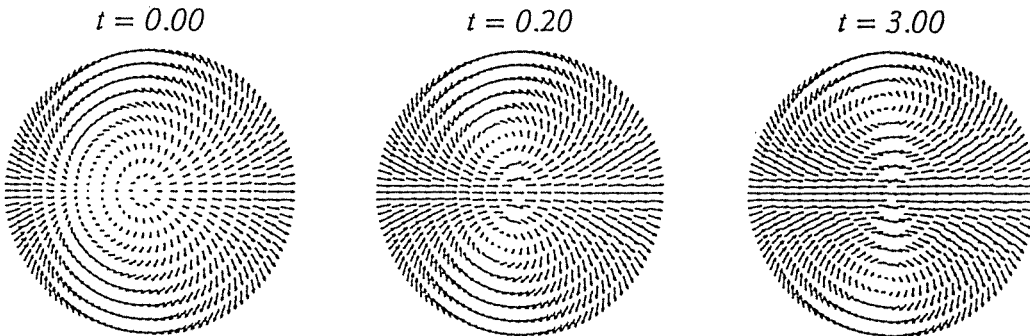
THEOREM 1.3. The limit function  $u_{\infty}$  is a minimizer of the functional

$$J_{\infty}(u) = \int_{\Omega} \left( \frac{|u - u_{\infty}|^2}{h} + |\nabla u|^2 + f(u) \right) dx$$

in  $\mathcal{K}$ , hence,  $u_{\infty}$  satisfies  $-\int_{\Omega} (2\nabla u_{\infty} \nabla \phi + f'(u_{\infty})\phi) dx = 0$  for any  $\phi \in C_0^{\infty}(\Omega)$ .

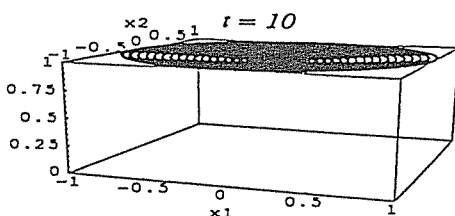
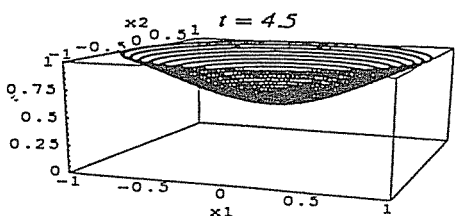
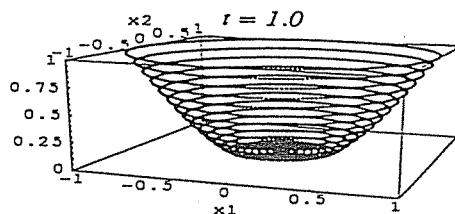
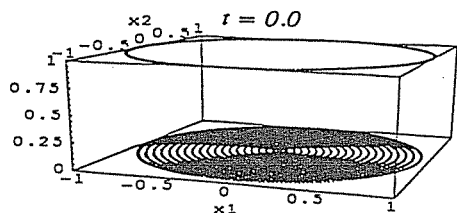
Here, we show numerical examples: The first case is Ginzburg-Landau type problem. We choose  $f(u) = \frac{1}{5}(|u|^2 - 1)^2$ . We treat the case when  $n = N = 2$ ,  $\Omega = \mathbb{B}^2$  in (1.2). We are interested in behavior of vortices.

Ex.1  $h = 0.02, \delta = 0.05$

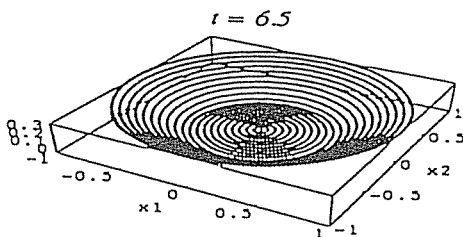
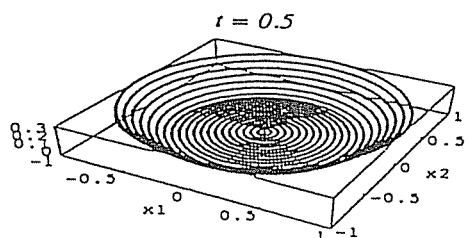
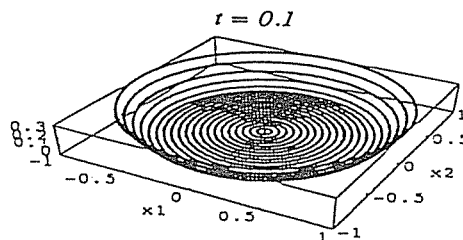
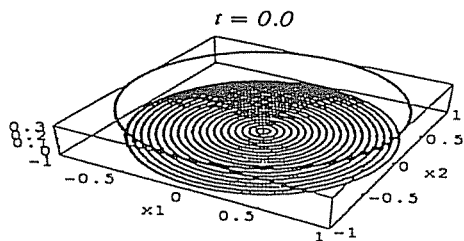


The second case is a free boundary problem. We choose  $f(u) = \chi_{u>0}^\epsilon = 1$  in  $\{x; \epsilon < u\}$  and 0 in  $\{x; u \leq 0\}$  with  $\|\nabla \chi_{u>0}^\epsilon\|_\infty \leq \frac{2}{\epsilon} \|\nabla u\|_\infty$  and  $\chi_{u>0}^\epsilon \in C^2(\mathbb{R})$ . We treat the case  $n = 2, N = 1$  and  $\Omega = B^2$ .

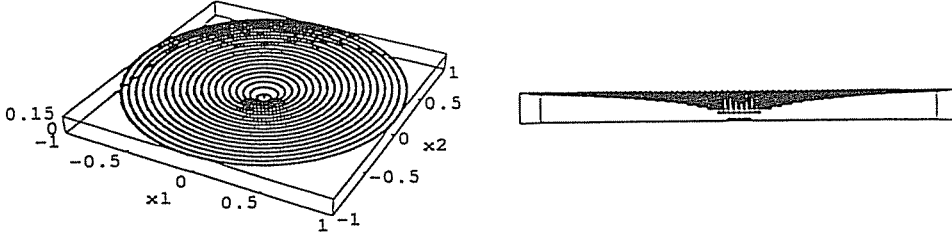
Ex.2  $h = 0.005, \epsilon = 0.05, u = 1.0$  on the boundary



Ex.3  $h = 0.005, \epsilon = 0.05, u = 0.35$  on the boundary



Ex.4  $h = 0.005, \epsilon = 0.005, u = 0.15$  on the boundary



## 2. Hyperbolic case

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary. We define a functional on  $H_0^1(\Omega; \mathbb{R}^N) \cap L^4(\Omega; \mathbb{R}^N)$  by

$$J(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{\delta} (|u|^2 - 1)^2 \right) dx, \quad \delta > 0.$$

We assume  $v_0 \in H_0^1(\Omega; \mathbb{R}^N) \cap L^4(\Omega; \mathbb{R}^N)$ , and redefine  $\{u_n\}$  by

$$\begin{cases} u_1 = u_0 + h v_0, \\ J_n(u_n) = \inf_{u \in H_0^1(\Omega; \mathbb{R}^N) \cap L^4(\Omega; \mathbb{R}^N)} J_n(u) \quad \text{for } n \geq 2, \end{cases}$$

where

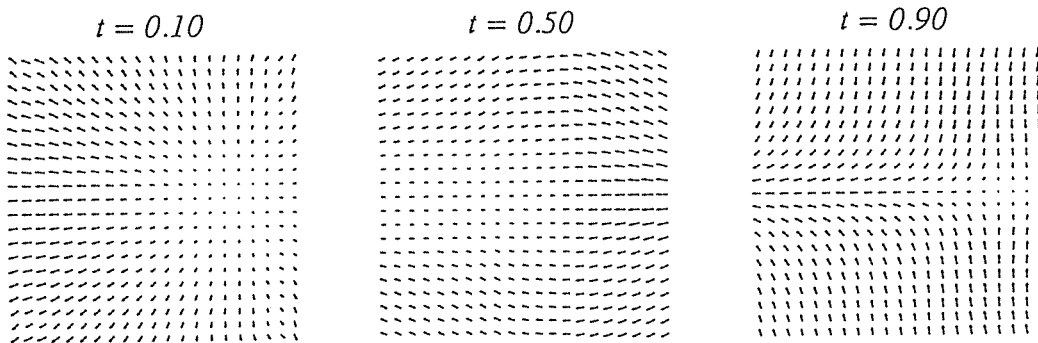
$$J_n(u) = \frac{\|u - 2u_{n-1} + u_{n-2}\|^2}{2h^2} + J(u).$$

PROPOSITION 2.1. (Nagasawa-Omata) If  $h \in (0, C_1^{-\frac{1}{2}})$ , then

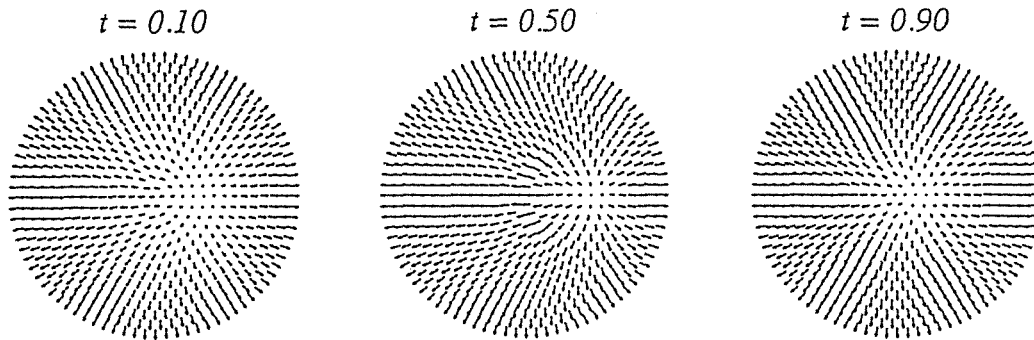
$$\frac{\|u_n - u_{n-1}\|^2}{2h^2} + J(u_n) \leq \frac{1}{1 - C_1 h^2} \exp \left\{ \frac{(n-2)_+ C_1 h^2}{1 - C_1 h^2} \right\} \left( \frac{1}{2} \|v_0\|^2 + J(u_1) \right),$$

where  $(n-2)_+ = \max\{n-2, 0\}$ .

Ex.5  $h = 0.01, \delta = 0.1$



Ex.6  $h = 0.01, \delta = 0.1$



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# DETERMINATION OF THE SOLUTIONS FOR THE MHD EQUATIONS BY FINITE ELEMENTS

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Dedicated to Professor Hideo Kawarada on the occasion of his 60th birthday

We consider the magnetohydrodynamic (MHD) equations in  $\mathbf{R}^2$  with periodic boundary conditions on the square  $\Omega = (0, L) \times (0, L)$  :

$$(1) \quad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \frac{1}{\rho\mu}(B \cdot \nabla)B + \frac{1}{2\rho\mu}\nabla(|B|^2) + \frac{1}{\rho}\nabla p = \nu\Delta u + f \\ \frac{\partial B}{\partial t} + (u \cdot \nabla)B - (B \cdot \nabla)u = \frac{1}{\mu\sigma}\Delta B \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0 \\ u(x, t) = u(x + Le, t), \quad B(x, t) = B(x + Le, t) \quad e \in \mathbf{Z} \times \mathbf{Z}, \end{cases}$$

where the variables  $u$ ,  $B$  and  $p$  denote the velocity vector, the magnetic field and the pressure, respectively. The constants  $\rho$ ,  $\mu$ ,  $\sigma$  and  $\nu$  represent the unit mass density, the magnetic permeability, the electric conductivity and the kinematic viscosity, respectively.  $f$  is the given external volume force to the fluid. Moreover, we assume that the integrals of  $u$ ,  $B$ , and  $f$  vanish on  $\Omega$  for all  $t > 0$ . Namely

$$\int_{\Omega} u \, dx = \int_{\Omega} B \, dx = \int_{\Omega} f \, dx = 0.$$

For the derivation and known properties of the MHD equations, we refer to, for instance, Z. Yoshida [3].

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Our interest is in the long time behaviour of the solutions to (1). It is well known that the long time dynamics of (1) is monitored by a finite number of degrees of freedom. See for instance R.Temam [2]. We want to examine the mechanism that what kind of finiteness can determine the system as time goes to infinity.

Now, following [1], let us define

$$\begin{aligned} \mathcal{V} &= \{u : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \text{ vector-valued trigonometric polynomials} \\ &\quad \text{with period } L, \operatorname{div} u = 0, \int_{\Omega} u \, dx = 0\} \\ H &= \text{the closure of } \mathcal{V} \text{ in } (L^2(\Omega))^2, \\ V &= \text{the closure of } \mathcal{V} \text{ in } (H^1(\Omega))^2, \end{aligned}$$

where  $H^l(\Omega)$  ( $l = 1, 2, \dots$ ) denote the usual Sobolev spaces. Let  $P$  denote the orthogonal projection in  $L^2(\Omega) \times L^2(\Omega)$  onto  $H$  and define the Stokes operator:

$$Au = -P\Delta u,$$

with the domain  $D(A) = V \cap (H^2(\Omega))^2$ . The norms of  $H$  and  $V$  will be defined usually and represented by  $|\cdot|_H$  and  $\|\cdot\|_V$ , respectively. There exists a complete orthonormal set  $w_j$  of eigenfunctions of  $A$  with the corresponding eigenvalues  $0 < \lambda_1 \leq \lambda_2 \leq \dots$ . We denote by  $P_m$  the orthogonal projection onto the linear space  $\operatorname{Span}\{w_1, \dots, w_m\}$ . Further, from now on, we assume that  $f = Pf \in L^\infty((0, \infty); H)$  and put

$$F = \limsup_{t \rightarrow \infty} \left( \int_{\Omega} |f(t, x)|^2 \, dx \right)^{1/2}.$$

Next let  $\mathcal{E} = \{x^1, x^2, \dots, x^N\}$  be a given finite set of points in  $\Omega$ , which is associated with

$$d_{\mathcal{E}} = \sup_{x \in \bar{\Omega}} \min_{i=1, \dots, N} |x - x^i|.$$

$d_{\mathcal{E}}$  reflects the density of  $\mathcal{E}$  in  $\Omega$ .

We now state our main result, which shows that the information about finite points of finite modes determines the whole system.

**Theorem.** Let  $(u_1, B_1, p_1)$ ,  $(u_2, B_2, p_2)$  solve (1) with the forcing term  $f_1, f_2$ , respectively, which satisfies  $|f_1(t) - f_2(t)|_H \rightarrow 0$  as  $t \rightarrow \infty$ . If there holds

$$\lim_{t \rightarrow \infty} (P_m u_1(x^j, t) - P_m u_2(x^j, t)) = \lim_{t \rightarrow \infty} (P_m B_1(x^j, t) - P_m B_2(x^j, t)) = 0$$

for all  $x^j \in \mathcal{E}$ , then we have

$$\lim_{t \rightarrow \infty} \|u_1(t) - u_2(t)\|_V = \lim_{t \rightarrow \infty} \|B_1(t) - B_2(t)\|_V = 0,$$

provided  $m \geq C(F)$  and  $d_\varepsilon \leq 1/(4\sqrt{6}L\lambda_m)$ , where  $C(F)$  denotes a computable function of  $F$ .

Remark that our theorem seems to be new also in the case of  $B \equiv 0$ ; i.e., Navier-Stokes equations. In this case,  $m$  is estimated by the relation

$$\lambda_{m+1} \geq 15\sqrt{6} \frac{F}{\nu^2}.$$

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## Topics in Double-Layer Convection

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### Abstract

Two different immiscible liquids lie in layers at rest between horizontal walls and are heated from below. When the temperature difference between the walls reaches a critical value, new solutions bifurcate from the solution at rest. In the first part of this paper, we focus on instabilities that involve interfacial deformations and report on a particular critical situation with a pair of oscillatory modes at wavenumber  $\alpha$  and a steady mode at wavenumber  $2\alpha$ . In the second part, the focus is on the case where the interfacial mode is strongly stabilized by surface tension and a suitable density stratification. A mechanism for a Hopf bifurcation is the competition between the least stable of the bulk modes in each fluid. The well known criterion for balancing the effective Rayleigh numbers in both fluids is augmented with a criterion for non-self-adjointness of the system, yielding a heuristic method for picking suitable fluids when Hopf modes are desired. The pattern formation problem in three dimensions is addressed for the case of doubly periodic solutions on a hexagonal lattice. Of the solutions with maximal symmetry, the traveling rolls are found to be stable. This is in contrast with past results of the qualitatively different mechanism of interfacial instability.

### Introduction

We consider the onset of thermal convection in a system with two fluids filling the space between two plates, where the lower plate is kept at a higher temperature than the upper plate. The two fluids have different thermal and mechanical properties. (Two-layer systems appear in various applications [2,3,29].) The destabilizing mechanism is the temperature difference, as in the single-fluid problem (the classical Bénard problem). The presence of two liquids and an interface introduces other destabilizing or stabilizing mechanisms, depending on the stratification in the fluid properties.

Asymptotic formulas for the interfacial mode for long-wave disturbances, short-wave disturbances, and when the liquids have similar physical properties, are available [16, 22 - 26, 28]. Surface tension is stabilizing in the short-wave limit. Differences in density and coefficients of cubical expansion contribute toward the buoyancy effect at the interface, and can be manipulated to either stabilize or destabilize the interfacial mode. In the limit of long waves, the volume ratio and differences in thermal conductivity and the buoyancy parameters are important. There is a preference for the placement of the more conductive liquid in the thicker layer as the system attempts to optimize heat transfer. Oscillations may be set up due to the competition between the destabilizing temperature gradient and a stabilizing influence of the interface. Some analysis and experiments have been done [1,4, 6, 10, 19, 21, 24] in regimes where the interfacial mode is not important and criticality is attained by bulk mode(s).

In two dimensions, a steady onset on its own would lead to rolls. For the oscillatory onset case, there is a twofold degeneracy of the critical eigenvalue (eigenvalues  $\pm i\omega$  for a critical wavenumber  $\alpha$ , and also for  $-\alpha$ ). The possible bifurcating solutions are standing rolls and travelling rolls [27]. If either is subcritical, then both are unstable. If both are supercritical, then one of them is stable. A limited number of critical situations were investigated numerically in [22], and found to be unstable.

### Governing Equations

The equations governing the problem are written in full in Chapter III of [16]. There are at least nine dimensionless ratios: the ratio of viscosities, densities, thermal conductivities, diffusivities, coefficients of thermal expansion, surface tension, the dimensionless depth of the lower fluid, a Rayleigh number  $R$  and Prandtl number  $P$  defined for the lower fluid. At the interface, we must have continuity of velocity, temperature and heat flux, and balance of tractions. In each fluid, the governing equations are the usual ones

for the Benard problem: the heat transport equation and the Navier-Stokes equations with the Oberbeck-Boussinesq approximation. The base state consists of zero velocity, and temperature that depends linearly on the vertical coordinate. We are concerned with the stability of this basic solution and with solutions bifurcating from it [9, 16]. In order to carry out the derivation of the amplitude equations, it is sufficient to keep up to the cubic nonlinearities. The interface conditions, when expanded in Taylor series about the unperturbed position for small deviations, contribute the cubic terms.

### The 2:1 steady/Hopf mode interaction

Multiple bifurcations associated with mode interactions in the presence of  $O(2)$  symmetry have been attracting much attention. Golubitsky, Stewart and Schaeffer [13] give a review of steady-state/steady-state, Hopf/steady-state, and Hopf/Hopf mode interactions. In particular, the Hopf/steady-state mode interaction has been investigated with applications to Couette flow between counter rotating cylinders. In this flow, the Taylor vortices and spirals bifurcate from the steady Couette flow simultaneously for a specific set of parameters [7, 11, 12, 17, 18]. The interaction considered there is between a pair of Hopf modes and a steady mode with the same wavenumber. The effect of resonance appears at the cubic order. In the present paper, on the other hand, we consider the interaction with the wavenumber ratio of 1:2, so that the resonant interaction sets in at the quadratic order [15].

Analyses of the Hopf/steady-state mode interaction with wavenumber ratio of 1:2 have a rather short history. A situation where the modes are neutral but not critical arises in free convection between vertical parallel plates [8]. In contrast, one may interpret the critical situation here as being more easily attainable under experimental conditions than a neutral situation. Hopf/steady-state interaction with general wavenumber ratio  $l:m$ , involving the case 2:1 as a subset, in a system with  $O(2)$  symmetry is studied by Hill and Stewart [14]. They give a partial geometric analysis.

**Linearized Stability Problem** For the linear stability analysis, 2D disturbances that depend on  $\exp(\sigma t)$  are examined, with wavenumbers  $\alpha$ . There are an infinite number of discrete eigenvalues  $\sigma$  [16, 22, 25, 29]. In the special case when the two fluids are the same, the interfacial mode merely allows the interface to be wavy, leaving the velocity and temperature fields as they are. This is neutrally stable and is not an active mode, and stability is determined by the one-fluid bulk modes. When the fluids have different properties, the interfacial mode begins to play an active role in the stability problem.

In [9], we depict a particular situation where there is a Hopf mode at wavenumber  $\alpha = 3.08$  and a real mode at wavenumber  $2\alpha = 6.16$  are simultaneously at criticality. We describe the bifurcating solutions.

**Amplitude Evolution Equations** We regard two of the parameters as bifurcation parameters  $(\lambda_1, \lambda_2)$ , e.g.,  $\lambda_1 = R - R_c$ , where  $R_c$  is the critical value of the Rayleigh number, and  $\lambda_2$  is one of the other dimensionless parameters. We denote by  $\lambda$  the vector  $(\lambda_1, \lambda_2)$ . We denote by  $\zeta_1(\lambda)$  the eigenfunction belonging to  $-\mu_1(\lambda)$  at wavenumber  $\alpha$ , and by  $\zeta_2(\lambda)$  the eigenfunction belonging to  $-\mu_1(\lambda)$  at wavenumber  $-\alpha$  (i.e.,  $\mu_2 = \mu_1$ ). The eigenfunctions belonging to  $-\bar{\mu}_k(\lambda)$  are the complex conjugates of  $\zeta_k$ . The spatial structure of the eigenfunctions is  $\zeta_1 = \tilde{\zeta}_1(z) \exp(i\alpha x)$ ,  $\zeta_2 = \tilde{\zeta}_2(z) \exp(-i\alpha x)$ ,  $\zeta_3 = \tilde{\zeta}_3(z) \exp(2i\alpha x)$ . For  $\lambda$  near 0, we denote by  $\mu_3(\lambda)$  the eigenvalue associated with  $\zeta_3$ .

The perturbation solution is of the form  $\Phi = \Phi_1 + \Psi$ ,  $\Phi_1 = \sum_{i=1}^3 z_i \zeta_i + \sum_{i=1}^3 \bar{z}_i \bar{\zeta}_i$ , where  $z_i$  are complex time-dependent amplitudes. In the transformed coordinate system of the Birkhoff normal form,

$$(1) \quad \frac{d\hat{z}_i}{dt} + \hat{F}_i(\hat{z}_1, \hat{z}_2, \hat{z}_3, \lambda) = 0, \quad i = 1, 2, 3,$$

$$\begin{aligned} \hat{F}_1(\hat{z}_1, \hat{z}_2, \hat{z}_3, \lambda) &= \mu_1(\lambda) \hat{z}_1 + \beta_{12}(\lambda) \hat{z}_2 \hat{z}_3 + \tilde{\gamma}_{11}(\lambda) |\hat{z}_1|^2 \hat{z}_1 + \gamma_{12}(\lambda) |\hat{z}_2|^2 \hat{z}_1 + \tilde{\gamma}_{17}(\lambda) |\hat{z}_3|^2 \hat{z}_1, \\ \hat{F}_2(\hat{z}_1, \hat{z}_2, \hat{z}_3, \lambda) &= \mu_1(\lambda) \hat{z}_2 + \beta_{12}(\lambda) \hat{z}_1 \hat{z}_3 + \tilde{\gamma}_{11}(\lambda) |\hat{z}_2|^2 \hat{z}_2 + \gamma_{12}(\lambda) |\hat{z}_1|^2 \hat{z}_2 + \tilde{\gamma}_{17}(\lambda) |\hat{z}_3|^2 \hat{z}_2, \\ \hat{F}_3(\hat{z}_1, \hat{z}_2, \hat{z}_3, \lambda) &= \mu_3(\lambda) \hat{z}_3 + \beta_{33}(\lambda) \hat{z}_1 \hat{z}_2 + \gamma_{31}(\lambda) |\hat{z}_3|^2 \hat{z}_3 + \tilde{\gamma}_{33}(\lambda) |\hat{z}_1|^2 \hat{z}_3 + \tilde{\gamma}_{33}(\lambda) |\hat{z}_2|^2 \hat{z}_3. \end{aligned}$$

There are 7 Landau coefficients  $\beta_{12}, \tilde{\gamma}_{11}, \gamma_{12}, \tilde{\gamma}_{17}, \beta_{33}$  (real),  $\gamma_{31}$  (real) and  $\tilde{\gamma}_{33}$ , which are computed at criticality [9].

We let  $\mu_1 = \mu_2 = i\omega + \epsilon_1$  and  $\mu_3 = \epsilon_2$  where  $\omega$ ,  $\epsilon_1$  and  $\epsilon_2$  are real. This yields a two-parameter bifurcation with  $\epsilon_1$  and  $\epsilon_2$  as bifurcation parameters. The equilibrium solutions ( $d/dt = 0$ ) of (1), hereafter dropping the hats on the  $z_i$ , are: *the steady solution*  $z_1 = 0$ ,  $z_2 = 0$ , *the traveling waves*  $z_2 = 0$ ,  $z_3 = 0$ , and *the (mixed) standing waves*. The standing waves are generated by equal and opposite traveling wave components, which excite the steady  $z_3$ -component.

Section 6 of [9] shows results of numerical integration of the amplitude equations. We show explicit formulas for determining the stability of many of these solutions. A comparison with the work of Hill and Stewart [14] is given in [9], with more details in the ICAM report, including corrections. There is a sector in the  $\epsilon_1 - \epsilon_2$  plane where the mixed standing wave solution is stable. Along the lower boundary of this sector, we have a Hopf bifurcation to a modulated wave solution. The transition from stability to instability across the upper boundary of the sector can be shown to arise from an odd mode of perturbation, and the new solution is termed the asymmetric mixed mode.

**Asymmetric Modes** We examine a new class of equilibrium solutions called an asymmetric mixed mode where  $|z_1| \neq |z_2|$  and  $|z_1 z_2 z_3| \neq 0$ . Results from numerical integration of the amplitude equations based on the fourth order Runge-Kutta scheme are given in [9]. We set  $z_n(t) \rightarrow z_n(t) \exp(i\omega t)$  for  $n = 1, 2$  in (1) and represent the asymmetric mixed mode by  $z_1 = r_1 e^{i\theta_1(t)}$ ,  $z_2 = r_2 e^{i\theta_2(t)}$ ,  $z_3 = r_3 e^{i\theta_3(t)}$ ,  $r_1 \neq r_2$ . The complete bifurcation diagram is discussed in [9, 30].

### An Oscillatory Bulk-Mode Competition

There has been recent interest among experimentalists [1, 4, 21] in observing an oscillatory onset that has been theoretically studied in the two-layer Benard problem. The weakly nonlinear analysis in the literature applies to conditions close to criticality and this is pursued here. In order to set up an oscillatory onset, there needs to be a competition between two modes. Two such mechanisms are as follows. First, a bulk mode which is destabilized by the temperature difference between the plates may compete with a stabilizing interfacial mode due to a favorable stratification in the fluid properties [9, 22, 25, 26, 29]. Secondly, when the interfacial mode is strongly stabilized, the competition between the least stable of the bulk modes associated with each fluid can lead to an oscillatory onset [6, 21, 31]. This case appears at the moment to be more accessible to experiments and is the subject of this section.

The experiments of [1] concern a layer of Fluorinert lying below a layer of silicone oil 47v10. The properties of these fluids are discussed in [24]. In choosing fluids for the purpose of observing an oscillatory onset, it is useful to have a rule of thumb so that a larger window of parameters would be involved, and with desirable short periods for ease of measurements. This is developed in [24] together with a linear stability analysis of the Fluorinert - 47v10 system and the fictitious Anderinert - 47v10 system, followed by the weakly nonlinear analysis for doubly periodic solutions on a hexagonal lattice [32]. The methodology has been applied in [22] to the interfacial instability case. The results for the bulk-mode competition case are found to be qualitatively different from those of the interfacial instability case. For solutions with two degrees of freedom, there are eleven classes of solutions. It is found that the traveling rolls are stable while the other solutions are supercritical but unstable.

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# Blowup of solutions of dissipative nonlinear wave equations

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In this talk, we consider the blowup problem (non-existence of global solutions) for nonlinear evolution equations of the form:

$$u''(t) + \partial\phi(u'(t)) + \partial\varphi(u(t)) - \partial\psi(u(t)) = 0, \quad t \geq 0 \quad (1)$$

in a real Hilbert space  $H$ . Here,  $\partial\phi$ ,  $\partial\varphi$  and  $\partial\psi$  are single valued proper subdifferentials of lower semicontinuous convex functions  $\phi$ ,  $\varphi$  and  $\psi$  from  $H$  to  $[0, +\infty]$ .

Typical examples of (1) considered here are as follows.

$$u_{tt} + \delta u_t - \Delta u - |u|^{q-2}u = 0, \quad t \geq 0, \quad x \in \Omega, \quad (2)$$

$$u_{tt} - \Delta u_t - \|\nabla u\|_{L^2(\Omega)}^2 \Delta u - |u|^{q-2}u = 0, \quad t \geq 0, \quad x \in \Omega, \quad (3)$$

$$u_{tt} + |u_t|^{m-2}u_t - \Delta u - |u|^{q-2}u = 0, \quad t \geq 0, \quad x \in \Omega, \quad (4)$$

with the boundary condition

$$u(t, x) = 0, \quad t \geq 0, \quad x \in \partial\Omega, \quad (5)$$

where  $\delta \geq 0$ ,  $m, q > 2$  and  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ .

For the case of  $\partial\phi \equiv 0$  in (1), the blowup problem has been studied by many authors. In particular, under appropriate conditions on  $\varphi$  and  $\psi$ , it is shown that if initial data belongs to the so called unstable set  $V$  defined by (7) below, then the solution of (1) blows up in a finite time (see, e.g., [8] and [3]).

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Recently, for (2) and (5), Ikehata and Suzuki [2] showed that if  $\delta > 0$  is small and initial data belongs to some subset  $V_\delta$  of the unstable set  $V$ , then the solution blows up in a finite time. Here, we note that  $V_\delta$  is not equal to  $V$  if  $\delta > 0$ . The blowup problem for (3) and (5) was studied by Ono [7], and he showed that if  $q \geq 4$  and initial energy is negative, then the solution blows up in a finite time (see Remark 1 below). The blowup problem for (4) and (5) was studied by Georgiev and Todorova [1], and they proved that if  $m < q$  and initial energy is sufficiently negative, then the solution blows up in a finite time. Our purpose in this talk is to generalize the above results in [2], [7] and [1]. That is, under appropriate conditions on  $\phi$ ,  $\varphi$  and  $\psi$ , we show that if initial data belongs to the unstable set  $V$ , then the solution of (1) blows up in a finite time as well as the case of  $\partial\phi \equiv 0$ .

In what follows, we always assume the following (A1) and (A2).

(A1)  $\varphi$  and  $\psi$  are homogeneous functions of degree  $p$  and  $q$ , respectively.

(A2)  $1 < p \leq q$ ,  $q > 2$ , and  $\varphi(u) > 0$  for  $u \in H \setminus \{0\}$ .

For  $u \in D(\varphi) \cap D(\psi)$  and  $v \in H$ , we put

$$J(u) = \varphi(u) - \psi(u), \quad K(u) = p\varphi(u) - q\psi(u), \quad E(u, v) = (1/2)|v|_H^2 + J(u), \quad (6)$$

and we define the unstable set  $V$  as follows:

$$V = \{(u, v) \in [D(\varphi) \cap D(\psi)] \times H; E(u, v) < d, K(u) < 0\}, \quad (7)$$

$$d = \inf\{J(u); u \in D(\varphi) \cap D(\psi), K(u) = 0, u \neq 0\}. \quad (8)$$

**Remark 1** It follows from (A1) and (A2) that  $d = \inf\{(1 - p/q)\varphi(u); u \in D(\varphi) \cap D(\psi), K(u) = 0, u \neq 0\} \geq 0$  and  $(1/q)K(u) \leq J(u) \leq E(u, v)$ , so we see that  $\{(u, v) \in [D(\varphi) \cap D(\psi)] \times H; E(u, v) < 0\} \subset V$ .

**Remark 2** From (A1) and (A2), it is shown that (\*)  $d = \inf\{(1 - p/q)\varphi(u); u \in D(\varphi) \cap D(\psi), K(u) < 0\}$  and the set  $V$  is invariant under the flow of (1), that is, if  $(u(t_0), u'(t_0)) \in V$  then  $(u(t), u'(t)) \in V$  for all  $t \geq t_0$ . The invariance of  $V$  and (\*), which are not used in [7] and [1], play important roles in the proofs of the following theorems.

**Theorem 1** *Suppose that  $\partial\phi$  is a non-negative self-adjoint operator in  $H$ , and assume (A1) and (A2). Then any strong solution  $u(t)$  of (1) satisfying  $(u(0), u'(0)) \in V$  does not exist for all  $t \geq 0$ .*

**Theorem 2** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . Assume that  $2 < m < q$ . Then any solution  $u(t)$  of (4) and (5) satisfying  $(u(0), u'(0)) \in V$  does not exist for all  $t \geq 0$ .*

The boundedness of domain  $\Omega$  is essential in the proof of Theorem 2, because we use there again and again the continuous imbedding  $L^r(\Omega) \hookrightarrow L^s(\Omega)$  if  $r \geq s$ . We are now trying to extend Theorem 2 to more general case including the case for unbounded domain  $\Omega$ . Theorem 1 was first proved in [6] for the case of  $\partial\phi(u') = \delta u'$  with  $\delta > 0$ . However, the proof in [6] is not applicable to the general case as in Theorem 1. So, following and modifying the argument in Ono [7], we introduce a function

$$P(t) = \frac{1}{2}|u(t)|_H^2 + \frac{1}{2} \int_0^t |B^{1/2}u(s)|_H^2 ds + \frac{1}{2}|B^{1/2}u(0)|_H^2(T_0 - t), \quad (9)$$

where we put  $B = \partial\phi$  and  $T_0$  is a positive constant determined only by the initial data (compare (9) with (5.5) in [7]), and we use the so called concavity argument for  $P(t)$ . More generally, Theorem 1 is also valid when  $u''(t)$  is replaced by  $Au''(t)$  in (1), where  $A$  is a positive self-adjoint operator in  $H$ , so our abstract result is applicable to a problem arising in a paper [5] by Mizoguchi, Ninomiya and Yanagida.

At the rest, we give a sketch and a remark of the proof for the case of  $\partial\phi(u') = \delta u'$ . The following lemma plays a crucial role there.

**Lemma 1** *Suppose that  $\partial\phi(u') = \delta u'$  with  $\delta > 0$ , and assume (A1) and (A2). Suppose that a strong solution  $u(t)$  of (1) satisfying  $(u(0), u'(0)) \in V$  exists globally in time. Then, there exists  $t_1 > 0$  such that  $I(t) > 0$  and*

$$\delta I'(t) \geq (1 + q/2)(d - E(t)), \quad \frac{d}{dt} \frac{E(t) - d}{I^\gamma(t)} \leq 0 \quad (10)$$

for almost all  $t \in [t_1, +\infty)$ , where  $\gamma = (q + 2)/4$ , and we put  $I(t) = (1/2)|u(t)|_H^2$  and  $E(t) = E(u(t), u'(t))$ .

For the proof of Lemma 1, see Lemma 2.4 in [6].

*Proof of Theorem 1 for case of  $\partial\phi(u') = \delta u'$  with  $\delta > 0$ .* Suppose that a strong solution  $u(t)$  of (1) satisfying  $(u(0), u'(0)) \in V$  exists globally in time. Then, from Lemma 1, there exists  $t_1 > 0$  such that

$$\delta I'(t) \geq (1 + q/2)(d - E(t)) \geq (1 + q/2)C_1 I^\gamma(t), \quad t \geq t_1, \quad (11)$$

where we put  $C_1 = (d - E(t_1))/I^\gamma(t_1)$ . Since  $\gamma = (q + 2)/4 > 1$ ,  $C_1 > 0$  and  $I(t_1) > 0$ ,  $I(t)$  can not exist for all  $t > 0$ . However, this contradicts the assumption that the strong solution  $u(t)$  exists globally in time. Hence, we obtain Theorem 1.  $\heartsuit$

**Remark 3** To prove Theorem 1 for the case of  $\partial\phi(u') = \delta u'$ , we used in [6] a blowup result for some ordinary differential inequalities of second order proved by Li and Zhou [4] and Souplet [9]. But, as we saw above, it is not needed in the proof.

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## Mathematical Issues in Viscoelastic Flows

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The lecture will discuss two classes of problems arising in the analysis of viscoelastic flows: flow stability and the asymptotics near reentrant corners.

In the study of hydrodynamics stability, two premises are often taken for granted: First, that linear stability implies stability to small disturbances, and second, that linear stability can be linked to the spectrum. While this is well justified for Newtonian fluid mechanics, viscoelastic flows raise new issues. Indeed, it is well known that, on the general level of  $C_0$ -semigroups, linear stability cannot always be inferred from the spectrum. A number of counterexamples have been given in the literature; in the lecture, I shall discuss a very natural example which is just a lower order perturbation of the wave equation. On the other hand, some positive results will be discussed. Assuming a spectral “separation condition,” it is possible to establish a link between linear stability and spectrum; such a result is applicable to hyperbolic PDEs in one space dimension. Finally, I shall discuss viscoelastic flows for fluids with constitutive relations of Jeffreys type. I shall discuss parallel shear flows at arbitrary Weissenberg numbers and general flows at sufficiently small Weissenberg numbers. In both cases, it can be shown that spectral stability implies linear stability, and, in the case of small Weissenberg number, also nonlinear stability.

The flow of a viscoelastic fluid through a contraction is an important problem in industrial processing and has been designated a standard benchmark problem for numerical simulation. The singularities arising at the reentrant corner have caused significant difficulty, especially for the upper convected Maxwell model. The lecture will discuss recent analytical progress in elucidating the nature of the corner singularity. A matched asymptotic solution can be constructed, which involves a “potential flow” region in the core, and cusp-shaped boundary layers near the walls. These boundary layers are a particular instance of a more general phenomenon of formation of stress boundary layers on walls at high Weissenberg numbers, which has also been observed in other flows. Apart from the issue of resolving the boundary layers, we shall also identify a downstream instability, which can lead to amplification of errors and the breakdown of numerical simulations. Finally, we present a comparison between the upper convected Maxwell model and the Phan-Thien Tanner model. The latter is distinguished by a limitation of stress growth at high deformation rates. For the corner flow, this leads to less singular stresses and less pronounced boundary layers. These features combine to make numerical simulation far less problematic.

# ON SPHERICALLY SYMMETRIC STELLAR MODELS IN GENERAL RELATIVITY

Tetu Makino

1. In general theory of relativity, the spherically symmetric equilibrium of a self-gravitating gas is governed by the Tolman-Oppenheimer-Volkoff equation:

$$\begin{aligned}\frac{dm}{dr} &= 4\pi r^2 \rho, \\ \frac{dp}{dr} &= -(\rho + p/c^2) \frac{G(m + 4\pi r^3 p/c^2)}{r^2(1 - 2Gm/c^2 r)}\end{aligned}\quad (1)$$

The metric is

$$ds^2 = e^{\nu(\tau)} c^2 dt^2 - e^{\lambda(\tau)} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where  $e^{-\lambda} = 1 - 3Gm/c^2 r$  and  $d\nu = -2dp/(p + \rho c^2)$ .  $G$  and  $c$  are positive constants. We assume

(A0)  $p = p(\rho)$  is a sufficiently smooth function of  $\rho > 0$  such that  $p > 0$  and  $dp/d\rho > 0$  for  $\rho > 0$  and  $p \rightarrow 0$  as  $\rho \rightarrow 0$ .

Given the central density  $\rho_0 > 0$ , we put  $p_0 = p(\rho_0)$  and consider the initial value problem for (1) with

$$m = 0, p = p_0 \text{ at } r = 0 \quad (2)$$

By transforming (1)(2) to an integral equation, we can show that for a sufficiently small  $\delta$  there exists a unique solution  $(m(r), p(r))$  in  $C[0, \delta]$  such that

$$\begin{aligned}m &= \frac{4\pi}{3} \rho_0 r^3 + O(r^5) \\ p &= p_0 - (\rho_0 + p_0/c^2) G(4\pi \rho_0/3 + 4\pi p_0/c^2) \frac{r^2}{2} + O(r^4)\end{aligned}$$

as  $r \rightarrow +0$ . Let us prolong this solution to the right as long as possible in the domain  $\{(r, m, p) | 0 < r < +\infty, 0 < p < +\infty, 2Gm/c^2 r < 1\}$ . Let  $(0, R)$

be the right maximal interval of existence.

It is easy to see that  $p \rightarrow 0$  as  $r \rightarrow R$ .

We assume

(A1)  $\frac{p}{\rho} \frac{dp}{d\rho} = \gamma + O(\rho^{\gamma-1})$  as  $\rho \rightarrow +0$  and  $4/3 < \gamma < 2$ .

Then we have

Theorem 1. Under (A0)(A1),  $R$  is finite for any central density.

2. The equation of state for neutron stars is given by

$$p = K_1 c^5 f(s), \rho = K_2 c^3 g(s)$$

where

$$\begin{aligned} f(s) &= s(s^2 - 1)^{1/2} + 3 \log(s + (s^2 - 1)^{1/2}) \\ g(s) &= 8s(s^2 - 1)^{3/2} - f(s). \end{aligned}$$

So, (A1) holds with  $\gamma = 5/3$ .

3. We assume

(A1)'  $\frac{p}{\rho} \frac{dp}{d\rho} = \gamma + \Omega(\rho^{\gamma-1}, 1/c^2)$ ,  $4/3 < \gamma < 2$ ,  $\Omega(\mu, \epsilon)$  is continuous in  $\mu \geq 0$ ,  $\epsilon \geq 0$ ,  $\Omega(0, \epsilon) = 0$  and Lipschitz continuous re  $\mu$  uniformly in  $\epsilon$ .

Then we have

Theorem 2. Under (A0)(A1)', there is a constant  $C$  independent of  $c$  such that

$$p/\rho, r^3/m, m/r \leq C \quad (3)$$

for  $0 \leq r \leq R$  and  $c \geq c_0$ .

4. The linearized equation for a spherically symmetric perturbation from the equilibrium is of the form

$$\frac{\partial^2 \xi}{\partial t^2} + \mathcal{A}\xi = 0, (0 < r < R) \quad (4)$$

with

$$\mathcal{A}\xi = \frac{1}{a_0(r)} \left[ -\frac{\partial}{\partial r} (a_1(r) \frac{\partial \xi}{\partial r}) + a_2(r) \xi \right],$$

where  $\xi = e^{-\nu/2} \delta r / r$ . The quantity  $\Gamma = \frac{\rho + p/c^2}{\rho} \frac{dp}{d\rho}$  appears in  $a_1$  and  $a_2$ . We assume

(A2) As  $\rho \rightarrow 0$ ,  $\frac{p}{\Gamma} d\Gamma dp, p \frac{d}{d\rho} (\frac{p}{\Gamma} \frac{d\Gamma}{d\rho}), p \frac{d}{d\rho} (\frac{p}{\rho} \frac{d\rho}{d\rho}) \rightarrow 0$ .

The equation (4) can be transformed to the standard form

$$z_{tt} - z_{hh} + Q(h)z = 0, (0 < h < H) \quad (5)$$

where

$$h = \int_0^r (a_0/a_1)^{1/2} dr, z = (a_0)^{1/4} \xi.$$

We see  $H$  is finite, and  $Q(h) \sim 2/h^2$  as  $h \rightarrow +0$ , and

$$Q(h) \sim \frac{1}{4} \frac{(3-\gamma)(1+\gamma)}{(\gamma-1)^2} \frac{1}{(H-h)^2}$$

as  $h \rightarrow H - 0$ . Thus we have

Theorem 3. Under (A0)(A1)(A2), the adjoint operator  $A$  of  $A_0$ , which realizes  $\mathcal{A}$  with domain  $C_0^2(0, R)$ , is the self-adjoint Friedrichs extension, and  $A$  has a purely discrete spectrum.

5. Assume

$$(A3) \inf_{0 \leq p \leq p_0} [3\Gamma - 4 + 3p \frac{d\Gamma}{dp}] > 0.$$

Then, using (3), we can show

Theorem 4. Under (A0)(A1)(A2)(A3), the equilibrium is neutral stable in the linearized sense, say, the minimum eigenvalue of  $A$  is positive.

6. For the equation of state of neutron stars, we see  $p \sim c^2 \rho/3$  as  $\rho \rightarrow +\infty$ . If we take  $p = c^2 \rho/3$  exactly, we can show that for any central density,  $R = +\infty$  and  $\rho \sim \frac{3}{56\pi} \frac{1}{r^2}$  as  $r \rightarrow +\infty$ . (Here we take  $G = c = 1$  for the simplicity.)

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