ON THE UNIVALENCY OF CERTAIN POWER SERIES

By

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I.

In this paragraph we consider a family of functions

$$f(z) = a_1z + a_2z^2 + \dots + a_nz^n + \dots$$

with the following properties:

(1) $a_1z + a_2z^2 + \dots + a_nz^n + \dots$ is convergent for $|z| < 1$,

(2) $|a_1| \geqq a > 0$, (a is fixed, positive),

(3) $\sum_{n=1}^{\infty} \gamma_n |a_n|^2 \leqq 1$, $(\gamma_1 a^2 < 1)$, where the γ_n's are given and positive, $(n = 1, 2, 3, \dots)$, and $\overline{\lim_{n=\infty}} \sqrt[2n-2]{\dfrac{n^2}{\gamma_n}} \leqq 1$.

Our object is to give some complements to a theorem of T. Itihara.[1] This theorem can be stated as follows: $f(z)$ *is univalent* (schlicht) *for* $|z| < \rho_0$, *if the equation* $\dfrac{a^2}{1-\gamma_1 a^2} = \psi(\rho)$ *has a root* ρ_0 *in the interval* (0, 1), *where* $\psi(\rho) = \sum_{n=2}^{\infty} \dfrac{n^2 \rho^{2n-2}}{\gamma_n}$.

Theorem 1. *The derivative* $f'(z)$ *does not vanish for* $|z| < 1$, *if the equation* $\dfrac{a^2}{1-\gamma_1 a^2} = \psi(\rho)$ *has no root in the interval* (0, 1) *and for* $|z| < \rho_0$, *if it has a root* ρ_0 *there. In the latter case* ρ_0 *can not be replaced by any greater number, for*

(4) $f_0(z) = -az + \lambda \sum_{n=2}^{\infty} \dfrac{\rho_0^{n-1}}{\gamma_n} nz^n$, where $\lambda = \dfrac{1-\gamma_1 a^2}{a}$,

(1) T. ITIHARA: Eine Bemerkung über die Schlichtheit der Potenzreihe und ihrer Abschnitte. Japan. Jour. of Math., Vol. 6, 1929.

Journal of Faculty of Science, Hokkaido Imperial University, Ser. I, Vol. I, No. 3, 1932.

belongs to the family and $f_0'(\rho_0)=0$. *And moreover* $f'(z)$ *has a zero on the circle* $|z|=\rho_0$ *when and only when*

$$f(z)=e^{i\theta}f_0(e^{-i\varphi}z)\ . \tag{5}$$

Proof.

$$
\begin{aligned}
|f'(z)| &= \left|\sum_{n=1}^{\infty} na_n z^{n-1}\right| \\
&\geqq |a_1|-\sum_{n=2}^{\infty} n\,|a_n|\,\rho^{n-1} \quad (0\leqq |z|=\rho<1) \\
&= |a_1|-\sum_{n=2}^{\infty}\sqrt{\gamma_n}\,|a_n|\cdot\frac{n\rho^{n-1}}{\sqrt{\gamma_n}} \\
&\geqq |a_1|-\sqrt{\sum_{n=2}^{\infty}\gamma_n|a_n|^2}\sqrt{\sum_{n=2}^{\infty}\frac{n^2\rho^{2n-2}}{\gamma_n}} \qquad (6) \\
&\geqq |a_1|-\sqrt{1-\gamma_1 a^2}\sqrt{\psi(\rho)} \\
&\geqq a-\sqrt{1-\gamma_1 a^2}\sqrt{\psi(\rho)}\ .
\end{aligned}
$$

Since $\psi(\rho)$ is an increasing function of ρ in the interval (0, 1) its value at the origin being zero, either the equation $\dfrac{a^2}{1-\gamma_1 a^2}=\psi(\rho)$ has no root in the interval (0, 1), or it has only one there. In the former case $f'(z)$ never vanishes for $|z|<1$ and in the latter for $|z|<\rho_0$, as $\psi(\rho)<\psi(\rho_0)=\dfrac{a^2}{1-\gamma_1 a^2}$, if $\rho<\rho_0$.

Next we suppose that there exists a function $f(z)$ of the family whose derivative has a zero on the circle $|z|=\rho_0$, which may be assumed without loss of generality to be $z=\rho_0$. Put $z=\rho_0$ in (6). Then, from $f'(\rho_0)=0$, it is easily seen that the signs of inequality must all be removed from the relations (6). Therefore in order that $f'(\rho_0 e^{i\varphi})=0$, it is necessary that $f(z)$ should be of the form (5). Conversely $f(z)=e^{i\theta}f_0(e^{-i\varphi}z)$ belongs to the family, for it is convergent for $|z|<\dfrac{1}{\rho_0}$, by $\overline{\lim\limits_{n\to\infty}}\sqrt[2n-2]{\dfrac{n^2}{\gamma_n}}\leqq 1$ and it is clear that $\sum\limits_{n=1}^{\infty}\gamma_n|a_n|^2\leqq 1$ and $f'(\rho_0 e^{i\varphi})=0$.

From theorem 1 it follows at once:

Theorem 2. *In Itihara's theorem* ρ_0 *can not be replaced by any greater number.*

Some special cases of theorem 2 will be mentioned:

Theorem 3. $\gamma_n = 1$ $(n = 1, 2, \dots)$. *Then*

$$\rho_0 = \left[1-(1-a^2)^{\frac{1}{3}}\left\{\left(1+\sqrt{1+\frac{1-a^2}{27}}\right)^{\frac{1}{3}}+\left(1-\sqrt{1+\frac{1-a^2}{27}}\right)^{\frac{1}{3}}\right\}\right]^{\frac{1}{2}}$$

and

$$f_0(z) = \frac{z}{a}\left\{\frac{1-a^2}{(1-\rho_0 z)^2}-1\right\}.$$

Theorem 4. $\gamma_n = n$ $(n = 1, 2, \dots)$. *Then*

$$\rho_0 = (1-\sqrt{1-a^2})^{\frac{1}{2}} \quad \text{and} \quad f_0(z) = -\frac{z(a^2-\rho_0 z)}{a(1-\rho_0 z)}.$$

Theorem 5. $\gamma_n = n^2$ $(n = 1, 2, \dots)$. *Then*

$$\rho_0 = a \quad \text{and} \quad f_0(z) = \int_0^z \frac{z-a}{1-az}dz.$$

II.

Recently J. Dieudonné has proved the following

Theorem 6. *Let* $w = f(z) = z + \dots$ *be convergent for* $|z| < 1$ *and* $|f(z)| < M$ *for* $|z| < 1$. *Then*

(1) $f(z)$ *is univalent for* $|z| < M-\sqrt{M^2-1}$ (1),

(2) $f(z)$ *maps* $|z| < M-\sqrt{M^2-1}$ *on a star-domain with respect to* $w = 0$ (2),

(1) J. DIEUDONNÉ: Sur quelques applications du lemme de Schwarz. Comptes Rendus, t. 190, 1930, p. 716–717. Recherches sur quelques problèmes relatifs aux polynomes et aux fonctions bornées d'une variable complexe. Ann. Sc. d. l'École Norm. Sup., t. 48, 1931.

(2) J. DIEUDONNÉ: l. c.

(3) *the inverse function* $z = w + \dots$ *is regular for*

$|w| < M(M-\sqrt{M^2-1})^2$ [1],

(4) *the limits in* (1), (2) *and* (3) *are attained by the function*

$$f(z) = \frac{Mz(1-Mz)}{M-z} .$$

Replacing the hypothesis $|f(z)| < M$ by $|f'(z)| < M$, we obtain

Theorem 7. *Let* $w = f(z) = z + \dots$ *be convergent for* $|z| < 1$ *and* $|f'(z)| < M$ *for* $|z| < 1$. *Then*

(1) $f(z)$ *is univalent for* $|z| < \dfrac{1}{M}$,

(2) $f(z)$ *maps* $|z| < M - \sqrt{M^2-1}$ *on a convex domain with respect to* $w = 0$.

(3) *the inverse function* $z = w + \dots$ *is regular for*

$$|w| < M\left(1 + (M^2-1)\log\left(1 - \frac{1}{M^2}\right)\right),$$

(4) *the limits in* (1), (2) *and* (3) *are attained by the function*

$$f(z) = M\int_0^z \frac{1-Mz}{M-z}dz = M\left(Mz + (M^2-1)\log\left(1-\frac{z}{M}\right)\right).$$

Proof. (1) Consider $\varphi(z) = \dfrac{f(z)}{M} = \dfrac{1}{M}z + \dfrac{a_2}{M}z^2 + \dots + \dfrac{a_n}{M}z^n + \dots$.

Then $\varphi(z)$ satisfies the conditions of theorem 5, for $\varphi(z)$ is regular and $|\varphi'(z)| < 1$ for $|z| < 1$ and Gutzmer's inequality gives $\sum_{n=1}^{\infty} n^2 \left|\dfrac{a_n}{M}\right|^2 \leqq 1$, $(a_1 = 1)$. Hence $\varphi(z)$ is univalent for $|z| < \dfrac{1}{M}$.

(2) The proof is entirely similar to Dieudonné's for (2) in theorem 6 [2],

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(1) E. LANDAU: Der Picard-Schottkysche Satz und die Blochsche Konstante. Sitzung. d. Preuss. Akad. Wiss., Jg. 1926, S. 467–494.

(2) J. DIEUDONNÉ: l. c.

(3) By (1) $f(z)$ is univalent for $|z|<\frac{1}{M}$ and

$$|f(z)| \geqq M\left(1+(M^2-1)\log\left(1-\frac{1}{M^2}\right)\right),$$

if $|z|=\frac{1}{M}$. Hence the inverse function $z=w+\dots.$ is regular for $|w|<M\left(1+(M^2-1)\log\left(1-\frac{1}{M^2}\right)\right)$.(1)

Lastly $f(z)=M\left(Mz+(M^2-1)\log\left(1-\frac{z}{M}\right)\right)$ has a vanishing derivative at $z=\frac{1}{M}$ and $f\left(\frac{1}{M}\right)=M\left(1+(M^2-1)\log\left(1-\frac{1}{M^2}\right)\right)$.

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(1) E. LANDAU: l. c.