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ON THE UNIVALENCY OF CERTAIN ANALYTIC FUNCTIONS

By

Kiyoshi NOSHIRO

The object of this paper is to give some generalizations on and complements to Mr. J. DIEUDONNÉ's results concerning the univalence (*Schlichtheit*) of a bounded function⁽¹⁾. First, assuming that $\varphi(z) = c_0 + \dots$, $c_0 \neq 0$, is regular and $\varphi(z) < D$ in the unit circle⁽²⁾, where D is a given connected domain, we obtain a general theorem on the univalence of the function $f(z) \equiv z\varphi(z)$, with the aid of Prof. KAKEYA's principle⁽³⁾. Next we state some applications of this theorem, taking some special domains as D . Finally we investigate the univalence of the function $f(z) = c_0z + c_1z^2 + \dots$, c_0, c_1 given, $c_0 \neq 0$, which is regular and bounded in the unit circle: $|f(z)| < 1$.

§ I.

Let $\varphi(z) = c_0 + \dots$, $c_0 \neq 0$, be regular and $\varphi(z) < D$ in the unit circle, where D is a given connected domain. We will here enunciate a theorem on the univalence of the function $f(z) \equiv z\varphi(z)$.

First we consider a function

$$\psi(z) = \alpha + \beta z + \dots, \quad (1)$$

which is regular and $\psi(z) < D$ in the unit circle. Let us denote by $g(z)$ a function, which maps D conformally on the unit circle. If we

(1) J. DIEUDONNÉ: Ann. d. l'Éc. Norm. Sup., **48**, 1931. Cf. p. 349-351.

(2) The notation $\varphi(z) < D$ means that the set of values taken by $\varphi(z)$ in the unit circle belongs to the domain D .

(3) S. KAKEYA: Sci. Rep. of Tokyo Bunrika Daigaku, Section A, I, No. 14, 1931, p. 159-165.

take a branch of $g(z)$, we obtain a function $G(z)$ defined by

$$G(z) \equiv g \{ \psi(z) \} , \quad (2)$$

which is uniform, regular in the unit circle and has the following properties:

$$(i) \quad |G(z)| < 1 \quad \text{for} \quad |z| < 1, \quad (ii) \quad G(0) = g(\alpha), \quad G'(0) = \beta g'(\alpha).$$

Hence, applying a theorem of E. LANDAU⁽⁴⁾, we have

$$|g(\alpha)|^2 + |\beta| |g'(\alpha)| \leq 1$$

$$\text{or} \quad |\beta| \leq \frac{1 - |g(\alpha)|^2}{|g'(\alpha)|} \equiv \Omega(\alpha, D), \quad (3)$$

since $g'(\alpha) \neq 0$. Here we must remark that the positive quantity $\Omega(\alpha, D)$ depends only on α and D , and does not depend either on the selection of a mapping function $g(z)$ or on the selection of a branch of $g(z)$ ⁽⁵⁾.

Next consider the function

$$\psi(\zeta) = \varphi\left(\frac{z_0 - \zeta}{1 - \bar{z}_0 \zeta}\right) = \varphi(z_0) - (1 - |z_0|^2) \varphi'(z_0) \zeta + \dots, \quad (|z_0| < 1). \quad (4)$$

Since $\psi(\zeta)$ is regular and $\psi(\zeta) < D$ for $|\zeta| < 1$, if we put

$$\alpha = \varphi(z_0), \quad \beta = -(1 - |z_0|^2) \varphi'(z_0),$$

we obtain, from (3),

$$|\varphi'(z_0)(1 - |z_0|^2)| \leq \Omega(\varphi(z_0), D).$$

(4) S. KAKEGA: l. c. p. 161.

(5) Let $g_1(z)$ and $g_2(z)$ be two mapping functions. Then there exists a relation $g_2(z) = L(g_1(z))$, where $L(z)$ is a function of the form $e^{i\theta} \frac{z - \gamma}{\bar{\gamma}z - 1}$, $|\gamma| < 1$, and so $\frac{1 - |g_1(\alpha)|^2}{|g_1'(\alpha)|} = \frac{1 - |g_2(\alpha)|^2}{|g_2'(\alpha)|}$. By an entirely similar discussion, we see that $\Omega(\alpha, D)$ does not also depend on the selection of a branch of $g(z)$. Cf. BIEBERBACH: Lehrbuch der Funktionentheorie, II, 2. Aufl., p. 45-46.

Consequently, if $\varphi(z_0) \neq 0$, we have

$$\left| z_0 \frac{\varphi'(z_0)}{\varphi(z_0)} \right| \leq \frac{|z_0|}{1-|z_0|^2} \frac{\Omega(\varphi(z_0), D)}{|\varphi(z_0)|}. \quad (5)$$

On the other hand, it is sufficient that

$$\left| z \frac{\varphi'(z)}{\varphi(z)} \right| < 1 \quad \text{for } |z| < R, \quad (6)$$

to show that $f(z) \equiv z\varphi(z)$ is univalent and starshaped for $|z| < R \leq 1^{(6)}$.

Thus we obtain a general theorem:

Theorem 1. Let $\varphi(z) = c_0 + \dots$, $c_0 \neq 0$, be regular and $\varphi(z) < D$ in the unit circle, where D is a given connected domain. Let $g(z)$ be an arbitrary branch of a function mapping D conformally on the unit circle, and put $\Omega(\alpha, D) = \frac{1-|g(\alpha)|^2}{|g'(\alpha)|}$, where α is a point in D . Then $f(z) \equiv z\varphi(z)$ is univalent and starshaped with respect to the origin for $|z| < R$, provided that

$$\frac{|z|}{1-|z|^2} \frac{\Omega(\varphi(z), D)}{|\varphi(z)|} < 1 \quad \text{for } |z| < R. \quad (7)$$

§ II.

We consider a circular domain $D: |z| < M$. Since $g(z) = \frac{z}{M}$ is a mapping function, it is easy to see that

$$\Omega(\alpha, D) = \frac{M^2 - |\alpha|^2}{M} \quad \text{and} \quad \left| \frac{z\varphi'(z)}{\varphi(z)} \right| \leq \frac{r}{1-r^2} \frac{1 - \left| \frac{\varphi(z)}{M} \right|^2}{\left| \frac{\varphi(z)}{M} \right|}$$

for $|z| = r^{(7)}$, (8)

- (6) The following theorem is known: Suppose that $f(z) = c_0 z + \dots$, $c_0 \neq 0$, is regular in the unit circle. Then $f(z)$ is univalent and starshaped with respect to the origin for $|z| < R$, provided that

$$\Re \left[z \frac{f'(z)}{f(z)} \right] > 0 \quad \text{for } |z| < R \leq 1.$$

Cf. A. KOBORI: *Memoirs of the College of Science, Kyoto Imperial University*, s. A, 1932, p. 279-291.

- (7) Hereafter we often denote $|z|$ by r .

if $\varphi(z)$ does not vanish at a point z within the unit circle. Thus

Theorem 2. Let $\varphi(z) = c_0 + \dots$, $c_0 \neq 0$, be regular and bounded in the unit circle: $|\varphi(z)| < M$. Then $f(z) \equiv z\varphi(z)$ is univalent and starshaped with respect to the origin for $|z| < R$, if

$$\left| \frac{\varphi(z)}{M} \right| > |z| \quad \text{for} \quad |z| < R.$$

Particularly, if we put $c_0 = 1$ in theorem 2, then we easily get

$$\frac{1-Mr}{M-r} \leq \left| \frac{\varphi(z)}{M} \right| \leq \frac{1+Mr}{M+r} \quad \text{for} \quad |z| = r \quad (9)$$

and so it is sufficient that r should be less than $R = M - \sqrt{M^2 - 1}$ in order that $\left| \frac{\varphi(z)}{M} \right| > |z|$. Thus we obtain DIEUDONNÉ's theorem:

Theorem 3. Suppose that $f(z) = z + \dots$ is regular and bounded in the unit circle: $|f(z)| < M$. Then $f(z)$ is univalent and starshaped with respect to the origin for $|z| < M - \sqrt{M^2 - 1}$. This limit can be attained by the function $f_0(z) = \frac{Mz(1-Mz)}{M-z}$.

Next consider a half-plane $D: \Re[z] > 0$. Taking a mapping function $g(z) = \frac{1-z}{1+z}$, we have

$$\Omega(a, D) = 2\Re[a] \quad \text{and} \quad \left| z \frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{2r \Re[\varphi(z)]}{1-r^2 |\varphi(z)|} \leq \frac{2r}{1-r^2}$$

for $|z| = r$. (10)

From this we get

Theorem 4. Let $\varphi(z)$ be regular and $\Re[\varphi(z)] > 0$ for $|z| < 1$. Then $f(z) \equiv z\varphi(z)$ is univalent and starshaped for $|z| < \sqrt{2}-1$. This limit can be attained by the function $f_0(z) = cz \frac{1-z}{1+z}$, ($c > 0$).

§ III.

We now consider a pricked (*punktiert*) circular domain D : $0 < |z| < M$. A function which maps D on the unit circle is

$$g(z) = \frac{1 + \log \frac{z}{M}}{1 - \log \frac{z}{M}}.$$

By a short calculation, we have

$$\Omega(\alpha, D) = 2 |\alpha| \log \frac{M}{|\alpha|} \quad \text{and} \quad \left| z \frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{2r}{1-r^2} \log \left| \frac{M}{\varphi(z)} \right|$$

for $|z| = r$. (11)

Hence we easily obtain

Theorem 5. Let $\varphi(z) = c_0 + \dots$ be regular for $|z| < 1$ and $0 < |\varphi(z)| < M$ for $0 < |z| < 1$. Then $f(z) \equiv z\varphi(z)$ is univalent and starshaped with respect to the origin for $|z| < r$, if

$$\left| \frac{\varphi(z)}{M} \right| \geq e^{-\frac{1-r^2}{2r}} \quad \text{for } |z| = r.$$

Put $c_0 = 1$ in theorem 5 and consider the function

$$F(z) = \frac{\log \frac{\varphi(z)}{M} - \log \frac{1}{M}}{\log \frac{\varphi(z)}{M} + \log \frac{1}{M}},$$

taking a branch of $\log z$ such that $\log 1 = 0$. Since $F(z)$ is regular and $|F(z)| < 1$ in the unit circle, $F(0)$ being equal to 0, SCHWARZ'S lemma gives $|F(z)| \leq |z|$, whence we have

$$M^{-\frac{1+r}{1-r}} \leq \left| \frac{\varphi(z)}{M} \right| \leq M^{-\frac{1-r}{1+r}} \quad \text{for } |z| = r. \quad (12)$$

Thus we can enunciate a theorem analogous to DIEUDONNÉ's.

Theorem 6. *Let $f(z) = z + \dots$ be regular for $|z| < 1$ and $0 < |f(z)| < M$ for $0 < |z| < 1$. Then*

(1) *$f(z)$ maps $|z| < R = \log eM - \sqrt{(\log eM)^2 - 1}$ on a starshaped domain Δ with respect to the origin.*

(2) *If we denote by d the shortest distance of the origin from the boundary of Δ , then $d \geq \left\{ \log eM - \sqrt{(\log eM)^2 - 1} \right\} M^{-\frac{\log M + \sqrt{\log M (2 + \log M)}}{2}}$.*

(3) *These limits (1) and (2) are attained by the function $f_0(z) = zM^{\frac{2e^{i\theta}z}{1+e^{i\theta}z}}$.*

Proof. From (11) and (12) we have

$$\left| z \frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{2r}{(1-r)^2} \log M \quad \text{for } |z| = r. \quad (13)$$

Since the right hand side is an increasing function of r for $0 < r < 1$, the equation $\frac{2r}{(1-r)^2} \log M = 1$ has just one root $R = \log eM - \sqrt{(\log eM)^2 - 1}$ there. From (12) we have $|f(Re^{i\theta})| \geq RM^{-\frac{2R}{1-R}}$. Our theorem is completely proved, considering the function

$$f_0(z) = zM^{\frac{2e^{i\theta}z}{1+e^{i\theta}z}},$$

which has

$$f_0(-Re^{-i\theta}) = -Re^{-i\theta}M^{-\frac{2R}{1-R}}, \quad f_0'(-Re^{-i\theta}) = \left(1 - \frac{2R}{(1-R)^2} \log M\right) M^{-\frac{2R}{1-R}} = 0.$$

Lastly we consider a ring-domain $D: m < |z| < M$, which can be mapped on the unit circle by a function

$$g(z) = \left(e^{i\frac{\pi}{2} \frac{\log \frac{z}{\sqrt{mM}}}{\log \sqrt{\frac{M}{m}}} - 1} \right) : \left(e^{i\frac{\pi}{2} \frac{\log \frac{z}{\sqrt{mM}}}{\log \sqrt{\frac{M}{m}}} + 1} \right).$$

In this case our discussion gives

$$\left| z \frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{2}{\pi} \frac{r}{1-r^2} \left(\log \frac{M}{m} \right) \cos \frac{\pi \log \frac{|\varphi(z)|}{\sqrt{mM}}}{\log \frac{M}{m}} \quad \text{for } |z| = r.$$

Consequently we obtain

Theorem 7. Let $\varphi(z)$ be regular and $m < |\varphi(z)| < M$ for $|z| < 1$. Then $f(z) \equiv z\varphi(z)$ is univalent and starshaped for $|z| < r$ if

$$\cos \frac{\pi}{2} \frac{\log \frac{|\varphi(z)|^2}{mM}}{\log \frac{M}{m}} \leq \frac{\pi}{2} \frac{r}{\log \frac{M}{m}} \quad \text{for } |z| = r.$$

§ IV.

DIEUDONNÉ's theorem (see theorem 3) can be enunciated, with a slight modification, as follows: let $f(z) = c_0 z + \dots$, c_0 given and $\neq 0$, be regular and $|f(z)| < 1$ for $|z| < 1$. Then $f(z)$ is univalent and starshaped with respect to the origin for

$$|z| < \frac{1}{|c_0|} - \sqrt{\frac{1}{|c_0|^2} - 1}.$$

It seems to me very difficult to generalize this theorem for the case where n initial coefficients $c_0, c_1, \dots, c_{n-1}$ ($c_0 \neq 0$) are given. In this paragraph we will treat the case where only two initial coefficients c_0, c_1 ($c_0 \neq 0$) are given. First suppose that

$$\varphi(z) = c_0 + c_1 z + \dots \quad (c_0, c_1 \text{ given, } c_0 \neq 0) \quad (14)$$

is regular and $|\varphi(z)| < 1$ for $|z| < 1$. Then the function

$$g(z) \equiv \frac{\varphi(z) - c_0}{1 - \bar{c}_0 \varphi(z)} = \frac{c_1}{1 - |c_0|^2} z + \dots \quad (15)$$

is regular and $|g(z)| < 1$ for $|z| < 1$. Hereafter, for brevity, we put

$$\alpha_0 = \frac{c_1}{1 - |c_0|^2} \quad (8)$$

From (15) we have

$$\varphi(z) = \frac{c_0 + g(z)}{1 + \bar{c}_0 g(z)} \quad (17)$$

Considering $|g(z)| < 1$ for $|z| < 1$, we have

$$\frac{|c_0| - |g(z)|}{1 - |c_0||g(z)|} \leq |\varphi(z)| \leq \frac{|c_0| + |g(z)|}{1 + |c_0||g(z)|} \quad (18)$$

for $|z| < 1$. Since the expression on the left hand side decreases and that on the right increases as $|g(z)|$ increases, using a known inequality

$$|g(z)| \leq r \frac{|\alpha_0| + r}{1 + |\alpha_0|r}, \quad (19)$$

we obtain the following inequality

$$\frac{|c_0| - |\alpha_0|(1 - |c_0|)r - r^2}{1 + |\alpha_0|(1 - |c_0|)r - |c_0|r^2} \leq |\varphi(z)| \leq \frac{|c_0| + |\alpha_0|(1 + |c_0|)r + r^2}{1 + |\alpha_0|(1 + |c_0|)r + |c_0|r^2} \quad (20)$$

for $|z| = r$. In order that the left expression of (20) be positive, we have only to take r less than

$$\rho = \frac{1}{2} \left\{ -|\alpha_0|(1 - |c_0|) + \sqrt{|\alpha_0|^2(1 - |c_0|)^2 + 4|c_0|} \right\}, \quad (21)$$

where ρ is the root between 0 and 1 of the equation

$$|c_0| - |\alpha_0|(1 - |c_0|)r - r^2 = 0. \quad (22)$$

(8) Of course $|\alpha_0| \leq 1$ and the equality holds if and only if $\varphi(z) = \frac{c_0 + e^{i\theta}z}{1 + \bar{c}_0 e^{i\theta}z}$.

Thus we see that $\varphi(z)$ does not vanish for $|z| < \rho^{(9)}$. This limit cannot be replaced by any greater number, for the lower bound of $|\varphi(z)|$ of (20) can be attained by the function

$$\varphi_0(z) = \frac{c_0 + (c_0 \bar{a}_0 e^{i\theta} + a_0)z + e^{i\theta} z^2}{1 + (\bar{a}_0 e^{i\theta} + \bar{c}_0 a_0)z + \bar{c}_0 e^{i\theta} z^2}$$

$$(\theta \equiv 2 \text{ amp } c_1 - \text{amp } c_0 + \pi \pmod{2\pi}) \quad (23)$$

at a point $z_0 = r e^{i\varphi_0}$, where $\varphi_0 \equiv \text{amp } c_0 - \text{amp } c_1 + \pi \pmod{2\pi}$, and $\varphi_0(\rho e^{i\varphi_0}) = 0$. From (17)

$$|\varphi'(z)| = (1 - |c_0|^2) \left| \frac{g'(z)}{(1 + \bar{c}_0 g(z))^2} \right|. \quad (24)$$

From (8)

$$\left| \frac{g'(z)}{z} - \frac{g(z)}{z^2} \right| \leq \frac{1 - \left| \frac{g(z)}{z} \right|^2}{1 - r^2}, \quad (25)$$

or

$$|g'(z)| \leq \frac{(1 - |g(z)|)(r^2 + |g(z)|)}{r(1 - r^2)}. \quad (26)$$

From

$$\left| \frac{z\varphi'(z)}{\varphi(z)} \right| = (1 - |c_0|^2) \left| \frac{zg'(z)}{(c_0 + g(z))(1 + \bar{c}_0 g(z))} \right|,$$

using (26), we have

$$\left| \frac{z\varphi'(z)}{\varphi(z)} \right| \leq \frac{1 - |c_0|^2}{1 - r^2} \frac{(1 - |g(z)|)(r^2 + |g(z)|)}{(|c_0| - |g(z)|)(1 - |c_0| |g(z)|)}, \quad (27)$$

provided that $|c_0| > |g(z)|$.

(9) This result has been obtained by M. KÖSSLER: Cf. Ueber die α -Stellen von beschränkten Potenzreihen. Mémoires de la Société Royale des Sciences de Bohême, Année 1930.

Since $\frac{1-t}{|c_0|-t} \frac{r^2+t}{1-|c_0|t}$ is an increasing function of t for $0 < t < |c_0|$, using the inequality (19), we obtain

$$\left| \frac{z\varphi'(z)}{\varphi(z)} \right| \leq (1-|c_0|^2) \frac{r(|\alpha_0|+2r+|\alpha_0|r^2)}{(|c_0|-|\alpha_0|(1-|c_0|)r-r^2)(1+|\alpha_0|(1-|c_0|)r-|c_0|r^2)} \quad (28)$$

$$\text{for } |z| < \rho = \frac{1}{2} \left\{ -|\alpha_0|(1-|c_0|) + \sqrt{|\alpha_0|^2(1-|c_0|)^2 + 4|c_0|} \right\}. \quad (10)$$

Since the right hand side of (28) is an increasing function of r in the interval $(0, \rho)$, the equation

$$(1-|c_0|^2) \frac{r(|\alpha_0|+2r+|\alpha_0|r^2)}{(|c_0|-|\alpha_0|(1-|c_0|)r-r^2)(1+|\alpha_0|(1-|c_0|)r-|c_0|r^2)} = 1 \quad (29)$$

or

$$c - 2\alpha(1-c)r - \{3 - c^2 + \alpha^2(1-c)^2\}r^2 - 2\alpha(1-c)r^3 + cr^4 = 0, \quad (30)$$

$$\text{where } c = |c_0|, \quad \alpha = |\alpha_0| = \frac{|c_1|}{1-|c_0|^2},$$

has only one root R between 0 and $\rho^{(11)}$.

Thus we obtain the following

Theorem 8. Suppose that

$$f(z) = c_0z + c_1z^2 + \dots \quad (c_0, c_1 \text{ given, } c_0 \neq 0)$$

is regular and $|f(z)| < 1$ for $|z| < 1$. Then

(10) By (19), we see that

$$|c_0| > r \frac{|\alpha_0|+r}{1+|\alpha_0|r} \geq |g(z)| \text{ for } |z| < \rho.$$

(11) The equation (30) has 4 real roots, of which two are positive and the others negative. If we denote a negative root by τ , all roots can be written as follows: $R, \frac{1}{R}, \tau, \frac{1}{\tau}$.

(1) $f(z)$ is univalent for $|z| < R$ and maps the circle $|z| < R$ on a starshaped domain Δ with respect to the origin, if R is the root between 0 and 1 of the equation

$$c - 2a(1-c)r - \{3 - c^2 + a^2(1-c)^2\}r^2 - 2a(1-c)r^3 + cr^4 = 0,$$

where

$$c = |c_0|, \quad a = \frac{|c_1|}{1 - |c_0|^2}.$$

(2) If we denote by d the shortest distance of the origin from the boundary of Δ , then

$$d \geq R \frac{c - a(1-c)R - R^2}{1 + a(1-c)R - cR^2}.$$

(3) These limits (1) and (2) can be attained by the function

$$f_0(z) = z \frac{c_0 + (c_0 \bar{a}_0 e^{i\theta} + a_0)z + e^{i\theta} z^2}{1 + (\bar{a}_0 e^{i\theta} + \bar{c}_0 a_0)z + \bar{c}_0 e^{i\theta} z^2},$$

where $\theta \equiv 2 \text{ amp } c_1 - \text{amp } c_0 + \pi \pmod{2\pi}$, $a_0 = \frac{c_1}{1 - |c_0|^2}$.

Proof. The first assertion is proved above and the second follows directly from (20). Finally, by an elementary calculation, we see that

$$f'_0(Re^{i\varphi_0}) = 0 \quad \text{and} \quad |f_0(Re^{i\varphi_0})| = R \frac{c - a(1-c)R - R^2}{1 + a(1-c)R - cR^2},$$

where $\varphi_0 \equiv \text{amp } c_0 - \text{amp } c_1 + \pi \pmod{2\pi}$.

Remark. The root R of the equation (30) can be written explicitly:

$$R = \frac{1}{2c} \left[a(1-c) + \sqrt{(1+c)(1-c)^2 a^2 + c(3-c)(1+c)} \right. \\ \left. - \left\{ (2+c)(1-c)^2 a^2 + c(3+c)(1-c) \right. \right. \\ \left. \left. + 2a(1-c) \sqrt{(1+c)(1-c)^2 a^2 + c(3-c)(1+c)} \right\}^{\frac{1}{2}} \right].$$

When c is fixed, R is a decreasing function of α in the interval $(0,1)$ (see footnote (8)) and especially

$$R = \frac{1}{c} - \sqrt{\frac{1}{c^2} - 1}, \quad \text{if } \alpha = 1,$$

$$R = \frac{1}{2} \left\{ \sqrt{\frac{(3-c)(1+c)}{c}} - \sqrt{\frac{(3+c)(1-c)}{c}} \right\}, \quad \text{if } \alpha = 0.$$

§ V.

In this paragraph we shall make some remarks on the univalence of certain analytic functions. Suppose that $f(z) = \alpha + \beta z + \dots$ (α, β given and $\beta \neq 0$) is regular and $f(z) \in D$ in the unit circle, where D is a given *simply* connected domain. It is well known that there exists one and only one function $\varphi(z)$ which maps D in a one-to-one and conformal manner on a circle under the condition: $\varphi(\alpha) = 0$, $\varphi'(\alpha) = \frac{1}{\beta}$. We denote the radius of the mapped circle by M . Next consider the function $F(z) = \varphi(f(z))$. Then $F(z) = z + \dots$ is regular and bounded in the unit circle: $|F(z)| < M$. Here we can, without difficulty, assert the following:

1°. The existence of $F(z)$ follows from that of $f(z)$, and conversely. Hence, it is necessary and sufficient that $M \geq 1$ ⁽¹²⁾ for the existence of $f(z)$.

2°. If $f(z)$ takes the same value at n given points $z_1, z_2, \dots, z_n$ within the unit circle, then $F(z)$ also takes the same value with the same multiplicity, its converse being also true.

Hence, as an immediate result from a theorem due to DIEUDONNÉ⁽¹³⁾, we can state the following theorem: $f(z)$ assumes at most p times

(12) The equality holds if and only if $f(z) = \varphi^{-1}(z)$ where φ^{-1} is an inverse function of φ .

(13) DIEUDONNÉ has proved that $F(z)$ assumes at most p times the same value for $|z| < \rho_p$ and ρ_p can be attained. Cf. l.c. p. 348. Or my paper: Proc. of Phys.-Math. Soc. of Japan, 3rd Ser., Vol. 14, 1932, p. 304-309. Cf. p. 308.

the same value for $|z| < \rho_p$, where ρ_p is the root between 0 and 1 of the equation $1 + r^2 + r^4 + \dots + r^{2p} - (p+1)Mr^p = 0$ and moreover ρ_p cannot be replaced by any greater number.

For example, we consider a half-plane $D: \Re[z] > 0$. Without difficulty, we have

$$\varphi(z) = \frac{2\Re[a]}{\beta} \frac{z-a}{z+\bar{a}} \quad \text{and} \quad M = \frac{2\Re[a]}{|\beta|}.$$

Hence, it is necessary and sufficient that $2\Re[a] \geq |\beta|$ for the existence of $f(z)$. By the theorem above we get at once

Theorem 9.⁽¹⁴⁾ Let $f(z) = a + \beta z + \dots$ (a, β given and $\beta \neq 0$) be regular and $\Re[f(z)] > 0$ for $|z| < 1$. Then $f(z)$ is univalent for

$$|z| < \rho_1 = \frac{2\Re[a]}{|\beta|} - \sqrt{\left(\frac{2\Re[a]}{|\beta|}\right)^2 - 1},$$

and the limiting case can be attained by the function

$$f_0(z) = \frac{Ma e^{-i\theta} + (\bar{a} - e^{-i\theta}a)z - \bar{a}Mz^2}{Me^{-i\theta} - (1 + e^{-i\theta})z + Mz^2} \quad \left(\theta = \text{amp } \beta, \quad M = \frac{2\Re[a]}{|\beta|}\right),$$

which has a derivative vanishing at a point on the circle $|z| = \rho_1$.

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Mathematical Institute
Hokkaido Imperial University..

(14) Compare with theorem 4 in KOBORI's paper: *Memoirs of the College of Science, Kyoto Imperial University, Ser. A. Vol. XIV, 1931, p. 251-255.*