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CONNECTIONS IN THE MANIFOLD ADMITTING CONTACT TRANSFORMATIONS

By

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The theory of connections in the manifold admitting the generalized transformations has been developed by the present author.⁽¹⁾ As an application of the theory, it is proposed now to consider some linear displacements in the general manifold preserving a contact transformation.

Consider an n -dimensional manifold X_n with coordinates x^ν ($\nu = a_1, a_2, \dots, a_n$), and a covariant vector field of components p_λ osculating at each point of X_n . The new manifold obtained in this manner is called the general manifold T_n . However in this general manifold T_n there is no *a priori* basis for the comparison of the covariant vectors at different points. Hence we shall define the relation between an osculating covariant vector p_λ at a given point $P(x_0^\lambda)$ and $p_\lambda + dp_\lambda$ at any nearly point $P'(x_0^\lambda + dx^\lambda)$, by the following equations:

$$(1) \quad dp_\lambda = \omega_{\lambda\mu} dx^\mu \quad \lambda = a_1, a_2, \dots, a_n,$$

where parameters $\omega_{\lambda\mu}$ are arbitrary functions of x^ν as well as p_λ . Consequently we see that if at any point $P(x_0^\lambda)$ of X_n we let osculate a covariant vector p_λ , then we get an osculating covariant vector at every point of X_n , so that our manifold T_n is completely determined. The connection so defined is a generalization of that developed some-

(1) T. HOSOKAWA: Connections in the Manifold Admitting Generalized Transformations, Proc. of the Imperial Acad., vol. 8 (1932), p. 384-351.

what by several authors⁽¹⁾ as may be seen. The curves defined by equations (1) are called the *base paths*.

Let us now consider the transformations of the form

$$(2) \quad 'x^\nu = 'x^\nu(x^\lambda; p_\lambda), \quad 'p_\lambda = 'p_\lambda(x^\nu; p_\nu)$$

in the $2n$ variables x^ν and p_λ , such that the following equations hold good

$$(3) \quad d'x^\nu 'p_\nu = \left(\frac{\partial 'x^\nu}{\partial x^\lambda} dx^\lambda + \frac{\partial 'x^\nu}{\partial p_\tau} dp_\tau \right) 'p_\nu = dx^\nu p_\nu$$

for arbitrary values of the differentials dx^λ and dp_λ , followingly for arbitrary functions $\omega_{\lambda\mu}$.

A transformation (2) satisfying this condition is a contact transformation. From (3) are derived the equations

$$(4) \quad 'p_\nu \frac{\partial 'x^\nu}{\partial x^\lambda} = p_\lambda, \quad 'p_\nu \frac{\partial 'x^\nu}{\partial p_\lambda} = 0.$$

Then one may see that a necessary and sufficient condition that a set of functions $'x^\nu(x; p)$ may determine a contact transformation (2) for which the $'p_\lambda(x; p)$ are uniquely determined is that the functions $'x^\nu(x; p)$ be homogeneous of degree zero in p 's, that the Jacobian of the $'x^\nu(x; p)$ with respect to the x 's be of rank n and that the identities

$$\frac{\partial 'x^\nu}{\partial x^\lambda} \frac{\partial 'x^\mu}{\partial p_\lambda} - \frac{\partial 'x^\nu}{\partial p_\lambda} \frac{\partial 'x^\mu}{\partial x^\lambda} = 0$$

be satisfied.⁽²⁾ Also every contact transformation admits a unique inverse contact transformation:⁽³⁾

$$(5) \quad x^\lambda = x^\lambda('x^\nu; 'p_\nu), \quad p_\lambda = p_\lambda('x^\nu; 'p_\nu).$$

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- (1) T. LEVI-CIVITA: Nozione di parallelismo in una varietà qualunque e conseguente specificazione geometrica della curvatura riemanniana, *Rendiconti di Palermo*, vol. 42 (1917), p. 173-205. L. BERWALD: Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus, *Math. Zeit.*, vol. 25 (1926), p. 40-73. E. BORTOLLOTTI: Differential invariants of direction and point displacements, *Annals of Math.*, vol. 32 (1931), p. 361-377.
- (2) L. P. EISENHART: Continuous Groups of Transformations, Princeton University Press, (1933), p. 242.
- (3) L. P. EISENHART: loc. cit., p. 249.

By differentiation of the first set of (2) and (5), we have

$$(6) \quad d'x^\nu = \left(\frac{\partial'x^\nu}{\partial x^\lambda} + \frac{\partial'x^\nu}{\partial p_\tau} \omega_{\tau\lambda} \right) dx^\lambda, \quad dx^\lambda = \left(\frac{\partial x^\lambda}{\partial'x^\mu} + \frac{\partial x^\lambda}{\partial'p_\tau} \bar{\omega}_{\tau\mu} \right) d'x^\mu,$$

where

$$d'p_\nu = \bar{\omega}_{\nu\mu} d'x^\mu.$$

Any set of n quantities $v^\nu(x; p)$, which are transformed by the transformation (2) into n new quantities $'v^\nu('x; 'p)$ in such a way that

$$(7) \quad 'v^\nu = u_\lambda^\nu v^\lambda,$$

will be called a *contravariant vector*; a *covariant vector* is a set of n quantities w_λ which are transformed by (2)

$$(8) \quad 'w_\mu = v_\mu^\lambda w_\lambda,$$

where

$$(9) \quad u_\lambda^\nu = \frac{\partial'x^\nu}{\partial x^\lambda} + \frac{\partial'x^\nu}{\partial p_\sigma} \omega_{\sigma\lambda}, \quad v_\mu^\lambda = \frac{\partial x^\lambda}{\partial'x^\mu} + \frac{\partial x^\lambda}{\partial'p_\sigma} \bar{\omega}_{\sigma\mu}.$$

Let it now be assumed that the following relations are satisfied:

$$(10) \quad \frac{\partial x^\nu}{\partial'p_\sigma} \bar{\omega}_{\sigma\kappa} \omega_{\lambda\nu} - \frac{\partial p_\lambda}{\partial'p_\sigma} \bar{\omega}_{\sigma\kappa} + \frac{\partial x^\nu}{\partial'x^\kappa} \omega_{\lambda\nu} = \frac{\partial p_\lambda}{\partial'x^\kappa}$$

and

$$(11) \quad \frac{\partial x^\nu}{\partial p_\sigma} \omega_{\sigma\kappa} \bar{\omega}_{\lambda\nu} - \frac{\partial'p_\lambda}{\partial p_\sigma} \omega_{\sigma\kappa} + \frac{\partial'x^\mu}{\partial x^\kappa} \omega_{\lambda\mu} = \frac{\partial'p_\lambda}{\partial x^\kappa}.$$

But from (9) is obtained

$$u_\lambda^\nu v_\mu^\lambda = \frac{\partial'x^\nu}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial'x^\mu} + \frac{\partial'x^\nu}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial'p_\sigma} \bar{\omega}_{\sigma\mu} + \frac{\partial x^\kappa}{\partial'x^\mu} \frac{\partial'x^\nu}{\partial p_\lambda} \omega_{\lambda\kappa} + \frac{\partial'x^\nu}{\partial p_\lambda} \frac{\partial x^\kappa}{\partial'p_\sigma} \omega_{\lambda\kappa} \bar{\omega}_{\sigma\mu},$$

and on the other hand

$$\frac{\partial'x^\nu}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial'p_\mu} + \frac{\partial'x^\nu}{\partial p_\sigma} \frac{\partial p_\sigma}{\partial'p_\mu} = 0$$

holds good. Therefore from (10), is obtained

$$(12) \quad u_{\lambda}^{\nu} v_{\mu}^{\lambda} = \delta_{\mu}^{\nu},$$

and in like manner from (11)

$$(13) \quad v_{\lambda}^{\nu} u_{\mu}^{\lambda} = \delta_{\mu}^{\nu},$$

where the δ 's are KRONECKER's deltas.

By means of these new definitions it is to be seen that the p_{λ} is a covariant vector, because from (4) we get

$$'p_{\nu} \frac{\partial' x^{\nu}}{\partial x^{\lambda}} + 'p_{\nu} \frac{\partial' x^{\nu}}{\partial p_{\sigma}} \omega_{\sigma\lambda} = p_{\lambda}$$

i. e.

$$u_{\lambda}^{\nu} 'p_{\nu} = p_{\lambda},$$

which becomes by (2)

$$'p_{\mu} = v_{\mu}^{\lambda} p_{\lambda}.$$

The equations (6) show that the differential dx^{λ} is a contravariant vector.

A tensor of the higher order is defined by the following equations:

$$'v_{\lambda_1 \dots \lambda_s}^{\nu_1 \dots \nu_t} = v_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_t} u_{\alpha_1}^{\nu_1} \dots u_{\alpha_t}^{\nu_t} v_{\lambda_1}^{\beta_1} \dots v_{\lambda_s}^{\beta_s}.$$

When a quantity is invariant by the transformation (2), it is called a *scalar*. Then from (13) it can be shown that $v^{\nu} w_{\nu}$ is a scalar.

Now let "metrics" be introduced in our manifold. The metrics must be an invariance by means of the transformation (2). We consider one parameter continuous group G_1 of the contact transformations. An infinitesimal transformation of the group G_1 is defined by equations of the form

$$(14) \quad 'x^{\lambda} = x^{\lambda} + \frac{\partial C}{\partial p_{\lambda}} \delta t, \quad 'p_{\lambda} = p_{\lambda} - \frac{\partial C}{\partial x_{\lambda}} \delta t,$$

where

$$(15) \quad C = p_\lambda \frac{\partial C}{\partial p_\lambda} \quad (1)$$

The function C is called the characteristic function of the contact transformation, and is an invariant function by means of the contact transformation (2).

From (14) are derived

$$(16) \quad \frac{dx^\lambda}{dt} = \frac{\partial C}{\partial p_\lambda}, \quad \frac{dp_\lambda}{dt} = -\frac{\partial C}{\partial x^\lambda},$$

and by integration of the above equations we get the finite equations of G_1 :

$$'x^\lambda = 'x^\lambda(x; p, t), \quad 'p_\lambda = 'p_\lambda(x; p, t).$$

If equations (16) are transformed by means of a contact transformation (2), we obtain

$$\frac{d'x^\lambda}{dt} = \frac{\partial \bar{C}}{\partial 'p_\lambda}, \quad \frac{d'p_\lambda}{dt} = -\frac{\partial \bar{C}}{\partial 'x^\lambda},$$

where $\bar{C}$ is the transform of the characteristic function of the group $G_1^{(2)}$. Accordingly we see that the $\frac{\partial C}{\partial p_\lambda}$ is a contravariant vector.

In particular we put

$$(17) \quad C = \sqrt{g^{\lambda\mu} p_\lambda p_\mu},$$

where the $g^{\lambda\mu}$'s are functions of the x 's as well as p 's, and are homogeneous of zero-th degree in the p 's, and the rank of the matrix of the $g^{\lambda\mu}$'s is n . But it is evident that the $g^{\lambda\mu}$'s are components of a contravariant tensor of the second order. We shall take $g^{\lambda\mu}$ as the *fundamental tensor* of the metrics.

(1) L. P. EISENHART: loc. cit., p. 252.

(2) L. P. EISENHART: loc. cit., p. 254.

If the functions $g^{\lambda\mu}$ be defined by the following equations:

$$(18) \quad g^{\lambda\mu} = \frac{1}{2} \frac{\partial^2 C^2}{\partial p_\lambda \partial p_\mu},$$

then by EULER's theorem we get

$$\frac{\partial g^{\lambda\mu}}{\partial p_\nu} p_\lambda p_\mu = \frac{1}{2} \frac{\partial^3 C^2}{\partial p_\lambda \partial p_\mu \partial p_\nu} p_\lambda p_\mu = 0.$$

Hence from (16), we have

$$(19) \quad \frac{dx^\lambda}{dt} = hg^{\lambda\mu} p_\mu, \quad \frac{dp_\lambda}{dt} = -\frac{h}{2} \frac{\partial g^{\nu\mu}}{\partial x^\lambda} p_\mu p_\nu,$$

where $h^{-1} = C$.

From (1) and the first set of (19),

$$\frac{dp_\lambda}{dt} = hg^{\nu\mu} \omega_{\lambda\mu} p_\nu.$$

If we define arbitrary functions $\omega_{\lambda\mu}$ by the following equations:

$$(20) \quad \omega_{\lambda\sigma} = -\frac{1}{2} g_{\nu\sigma} \frac{\partial g^{\nu\mu}}{\partial x^\lambda} p_\mu,$$

then equations (1) are reduced to the second set of (19).

We shall now define a *linear displacement* for contravariant and covariant vectors v^ν and w_λ :

$$(21) \quad \begin{cases} \delta v^\nu = dv^\nu + \Gamma_{\lambda\mu}^\nu v^\lambda dx^\mu + A_{\lambda}^{\nu\sigma} v^\lambda dp_\sigma, \\ \delta w_\nu = dw_\nu - \Gamma_{\nu\mu}^\lambda w_\lambda dx^\mu - A_{\nu}^{\lambda\sigma} w_\lambda dp_\sigma, \end{cases}$$

where $\Gamma_{\lambda\mu}^\nu$ and $A_{\lambda}^{\nu\sigma}$ are the functions of x 's as well as p 's. If the linear displacement is taken along the base paths satisfying (1), we get from the above equations

$$(22) \quad \begin{cases} \delta v^\nu = dv^\nu + \overset{*}{\Gamma}_{\lambda\mu}^\nu v^\lambda dx^\mu, \\ \nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \frac{\partial v^\nu}{\partial p_\sigma} \omega_{\sigma\mu} + \overset{*}{\Gamma}_{\lambda\mu}^\nu v^\lambda \end{cases}$$

and

$$(22') \quad \begin{cases} \delta w_\nu = dw_\nu - \overset{*}{\Gamma}_{\nu\mu}^\lambda w_\lambda dx^\mu, \\ \nabla_\mu w_\nu = \frac{\partial w_\nu}{\partial x^\mu} + \frac{\partial w_\nu}{\partial p_\sigma} \omega_{\sigma\mu} - \overset{*}{\Gamma}_{\nu\mu}^\lambda w_\lambda, \end{cases}$$

where

$$\overset{*}{\Gamma}_{\lambda\mu}^\nu = \Gamma_{\lambda\mu}^\nu + A_\lambda^\sigma \omega_{\sigma\mu}.$$

In order that $\nabla_\mu v^\nu$ may be the components of a mixed tensor, $\overset{*}{\Gamma}_{\lambda\mu}^\nu$ must satisfy the following equation:

$$(23) \quad \frac{\partial u_\lambda^\nu}{\partial x^\mu} + \frac{\partial u_\lambda^\nu}{\partial p_\sigma} \omega_{\sigma\mu} + \overset{*}{\Gamma}_{\lambda\sigma}^\nu u_\lambda^\sigma u_\mu^\sigma = u_\lambda^\nu \overset{*}{\Gamma}_{\lambda\mu}^\sigma,$$

where $\overset{*}{\Gamma}_{\mu\nu}^\lambda$ are functions of x 's as well as p 's and $\overset{*}{\Gamma}_{\mu\nu}^\lambda$ of x 's as well as p 's.

In the same manner as that of the general linear displacements,⁽¹⁾ we can calculate the *curvature tensor*:

$$R_{\nu\mu\rho}^{\dots\lambda} = \frac{\partial \overset{*}{\Gamma}_{\rho\nu}^\lambda}{\partial x^\mu} - \frac{\partial \overset{*}{\Gamma}_{\rho\mu}^\lambda}{\partial x^\nu} + \overset{*}{\Gamma}_{\omega\mu}^\lambda \overset{*}{\Gamma}_{\rho\nu}^\omega - \overset{*}{\Gamma}_{\omega\nu}^\lambda \overset{*}{\Gamma}_{\rho\mu}^\omega + \frac{\partial \overset{*}{\Gamma}_{\rho\nu}^\lambda}{\partial p_\tau} \omega_{\tau\mu} - \frac{\partial \overset{*}{\Gamma}_{\rho\mu}^\lambda}{\partial p_\tau} \omega_{\tau\nu}.$$

When the p 's are such that $C \neq 0$ and $h = \text{const.}$, we can normalize $C = 1$, by replacing p_λ by $h^{-1} p_\lambda$. Since C is homogeneous of degree one in the p 's. Hence from (19) we get

$$(24) \quad \frac{dx^\lambda}{dt} = g^{\lambda\mu} p_\mu, \quad \frac{dp_\lambda}{dt} = -\frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\lambda} p_\mu p_\nu.$$

When the rank of the hessian of C with respect to p 's is $n-1$, the first set of the above equations can be solved with respect to p 's as

(1) T. HOSOKAWA: On the Various Linear Displacements in the Berwald-Finsler's Manifold, Science Reports, Tôhoku Imp. University, vol. 19 (1930), p. 37-51.

functions of the x 's and $\dot{x}$'s, where $\dot{x}^\lambda = \frac{dx^\lambda}{dt}$. We denote by $\hat{C}$ the function resulting from the substitution in C of these expressions for p_λ . Then we get

$$(25) \quad \hat{g}_{\lambda\mu} = \frac{1}{2} \frac{\partial^2 \hat{C}}{\partial \dot{x}^\lambda \partial \dot{x}^\mu}, \quad p_\lambda = \hat{g}_{\lambda\mu} \dot{x}^\mu.$$

From the second set of (24), we have

$$(26) \quad \frac{d^2 x^\lambda}{dt^2} + \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0,$$

where $\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\}$ are CHRISTOFFEL's symbol with respect to $\hat{g}_{\mu\lambda}$. Thus the paths defined by (24) are the geodesics of BERWALD-FINSLER's manifold.⁽¹⁾ In assumption (20), the parallelism defined by equations (1) is reduced to that by (26). Accordingly from the first set of (25), we have

$$dp_\lambda = - \left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\} \hat{g}_{\nu\sigma} \dot{x}^\sigma dx^\mu.$$

Hence from (1), we obtain

$$\omega_{\lambda\mu} = - \left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\} \hat{g}_{\nu\sigma} \dot{x}^\sigma.$$

Consequently from the linear displacement (22) and (22') we can reduce the connections which has already been studied by the present author.⁽²⁾

(1) M. S. KNEBELMEN: Collineations and Motions in Generalized Space, American Journal of Mathematics, vol. 51 (1928), p. 527-564.

(2) T. HOSOKAWA: loc. cit., (1930), p. 42.