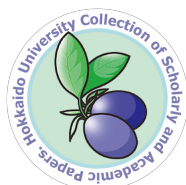




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ON THE THEORY OF SCHLICHT FUNCTIONS

By

Kiyoshi NOSHIRO

Let us denote by k a certain positive integer. If

$$f(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$$

is regular and schlicht in the unit circle, then we may say, for convenience, that $f(z)$ is a function of the class k . If $f(z)$ is a function of the class k and maps the unit circle on a starshaped domain with centre at the origin, then we may say, for brevity, that $f(z)$ is a starshaped function of the class k . Similarly, if $f(z)$ is a function of the class k and maps the unit circle on a convex domain, we may call $f(z)$ a convex function of the class k . Our main object in this paper is to state some theorems concerning a starshaped resp. convex function of the class $k^{(1)}$, which are known when $k = 1$ or 2 . First we will obtain some results on the coefficients, using an easy lemma. Next, some extensions of STROHHÄCKER's theorems⁽²⁾ will be mentioned. Applying the above results, we can extend SZEGÖ's theorem⁽³⁾ on the polynomial sections of a starshaped resp. convex function.

It is well known that under the condition that $f(z)$ is regular in D and that $f'(z)$ never vanishes there, we cannot necessarily assert $f(z)$ to be schlicht in D . In § V, imposing a further condition on

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- (1) Recently Mr. CHEN has obtained some results concerning a (schlicht) function of the class k . See Kien Kwong CHEN: Proc. Imp. Acad. Japan, 1933, vol. 9, p. 465-467.
 - (2) E. STROHHÄCKER: Math. Zeits., Bd. 37, 1933, p. 350-380.
 - (3) G. SZEGÖ: Math. Ann., Bd. 100, 1928, p. 188-211. See also S. TAKAHASHI: Proc. Physico-Math. Soc. Japan, 3rd ser., vol. 16, 1934, p. 7-15; L. BIEBERBACH: Bulletin, Calcutta Math. Soc., vol. 20, 1930, p. 17-20; and also A. KOBORI: Memoires, College of Science, Kyoto Imp. Univ., ser. A, vol. 16, 1933, p. 127-135.

$f(z)$, we will study the univalence (*Schlichtheit*) of $f(z)$. In the last paragraph we will state a theorem on a schlicht meromorphic function in the unit circle, which is of some interest in itself.

§ I. AN IMPORTANT LEMMA.

Suppose that

$$(1) \quad \phi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$$

is a starshaped function of the class k . If we put $\zeta = z^k$ and define a function

$$f(\zeta) = [\phi(z)]^k,$$

then $f(\zeta)$ is regular and starshaped⁽¹⁾ for $|\zeta| < 1$. For the proof, since $f(\zeta) = \zeta + \dots$ is regular and does not vanish for $0 < |\zeta| < 1$, we have only to show⁽²⁾

$$\Re \left(\zeta \frac{f'(\zeta)}{f(\zeta)} \right) > 0 \quad \text{for} \quad |\zeta| < 1.$$

But this inequality follows from

$$\Re \left(z \frac{\phi'(z)}{\phi(z)} \right) > 0 \quad \text{for} \quad |z| < 1,$$

since we easily get

$$\zeta \frac{f'(\zeta)}{f(\zeta)} = z \frac{\phi'(z)}{\phi(z)}, \quad (\zeta = z^k).$$

Thus we obtain

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- (1) We say that $f(z)$ is starshaped for $|z| < \rho$, if $f(z)$ is regular and schlicht and maps $|z| < \rho$ on a starshaped domain with centre at the origin.
 (2) Here we use a well known theorem: Suppose that $f(z) = z + \dots$ is regular in the unit circle. Then $f(z)$ is starshaped for $|z| < 1$, if and only if

$$f(z) \neq 0 \quad \text{for} \quad 0 < |z| < 1 \quad \text{and} \quad \Re \left(z \frac{f'(z)}{f(z)} \right) > 0 \quad \text{for} \quad |z| < 1.$$

Lemma. If $\phi(z)$ is a starshaped function of the class k , then

$$(2) \quad f(\zeta) = [\phi(z)]^k, \quad (\zeta = z^k)$$

is a starshaped function of the class 1.

§ II. ON THE COEFFICIENTS OF A STARSHAPED RESP. CONVEX FUNCTION.

In this paragraph we will enunciate a theorem on the coefficients of a starshaped resp. convex function of the class k .

Theorem 1. If $\phi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$ is a starshaped function of the class k , then

$$(3) \quad |c_n| \leq \frac{1}{n!} \frac{2}{k} \left(\frac{2}{k} + 1 \right) \left(\frac{2}{k} + 2 \right) \dots \left(\frac{2}{k} + (n-1) \right) \\ (n = 1, 2, 3, \dots).$$

The right hand side of (3) cannot be replaced by any smaller number. This extremal case can be given by the function

$$(4) \quad \phi_0(z) = \frac{z}{(1-z^k)^{\frac{2}{k}}} = z e^{-\frac{2}{k} \log(1-z^k)},$$

taking a branch of $\log$ such that $\log 1 = 0$.

Proof. Consider the function $f(\zeta) = [\phi(z)]^k$, $(\zeta = z^k)$. Since the function $\zeta \frac{f'(\zeta)}{f(\zeta)}$ is regular for $|\zeta| < 1$, this can be expanded in a power series:

$$\zeta \frac{f'(\zeta)}{f(\zeta)} = 1 + k \zeta \frac{c_1 + 2c_2 \zeta + \dots + nc_n \zeta^{n-1} + \dots}{1 + c_1 \zeta + c_2 \zeta^2 + \dots + c_n \zeta^n + \dots} \\ = 1 + b_1 \zeta + b_2 \zeta^2 + \dots + b_n \zeta^n + \dots$$

Using CARATHÉODORY's theorem⁽¹⁾ and the lemma, we obtain

$$|b_n| \leq 2 \quad (n = 1, 2, 3, \dots),$$

whence follows that

$$\zeta \frac{f'(\zeta)}{f(\zeta)} \ll 1 + \frac{2\zeta}{1-\zeta},^{(2)}$$

that is,

$$\frac{c_1 + 2c_2\zeta + \dots + nc_n\zeta^{n-1} + \dots}{1 + c_1\zeta + c_2\zeta^2 + \dots + c_n\zeta^n + \dots} \ll \frac{2}{k} \frac{1}{1-\zeta},$$

consequently

$$\log(1 + c_1\zeta + c_2\zeta^2 + \dots + c_n\zeta^n + \dots) \ll -\frac{2}{k} \log(1-\zeta),$$

taking a branch of $\log$ such that $\log 1 = 0$. From

$$1 + c_1\zeta + c_2\zeta^2 + \dots + c_n\zeta^n + \dots \ll \frac{1}{(1-\zeta)^{\frac{2}{k}}},$$

follows the inequality (3). Our theorem is completely proved, considering the function

$$\phi_0(z) = \frac{z}{(1-z^k)^{\frac{2}{k}}} = z + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{2}{k} \left(\frac{2}{k} + 1\right) \dots \left(\frac{2}{k} + (n-1)\right) z^{nk+1}$$

which is a starshaped function of the class k , because

$$\Re\left(z \frac{\phi_0'(z)}{\phi_0(z)}\right) = \Re\left(\frac{1+z^k}{1-z^k}\right) > 0 \quad \text{for } |z| < 1.$$

Remark. $w = \phi_0(z)$ maps the unit circle on a whole w -plane cut from $w = \frac{1}{k\sqrt[k]{4}} e^{\frac{i(2\nu+1)\pi}{k}}$ to $w = \infty$ along each ray which starts from $w = 0$ and passes through $w = \frac{1}{k\sqrt[k]{4}} e^{\frac{i(2\nu+1)\pi}{k}}$, where $\nu = 0, 1, 2, \dots, k-1$.

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- (1) CARATHÉODORY's theorem states that if $\varphi(\zeta) = 1 + \sum_{n=1}^{\infty} b_n \zeta^n$ is regular and $\Re(\varphi(\zeta)) > 0$ for $|\zeta| < 1$, then $|b_n| \leq 2$.
- (2) Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$ be two power series and let all the coefficients b_n of $B(z)$ be non-negative. Then $A(z) \ll B(z)$ means that $|a_n| \leq b_n$ for every n .

Theorem 2. Suppose that $\varphi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$ is a convex function of the class k . Then

$$(5) \quad |c_n| \leq \frac{1}{nk+1} \frac{1}{n!} \frac{2}{k} \left(\frac{2}{k} + 1\right) \left(\frac{2}{k} + 2\right) \dots \left(\frac{2}{k} + (n-1)\right) \\ (n = 1, 2, 3, \dots).$$

The right hand side of (5) cannot be replaced by any smaller number. This extremal case can be given by the function

$$(6) \quad \varphi_0(z) = \int_0^z \frac{dz}{(1-z^k)^{\frac{2}{k}}}.$$

Proof. By ALEXANDER's theorem⁽¹⁾

$$\Phi(z) = z\varphi'(z) = z + \sum_{n=1}^{\infty} (nk+1)c_n z^{nk+1}$$

is a starshaped function of the class k . Hence, applying theorem 1, we have

$$(nk+1)|c_n| \leq \frac{1}{n!} \frac{2}{k} \left(\frac{2}{k} + 1\right) \left(\frac{2}{k} + 2\right) \dots \left(\frac{2}{k} + (n-1)\right).$$

§ III. SOME EXTENSIONS OF STROHHÄCKER'S THEOREMS.

Here some extensions of STROHHÄCKER's theorems will be mentioned.

Theorem 3. If $\Phi(z)$ is a starshaped function of the class k and if $|z_0| \leq r < 1$, then the point $\frac{\Phi(z_0)}{z_0}$ belongs to the closed domain D , which is the image of the circle $|\zeta| \leq r^k$ by $\frac{1}{(1-\zeta)^{\frac{2}{k}}}$.

Proof. Since, by the lemma, the function $f(\zeta) = [\Phi(z)]^k$, ($\zeta = z^k$) is a starshaped function of the class 1, the point $\frac{f(\zeta_0)}{\zeta_0} = \left[\frac{\Phi(z_0)}{z_0}\right]^k$, ($\zeta_0 = z_0^k$)

(1) The following theorem is due to Mr. J. ALEXANDER. Let $\varphi(z) = z + \dots$ be regular in the unit circle. Then $\varphi(z)$ maps $|z| < 1$ on a convex domain, if and only if $\Phi(z) = z\varphi'(z)$ maps $|z| < 1$ on a starshaped domain with centre at the origin. Cf. J. ALEXANDER: Ann. of Math., 2. ser., vol. 17, 1915-16, p. 12-22.

belongs, by STROHHÄCKER's theorem⁽¹⁾, to the domain D' , which is the image of the circle $|\zeta| \leq r^k$ by $\frac{1}{(1-\zeta)^2}$. Therefore the point $\frac{\phi(z_0)}{z_0}$ belongs to the domain D , which is the image of the circle $|\zeta| \leq r^k$ by $\frac{1}{(1-\zeta)^{\frac{2}{k}}}$.

Remark. D cannot be replaced by any smaller domain, for the function $\phi_0(z)$ gives the extremal case; more precisely, the set of values taken by $\frac{\phi_0(z)}{z}$ for $|z| \leq r$ is identical with the domain D .

As an immediate result of the above theorem, we have

Theorem 4. *If $\phi(z)$ is a starshaped function of the class k , then*

$$(7) \quad \frac{1}{(1+r^k)^{\frac{2}{k}}} \leq \left| \frac{\phi(z)}{z} \right| \leq \frac{1}{(1-r^k)^{\frac{2}{k}}} \quad (|z| \leq r < 1).$$

Proof. The circle $|\zeta| \leq r^k$ can be mapped by $s = \frac{1}{1-\zeta}$ on a circle K , which has the segment $\frac{1}{1+r^k} \dots \frac{1}{1-r^k}$ as diameter. Since the domain D (in theorem 3) is the image of K by $w = s^{\frac{2}{k}}$, it is clear that D lies on the ring: $\frac{1}{(1+r^k)^{\frac{2}{k}}} \leq |w| \leq \frac{1}{(1-r^k)^{\frac{2}{k}}}$. Therefore our assertion is true, by theorem 3.

Remark. The limits of (7) can be also attained by the function $\phi_0(z)$.

Theorem 5. *If $\phi(z)$ is a starshaped function of the class k , and if $|z_0| \leq r < 1$, then the point $z_0 \frac{\phi'(z_0)}{\phi(z_0)}$ lies on the circular domain which has the segment $\frac{1-r^k}{1+r^k} \dots \frac{1+r^k}{1-r^k}$ as diameter.*

(1) Mr. STROHHÄCKER has proved that if $f(z)$ is starshaped for $|z| < 1$ and if $|z_0| \leq r < 1$, then the point $\frac{f(z_0)}{z_0}$ lies in the closed domain, which is the image of the circle $|z| \leq r$ by $\frac{1}{(1-z)^2}$. Cf. E. STROHHÄCKER: loc. cit.

Proof. Since

$$g(z) \equiv z \frac{\Phi'(z)}{\Phi(z)} = 1 + kc_1 z^k + \dots$$

is regular and $\Re(g(z)) > 0$ for $|z| < 1$, the function

$$\psi(z) \equiv \frac{1}{z^k} \frac{1-g(z)}{1+g(z)}$$

is regular and $|\psi(z)| < 1$ for $|z| < 1$. Hence

$$\left| \frac{1-g(z)}{1+g(z)} \right| < |z|^k \quad (|z| < 1),$$

whence follows our assertion.

Theorem 6. *Let $\Phi(z)$ be a starshaped function of the class k . Then we have*

$$(8) \quad \frac{1-r^k}{1+r^k} \leq \left| z \frac{\Phi'(z)}{\Phi(z)} \right| \leq \frac{1+r^k}{1-r^k} \quad (|z| \leq r < 1)$$

$$(9) \quad \frac{1-r^k}{1+r^k} \leq \Re \left(z \frac{\Phi'(z)}{\Phi(z)} \right) \leq \frac{1+r^k}{1-r^k} \quad (|z| \leq r < 1).$$

Proof. This theorem comes directly from the above.

Remark. The limits of (8) and (9) can be also attained by the function $\Phi_0(z) = \frac{z}{(1-z^k)^{\frac{2}{k}}}$, for $z \frac{\Phi'_0(z)}{\Phi_0(z)} = \frac{1+z^k}{1-z^k}$.

Theorem 7. *If $\varphi(z)$ is a convex function of the class k , and if $|z_0| \leq r < 1$, then the point $\varphi'(z_0)$ belongs to D , where D denotes the same domain as in theorem 3.*

Proof. Since $\Phi(z) = z\varphi'(z)$ is a starshaped function of the class k , $\frac{\Phi(z_0)}{z_0} = \varphi'(z_0)$ lies, by theorem 3, on the domain D .

§ IV. AN EXTENSION OF SZEGÖ'S THEOREM.

Mr. G. SZEGÖ⁽¹⁾ has proved that if

$$f(z) = z + c_1 z^2 + c_2 z^3 + \dots + c_n z^{n+1} + \dots$$

is regular and starshaped for $|z| < 1$, then every section

$$f_n(z) = z + c_1 z^2 + c_2 z^3 + \dots + c_n z^{n+1}$$

is starshaped for $|z| < \frac{1}{4}$. Recently Mr. S. TAKAHASHI⁽²⁾ has shown that if $f(z)$ is an odd function, then SZEGÖ's theorem can be mentioned in the form: Every section $f_n(z)$ is starshaped for $|z| < \frac{1}{\sqrt{3}}$. Now we state a theorem which contains SZEGÖ-TAKAHASHI's theorem as its special case.

Theorem 8. *Let k be any positive integer and*

$$\Phi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$$

be a starshaped function of the class k . Then every section

$$z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1}$$

is starshaped for

$$|z| < \sqrt[k]{\frac{k}{2(k+1)}},$$

where the number $\sqrt[k]{\frac{k}{2(k+1)}}$ cannot be replaced by any greater one.

And the extremal case can be given by the function $\Phi_0(z)$ of (4).

Proof. For the proof we apply the elegant method which was used by Mr. G. SZEGÖ in the case $k = 1$. We put, for $n \geq 2$,

(1) G. SZEGÖ: loc. cit.

(2) S. TAKAHASHI: loc. cit.

$$s_n(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_{n-1} z^{(n-1)k+1},$$

$$\rho_n(z) = c_n z^{nk+1} + c_{n+1} z^{(n+1)k+1} + \dots; \text{ that is, } \phi(z) = s_n(z) + \rho_n(z),$$

and

$$R = \sqrt[k]{\frac{k}{2(k+1)}}.$$

In order that $s_n(z)$ should be starshaped for $|z| < R$, it is sufficient that for $0 < |z| < R$

$$(10) \quad s_n(z) \neq 0$$

and

$$(11) \quad \Re\left(z \frac{s'_n(z)}{s_n(z)}\right) = \Re\left(z \frac{\phi'(z) - \rho'_n(z)}{\phi(z) - \rho_n(z)}\right) \\ = \Re\left(z \frac{\phi'(z)}{\phi(z)}\right) + \Re\left(z \frac{\rho_n(z) \frac{\phi'(z)}{\phi(z)} - \rho'_n(z)}{\phi(z) - \rho_n(z)}\right) > 0.$$

Hence, for our object, it will suffice to show that for $|z| = R$

$$(12) \quad |\phi(z)| > |\rho_n(z)|$$

and

$$(13) \quad \frac{1 - |z|^k}{1 + |z|^k} \geq |z| \frac{|\rho_n(z)| \left| \frac{\phi'(z)}{\phi(z)} \right| + |\rho'_n(z)|}{|\phi(z)| - |\rho_n(z)|} \\ = \frac{\left| \frac{\rho_n(z)}{z} \right| \left| z \frac{\phi'(z)}{\phi(z)} \right| + |\rho'_n(z)|}{\left| \frac{\phi(z)}{z} \right| - \left| \frac{\rho_n(z)}{z} \right|}.$$

In fact $\frac{\rho_n(z)}{\phi(z)}$ is regular for $|z| < 1$, since $\phi(z) \neq 0$ for $0 < |z| < 1$. Hence, if we have (12) for $|z| = R$, then the same inequality holds

for $0 < |z| < R$, whence follows (10) for $0 < |z| < R$. Consequently $z \frac{s'_n(z)}{s_n(z)}$ is regular for $|z| \leq R$ and if $\Re\left(z \frac{s'_n(z)}{s_n(z)}\right) \geq 0$ for $|z| = R$, then $\Re\left(z \frac{s'_n(z)}{s_n(z)}\right) > 0$ for $|z| < R$. But the inequality $\Re\left(z \frac{s'_n(z)}{s_n(z)}\right) \geq 0$ for $|z| = R$ follows from (13) for $|z| = R$, by (9) and (12).

We put

$$(14) \quad \rho = R^k = \frac{k}{2(k+1)} \quad \text{and} \quad a = \frac{2}{k}.$$

Thus we have obtained the following inequalities:

$$(3) \quad |c_n| \leq \frac{a(a+1)(a+2)\dots(a+n-1)}{n!}$$

$$(7) \quad \left| \frac{\phi(z)}{z} \right| \geq \frac{1}{(1+\rho)^{\frac{2}{k}}} \quad (|z| = R),$$

$$(8) \quad \left| z \frac{\phi'(z)}{\phi(z)} \right| \leq \frac{1+\rho}{1-\rho} \quad (|z| = R).$$

On the other hand,

$$(15) \quad \left| \frac{\rho_n(z)}{z} \right| \leq \sum_{\nu=n}^{\infty} |c_\nu| \rho^\nu \\ \leq \sum_{\nu=n}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^\nu \quad (|z| = R).$$

Similarly,

$$(16) \quad |\rho'_n(z)| \leq \sum_{\nu=n}^{\infty} \frac{(\nu k + 1)a(a+1)\dots(a+\nu-1)}{\nu!} \rho^\nu \quad (|z| = R).$$

I. Let us consider the case where $n \geq 4$. First we prove the inequality (12) for $|z| = R$. By (7) and (15),

$$\begin{aligned}
(17) \quad \left| \frac{\phi(z)}{z} \right| - \left| \frac{\rho_n(z)}{z} \right| &\geq \frac{1}{(1+\rho)^a} - \sum_{\nu=n}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^\nu \\
&\geq \frac{1}{(1+\rho)^a} - \sum_{\nu=4}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^\nu \\
&= \left\{ \frac{1}{(1-\rho)^a} - \sum_{\nu=4}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^\nu \right\}^{(1)} \\
&\quad - \left\{ \frac{1}{(1-\rho)^a} - \frac{1}{(1+\rho)^a} \right\} \\
&= \frac{(3k+4)(k+2)}{3(k+1)^2} - \left\{ \left(\frac{2k+2}{k+2} \right)^{\frac{2}{k}} - \left(\frac{2k+2}{3k+2} \right)^{\frac{2}{k}} \right\} \equiv A(k).
\end{aligned}$$

Since $\frac{(3k+4)(k+2)}{3(k+1)^2}$ and $\left(\frac{2k+2}{k+2} \right)^{\frac{2}{k}} - \left(\frac{2k+2}{3k+2} \right)^{\frac{2}{k}}$ are decreasing functions of k in the interval $(1, \infty)$, it is clear that for $k \geq 2$

$$\frac{(3k+4)(k+2)}{3(k+1)^2} > \lim_{k \rightarrow \infty} \frac{(3k+4)(k+2)}{3(k+1)^2} = 1$$

and

$$\left(\frac{2k+2}{k+2} \right)^{\frac{2}{k}} - \left(\frac{2k+2}{3k+2} \right)^{\frac{2}{k}} < \frac{2 \cdot 2 + 2}{2 + 2} - \frac{2 \cdot 2 + 2}{3 \cdot 2 + 2} = \frac{3}{4}.$$

Hence

$$A(k) > 1 - \frac{3}{4} = \frac{1}{4} \quad \text{for } k \geq 2.$$

On the other hand,

$$A(1) = \frac{7 \cdot 3}{3 \cdot 2^2} - \left(\frac{16}{15} \right)^2 > 0.$$

$$\begin{aligned}
(1) \quad \frac{1}{(1-\rho)^a} &= 1 + \sum_{\nu=1}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^\nu \\
&= 1 + a\rho + \frac{1}{2!} a(a+1) \rho^2 + \frac{1}{3!} a(a+1)(a+2) \rho^3 + \sum_{\nu=4}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^\nu.
\end{aligned}$$

Here we prove the inequality (13) for $|z| = R$. For this purpose, it will suffice to show that, by (7), (8), (15) and (16),

$$(18) \quad \frac{1-\rho}{1+\rho} \geq \frac{\frac{1+\rho}{1-\rho} \sum_{\nu=n}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^{\nu} + \sum_{\nu=n}^{\infty} (\nu k+1) \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^{\nu}}{\frac{1}{(1+\rho)^{\frac{2}{k}}} - \sum_{\nu=n}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^{\nu}}$$

or that

$$(19) \quad \frac{1-\rho}{1+\rho} \geq \frac{\frac{1+\rho}{1-\rho} \sum_{\nu=4}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^{\nu} + \sum_{\nu=4}^{\infty} (\nu k+1) \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^{\nu}}{\frac{1}{(1+\rho)^{\frac{2}{k}}} - \sum_{\nu=4}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^{\nu}},$$

since the right hand side of (18) increases as n decreases for $4 \leq n < \infty$. In (17) we have shown that

$$\frac{1}{(1+\rho)^{\frac{2}{k}}} > \sum_{\nu=4}^{\infty} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^{\nu}.$$

Hence (19) is equivalent to

$$(20) \quad \frac{1-\rho}{1+\rho} \cdot \frac{1}{(1+\rho)^{\frac{2}{k}}} \geq \sum_{\nu=4}^{\infty} \left\{ \frac{1+\rho}{1-\rho} + \frac{1-\rho}{1+\rho} + \nu k + 1 \right\} \frac{a(a+1)\dots(a+\nu-1)}{\nu!} \rho^{\nu}.$$

At first we prove (20) for $k \geq 2$. Obviously $\rho = \frac{k}{2(k+1)}$ is an increasing function of k for $2 \leq k < \infty$ and the values taken by ρ belongs to the interval $\frac{1}{3} \leq \rho < \frac{1}{2}$. k and a can be written, as functions of ρ , such that $k = \frac{2\rho}{1-2\rho}$ and $a = \frac{1}{\rho} - 2$ for $\frac{1}{3} \leq \rho < \frac{1}{2}$. Hence, for the proof of (20), it suffices to show that for $\frac{1}{3} \leq \rho < \frac{1}{2}$

$$(21) \quad \frac{1-\rho}{1+\rho} \cdot \frac{1}{(1+\rho)^{\frac{1}{\rho}-2}} \geq \sum_{\nu=4}^{\infty} \left\{ \frac{1+\rho}{1-\rho} + \frac{1-\rho}{1+\rho} + \frac{2\nu\rho}{1-2\rho} + 1 \right\} \\ \times \frac{(1-2\rho)(1-\rho)(1+\rho)\dots(1+\rho(\nu-3))}{\nu!}$$

or

$$(22) \quad \frac{1}{(1+\rho)^{\frac{1}{\rho}-2}} \geq \sum_{\nu=4}^{\infty} \left\{ \frac{3+2(\nu-3)\rho+\rho^2-2(\nu+1)\rho^3}{1-\rho} \right\} \\ \times \frac{(1+\rho)\dots(1+\rho(\nu-3))}{\nu!}.$$

Put

$$Q(\rho) = \frac{3+2(\nu-3)\rho+\rho^2-2(\nu+1)\rho^3}{1-\rho} \quad \left(\begin{array}{c} \nu \geq 4 \\ \frac{1}{3} \leq \rho \leq \frac{1}{2} \end{array} \right).$$

Then

$$(1-\rho)^2 Q'(\rho) = (2\nu-3) + 2\rho - (6\nu+7)\rho^2 + 4(\nu+1)\rho^3 > 0$$

for $\frac{1}{3} \leq \rho \leq \frac{1}{2}$ ⁽¹⁾. Consequently the right hand side of (22) (say $\equiv B(\rho)$) is an increasing function of ρ for $\frac{1}{3} \leq \rho \leq \frac{1}{2}$. Hence, for $\frac{1}{3} \leq \rho < \frac{1}{2}$

$$(23) \quad B(\rho) < B\left(\frac{1}{2}\right) = \sum_{\nu=4}^{\infty} \frac{3\nu}{2} \cdot \frac{\left(1+\frac{1}{2}\right)\dots\left(1+\frac{\nu-3}{2}\right)}{\nu!} \\ = 3 \sum_{\nu=4}^{\infty} \frac{1}{2^{\nu-1}} = \frac{3}{4}.$$

On the other hand,

$$(24) \quad \frac{1}{(1+\rho)^{\frac{1}{\rho}-2}} > \frac{1}{\left(1+\frac{1}{3}\right)^{\frac{1}{3}-2}} = \frac{3}{4} \quad \text{for} \quad \frac{1}{3} \leq \rho < \frac{1}{2},$$

(1) Put $\lambda(\rho) = (1-\rho)^2 Q'(\rho)$. Then $\lambda'(\rho) = 2(1-\rho)(1-6(\nu+1)\rho) < 0$ for $\frac{1}{3} \leq \rho \leq \frac{1}{2}$.

Consequently $\lambda(\rho) \geq \lambda\left(\frac{1}{2}\right) = \nu-3-\frac{1}{4} \geq \frac{3}{4} > 0$ for $\frac{1}{3} \leq \rho \leq \frac{1}{2}$.

because $\frac{1}{(1+\rho)^{\frac{1}{p}-2}}$ is increasing in the interval $\frac{1}{3} \leq \rho < \frac{1}{2}$. From

(23) and (24) follows the required inequality (22). Thus, for $k \geq 2$ our proof of (20) is completed.

But there remains the case $k = 1$. If $k = 1$, then, by (14), $\rho = \frac{1}{4}$ and $\alpha = 2$; in this case,

$$\text{The left hand side of (20)} = \frac{1 - \frac{1}{4}}{1 + \frac{1}{4}} \frac{1}{\left(1 + \frac{1}{4}\right)^2} = \frac{48}{125} > \frac{1}{4}$$

$$\begin{aligned} \text{The right hand side of (20)} &= \sum_{\nu=4}^{\infty} \left\{ \frac{1 + \frac{1}{4}}{1 - \frac{1}{4}} + \frac{1 - \frac{1}{4}}{1 + \frac{1}{4}} + \nu + 1 \right\} (\nu + 1) \left(\frac{1}{4}\right)^{\nu} \\ &= \frac{49}{15} \sum_{\nu=4}^{\infty} (\nu + 1) \left(\frac{1}{4}\right)^{\nu} + \sum_{\nu=4}^{\infty} \nu(\nu + 1) \left(\frac{1}{4}\right)^{\nu} \quad (1) \\ &= \frac{49}{15} \cdot \frac{1}{36} + \frac{53}{27 \cdot 16} < \frac{1}{4}. \end{aligned}$$

Therefore the inequality (20) is also true for $k = 1$.

Thus it is completely proved that *every section*

$$s_n(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_{n-1} z^{(n-1)k+1}$$

is starshaped for

$$|z| < R = \sqrt[k]{\frac{k}{2(k+1)}},$$

as long as n is greater than or equal to 4.

II. Let us consider the section $s_n(z)$ when $n = 2$; that is, $s_2(z) = z + c_1 z^{k+1}$. If $c_1 = 0$, then $s_2(z) = z$. Next suppose that $c_1 \neq 0$.

(1) Here we use the equalities: $\sum_4^{\infty} (\nu + 1) \rho^{\nu} = \frac{1}{(1-\rho)^2} - (1 + 2\rho + 3\rho^2 + 4\rho^3)$,

$$\sum_4^{\infty} \nu(\nu + 1) \rho^{\nu} = \frac{2\rho}{(1-\rho)^3} - (2\rho + 6\rho^2 + 12\rho^3).$$

$$w = z \frac{s_2'(z)}{s_2(z)} = \frac{1 + (k+1)c_1 z^k}{1 + c_1 z^k} = \frac{1 + (k+1)\xi}{1 + \xi},$$

putting $\xi = c_1 z^k$.

The circle $|\xi| \leq \rho (< 1)$ can be mapped by $\frac{1 + (k+1)\xi}{1 + \xi}$ on a circle K having the segment $\frac{1 - (k+1)\rho}{1 - \rho} \dots \frac{1 + (k+1)\rho}{1 + \rho}$ as diameter, hence K lies on the half plane: $\Re(w) > 0$, provided that $\rho < \frac{1}{k+1}$. Therefore, $\Re\left(z \frac{s_2'(z)}{s_2(z)}\right) > 0$ for $|z| < \sqrt[k]{\frac{1}{|c_1|(k+1)}}$. Since $|c_1| \leq \frac{2}{k}$, by theorem 1, it is easy to see that $s_2(z)$ is starshaped for $|z| < R = \sqrt[k]{\frac{k}{2(k+1)}}$.

Next consider the section $s_2(z)$ of $\Phi_0(z) = \frac{z}{(1 - z^k)^{\frac{2}{k}}} = z + \frac{2}{k} z^{k+1} + \dots$. Then $s_2(z) = z + \frac{2}{k} z^{k+1}$, $s_2'(z) = 1 + \frac{2}{k}(k+1) z^k$. The section $s_2(z) = z + \frac{2}{k} z^{k+1}$ cannot be starshaped for $|z| < R'$, if $R' > R$, because $s_2'(z)$ has zero-points on the circumference $|z| = R = \sqrt[k]{\frac{k}{2(k+1)}}$. Thus it is proved that every section $s_2(z)$ is starshaped for $|z| < R = \sqrt[k]{\frac{k}{2(k+1)}}$ and R cannot be replaced by any greater number, provided that k is fixed, and the extremal case can be given by the section $s_2(z)$ of $\Phi_0(z)$.

III. Lastly we must prove that

$$s_3(z) = z + c_1 z^{k+1} + c_2 z^{2k+1}$$

is starshaped for $|z| < R = \sqrt[k]{\frac{k}{2(k+1)}}$. We will prove for $|z| \leq R$

$$(25) \quad \Re \frac{1 + (k+1)c_1 z^k + (2k+1)c_2 z^{2k}}{1 + c_1 z^k + c_2 z^{2k}} > 0.$$

The denominator does not vanish there, because

$$|c_1 z^k + c_2 z^{2k}| \leq \frac{2}{k} R^k + \frac{1}{2} \frac{2}{k} \left(\frac{2}{k} + 1\right) R^{2k} = \frac{5k+6}{4(k+1)^2} < 1.$$

Hence we can assume $|z| = R$. And, further, it suffices to show this inequality for $z = R$. (Consider $\bar{\varepsilon}\phi(\varepsilon z)$ with proper ε , $|\varepsilon| = 1$). Hence our assertion can be stated as follows:

$$(26) \quad \Re \frac{1 + (k+1)c_1 \frac{k}{2(k+1)} + (2k+1)c_2 \left(\frac{k}{2(k+1)}\right)^2}{1 + c_1 \frac{k}{2(k+1)} + c_2 \left(\frac{k}{2(k+1)}\right)^2} > 0.$$

if we put $\zeta = z^k$, the function

$$(27) \quad F(\zeta) = z \frac{\phi'(z)}{\phi(z)} = 1 + k\zeta \frac{c_1 + 2c_2\zeta + \dots}{1 + c_1\zeta + c_2\zeta^2 + \dots} \\ = 1 + 2C_1\zeta + 2C_2\zeta^2 + \dots$$

has for $|\zeta| < 1$ a positive real part, so that by CARATHÉODORY-TOEPLITZ's theorem we have

$$(28) \quad |C_1| \leq 1, \quad |C_2 - C_1^2| \leq 1 - |C_1|^2,$$

where

$$c_1 = \frac{2}{k}C_1, \quad c_2 = \frac{2}{k^2}C_1^2 + \frac{1}{k}C_2.$$

Hence the inequality (26) can be written as follows:

$$(29) \quad \Re \frac{1 + C_1 + \frac{(2k+1)(k+2)}{4(k+1)^2}C_1^2 + \frac{k(2k+1)}{4(k+1)^2} \cdot \eta \cdot (1 - |C_1|^2)}{1 + \frac{1}{k+1}C_1 + \frac{k+2}{4(k+1)^2}C_1^2 + \frac{k}{4(k+1)^2} \cdot \eta \cdot (1 - |C_1|^2)} > 0 \\ (|\eta| \leq 1).$$

When C_1 is fixed, the fraction of (29) can be considered as a regular function of η for $|\eta| \leq 1$, because the denominator never vanishes there.⁽¹⁾ Hence we can assume $|\eta| = 1$. Consequently, if we put, for the sake of simplicity,

$$(1) \quad \left| \frac{1}{k+1}C_1 + \frac{k+2}{4(k+1)^2}C_1^2 + \frac{k}{4(k+1)^2} \cdot \eta \cdot (1 - |C_1|^2) \right| \leq \frac{1}{k+1} + \frac{k+2}{4(k+1)^2} + \frac{k}{4(k+1)^2} \\ = \frac{3}{2(k+1)} \leq \frac{3}{4} < 1.$$

$$\begin{aligned}
(2k+1)w_1 &= 1 + C_1 + \frac{(2k+1)(k+2)}{4(k+1)^2} C_1^2, \\
(30) \quad w_2 &= 1 + \frac{C_1}{k+1} + \frac{k+2}{4(k+1)^2} C_1^2, \\
Z &= \frac{k}{4(k+1)^2} \eta(1 - |C_1|^2) \quad (|\eta| = 1),
\end{aligned}$$

we have only to show that

$$\Re \frac{w_1 + Z}{w_2 + Z} > 0$$

or

$$\Re (w_1 + Z)(\overline{w_2 + Z}) = \Re w_1 \overline{w_2} + \Re (w_1 + w_2) \overline{Z} + |Z|^2 > 0,$$

or

$$\Re w_1 \overline{w_2} + |Z|^2 > |Z| |w_1 + w_2|,$$

which is equivalent to

$$(31) \quad \left(\frac{|w_1 + w_2|}{2} - |Z| \right)^2 > \frac{|w_1 - w_2|^2}{4}.$$

Since

$$\begin{aligned}
|w_1 + w_2| &= \left| \frac{2k+2}{2k+1} + \frac{3k+2}{(2k+1)(k+1)} C_1 + \frac{k+2}{2(k+1)^2} C_1^2 \right| \\
&\geq \frac{2k+2}{2k+1} - \frac{3k+2}{(2k+1)(k+1)} - \frac{k+2}{2(k+1)^2} \\
&= \frac{2(k^2-1)+k}{2(k+1)^2} \geq \frac{k}{2(k+1)^2},
\end{aligned}$$

(31) is equivalent to

$$(32) \quad |w_1 + w_2| - |w_1 - w_2| > 2|Z|.$$

If we put $C_1 = \zeta$, then (32) can be written such that

$$\begin{aligned}
(33) \quad & \left| 2(k+1)^2 + (3k+2)\zeta + \frac{(k+2)(2k+1)}{2(k+1)} \zeta^2 \right| - k \left| 2(k+1) + \zeta \right| \\
& > \frac{k(2k+1)}{2(k+1)} (1 - |\zeta|^2) \quad (|\zeta| \leq 1).
\end{aligned}$$

We write

$$\zeta = -2(k+1) + \zeta_1 = -2(k+1) + r e^{i\varphi} ;$$

it is geometrically clear that $2k+1 \leq r \leq 2k+3$ and

$$(34) \quad -\varphi_0(r) \leq \varphi \leq \varphi_0(r) ,$$

if r is fixed, where $\varphi_0(r)$ can be determined from the equation $|-2(k+1) + r e^{i\varphi}| = 1$ such that $0 < \varphi_0(r) < \frac{\pi}{2}$. The point $-2(k+1) + r e^{i\varphi_0(r)}$ lies on the circumference $|\zeta| = 1$. We obtain

$$(35) \quad \cos \varphi_0(r) = \frac{r^2 + (4(k+1)^2 - 1)}{4(k+1)r} .$$

Now (33) can be written in the form:

$$(33)' \quad |a - \beta \zeta_1 + \gamma \zeta_1^2| > \delta + Q ,$$

where

$$\begin{aligned} Q &= kr - \frac{k(2k+1)}{2(k+1)} \left| \zeta_1 - 2(k+1) \right|^2 \\ &= -2k(k+1)(2k+1) + k(1 + 2(2k+1) \cos \varphi) r - \frac{k(2k+1)}{2(k+1)} r^2 \end{aligned}$$

and

$$\begin{aligned} \alpha &= 2(k+1)^2(2k+1) , & \beta &= 4k^2 + 7k + 2 , \\ \gamma &= \frac{(k+2)(2k+1)}{2(k+1)} , & \delta &= \frac{k(2k+1)}{2(k+1)} . \end{aligned}$$

Considering (34) and (35), we have

$$\begin{aligned} (36) \quad Q &\geq -2k(k+1)(2k+1) + k(1 + 2(2k+1) \cos \varphi_0(r)) r - \frac{k(2k+1)}{2(k+1)} r^2 \\ &= -\delta + kr . \end{aligned}$$

From (33)'

$$(37) \quad \left\{ (a + \gamma r^2) \cos \varphi - \beta r \right\}^2 + (a - \gamma r^2)^2 (1 - \cos^2 \varphi) - (\delta + Q)^2 > 0 .$$

Put in (37)

$$\cos \varphi = \frac{Q + 2k(k+1)(2k+1) - kr + \delta r^2}{2k(2k+1)r},$$

and consider the left hand side $f(r, Q)$ of (37) as a function of r and Q , where r varies in the interval $2k+1 \leq r \leq 2k+3$ and Q in a certain interval $Q_1(r) \leq Q \leq Q_2(r)$, $Q_1(r)$ being equal to $-\delta + kr$. We show that $f(r, Q)$ is an increasing function of Q for $Q_1(r) \leq Q$, when r is fixed in the interval $2k+1 \leq r \leq 2k+3$.

$$\begin{aligned} \frac{\partial f(r, Q)}{\partial Q} &= \frac{\alpha + \gamma r^2}{k(2k+1)r} \{ (\alpha + \gamma r^2) \cos \varphi - \beta r \} - \frac{(\alpha - \gamma r^2)^2}{k(2k+1)r} \cos \varphi - 2(\delta + Q) \\ &= \frac{4\alpha\gamma}{k(2k+1)} r \cos \varphi - \frac{\beta(\alpha + \gamma r^2)}{k(2k+1)} - 2(\delta + Q) \\ &= \frac{4\alpha\gamma}{k(2k+1)} \cdot \frac{Q + 2k(k+1)(2k+1) - kr + \delta r^2}{2k(2k+1)} - \frac{\beta(\alpha + \gamma r^2)}{k(2k+1)} \\ &\quad - 2(\delta + Q), \\ \frac{\partial^2 f(r, Q)}{\partial Q^2} &= \frac{2\alpha\gamma}{k^2(2k+1)^2} - 2 = \frac{6k+4}{k^2} > 0. \end{aligned}$$

Hence $\frac{\partial f(r, Q)}{\partial Q}$ is positive for $Q_1(r) \leq Q$, if it is positive for $Q = Q_1(r)$.

However, when $Q = Q_1(r)$, by an easy calculation, we get

$$(38) \quad \frac{\partial f(r, Q)}{\partial Q} = \frac{1}{k} (6k^3 + 14k^2 + 9k + 2) - 2kr - \frac{k+2}{2(k+1)} r^2,$$

which is positive for $2k+1 \leq r \leq 2k+3$, as it is true⁽¹⁾ for $r = 2k+3$.

Therefore it is shown that $f(r, Q)$ is minimum for $Q = Q_1(r)$, that is, $\cos \varphi = \cos \varphi_0(r)$ or $|\zeta| = 1$, $\zeta = e^{i\theta}$. Then our assertion can be enunciated in the form:

$$(33)'' \quad \left| 2(k+1)^2 e^{-i\theta} + (3k+2) + \frac{(k+2)(2k+1)}{2(k+1)} e^{i\theta} \right| > k \left| 2(k+1) + e^{i\theta} \right|$$

(1) $\frac{\partial f(r, Q)}{\partial Q} = \frac{(k+2)^2}{2k(k+1)} > 0$ when $Q = Q_1(r)$ and $r = 2k+3$.

or

$$|ae^{-i\theta} + \beta + \gamma e^{i\theta}|^2 - k^2 |\delta + e^{i\theta}|^2 > 0,$$

putting

$$\alpha = 2(k+1)^2, \beta = 3k+2, \gamma = \frac{(k+2)(2k+1)}{2(k+1)}, \delta = 2(k+1),$$

or

$$(33)''' \quad \{\beta^2 - k^2 + (\alpha - \gamma)^2 - k^2 \delta^2\} + 2\{\beta(\alpha + \gamma) - k^2 \delta\} \cos \theta \\ + 4\alpha\gamma \cos^2 \theta > 0.$$

For the proof of (33)', we have only to obtain⁽¹⁾

$$(39) \quad 4\alpha\gamma\{\beta^2 - k^2 + (\alpha - \gamma)^2 - k^2 \delta^2\} - \{\beta(\alpha + \gamma) - k^2 \delta\}^2 > 0,$$

since $\alpha\gamma > 0$. Denoting by D the left hand side of (39), we easily have

$$D = \frac{(k+2)(2k+1)^2(16k^5 + 80k^4 + 147k^3 + 126k^2 + 52k + 8)}{4(k+1)^2} > 0. \\ \text{(Q.E.D.)}$$

Thus our theorem is completely proved.

Applying ALEXANDER's theorem⁽²⁾ and theorem 8, we obtain at once

Theorem 9. *Let*

$$\varphi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$$

be a convex function of the class k . Then every section

$$z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1}$$

is convex⁽³⁾ for

$$|z| < \sqrt[k]{\frac{k}{2(k+1)}},$$

(1) Clearly $\alpha + 2bx + cx^2$ is always positive, provided that $c > 0$ and $ca - b^2 > 0$.

(2) J. ALEXANDER: loc. cit.

(3) If $f(z)$ is regular for $|z| < \rho$ and maps $|z| < \rho$ on a convex domain, then we say that $f(z)$ is convex for $|z| < \rho$.

where $\sqrt[k]{\frac{k}{2(k+1)}}$ cannot be replaced by any greater number. The extremal case can be given by the function $\varphi_0(z)$ of (6).

Remark. When $k = 1$ and $k = 2$, this theorem reduces to SZEGÖ-TAKAHASHI'S.⁽¹⁾

Recently Mr. A. KOBORI⁽²⁾ has given a complement to SZEGÖ's theorem: Let $f(z) = z + c_1 z^2 + c_2 z^3 + \dots + c_n z^{n+1} + \dots$ be regular and starshaped for $|z| < 1$. Then every function $g(z) = z + b_1 z^2 + b_2 z^3 + \dots + b_n z^{n+1} + \dots$, with $|b_n| \leq |c_n|$ ($n = 1, 2, 3, \dots$), is starshaped for $|z| < R = 0.1646 \dots$, where R is the root between 0 and 1 of the equation $2(1-r)^3 = 1+r$.

KOBORI's result can be extended in the following form:

Theorem 10. *Let*

$$\phi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$$

be a starshaped function of the class k . Then every function

$$G(z) = z + b_1 z^{k+1} + b_2 z^{2k+1} + \dots + b_n z^{nk+1} + \dots,$$

such that

$$|b_n| \leq |c_n| \quad (n = 1, 2, 3, \dots),$$

is starshaped for $|z| < R$, where R is the root between 0 and 1 of the equation

$$2^k(1-r^k)^{k+2} = (1+r^k)^k.$$

Proof. It is well known that if $\psi(z) = z + \sum_1^\infty a_n z^{n+1}$ is regular for $|z| < 1$ and if $\sum_1^\infty (n+1) |a_n| r^n < 1$, then $\psi(z)$ is starshaped for $|z| < r < 1$. We have, applying theorem 1,

(1) G. SZEGÖ: loc. cit., S. TAKAHASHI: loc. cit.

(2) A. KOBORI: loc. cit. And see also L. BIERBERBACH: loc. cit.

$$\begin{aligned}
\sum_1^{\infty} (nk+1) |b_n| r^{nk} &\leq \sum_1^{\infty} (nk+1) |c_n| r^{nk} \\
&\leq \sum_1^{\infty} (nk+1) \frac{1}{n!} \frac{2}{k} \left(\frac{2}{k}+1\right) \dots \left(\frac{2}{k}+(n-1)\right) r^{nk} \\
&= \Phi'_0(r) - 1 = \frac{1+r^k}{(1-r^k)^{\frac{2}{k}+1}} - 1.
\end{aligned}$$

Hence, if we denote by R the root between 0 and 1 of the equation $\Phi'_0(r) = 2$ that is: $2^k(1-r^k)^{k+2} = (1+r^k)^k$, then every function $G(z)$ is starshaped for $|z| < R$.

Remark. The limit R can be attained by the function

$$G_0(z) = 2z - \phi_0(z) = 2z - \frac{z}{(1-z^k)^{\frac{2}{k}}}$$

for $G'_0(z) = 2 - \phi'_0(z)$ has zero-points on the circumference $|z| = R$.

As an immediate result of the above,

Theorem 11.⁽¹⁾ *Let*

$$\varphi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots,$$

be a convex function of the class k . Then every function

$$g(z) = z + b_1 z^{k+1} + b_2 z^{2k+1} + \dots + b_n z^{nk+1} + \dots,$$

such that

$$|b_n| \leq |c_n| \quad (n = 1, 2, 3, \dots),$$

is convex for $|z| < R$, where R denotes the same quantity in theorem 10.

Remark. It is evident that the limit R can be attained by the function

$$g_0(z) = \int_0^z \frac{G_0(z)}{z} dz = \int_0^z \left(2 - \frac{\phi_0(z)}{z}\right) dz = 2z - \phi_0(z).$$

(1) In the case $k = 1$, this theorem reduces also to KOBORI's. See, loc. cit.

§ V. ON THE UNIVALENCY OF A FUNCTION WITH A NON-VANISHING DERIVATIVE.

Supposing that $f(z)$ is a function which is regular in a certain domain D and whose derivative $f'(z)$ never vanishes there, we cannot necessarily assert $f(z)$ to be schlicht in D . For example, $f(z) = e^z$ has a non-vanishing derivative but is not schlicht in a circle of radius greater than π . Here we shall give some results on the univalence (*Schlichtheit*) of a function $f(z)$ with a non-vanishing derivative.

Theorem 12.⁽¹⁾ *Let $f(z)$ be regular in a convex domain D . Then $f(z)$ is schlicht, if the value-set of $f'(z)$ in D lies in a half-plane Ω not containing the origin in its interior.*

Proof. Let z_1 and z_2 ($z_1 \neq z_2$) be two arbitrary points in D . Since D is a convex domain, the segment $\overline{z_1 z_2}$, joining z_1 and z_2 , lies in D . Denote by M the set of values taken by $f'(z)$ on the segment $\overline{z_1 z_2}$, then it is clear that M is a bounded closed set lying in Ω . Hence we can describe a circle C which contains M in its interior and lies in Ω . If we denote by a the centre of C , then we have

$$|f'(z) - a| < |a|,$$

for every z on the segment $\overline{z_1 z_2}$.

Now

$$\begin{aligned} f(z_2) - f(z_1) &= \int_{z_1}^{z_2} f'(z) dz = \int_{z_1}^{z_2} a dz - \int_{z_1}^{z_2} (a - f') dz \\ &= a(z_2 - z_1) - \int_{z_1}^{z_2} (a - f') dz. \end{aligned}$$

(1) When I read this paper at the annual meeting of the Physico-Math. Soc. of Japan, held in April 1934, Prof. KAKEYA kindly remarked to me that this theorem can be easily proved by a geometrical consideration.

Taking the segment $z_1 z_2$ as the path of integration,

$$\left| \int_{z_1}^{z_2} (a-f') dz \right| \leq \int_0^{|z_1-z_2|} |a-f'| ds < |a| \int_0^{|z_1-z_2|} ds = |a| |z_1-z_2|.$$

Hence

$$\begin{aligned} \left| f(z_2) - f(z_1) \right| &\geq |a| |z_1 - z_2| - \left| \int_{z_1}^{z_2} (a-f') dz \right| \\ &> |a| |z_1 - z_2| - |a| |z_1 - z_2| = 0. \quad (\text{Q.E.D.}) \end{aligned}$$

Remark. For this proof we owe much to Prof. K. KUNUGUI.

As an immediate result

Theorem 13. *If $f(z) = z + \dots$ is regular and*

$$R(f'(z)) > 0 \quad \text{or} \quad |f'(z) - 1| < 1$$

for $|z| < R$, then $f(z)$ is schlicht for $|z| < R$.

Next an application of theorem 13 will be enunciated:

Theorem 14.⁽¹⁾ *If $f(z) = c_0 z + \frac{c_1}{2} z^2 + \dots$ (c_0, c_1 given, $c_0 \neq 0$) is regular for $|z| < 1$ and if $|f'(z)| < 1$ for $|z| < 1$, then $f(z)$ is schlicht for*

$$(40) \quad |z| < R = \frac{1}{2} \left[-|a_0| (1 - |c_0|) + \sqrt{|a_0|^2 (1 - |c_0|)^2 + 4|c_0|} \right],$$

where $a_0 = \frac{c_1}{1 - |c_0|^2}$. This limit can be attained by the function

$$(41) \quad f_0(z) = \int_0^z \frac{c_0 + (c_0 \bar{a}_0 e^{i\theta} + a_0)z + e^{i\theta} z^2}{1 + (\bar{a}_0 e^{i\theta} + \bar{c}_0 a_0)z + \bar{c}_0 e^{i\theta} z^2} dz,$$

where $\theta \equiv 2 \arg c_1 - \arg c_0 + \pi \pmod{2\pi}$.

(1) This is an extension of a theorem which I have already proved. See K. NOSHIRO: Journ. Fac. Sci. Hokkaido Imperial Univ., ser. I, vol. 1, 1932, p. 157-161, esp. p. 160. And also K. NOSHIRO: loc. cit., vol. 2, 1934, p. 89-101, esp. p. 98; Here I considered the case when $f(z) = c_0 z + c_1 z^2 + \dots$ (c_0, c_1 given, $c_0 \neq 0$) is regular and $|f'(z)| < 1$ for $|z| < 1$.

Proof. It is clear that

$$(42) \quad g(z) \equiv \frac{f'(z) - c_0}{1 - \bar{c}_0 f'(z)} = \frac{c_1}{1 - |c_0|^2} z + \dots$$

is regular and $|g(z)| < 1$ for $|z| < 1$. Hence

$$(43) \quad |f'(z) - c_0| = \left| \frac{(1 - |c_0|^2) g(z)}{1 + \bar{c}_0 g(z)} \right| \leq \frac{(1 - |c_0|^2) |g(z)|}{1 - |c_0| |g(z)|},$$

Hence, by theorem 13, $f(z)$ is schlicht for $|z| < R$, provided that

$$(44) \quad |g(z)| < |c_0| \quad \text{for} \quad |z| < R.$$

Using a known inequality $|g(z)| \leq r \frac{|a_0| + r}{1 + |a_0| r}$ for $|z| \leq r$, it is seen that the inequality (44) holds good, because R is the root between 0 and 1 of the equation $r \frac{|a_0| + r}{1 + |a_0| r} = |c_0|$. Thus our theorem is proved, considering the function $f_0(z)$ whose derivative vanishes at a point $z_0 = R e^{i\lambda_0}$, where $\lambda_0 \equiv \arg c_0 - \arg c_1 + \pi \pmod{2\pi}$.

§ VI. A THEOREM ON A SCHLICHT MEROMORPHIC FUNCTION IN THE UNIT CIRCLE.

We will here enunciate a theorem analogous to FEJÉR's⁽¹⁾ on a schlicht bounded function.

Theorem 15. *Suppose that*

$$f(z) = \frac{1}{z} + c_2 z + c_3 z^2 + \dots + c_n z^{n-1} + \dots$$

is meromorphic and schlicht for $|z| < 1$. Then

$$|1 + c_2 + c_3 + \dots + c_n| \leq 2 + \sqrt{\frac{241}{432}} = 2,7469\dots \quad (n = 2, 3, \dots).$$

1) Mr. FEJÉR has proved that if $f(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n + \dots$ is regular-schlicht and bounded in the unit circle: $|f(z)| < 1$, then

$$|c_0 + c_1 + \dots + c_n| \leq 1 + \frac{1}{\sqrt{2}} \quad (n = 0, 1, 2, \dots).$$

Cf. Acta Mathematica, Bd. 49, 1926, p. 183-190; Acta, Szeged, Bd. 4, 1928, p. 14-24.

Proof. We use two known results :

$$(45) \quad \sum_2^{\infty} (k-1) |c_k|^2 \leq 1 \quad (\text{BIEBERBACH})$$

$$|f(z)| \leq |z| + \frac{1}{|z|} \quad (\text{LÖWNER})$$

Since

$$F(z) = zf(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n + \dots \quad (c_0 = 1, c_1 = 0)$$

is regular and bounded in the unit circle : $|F(z)| \leq |z|^2 + 1 < 2$, we have by LANDAU's theorem⁽¹⁾

$$(46) \quad \left| \frac{(n+1)c_0 + nc_1 + \dots + 2c_{n-1} + c_n}{n+1} \right| \leq 2.$$

On the other hand, using BIEBERBACH's result,

$$\begin{aligned} (47) \quad |c_1 + 2c_2 + \dots + nc_n| &\leq 2|c_2| + \dots + n|c_n| \\ &= \frac{2}{\sqrt{1}} \cdot \sqrt{1} |c_2| + \dots + \frac{n}{\sqrt{n-1}} \cdot \sqrt{n-1} |c_n| \\ &\leq \sqrt{\sum_2^n (k-1) |c_k|^2} \sqrt{\sum_2^n \frac{k^2}{k-1}} \leq \sqrt{\sum_2^n \frac{k^2}{k-1}}. \end{aligned}$$

By (46) and (47),

$$\begin{aligned} |c_0 + c_1 + c_2 + \dots + c_n| &= \left| \frac{(n+1)c_0 + nc_1 + \dots + c_n}{n+1} + \frac{c_1 + 2c_2 + \dots + nc_n}{n+1} \right| \\ &\leq \left| \frac{(n+1)c_0 + nc_1 + \dots + c_n}{n+1} \right| \\ &\quad + \frac{1}{n+1} |c_1 + 2c_2 + \dots + nc_n| \\ &\leq 2 + \sqrt{\frac{1}{(n+1)^2} \sum_1^{n-1} \frac{(k+1)^2}{k}}. \end{aligned}$$

(1) E. LANDAU: Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, 2. Aufl., p. 22.

Here remark that

$$(48) \quad \phi(n) = \frac{1}{(n+1)^2} \sum_{k=1}^{n-1} \frac{(k+1)^2}{k} = \frac{(n-1)(n+4)}{2(n+1)^2} + \frac{1}{(n+1)^2} \sum_{k=1}^{n-1} \frac{1}{k}$$

decreases as n increases for $n \geq 5$. To prove that $\phi(n+1) < \phi(n)$ for $n \geq 5$, it suffices to show

$$(49) \quad \sum_{k=1}^{n-1} \frac{1}{k} > \frac{2+20n+11n^2-n^3}{2n(2n+3)} \quad \text{for } n \geq 5.$$

But this inequality (49) holds good, because, for $n \geq 5$,

$$\sum_{k=1}^{n-1} \frac{1}{k} \geq \frac{25}{12} \quad \text{and} \quad \frac{2+20n+11n^2-n^3}{2n(2n+3)} < 2.$$

Consequently it is seen that

$$\max_{n \geq 2} \phi(n) = \max_{5 \geq n \geq 2} \phi(n),$$

whence it follows that

$$\max_{n \geq 2} \phi(n) = \frac{241}{432}.$$

Thus we obtain

$$|1 + c_2 + c_3 + \dots + c_n| \leq 2 + \sqrt{\frac{241}{432}} = 2,7469\dots$$

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After I completed this paper, Mr. K. JOH kindly wrote to me to say that theorems 1 and 2 had been obtained by Mr. G. GOLUSIN in *Recueil Math. Moscou* **36**, p. 152-172.