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BASE CONNECTIONS IN A SPECIAL KAWAGUCHI SPACE

By

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Introduction. In a FINSLER space the arc-length of a curve, $x^i = x^i(t)$, is given by the integral

$$\int F(x^1, x^2, \ldots, x^n ; x'^1, x'^2, \ldots, x'^n)dt .$$

The function F is positively homogeneous of degree one with respect to $x'^1, x'^2, \ldots, x'^n$.

Prof. A. KAWAGUCHI[1] first investigated a space involving metric tensors whose components are functions of not only x^i and x'^i but of higher derivatives $x'', \ldots, x^{(r)}$ as well. Accordingly we give the name a special KAWAGUCHI space to the manifold associated with the integral $\int F(x, x', x'', \ldots, x^{(m)})\,dt$. He has developed the base connections of order r and of dimension n in his space $K_n^{(r)}$:

$$\overset{s+1}{p}{}^{\nu} = \frac{\overset{(s)(s)}{\delta}\, p^{\nu}}{dt} = \frac{d\overset{s}{p}{}^{\nu}}{dt} + \overset{(s)}{\Gamma}{}^{\nu}_{\lambda\mu}\, \overset{s}{p}{}^{\lambda}\, \overset{1}{p}{}^{\mu} - s\, \overset{(s)}{\Gamma}{}^{\nu}_{\mu}\, \overset{s}{p}{}^{\nu}\, \overset{1}{p}{}^{\mu}\ (2),$$

$$s = 1, 2, \ldots, \quad r-1 .$$

H. V. CRAIG[3] and J. L. SYNGE[4] have found various intrinsic vectors and covariant differentials in a special KAWAGUCHI space, where the

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(1) A. KAWAGUCHI, [I] Die Differentialgeometrie in der verallgemeinerten Mannigfaltigkeit, Rendiconti del Circolo Mat. di Palermo, vol. LVI (1932), pp. 245–276.

(2) See [I], p. 255.

(3) H. V. CRAIG, [II] On a generalized tangent vector, American Journal of Math., vol. LVII (1935), pp. 456-462; [III] On the solution of the EULER equation for their highest derivatives, Bulletin of the Amer. Math. Soc., vol. XXXVI (1930), pp. 558–562.

(4) J. L. SYNGE, [IV] Some intrinsic and derived vectors in a KAWAGUCHI space, American Journal of Math., vol. LVII (1935), pp. 677–691.

arc-length

$$s = \int F(x, x', x'', \ldots, x^{(m)})dt$$

of a curve, $x^i = x^i(t)$, is invariant under a transformation of parameter t.

Recently Prof. A. KAWAGUCHI has defined the various connections in a special KAWAGUCHI space whose metric is given by the integral

$$s = \int \left\{A_i(x, x')x''^i + B(x, x')\right\}^{\frac{1}{p}} dt$$

by applying "SYNGE vectors" of the function $A_i x''^i + B^{(1)}$.

The principal object of this paper is to determine the base connections for a special KAWAGUCHI space whose metric is given by the integral

$$s = \int \left\{A_i(x, x')x'''^i + B(x, x', x'')\right\}^{\frac{1}{p}} dt$$

by a similar method to that of Prof. A. KAWAGUCHI; a problem of equivalency between two connections will be solved and finally some generalization will be attempted.

The author wishes to express his best thanks to Prof. A. KAWAGUCHI and Mr. H. HOMBU for their kind advices.

1. Notations and preliminary formulae. The symbolism to be employed in this paper is set down in the following table:

$$x'^i = \frac{dx^i}{dt}, \quad x^{(p)i} = \frac{d^p x^i}{dt^p}, \quad F^{(\alpha)} = \frac{d^\alpha}{dt^\alpha} F,$$

$$F_{(1)i} = \frac{\partial F}{\partial x'^i}, \quad F_{(\alpha)i} = \frac{\partial F}{\partial x^{(\alpha)i}},$$

$$\dot{x}^i = \frac{dx^i}{d\bar{t}}, \quad \alpha = \frac{d\bar{t}}{dt}, \quad \alpha' = \frac{d\alpha}{dt}, \quad \alpha'' = \frac{d^2\alpha}{dt^2};$$

$$(1) \qquad s = \int \left\{F(x, x', x'', x''')\right\}^{\frac{1}{p}} dt,$$

$$(2) \qquad F = F(x, x', x'', x''') = A_i(x, x')x'''^i + B(x, x', x'').$$

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(1) A. KAWAGUCHI, [V] Die Geometrie des Integrals $\int (A_i x''^i + B)^{\frac{1}{p}} dt$, Proceedings of the Imperial Academy, Tokyo, vol. XII (1936), pp. 205–208.

On the functions, F, A_i, B, ... to be introduced bellow, notation will be modified by substituting (i) in place of $(\alpha)i$ for the differentiation with respect to the highest derivatives $x^{(\alpha)i}$ present.

The transformation of variables, $x^i = x^i(x^\lambda)$, $t = t(\bar{t})$, gives rise to the following equations:

$$
(3)\quad \begin{cases} x^i = x^i(x^\lambda)\,, \quad t = t(\bar{t})\,, \\ x'^i = P^i_\lambda \dot{x}^\lambda \alpha\,, \quad P^i_\lambda = \dfrac{\partial x^i}{\partial x^\lambda}\,, \quad Q^\lambda_i = \dfrac{\partial x^\lambda}{\partial x^i}\,, \quad P^i_{\lambda\mu} = \dfrac{\partial^2 x^i}{\partial x^\lambda \partial x^\mu}\,, \text{ etc.}, \\ x''^i = P^i_\lambda \ddot{x}^\lambda \alpha^2 + P^i_{\lambda\mu} \dot{x}^\lambda \dot{x}^\mu \alpha^2 + P^i_\lambda \dot{x}^\lambda \alpha'\,, \\ x'''^i = P^i_\lambda \dddot{x}^\lambda \alpha^3 + 3P^i_{\lambda\mu} \ddot{x}^\lambda \dot{x}^\mu \alpha^3 + 3P^i_\lambda \ddot{x}^\lambda \alpha\alpha' + P^i_{\lambda\mu\nu} \dot{x}^\lambda \dot{x}^\mu \dot{x}^\nu \alpha^3 \\ \qquad + 3P^i_{\lambda\mu} \dot{x}^\lambda \dot{x}^\mu \alpha\alpha' + P^i_\lambda \dot{x}^\lambda \alpha''. \end{cases}
$$

The transformation, $x^i = x^i(x^\lambda)$, gives rise to following equalities:

$$
(4)\quad \frac{\partial x^{(\alpha)i}}{\partial x^{(\beta)\lambda}} = \binom{\alpha}{\beta} \frac{\partial x^{(\alpha-\beta)i}}{\partial x^\lambda}\,.
$$

The EULER vector of the function $F(x, x', \ldots, x^{(m)})$ is

$$
(5)\quad E_i = \sum_{\alpha=0}^{m} (-1)^{\alpha+1} \frac{d^\alpha}{dt^\alpha} F_{(\alpha)i}\,,
$$

and the vectors introduced by SYNGE[1], referred to $F(x, x', \ldots, x^{(m)})$, are given by the following expressions:

$$
(6)\quad \overset{\alpha}{E}_i = \sum_{\beta=\alpha}^{m} (-1)^{\beta+1} \binom{\beta}{\alpha} \frac{d^{(\beta-\alpha)}}{dt^{(\beta-\alpha)}} F_{(\beta)i}\,, \quad \alpha = 0,\ 1,\ \ldots,\ m\,.
$$

Finally a KAWAGUCHI derivation[2] along a curve for a vector X^i referred to a vector $T_i(x, x', \ldots, x^{(p)})$ or SYNGE vectors $\overset{\alpha}{E}_i$ respectively are given by the following formulae:

$$
(7)\quad \overset{p-l}{D}_{ij}(T) X^j = \sum_{\alpha=l}^{p} \binom{\alpha}{l} T_{i(\alpha)j} X^{j(\alpha-l)}\,, \qquad l = 1,\ 2,\ \ldots,\ p\,;
$$

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(1) See [IV].

(2) A. KAWAGUCHI, [VI] Some intrinsic derivations in a generalized space, Proceedings of the Imperial Academy, Tokyo, vol. XII, pp. 149–151; [VII] Certain identities in a generalized space, ibid., pp. 152–155.

$$(8)\qquad \overset{2m-\alpha-l}{D_{ij}}(\overset{\alpha}{E})X^j = \sum_{\beta=l}^{2m-\alpha}\binom{\beta}{l}\overset{\alpha}{E}_{i(\beta)j}X^{j(\beta-l)}$$

and, in particular, for a tangent vector x'^i the latter reduces to:

$$(9)\qquad \overset{2m-p-\rho}{D_{ij}}(\overset{p}{E})x'^j = \delta^1_\rho \overset{p}{E}_i - \sum_{l=1}^{m-p+1}\delta^l_\rho\binom{\rho+l-1}{l}\overset{p+l-1}{E}_i .$$

2. Determination of base connections. It is proposed to deal with a special KAWAGUCHI space of order two and dimension n, in which space the arc-length may be taken as

$$(1\qquad s = \int F^{\frac{1}{p}}dt ,$$

where

$$(2)\qquad F = A_i(x, x')x'''^i + B(x, x', x'') .$$

CRAIG has showed that the invariance of the integral (1) for a change of the parameter t implies the following identities in x^i, x'^i, x''^i and x'''^i:

$$\sum_{\beta=\rho}^{3}\binom{\beta}{\rho}x^{(\beta-\rho+1)i}F_{(\beta)i} = p\delta^1_\rho F , \qquad \rho = 1,\ 2,\ 3,$$

δ^i_j being KRONECKER's delta. It may be interesting to write out these conditions explicitly:

$$A_i x'^i = 0 , \quad B_{(i)}x'^i + 3A_i x''^i = 0 ,$$

$$(A_{j(i)}x'''^j + B_{(1)i})x'^i + 2B_{(i)}x''^i + 3A_i x'''^i = p(A_i x'''^i + B) .$$

These are equivalent to:

$$(10)\qquad \begin{cases} A_i x'^i = 0 , \quad B_{(i)}x'^i + 3A_i x''^i = 0 , \\ A_{j(i)}x'^i = (p-3)A_j , \quad B_{(1)i}x'^i + 2B_{(i)}x''^i = pB . \end{cases}$$

According to equation (2), the transformation, $x^i = x^i(x^\lambda)$, gives rise to the following:

$$A_i x'''^i + B = A_i(P^i_\lambda x'''^\lambda + 3P^i_{\lambda\mu}x''^\lambda x'^\mu + P^i_{\lambda\mu\nu}x'^\lambda x'^\mu x'^\nu) + B$$

$$= A_\lambda x'''^\lambda + \bar{B} .$$

Hence we obtain:

$$A_i P^i_\lambda = A_\lambda ,$$

$$B + A_i(3P^i_{\lambda\mu} x''^\lambda x'^\mu + P^i_{\lambda\mu\nu} x'^\lambda x'^\mu x'^\nu) = \bar{B} ,$$

so that:

$$A_i P^i_\lambda = A_\lambda , \quad B_{(i)(j)} P^i_\lambda P^j_\mu = \bar{B}_{(\lambda)(\mu)} .$$

Hence one obtains:

The expressions A_i or $A_{i(j)}$ and $B_{(i)(j)}$ are respectively the components of a covariant vector or the components of covariant tensors.

According to equations (6), one obtains the SYNGE vectors

$$(11)\quad \begin{cases} \overset{2}{E}_i = 3A_{i(j)} x''^j + 3A_{i(0)j} x'^j - B_{(i)} , \\ \overset{1}{E}_i = (3A_{i(j)} + A_{j(i)} - 2B_{(i)(j)}) x'''^j + B_{(1)i} - 2B_{(i)(1)j} x''^j \\ \qquad - 2B_{(i)(0)j} x'^j + 3(A_{i(j)(k)} x''^j x''^k + 2A_{i(j)(0)k} x''^j x'^k \\ \qquad + A_{i(0)j(0)k} x'^j x'^k + A_{i(0)j} x''^j) , \\ \overset{0}{E}_i = (A_{i(j)} + A_{j(i)} - B_{(i)(j)}) x^{\mathrm{IV}\, j} - B_{(i)(j)(k)} x'''^j x'''^k \\ \qquad + (A_{j(i)(k)} + 3A_{i(j)(k)} - 2B_{(i)(j)(1)k}) x'''^j x''^k \\ \qquad + (2A_{j(i)(0)k} + A_{i(j)(0)k} - 2B_{(i)(j)(0)k}) x'''^j x'^k \\ \qquad + (A_{i(0)j} - A_{j(0)i} - B_{(i)(1)j} + B_{(1)i(j)}) x'''^j \\ \qquad + A_{i(j)(k)(l)} x''^j x''^k x''^l + 3A_{i(j)(k)(0)l} x''^j x''^k x'^l \\ \qquad + (3A_{i(j)(0)k} - B_{(i)(1)j(1)k}) x''^j x''^k + 3A_{i(j)(0)k(0)l} x''^j x'^k x'^l \\ \qquad + (3A_{i(0)j(0)k} - 2B_{(i)(1)j(0)k}) x''^j x'^k + (B_{(1)i(1)j} - B_{(i)(0)j}) x''^j \\ \qquad + (A_{i(0)j(0)k(0)l} x'^k x'^l - B_{(i)(0)j(0)k} x'^k + B_{(1)i(0)j}) x'^j - B_{(0)i} . \end{cases}$$

By a short calculation we verify:

$$\overset{2}{E}_i x'^i = 0 , \qquad \overset{1}{E}_i x'^i = pF .$$

Under the transformation of the parameter, $t = t(\bar{t})$, these vectors are not invariant but are transformed as follows:

$$(12)\qquad \begin{cases} \overset{2}{E}_i = {}'\overset{2}{E}_i a^{p-2} + 3(p-2)'A_i a^{p-4}a' , \\ \overset{1}{E}_i = {}'\overset{1}{E}_i a^{p-1} + (2p-3)'\overset{2}{E}_i a^{p-3}a' + 3(p-1)(p-4)'A_i a^{p-5}a'^2 \\ \qquad\qquad + (3p-4)'A_i a^{p-4}a'' . \end{cases}$$

As $A_{i(j)}$ or $B_{(i)(j)}$ are the components of a tensor, the expressions:

$$(13)\qquad G_{ij} = 3A_{i(j)} + A_{j(i)} - 2B_{(i)(j)}$$

are also the components of a covariant tensor. Therefore, if the determinants $|A_{i(j)}|$ and $|G_{ij}|$ are different from zero, the two following covariant vectors are obtained:

$$(14)\qquad \overset{2}{x}^i = x'''^i + \Lambda^i(x, x', x'') , \quad \overset{3}{x}^i = x''''^i + \Gamma^i(x, x', x'') ,$$

after rewriting (11) into the following forms:

$$(11')\qquad \begin{cases} \dfrac{1}{3}\overset{2}{E}_i = A_{i(j)}x'''^j + \Lambda_i(x, x', x'') , \\ \overset{1}{E}_i = G_{ij}x'''^j + \Gamma_i(x, x', x'') , \end{cases}$$

where

$$G_{ij}G^{il} = \delta^l_j , \quad A_{i(j)}A^{il} = \delta^l_j , \quad \Lambda^i = A^{ji}\Lambda_j , \quad \Gamma^i = G^{ji}\Gamma_j .$$

After a short calculation we get:

$$(15)\qquad \begin{cases} A_{i(j)}x'^i = -A_j , \quad A_{i(j)}x'^j = (p-3)A_i ; \\ G_{ij}x'^i = pA_j , \quad G_{ij}x'^j = (3p-4)A_i . \end{cases}$$

Accordingly one must exclude the case for

$$p = 0 , \quad 3 , \quad 4/3 .$$

By applying a KAWAGUCHI covariant derivation to a vector v^i referred to the vectors $\overset{2}{E}_i$, $\overset{1}{E}_i$ and $\overset{0}{E}_i$, we have:

$$\begin{cases} \overset{2\text{-}1}{D}_{ij}(\overset{2}{E})v^j = 2(3A_{i(j)} - B_{(i)(j)})\dfrac{dv^j}{dt} \\ \qquad + (3A_{i(j)(k)}x'''^k - B_{(i)(1)j} + 3A_{i(0)j} + 3A_{i(j)(0)k}x'^k)\, v^j , \end{cases}$$

$$
(16)\quad \begin{cases}
\overset{3-2}{D}_{ij}(\overset{1}{E})v^j = 3(3A_{i(j)}-2B_{(i)(j)}+A_{j(i)})\dfrac{dv^j}{dt} \\
\qquad +\{B_{(1)i(j)}-2B_{(i)(1)j}-2B_{(i)(1)k(j)}x''^k-2B_{(i)(0)k(j)}x'^k \\
\qquad -2B_{(i)(j)(k)}x'''^k+6A_{i(j)(k)}x''^k+3(2A_{i(j)(0)k}x'^k+A_{i(0)j})\}v^j, \\
\overset{4-3}{D}_{ij}(\overset{0}{E})v^j = 4(A_{i(j)}-B_{(i)(j)}+A_{j(i)})\dfrac{dv^j}{dt}+\{-2B_{(i)(j)(k)}x'''^k \\
\qquad +(A_{j(i)(k)}+3A_{i(j)(k)}-2B_{(i)(j)(1)k})x''^k+(2A_{j(i)(0)k}-2B_{(i)(j)(0)k} \\
\qquad +A_{i(j)(0)k})x'^k+(A_{i(0)j}-A_{j(0)i}-B_{(i)(1)j}+B_{(1)i(j)})\}v^j .
\end{cases}
$$

In these equations the coefficients of $\dfrac{dv^j}{dt}$ besides numerical factors are respectively:

$$
\xi_{ij} = 3A_{i(j)}-B_{(i)(j)}\,,\quad \zeta_{ij} = A_{i(j)}-B_{(i)(j)}+A_{j(i)}\,,
$$

$$
G_{ij} = 3A_{i(j)}+A_{j(i)}-2B_{(i)(j)}\,.
$$

From the above equations we have identically:

$$
\xi_{ij}x'^i = 0\,,\quad \zeta_{ij}x'^i = \zeta_{ji}x'^i = (p-1)A_j\,,
$$

hence without the cases for $p = 0,\ 1,\ 4/3$ and 3, if the determinant $|\zeta_{ij}|$ is not vanished identically we can solve the equations $(16)_2$ and $(16)_3$ with respect to $\dfrac{dv^j}{dt}$ in the forms:

$$
(17)\quad \begin{cases}
D(\overset{1}{E})v^i = \dfrac{dv^i}{dt}+\overset{1}{\Gamma}{}^i_j(x,\,x',\,x'')v^j\,, \\
D(\overset{0}{E})v^i = \dfrac{dv^i}{dt}+\overset{0}{\Gamma}{}^i_{jk}(x,\,x',\,x'')x'''^k v^j+\overset{0}{\Gamma}{}^i_j(x,\,x',\,x'')v^j\,.
\end{cases}
$$

Contrarily we cannot solve the first equation of (16), because the determinant $|\xi_{ij}|$ is identically vanished.

But unfortunately the vectors (17) are not invariant under the transformation of the parameter t. Now we propose the problem of how to define a covariant derivation of a vector invariant under the transformation of the parameter t. This problem is solved by using the elimination method.

We take any vector v^i in the special KAWAGUCHI space whose components are transformed by the transformation of variables, $x^i = x^i(x^\lambda)$, $t = t(\bar{t})$, as follows:

$$v^i = P^i_\lambda v^\lambda \alpha^{\mathfrak{k}},$$

where $\mathfrak{k}$ is an integer.

Now a covariant derivation of a vector in case for $p = 3/2$, can be defined, i.e., we have a

Theorem. *The expressions*

$$\delta v^i = dv^i + \Pi^i_{jk} v^j dx^k + \mathfrak{k}\,\Pi^i_j v^j dt, \tag{18}$$

or

$$\delta v^i = dv^i + \Pi^i_{jk} v^j dx^k + \mathfrak{k}\,\Pi v^i dt, \tag{18'}$$

are the components of a contravariant vector, when the determinant $|H_{ij}|$ *is not vanished, where*

$$\begin{cases} 3\Pi^i_{jk} = \Gamma^i_{(j)(1)k} - H^{rq}\Gamma^i_{(j)(r)}\Gamma^h_{(h)(q)(1)k}, \\ \Pi^i_j = \dfrac{2}{3}\Gamma^i_{(j)} - H^{ki}\Gamma^h_{(h)(k)(1)j}, \quad \Pi = \dfrac{1}{n}\left(\dfrac{1}{3}\Gamma^h_{(h)} - \Pi^i_{ik}x'^k\right)^{(1)}, \\ H_{ij} = \Gamma^h_{(h)(i)(j)}, \quad H_{ij}H^{ik} = \delta^k_j. \end{cases} \tag{19}$$

Proof. Under the transformation of a group (3), in the case for $p = 3/2$, the parameters Γ^i are transformed as follows:

$$\begin{cases} \Gamma'^i = P^i_\lambda \Gamma^\lambda \alpha^3 - 3P^i_{\lambda\mu}\ddot{x}^\lambda \dot{x}^\mu \alpha^3 - 3P^i_\lambda \ddot{x}^\lambda \alpha\alpha' - P^i_{\lambda\mu\nu}\dot{x}^\lambda \dot{x}^\mu \dot{x}^\nu \alpha^3 \\ \quad -3P^i_{\lambda\mu}\dot{x}^\lambda \dot{x}^\mu \alpha\alpha' - P^i_\lambda \dot{x}^\lambda \alpha'' + 3P^i_\lambda \dfrac{3(p-1)(p-4)}{3p-4}\left(\dot{x}^\lambda \dfrac{\alpha'^2}{\alpha} - \dot{x}^\lambda \alpha''\right). \end{cases} \tag{20}$$

When the equation (20) is differentiated with respect to $\ddot{x}^\lambda$, and then with respect to $\dot{x}^\mu$ or $\ddot{x}^\mu$, one gets respectively:

$$\Gamma^i_{(j)} P^j_\lambda \alpha = P^i_\nu \Gamma^\nu_{(\lambda)} \alpha^2 - 3P^i_{\lambda\nu}\dot{x}^\nu \alpha^2 - 3P^i_\lambda \alpha' \tag{21}$$

and

$$\Gamma^i_{(j)(1)k} P^j_\lambda P^k_\mu + \frac{1}{\alpha}\Gamma^i_{(j)(k)} P^j_\lambda (2P^k_{\mu\nu}\dot{x}^\nu \alpha^2 + P^k_\mu \alpha') = P^i_\nu \Gamma^\nu_{(\lambda)(1)\mu} - 3P^i_{\lambda\mu} \tag{22}$$

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1) This was introduced by Mr. H. HOMBU.

or

$$\Gamma^{i}_{(j)(k)}P^{j}_{\lambda}P^{k}_{\mu}=\frac{1}{a}P^{i}_{\nu}\Gamma^{\nu}_{(\lambda)(\mu)}\,. \tag{23}$$

Hence $\Gamma^{i}_{(j)(k)}$ are the components of a mixed tensor of order two for covariance and of order one for contravariance. Again differentiating both sides of equation (23) with respect to $\dot{x}^{\nu}$, we have the equalities:

$$\Gamma^{i}_{(j)(k)(1)l}P^{l}_{\nu}a^{2}+a\Gamma^{i}_{(j)(k)(l)}(2P^{l}_{\nu\sigma}\dot{x}^{\sigma}a^{2}+P^{l}_{\nu}a')=\Gamma^{\sigma}_{(\lambda)(\mu)(1)\nu}P^{i}_{\sigma}Q^{\lambda}_{j}Q^{\mu}_{k}\,.$$

By the hypothesis that the determinant $|H_{jk}|$ is different from zero the above equations can be solved with respect to $2P^{l}_{\nu\sigma}\dot{x}^{\sigma}a^{2}+2P^{l}_{\nu}a'$; i.e.,

$$2P^{l}_{\nu\sigma}\dot{x}^{\sigma}a^{2}+P^{l}_{\nu}a'=\frac{1}{a}H^{kl}(\Gamma^{\sigma}_{(\sigma)(\mu)(1)\nu}Q^{\mu}_{k}-\Gamma^{h}_{(h)(k)(1)q}P^{q}_{\nu}a^{2})\,. \tag{24}$$

When both sides of equation $v^{i}=P^{i}_{\lambda}v^{\lambda}a^{1}$ are differentiated, we have:

$$dv^{i}=P^{i}_{\lambda}dv^{\lambda}a^{1}+P^{i}_{\lambda\mu}v^{\lambda}dx^{\mu}a^{1}+1P^{i}_{\lambda}v^{\lambda}a^{1-1}a'dt\,. \tag{25}$$

Eliminate the quantities $P^{i}_{\lambda\mu}$ and a' in equations (21), (22), (23), (24) and (25) and replace the expressions $\frac{1}{3}(\Gamma^{i}_{(j)(1)k}-H^{rq}\Gamma^{i}_{(j)(r)}\Gamma^{h}_{(h)(q)(1)k})$ and $\frac{2}{3}\Gamma^{i}_{(j)}-H^{ki}\Gamma^{h}_{(h)(k)(1)j}$ by the notations Π^{i}_{jk} and Π^{i}_{j}; we obtain:

$$\begin{cases}\delta v^{i}=dv^{i}+\Pi^{i}_{jk}v^{j}dx^{k}+1\Pi^{i}_{j}v^{j}dt\\ \quad=P^{i}_{\lambda}a^{1}(dv^{\lambda}+\Pi^{\lambda}_{\mu\nu}v^{\mu}dx^{\nu}+1\Pi^{\lambda}_{\mu}v^{\mu}dt)\,.\end{cases} \tag{26}$$

Analogously the other part of the theorem can be proven.

Now we can modify the above theorem for $p\neq 3/2$. *That is, in this case if it be assumed that determinants* $|\pi_{ij}|$ *and* $|\Omega^{i}_{.j}|$ *are not vanished identically, it is sufficient that we put:*

$$\Pi^{i}_{kl}=\frac{1}{2}\Omega^{.i}_{j}(A^{j}_{(1)k(1)l}-A^{j}_{(1)k(h)}\pi_{(m)(1)l}\pi^{mh}-A^{j}_{(h)(1)l}\pi^{mh}\pi_{(m)(1)k})\,, \tag{19'}_1$$

$$\Pi=-\frac{3p-4}{9n(2p-3)(p-2)}(A^{i}_{(1)i}-\Omega^{i}_{.k}\pi^{lk}A_{(l)(1)i})\,, \tag{19'}_2$$

where

$$A^{j}=\Gamma^{j}_{(k)}x'^{k}\,,\quad \pi=A^{j}_{(j)}\,,\quad \pi_{lm}=\pi_{(l)(m)}\,,\quad \pi^{lm}\pi_{lp}=\delta^{m}_{p}\,,$$

$$\Omega^{i}_{.j}=A^{i}_{(j)}+3\delta^{i}_{j}\,,\qquad \Omega^{.j}_{k}\Omega^{k}_{.l}=\delta^{j}_{l}\,.$$

Proof. In general, under the transformation of a group (3) the parameters Γ^i are tranformed as follows:

$$(20') \quad \Gamma^i = P^i_\lambda \Gamma^\lambda \alpha^3 - 3P^i_{\lambda\mu} \ddot{x}^\lambda \dot{x}^\mu \alpha^3 - 3P^i_\lambda \ddot{x}^\lambda \alpha\alpha' - P^i_{\lambda\mu\nu} \dot{x}^\lambda \dot{x}^\mu \dot{x}^\nu \alpha^3 - 3P^i_{\lambda\mu} \dot{x}^\lambda \dot{x}^\mu \alpha\alpha' - P^i_\lambda \dot{x}^\lambda \alpha'' + P^i_\lambda \left\{ (2p-3) G^{\mu\lambda} \overset{2}{E}_\mu \alpha\alpha' + 3 \frac{(p-1)(p-4)}{3p-4} \dot{x}^\lambda \frac{\alpha'^2}{\alpha} - \dot{x}^\lambda \alpha'' \right\} .$$

When equation (20′) is differentiated with respect to $\ddot{x}^\lambda$ and $\dot{x}^\lambda$ is multiplied, we obtain:

$$(A) \quad \Gamma^j_{(k)} x'^k \alpha = P^j_\lambda \Gamma^\lambda_{(\sigma)} \dot{x}^\sigma \alpha^3 - 3P^i_{\sigma\mu} \dot{x}^\sigma \dot{x}^\mu \alpha^3 + P^j_\lambda \frac{6(3p^2-12p+11)}{3p-4} \dot{x}^\lambda \alpha\alpha' ,$$

i.e.,

$$(A') \quad \Lambda^j = P^j_\lambda \Lambda^\lambda \alpha^2 + \frac{6(3p^2-12p+11)}{3p-4} P^j_\lambda \dot{x}^\lambda \alpha' - 3P^j_{\sigma\mu} \dot{x}^\sigma \dot{x}^\mu \alpha^2 .$$

Differentiating equation (A′) with respect to $\ddot{x}^\nu$, $\ddot{x}^\rho$ and $\dot{x}^\nu$ successively, we get:

$$(B) \quad \Lambda^j_{(l)(m)(1)k} P^l_\mu P^m_\rho P^k_\nu \alpha^3 + \Lambda^j_{(l)(m)(k)} P^l_\mu P^m_\rho (2P^k_{\nu\sigma} \dot{x}^\sigma \alpha^2 + P^k_\nu \alpha') \alpha^2 = P^j_\lambda \Lambda^\lambda_{(\mu)(\rho)(1)\nu} .$$

When equation (B) is multiplied by Q^ρ_i and contracted over j and i, equation (B) can be immediately solved with respect to $2P^k_{\nu\sigma} \dot{x}^\sigma \alpha^2 + P^k_\nu \alpha'$:

$$(C) \quad 2P^k_{\nu\sigma} \dot{x}^\sigma \alpha^2 + P^k_\nu \alpha' = \pi^{lk} \left(Q^\mu_l \pi_{(\mu)(1)\nu} \frac{1}{\alpha^2} - \pi_{(l)(1)m} P^m_\nu \alpha \right) .$$

If equation (A′) be differentiated with respect to $\dot{x}^\lambda$ and $\dot{x}^\mu$ successively and relations (C) be applied we obtain:

$$\begin{aligned} (D) \quad & \Lambda^j_{(1)k(1)l} P^k_\lambda P^l_\mu \alpha^2 + \Lambda^j_{(1)k(l)} \pi^{ml} \left(Q^\nu_m \pi_{(\nu)(1)\mu} \frac{1}{\alpha^2} - \pi_{(m)(1)h} P^h_\mu \alpha \right) P^k_\lambda \alpha \\ & + \Lambda^j_{(k)(1)l} P^l_\mu \alpha \pi^{mk} \left(Q^\sigma_m \pi_{(\sigma)(1)\lambda} \frac{1}{\alpha^2} - \pi_{(m)(1)h} P^h_\lambda \alpha \right) \\ & + \Lambda^j_{(k)(l)} \pi^{mk} \left(Q^\sigma_m \pi_{(\sigma)(1)\lambda} \frac{1}{\alpha^2} - \pi_{(m)(1)h} P^h_\lambda \alpha \right) \pi^{ql} \\ & \qquad \times \left(Q^\nu_q \pi_{(\nu)(1)\mu} \frac{1}{\alpha^2} - \pi_{(q)(1)h} P^h_\mu \alpha \right) \\ & = -2(\Lambda^j_{(k)} + 3\delta^j_k) P^k_{\lambda\mu} \alpha^2 + P^j_\nu \Lambda^\nu_{(1)\lambda(1)\mu} \alpha^2 . \end{aligned}$$

After a short calculation, remembering that π_{ij} and $\Omega^k_{\cdot j}$ are the components of tensors of weight four and zero respectively, we get:

$$\text{(E)} \qquad P^i_{\lambda\mu} = -\Pi^i_{jk} P^j_\lambda P^k_\mu + \Pi^\sigma_{\lambda\mu} P^i_\sigma ,$$

where Π^i_{jk} are given by $(19')_1$.

On the other hand, differentiating equation (A') with respect to $\dot{x}^\lambda$ and applying relations (C) we have:

$$\frac{9(2p-3)(p-2)}{3p-4} n a' = -A^j_{(1)j} a - A^\nu_{(1)\nu} a^2 - \Omega^j_{\cdot k} \pi^{lk} A_{(l)(1)j} a + \Omega^\lambda_{\cdot\nu} \pi^{\mu\nu} A_{(\mu)(1)\lambda} a^2 ,$$

i.e.,

$$\text{(F)} \qquad a' = -\Pi a + {}'\Pi a^2 ,$$

where Π is given by $(19')_2$. Therefore a covariant differential is obtained of the same form as equation (18').

It is time to introduce the base connections in the special KAWAGUCHI space with which we are dealing. Two classes of vectors are defined, a vector belonging to one of the classes defined along the base curve and finite, and a vector belonging to the other class not necessarily defined along the base curve and infinitesimal, i.e.:

$$\text{(27)} \qquad \begin{cases} \overset{(2)}{x}{}^i = x''^i + \Pi^i_{jk} x'^j x'^k + \Pi x'^i , \\ \overset{(3)}{x}{}^i = \overset{(2)}{x}{}'^i + \Pi^i_{jk} \overset{(2)}{x}{}^j x'^k + 2\Pi \overset{(2)}{x}{}^i , \end{cases}$$

and

$$\text{(28)} \qquad \begin{cases} \overset{(1)}{\delta x}{}^i = dx'^i + \Pi^i_{jk} x'^j dx^k + \Pi x'^i dt , \\ \overset{(2)}{\delta x}{}^i = \overset{(2)}{dx}{}^i + \Pi^i_{jk} \overset{(2)}{x}{}^j dx^k + 2\Pi \overset{(2)}{x}{}^i dt . \end{cases}$$

Thus the components of the contravariant vectors $\overset{(1)}{\delta x}{}^i$ and $\overset{(2)}{\delta x}{}^i$ are invariant besides powers of a under the transformation of the parameter t, and along the base curve they reduce to (27). Hence one gets:

Equations (28) define the base connections in the KAWAGUCHI space of order two and dimension n in which a arc-length of a curve, $x^i = x^i(t)$, is given by the integral (1).

3. Curvature tensors. From equation (26) we have:

$$\begin{aligned} \delta v^i &= dv^i + \Pi^i_{jk} v^j dx^k + \mathfrak{f}\, \Pi v^i dt \\ &= \frac{\partial v^i}{\partial x^l} dx^l + \frac{\partial v^i}{\partial x'^l} dx'^l + \frac{\partial v^i}{\partial x''^l} dx''^l + \frac{\partial v^i}{\partial t} dt + \Pi^i_{jk} v^i dx^k + \mathfrak{f}\, \Pi v^i dt . \end{aligned}$$

In the above equations according to equations (28), we substitute for dx'^l and dx''^l the expressions $\overset{(1)}{\delta x^l}$ and $\overset{(2)}{\delta x^l}$; then we get the following form:

$$\delta v^i = \left\{\frac{\partial v^i}{\partial t}+\mathfrak{k}\,\Pi v^i-\frac{\partial v^i}{\partial x'^h}\Pi x'^h+\frac{\partial v^i}{\partial x''^l}\Omega_q^{\cdot l}\gamma_h^q\,\Pi x'^h-2\frac{\partial v^i}{\partial x''^l}\Omega_h^{\cdot l}\,\Pi\overset{(2)}{x^h}\right\}dt$$
$$+\left\{\frac{\partial v^i}{\partial x^h}-\frac{\partial v^i}{\partial x'^l}\Pi^l_{jh}x'^j-\frac{\partial v^i}{\partial x''^l}(\Omega_k^{\cdot l}\beta_h^k-\Omega_k^{\cdot l}\gamma_m^k\,\Pi^m_{jh}x'^j\right.$$
$$\left.+\Omega_k^{\cdot l}\,\Pi^k_{jh}\overset{(2)}{x^j})+\Pi^i_{jh}v^j\right\}dx^h+\left(\frac{\partial v^i}{\partial x'^h}-\frac{\partial v^i}{\partial x''^l}\Omega_k^{\cdot l}\gamma_h^k\right)\overset{(1)}{\delta x^h}$$
$$+\frac{\partial v^i}{\partial x''^l}\Omega_h^{\cdot l}\overset{(2)}{\delta x^h}\,,$$

where

$$\gamma_h^i=\frac{\overset{(2)}{\partial x^i}}{\partial x'^h}=\Pi^i_{jk(1)h}x'^jx'^k+\Pi^i_{hk}x'^k+\Pi^i_{jh}x'^j+\Pi_{(1)h}x'^i+\Pi\delta^i_h\,,$$

$$\beta_h^i=\frac{\overset{(2)}{\partial x^i}}{\partial x^h}=\Pi^i_{jk(0)h}x'^jx'^k+\Pi^i_{jh}\overset{(2)}{x^j}+\Pi_{(0)h}x'^i\,,$$

$$\Omega^i_{\cdot h}=\frac{\overset{(2)}{\partial x^i}}{\partial x''^h}=\delta^i_h+\Pi^i_{jk(h)}x'^jx'^k+\Pi_{(h)}x'^i\,,$$

and we assume that the determinant $|\Omega^i_{\cdot h}|$ is different from zero and put

$$\Omega_i^{\cdot h}\Omega^i_{\cdot k}=\delta^h_k\,,\quad \Omega_i^{\cdot h}\Omega^k_{\cdot h}=\delta^k_i\,.$$

We put:

(29) $$\nabla_h v^i=\frac{\partial v^i}{\partial x^h}-\frac{\partial v^i}{\partial x''^l}\Pi^l_{jh}x'^j-(\Omega_k^{\cdot l}\Pi^k_{jh}\overset{(2)}{x^j}-\Omega_k^{\cdot l}\gamma_m^k\Pi^m_{jh}x'^j$$
$$+\Omega_k^{\cdot l}\beta_h^k)\frac{\partial v^i}{\partial x''^l}+\Pi^i_{jh}v^j\,,$$

(30) $$\overset{1}{\nabla}_h v^i=\frac{\partial v^i}{\partial x'^h}-\frac{\partial v^i}{\partial x''^l}\Omega_j^{\cdot l}\gamma_h^j\,,$$

(31) $$\overset{2}{\nabla}_h v^i=\frac{\partial v^i}{\partial x''^l}\Omega_h^{\cdot l}\,,$$

and

$$(32)\qquad \overset{0}{\nabla} v^i = \frac{\partial v^i}{\partial t} - \frac{\partial v^i}{\partial x'^h}\Pi x'^h + \frac{\partial v^i}{\partial x''^l}\Omega_q^{\cdot l}\gamma_h^q \Pi x'^h - 2\frac{\partial v^i}{\partial x''^l}\Omega_h^{\cdot l}\Pi \overset{(2)}{x}{}^h + \mathfrak{k}\,\Pi v^i .$$

But we can prove that $\overset{0}{\nabla} v^i = 0$. In fact, v^i are the components of a contravariant vector of weight $\mathfrak{k}$, therefore we can conclude that v^i do not involve the parameter t explicitly. So that we have: $\dfrac{\partial v^i}{\partial t} = 0$. On the other hand the vectors v^i and $\overset{(2)}{x}{}^i$ are invariant besides powers of α, under the change of parameter t, so that we have:

$$\frac{\partial v^i}{\partial x'^j}x'^j + 2\frac{\partial v^i}{\partial x''^j}x''^j = \mathfrak{k}v^i, \quad \frac{\partial v^i}{\partial x''^j}x'^j = 0,$$

$$\frac{\partial \overset{(2)}{x}{}^i}{\partial x'^j}x'^j + 2\frac{\partial \overset{(2)}{x}{}^i}{\partial x''^j}x''^j = 2\overset{(2)}{x}{}^i, \quad \text{i. e.,}\quad \gamma_j^i x'^j + 2\Omega_{\cdot j}^{i} x''^j = 2\overset{(2)}{x}{}^i .$$

I.e. substituting the above relations in the equation (32) we have immediately:

$$(32')\qquad \overset{0}{\nabla} v^i = 0 .$$

Now let the calculation of the curvature tensors be considered. When equation (29) is written in the following form:

$$(29')\qquad \nabla_h v^i = \frac{\partial v^i}{\partial x^h} - \frac{\partial v^i}{\partial x'^l}A_h^l - \frac{\partial v^i}{\partial x''^l}B_h^l + \Pi_{jh}^i v^j ,$$

where

$$A_h^l = \Pi_{jh}^l x'^j, \quad B_h^l = \Omega_k^{\cdot l}\beta_h^k - \Omega_k^{\cdot l}\gamma_m^k \Pi_{jh}^m x'^j + \Omega_k^{\cdot l}\Pi_{jh}^k \overset{(2)}{x}{}^j$$
$$= \Omega_k^{\cdot l}\beta_h^k - \Omega_k^{\cdot l}\gamma_m^k A_h^m + \Omega_k^{\cdot l}\Pi_{jh}^k \overset{(2)}{x}{}^j ,$$

one obtains

$$(\nabla_k\nabla_h - \nabla_h\nabla_k)v^i = \frac{\partial v^i}{\partial x''^l}\left(-\frac{\partial B_{[h}^l}{\partial x^{k]}} + A_{[k}^m\frac{\partial B_{h]}^l}{\partial x'^m} + B_{[k}^m\frac{\partial B_{h]}^l}{\partial x''^m} + \Pi_{[hk]}^r B_r^l\right)$$
$$+ \frac{\partial v^i}{\partial x'^l}\left(-\frac{\partial A_{[h}^l}{\partial x^{k]}} + A_{[k}^m\frac{\partial A_{h]}^l}{\partial x'^m} + B_{[k}^m\frac{\partial A_{h]}^l}{\partial x''^m} + \Pi_{[hk]}^r A_r^l\right)$$
$$+ v^j\left(\frac{\partial \Pi_{j[h}^i}{\partial x^{k]}} - A_{[k}^m\frac{\partial \Pi_{|j|h]}^i}{\partial x'^m} - B_{[k}^m\frac{\partial \Pi_{|j|h]}^i}{\partial x''^m} + \Pi_{l[k}^i\Pi_{|j|h]}^l - \Pi_{jl}^i\Pi_{[hk]}^l\right),$$

so that it follows

$$(\nabla_k\nabla_h-\nabla_h\nabla_k)v^i = R_{jhk}^{\cdots\ i}v^j+S_{hk}^{\cdot\cdot\ l}\overset{1}{\nabla}_l v^i+T_{hk}^{\cdot\cdot\ l}\overset{2}{\nabla}_l v^i,$$

where

$$\frac{1}{2}R_{jhk}^{\cdots\ i} = \frac{\partial \Pi^i_{j[h}}{\partial x^{k]}}-A^m_{[k}\frac{\partial \Pi^i_{|j|h]}}{\partial x'^m}-B^m_{[k}\frac{\partial \Pi^i_{|j|h]}}{\partial x''^m}+\Pi^i_{l[k}\Pi^l_{|j|h]}-\Pi^i_{jl}\Pi^l_{[hk]},$$

$$\frac{1}{2}S_{hk}^{\cdot\cdot\ l} = -\frac{\partial A^l_{[h}}{\partial x^{k]}}+A^m_{[k}\frac{\partial A^l_{h]}}{\partial x'^m}+B^m_{[k}\frac{\partial A^l_{h]}}{\partial x''^m}+\Pi^r_{[hk]}A^l_r$$

and

$$\frac{1}{2}T_{hk}^{\cdot\cdot\ j} = \Omega^j_{\cdot l}\Big(-\frac{\partial B^l_{[h}}{\partial x^{k]}}+A^m_{[k}\frac{\partial B^l_{h]}}{\partial x'^m}+B^m_{[k}\frac{\partial B^l_{h]}}{\partial x''^m}+\Pi^r_{[hk]}B^l_r\Big)$$

$$+\frac{1}{2}\gamma^j_l S_{hk}^{\cdot\cdot\ l}.$$

Similarly one gets, when he puts $C^l_h = \Omega^{\cdot l}_{j}\gamma^j_h$,

$$\overset{1}{\nabla}_k\overset{1}{\nabla}_h v^i-\overset{1}{\nabla}_h\overset{1}{\nabla}_k v^i = K_{jkh}^{\cdots\ i}v^j+Q_{hk}^{\cdot\cdot\ l}\overset{1}{\nabla}_l v^i+W_{hk}^{\cdot\cdot\ l}\overset{2}{\nabla}_l v^i,$$

where

$$K_{jkh}^{\cdots\ i} = C^l_h\frac{\partial \Pi^i_{jk}}{\partial x''^l}-\frac{\partial \Pi^i_{jk}}{\partial x'^h},$$

$$Q_{hk}^{\cdot\cdot\ l} = -\Pi^l_{hk}+\frac{\partial A^l_k}{\partial x'^h}-C^m_h\frac{\partial A^l_k}{\partial x''^m}$$

and

$$W_{hk}^{\cdot\cdot\ l} = \Omega^{\cdot l}_{\cdot q}\Big\{-\frac{\partial C^q_h}{\partial x^k}+A^m_k\frac{\partial C^q_h}{\partial x'^m}+B^m_k\frac{\partial C^q_h}{\partial x''^m}+\frac{\partial B^q_k}{\partial x'^h}$$

$$-C^m_h\frac{\partial B^q_k}{\partial x''^m}+C^q_r\frac{\partial A^r_k}{\partial x'^h}-C^q_r C^m_h\frac{\partial A^r_k}{\partial x''^m}\Big\}.$$

Analogously one can get all the other curvature- and torsion-tensors. Among these tensors there exist many identities, for example, the tensor $R_{jhk}^{\cdots\ i}$ is skew-symmetric with respect to two exterior indices and also the tensors $S_{hk}^{\cdot\cdot\ l}$ and $T_{hk}^{\cdot\cdot\ l}$ are skew-symmetric with respect to two lower indices.

4. Problem of equivalency[1]. Now let the problem of equivalency of two systems be considered:

(33) Π^i_{jk}, Π (34) $\Pi^\lambda_{\mu\nu}$, $'\Pi$,

under the change of variables

$$x^i = x^i(x^\lambda), \quad t = t(\bar{t}). \tag{35}$$

Instead of (35) the prolonged group:

$$\begin{cases} x^i = x^i(x^\lambda), \quad y^i = P^i_\lambda y^\lambda \alpha, \\ z^i = P^i_\lambda z^\lambda \alpha^2 + P^i_{\lambda\mu} y^\lambda y^\mu \alpha^2 + P^i_\lambda y^\lambda \alpha', \\ t = t(\bar{t}), \quad dt = \alpha^{-1} d\bar{t}, \quad \alpha' = \dot{\alpha}\alpha \end{cases} \tag{36}$$

is considered, where

$$y^i = x'^i, \quad z^i = x''^i, \quad y^\lambda = \dot{x}^\lambda, \quad z^\lambda = \ddot{x}^\lambda.$$

In the first place, it is recalled that among the tensors associated to system (33) under the infinite group (36) there exist many vectors as $Q^{\cdot\cdot k}_{hk}$, $S^{\cdot\cdot k}_{hk}$, ... and the operators ∇_h, $\overset{1}{\nabla}_h$, $\overset{2}{\nabla}_h$, $\overset{0}{\nabla}$ permit the deduction from each of these vectors of a set of new covariant vectors as

$$\nabla Q^{\cdot\cdot k}_{hk}, \qquad \nabla\nabla Q^{\cdot\cdot k}_{hk}, \quad \ldots\ldots\ldots$$

We suppose that among these vectors there exist n linearly independent vectors which are indicated by the symbols

$$a^\alpha_i, \qquad \alpha = 1, 2, \ldots, n,$$

where the Latin index designates, as usual, the components of a vector and the Greek index serves to distinguish the vectors. We introduce the following Pfaffians each of which is the components of a contravariant vector:

$$\begin{cases} \overset{0}{\omega}{}^i = y^i dt, \quad \overset{1}{\omega}{}^i = dx^i - y^i dt, \\ \overset{2}{\omega}{}^i = dy^i - z^i dt + \Pi^i_{jk} y^j (dx^k - y^k dt), \\ \overset{3}{\omega}{}^i = \overset{(1)}{dx}{}^i + \overset{(1)}{\Pi}{}^i_{jk} x^j dx^k + \overset{(1)}{\Pi} x^i dt, \end{cases} \tag{37}$$

(1) Cf. W. SLEBODZIŃSKI, [VIII] Sur deux connexions affines generalisées, Prace Matematyczno-Fizyczne, vol. XLIII (1936), p. 20.

and

$$(38)\qquad \begin{cases} \overset{0}{\omega}{}^{\lambda} = y^{\lambda}\,d\bar{t}\,, \quad \overset{1}{\omega}{}^{\lambda} = dx^{\lambda} - y^{\lambda}\,d\bar{t}\,, \\ \overset{2}{\omega}{}^{\lambda} = dy^{\lambda} - z^{\lambda}\,d\bar{t} + \Pi^{\lambda}_{\mu\nu}\,y^{\mu}(dx^{\nu} - y^{\nu}\,d\bar{t})\,, \\ \overset{3}{\omega}{}^{\lambda} = \overset{(1)}{dx}{}^{\lambda} + \Pi^{\lambda}_{\mu\nu}\overset{(1)}{x}{}^{\mu}\,dx^{\nu} + {}'\Pi\overset{(1)}{x}{}^{\lambda}\,d\bar{t}\,, \end{cases}$$

and we put:

$$(39)\qquad \overset{0}{\Pi}{}^{\alpha} = a^{\alpha}_{i}\,\overset{0}{\omega}{}^{i}\,, \quad \overset{1}{\Pi}{}^{\alpha} = a^{\alpha}_{i}\,\overset{1}{\omega}{}^{i}\,, \quad \overset{2}{\Pi}{}^{\alpha} = a^{\alpha}_{i}\,\overset{2}{\omega}{}^{i}\,, \quad \overset{3}{\Pi}{}^{\alpha} = a^{\alpha}_{i}\,\overset{3}{\omega}{}^{i}\,.$$

Thus we have $4n$ invariant forms of the system (33) to which we add the parameter dt and a'.

Therefore if system (33) and (34) are equivalent under the transformation of form (36), by this change of variables each of the expressions $\overset{p}{\Pi}{}^{\alpha}$, $p = 0, 1, 2, 3$ corresponding to system (37) must be transformed into each of the expressions ${}'\overset{p}{\Pi}{}^{\alpha}$ corresponding to system (38). Hence one obtains:

$$(40)\qquad \begin{cases} \overset{0}{\Pi}{}^{\alpha} = {}'\overset{0}{\Pi}{}^{\alpha}, \quad \overset{1}{\Pi}{}^{\alpha} = {}'\overset{1}{\Pi}{}^{\alpha}, \quad \overset{2}{\Pi}{}^{\alpha} = {}'\overset{2}{\Pi}{}^{\alpha}a\,, \quad \overset{3}{\Pi}{}^{\alpha} = {}'\overset{3}{\Pi}{}^{\alpha}a\,, \\ dt = a^{-1}d\bar{t}\,, \quad a' = \dot{a}a\,. \end{cases}$$

Reciprocally suppose that when two systems (33) and (34) are given, there exists a transformation of variables x^i, y^i, z^i, t into $x^{\lambda}, y^{\lambda}, z^{\lambda}, \bar{t}$ which realises the equalities (40), then it can be proven that this transformation of the variables is of form (36) and transforms system (37) into system (38). Hence the problem of equivalency of the systems reduces to that of the equivalency of the Pfaffians.

In fact, because of the equations (37), (38) and the definition of the forms $\overset{p}{\Pi}{}^{\alpha}$ the equalities $\overset{0}{\Pi}{}^{\alpha} = {}'\overset{0}{\Pi}{}^{\alpha}$, $\overset{1}{\Pi}{}^{\alpha} = {}'\overset{1}{\Pi}{}^{\alpha}$, $\overset{2}{\Pi}{}^{\alpha} = {}'\overset{2}{\Pi}{}^{\alpha}a$, $\overset{3}{\Pi}{}^{\alpha} = {}'\overset{3}{\Pi}{}^{\alpha}a$ give rise to

$$(41)\qquad \begin{cases} a^{\alpha}_{i}\,y^{i}\,dt = {}'a^{\alpha}_{\lambda}\,y^{\lambda}\,d\bar{t}\,, \\ a^{\alpha}_{i}(dx^{i} - y^{i}\,dt) = {}'a^{\alpha}_{\lambda}(dx^{\lambda} - y^{\lambda}\,d\bar{t})\,. \end{cases}$$

Because of (41) it can be concluded that $t = t(\bar{t})$. A short calculation shows that in the change of variables which realises equalities (40), the variables x^i depend only on x^{λ} and t. But it can easily be proven that variables x^i do not involve t explicitly, hence one obtains:

$$x^i = x^i(x^\lambda)$$

and

$$dx^i = P^i_\lambda dx^\lambda .$$

By substituting the dx^i in equations (41) by the above expressions one obtains:

$$(a^\alpha_i P^i_\lambda - {}'a^\alpha_\lambda)dx^\lambda - (a^\alpha_i y^i - {}'a^\alpha_\lambda y^\lambda a)dt = 0 .$$

As the variables x^λ, t are independent, one gets:

$$(42) \qquad {}'a^\alpha_\lambda = a^\alpha_i P^i_\lambda , \quad {}'a^\alpha_\lambda y^\lambda a = a^\alpha_i y^i .$$

By the substitution of the expressions ${}'a^\alpha_i$ in the equation $(42)_2$ with the equation $(42)_1$, one obtains:

$$a^\alpha_i (y^i - P^i_\lambda y^\lambda a) = 0 .$$

As the vectors a^α_i are linearly independent by the hypothesis it follows that

$$(43) \qquad y^i = P^i_\lambda y^\lambda a .$$

Hence by means of equalities (40) and relations (43) one obtains:

$$(44) \quad \begin{cases} dx^i - y^i dt = P^i_\lambda (dx^\lambda - y^\lambda d\bar{t}) , \\ dy^i - z^i dt + \Pi^i_{jk} y^j (dx^k - y^k dt) \\ \qquad\qquad = P^i_\lambda a (dy^\lambda - z^\lambda d\bar{t} + \Pi^\lambda_{\mu\nu} y^\mu (dx^\nu - y^\nu d\bar{t}) , \\ \overset{(1)}{dx}{}^i + \Pi^i_{jk} \overset{(1)}{x}{}^j dx^k + \overset{(1)}{\Pi} x^i dt = P^i_\lambda a (\overset{(1)}{dx}{}^\lambda + \Pi^\lambda_{\mu\nu} \overset{(1)}{x}{}^\mu dx^\nu + {}'\overset{(1)}{\Pi} x^\lambda d\bar{t}) . \end{cases}$$

As $dx^i - y^i dt$ and dt can be regarded as mutually independent, by equating the coefficients of $dx^i - y^i dt$ and dt respectively in the both sides in the equation $(44)_2$, one obtains:

$$(45) \quad \begin{cases} \Pi^i_{jk} P^j_\lambda P^k_\mu + P^i_{\lambda\mu} = \Pi^\nu_{\lambda\mu} P^i_\nu , \\ z^i = P^i_\lambda z^\lambda a^2 + P^i_{\lambda\mu} y^\lambda y^\mu a^2 + P^i_\lambda y^\lambda a' . \end{cases}$$

Finally according to the equation $(44)_3$ and (45) one obtains:

$$a' = -\Pi a + {}'\Pi a^2 .$$

That is it follows that the systems of the equations (37) and (38) are equivalent under the transformation of the variables (36).

Thus the problem of equivalency of the systems (33) and (34) with respect to the transformation (35) has been reduced into the problem of equivalency of two systems of Pfaffians. As to the latter problem the general method introduced by Prof. E. CARTAN[1] can be applied. Therefore it may be unneccessary to repeat the arguments.

But our reasons are invalid when it does not give n linearly independent vectors of the system (33). In this case we need a special study.

5. Some generalization. Now the author will determine base connections in a special KAWAGUCHI space of order $m-1$ and dimension n where the arc-length of a curve, $x^i = x^i(t)$, is given by the integral

$$(1') \quad s = \int \left\{ A_i(x, x', \ldots, x^{(m-2)}) x^{(m)i} + B(x, x', \ldots, x^{(m-1)}) \right\}^{\frac{1}{p}} dt .$$

CRAIG's conditions for the integral (1′) are given by the following identities:

$$(10') \quad \sum_{\beta=\rho}^{m} \binom{\beta}{\rho} x^{(\beta-\rho+1)i} F_{(\beta)i} = p\delta^1_\rho F , \quad \rho = 1, 2, \ldots, m,$$

where

$$(2') \qquad F = A_i(x, x', \ldots, x^{(m-2)}) x^{(m)i} + B(x, x', \ldots, x^{(m-1)}) .$$

The equations (10′) are equivalent to

$$\sum_{\beta=1}^{m-2} \beta x^{(\beta)i} A_{j(\beta)i} = (p-m) A_j ,$$

$$\sum_{\beta=1}^{m-1} \beta x^{(\beta)i} B_{(\beta)i} = pB ,$$

$$\sum_{\beta=\rho}^{m-2} \binom{\beta}{\rho} x^{(\beta-\rho+1)i} A_{j(\beta)i} = 0 , \qquad \rho > 1 ,$$

$$\sum_{\beta=\rho}^{m-1} \binom{\beta}{\rho} x^{(\beta-\rho+1)i} B_{(\beta)i} + \binom{m}{\rho} x^{(m-\rho+1)i} A_i = 0 , \qquad \rho > 1 .$$

At first we have a

Lemma. *A_i, $A_{i(j)}$ and $B_{(i)(j)}$ are the components of a vector or tensors.*

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(1) E. CARTAN, [IX] Les sous-groupes continus de tranformations, Ann. Ec. Norm., vol. XXV (1908), pp. 57–197.

Proof. Under the transformation of coördinates from equation (2′) we have:

$$A_i x^{(m)i} + B = A_i(P^i_\lambda x^{(m)\lambda} + R^i) + B = A_\lambda x^{(m)\lambda} + \bar{B},$$

where we indicate by R^i the rest in expansion of $x^{(m)i}$. From the above equation we have:

$$A_\lambda = A_i P^i_\lambda ,$$

and also

$$B = \bar{B} - A_i R^i ,$$

therefore

$$B_{(i)(j)} = \bar{B}_{(\lambda)(\mu)} Q^\lambda_i Q^\mu_j .$$

Before determining the base connections we introduce SYNGE vectors which are defined by

$$(6') \qquad \overset{\alpha}{E}_i = \sum_{\beta=\alpha}^{m} (-1)^\beta \binom{\beta}{\alpha} F_{(\beta)i}^{\cdot\cdot(\beta-\alpha)} .$$

Particularly, in this paper,

$$(6'') \quad \overset{m-2}{E}_i = (-1)^m \left\{ F_{(m-2)i} - (m-1)\frac{d}{dt} F_{(m-1)i} + \frac{m(m-1)}{2}\frac{d^2}{dt^2} A_i \right\} + \dots .$$

$$= (-1)^m \left\{ A_{k(i)} - (m-1) B_{(k)(i)} + \frac{m(m-1)}{2} A_{i(k)} \right\} x^{(m)k} + \Gamma_i$$

plays an important role, Γ_i involving $x^i, x'^i, \dots, x^{(m-1)i}$ only.

We put:

$$G_{ij} = A_{i(j)} - (m-1) B_{(i)(j)} + \frac{m(m-1)}{2} A_{j(i)} .$$

Evidently G_{ij} are the components of a tensor and, in general, the determinant $|G_{ij}|$ is not vanished identically. And we assume this, then we can define G^{ij} by relations:

$$G_{il} G^{ij} = \delta^j_l , \qquad G_{ij} G^{lj} = \delta^l_i .$$

Owing to this assumption we can solve the equations (6″) with respect to $x^{(m)l}$, that is

$$(46) \qquad (-1)^m \overset{m-2}{E}_i G^{il} = x^{(m)l} + \Gamma_i G^{il} = x^{(m)l} + \Gamma^l(x, x', \dots, x^{(m-1)}) ;$$

we put

$$(46') \qquad \overset{(m)}{x^i} = x^{(m)i} + \Gamma^i(x, x', \ldots, x^{(m-1)}) .$$

The expressions (46) or (46′) are evidently the components of a contravariant vector.

Now the base connections can be defined. We have an important

Theorem. *The expressions*

$$(47) \qquad \delta^{(m-\rho)} x^i = \frac{1}{\binom{m}{\rho}} \sum_{\alpha=\rho}^{m} \binom{\alpha}{\rho} \overset{(m)}{x}{}^i_{(\alpha)j} dx^{(\alpha-\rho)j}$$

or

$$(47') \qquad \delta^{(m-\rho)} x^i = dx^{(m-\rho)i} + \frac{1}{\binom{m}{\rho}} \sum_{\alpha=\rho}^{m-1} \binom{\alpha}{\rho} \Gamma^i_{(\alpha)j} dx^{(\alpha-\rho)j}$$

$$(\rho = 1, 2, \ldots, m-1)$$

are the components of a contravariant vector.

Proof. According to (46′) by a transformation $x^i = x^i(x^\lambda)$, one obtains:

$$\sum_{\alpha=\rho}^{m} \binom{\alpha}{\rho} \overset{(m)}{x}{}^i_{(\alpha)j} dx^{(\alpha-\rho)j} = \sum_{\alpha=\rho}^{m} \binom{\alpha}{\rho} \sum_{\beta=\alpha}^{m} P^i_\tau \overset{(m)}{x}{}^\tau_{(\beta)\mu} \frac{\partial x^{(\beta)\mu}}{\partial x^{(\alpha)j}} dx^{(\alpha-\rho)j}$$

$$= \sum_{\beta=\rho}^{m} \sum_{\alpha=\rho}^{\beta} \binom{\alpha}{\rho} \binom{\beta}{\alpha} P^i_\tau \overset{(m)}{x}{}^\tau_{(\beta)\mu} \frac{\partial x^{(\beta-\alpha)\mu}}{\partial x^j} dx^{(\alpha-\rho)j}$$

$$= \sum_{\beta=\rho}^{m} \binom{\beta}{\rho} P^i_\tau \overset{(m)}{x}{}^\tau_{(\beta)\mu} \sum_{\alpha=\rho}^{\beta} \binom{\beta-\rho}{\alpha-\rho} \frac{\partial x^{(\alpha-\beta)\mu}}{\partial x^j} dx^{(\alpha-\rho)j}$$

$$= \sum_{\beta=\rho}^{m} \binom{\beta}{\rho} P^i_\tau \overset{(m)}{x}{}^\tau_{(\beta)\mu} \sum_{\alpha=\rho}^{\beta} \frac{\partial x^{(\beta-\rho)\mu}}{\partial x^{(\alpha-\rho)j}} dx^{(\alpha-\rho)j}$$

$$= P^i_\tau \sum_{\alpha=\rho}^{m} \binom{\alpha}{\rho} \overset{(m)}{x}{}^\tau_{(\alpha)\mu} dx^{(\alpha-\beta)\mu} .$$

It may be interesting to write out a few of the expressions (47′) explicitly:

$$(47'')\quad \begin{cases} \overset{(1)}{\delta x^i} = dx'^i + \dfrac{1}{m}\Gamma^i_{(j)}dx^j , \\ \overset{(2)}{\delta x^i} = dx''^i + \dfrac{m-1}{m}\Gamma^i_{(j)}dx'^j + \dfrac{1}{m}\Gamma^i_{(m-2)j}dx^j , \\ \overset{(3)}{\delta x^i} = dx'''^i + \dfrac{(m-1)(m-2)}{2m}\Gamma^i_{(j)}dx''^j + \dfrac{m-2}{m}\Gamma^i_{(m-2)j}dx'^j \\ \qquad\qquad + \dfrac{1}{m}\Gamma^i_{(m-3)j}dx^j , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \overset{(m-1)}{\delta x^i} = dx^{(m-1)i} + \dfrac{1}{m}\sum_{\alpha=1}^{m-1}\alpha\Gamma^i_{(\alpha)j}dx^{(\alpha-1)j} . \end{cases}$$

Hence: *The expressions (47′) or (47″) give the base connections in a special* KAWAGUCHI *space of order* $m-1$ *and of dimension* n $K_n^{(m-1)}$.

Along the base curve the equations (47″) become:

$$\overset{(2)}{x^i} = x''^i + \frac{1}{m}\Gamma^i_{(j)}x'^j ,$$

$$\overset{(3)}{x^i} = x'''^i + \frac{m-1}{m}\Gamma^i_{(j)}x''^j + \frac{1}{m}\Gamma^i_{(m-2)j}x'^j ,$$

$$\overset{(4)}{x^i} = x^{\mathrm{IV}i} + \frac{(m-1)(m-2)}{2m}\Gamma^i_{(j)}x'''^j + \frac{m-2}{m}\Gamma^i_{(m-2)j}x''^j + \frac{1}{m}\Gamma^i_{(m-2)j}x'^j ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\overset{(m)}{x^i} = x^{(m)i} + \frac{1}{m}\sum_{\alpha=1}^{m-1}\alpha\Gamma^i_{(\alpha)j}x^{(\alpha)j} .$$

The above expressions are the same vectors as those introduced by Prof. A. KAWAGUCHI[1].

Now a covariant differential of a vector in $K_n^{(m-1)}$ can be defined, namely we have a

Theorem. *Expressions*

$$(48)\qquad \delta v^i = dv^i + \frac{1}{m}v^k\sum_{\alpha=1}^{m-1}\alpha\Gamma^i_{(k)(\alpha)j}dx^{(\alpha-1)j}$$

(1) See [VI], p. 149 and also [VII], p. 153.

are the components of a vector, where v^i are the components of a vector in $K_n^{(m-1)}$.

Proof. From the equation (48), by transformation $x^i = x^i(x^\lambda)$, we have:

$$
\begin{aligned}
&dv^i + \frac{1}{m} v^k \sum_{\alpha=1}^{m-1} a \Gamma^i_{(k)(\alpha)j} dx^{(\alpha-1)j} \\
&\quad = P^i_\mu dv^\mu + P^i_{\mu\nu} v^\mu dx^\nu + \frac{1}{m} v^\pi \sum_{\alpha=1}^{m-1} a P^k_\pi \Gamma^i_{(k)(\alpha)j} dx^{(\alpha-1)j} \\
&\quad = P^i_\mu dv^\mu + P^i_{\mu\nu} v^\mu dx^\nu + \frac{1}{m} v^\pi \sum_{\alpha=1}^{m-1} a (P^i_\mu \Gamma^\mu_{(\pi)} - m P^i_{\pi\nu} x'^\nu)_{(\alpha)j} dx^{(\alpha-1)j} \\
&\quad = P^i_\mu dv^\mu + P^i_{\mu\nu} v^\mu dx^\nu + \frac{1}{m} v^\pi P^i_\mu \Gamma^\mu_{(\pi)(1)\nu} dx^\nu - v^\pi P^i_{\pi\nu} dx^\nu \\
&\qquad + \frac{1}{m} v^\pi \sum_{\alpha=2}^{m-1} a P^i_\mu \Gamma^\mu_{(\pi)(\alpha)j} dx^{(\alpha-1)j} \\
&\quad = P^i_\mu \left\{ dv^\mu + \frac{1}{m} v^\pi \sum_{\alpha=1}^{m-1} a \Gamma^\mu_{(\pi)(\alpha)\lambda} dx^{(\alpha-1)\lambda} \right\} .
\end{aligned}
$$

But unfortunately these expressions are not invariant by the change of parameter t.

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