ON THE THEORY OF THE CLUSTER SETS OF ANALYTIC FUNCTIONS

By

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F. IVERSEN[1], W. GROSS[2] and others[3] have obtained many important results on the cluster sets of analytic functions. The object of this paper is to give some generalizations of and some complements to their results.

First, in § 1, we state an extension of IVERSEN's theorem, using a method for which the writer owes much to J. L. DOOB.[4] Next, in § 2, applying NEVANLINNA's theory of harmonic measures[5], we give a more precise form of CARTWRIGHT's[6] extension of IVERSEN's theorem. Finally, in § 3, we enunciate some applications of BEURLING's[7] theorem and give some complements to the principle of maximum modulus.

§ 1. An extension of Iversen's theorem.

1. Suppose that $f(z)$ is a uniform function, meromorphic in a connected domain D. Let z_0 be a point on the boundary C of D. We associate with z_0 the following sets of values:

(1) The cluster set $S_{z_0}^{(D)}$. This is the set of all values a such that $\lim\limits_{\nu\to\infty} f(z_\nu) = a$ where $\{z_\nu\}$ is a sequence of points tending to z_0

(1) F. IVERSEN: Recherches sur les fonctions inverses des fonctions méromorphes, Thèse de Helsingfors, 1914.

(2) W. GROSS: Über die Singularitäten analytischer Funktionen, Monatshefte für Math. und Phys., Bd. 29 (1918), pp. 1–47.

(3) W. SEIDEL: On the cluster values of analytic functions, Transactions of American Mathematical Society, vol. 34 (1932), pp. 1–21. J. L. DOOB: On a theorem of Gross and Iversen, Annals of Mathematics, vol. 33 (1932), pp. 753–757. A. BEURLING: Études sur un problème de majoration, Thèse de Upsal, 1933. See pp. 100–103. M. L. CARTWRIGHT: On the asymptotic values of functions with a non-enumerable set of singuralities, Journal of London Math. Soc., vol. 11 (1936), pp. 303–305.

(4) Loc. cit.

(5) R. NEVANLINNA: Eindeutige analytische Funktionen, 1936. Cf. pp. 106–153.

(6), (7) Loc. cit.

inside D. In other words, $S_{z_0}^{(D)}$ is identical with the product of all $\bar{\mathfrak{D}}_n$ where $\bar{\mathfrak{D}}_n$ is the closed cover of the set $\mathfrak{D}_n$ of values taken by $w = f(z)$ inside the common part of K_n and D, the circle: $|z-z_0| < \frac{1}{n}$ being denoted by K_n. Hence, it is obvious that $S_{z_0}^{(D)}$ is a closed (non empty) set.

(2) The cluster set $S_{z_0}^{(C)}$. This is the product $\prod_{n=1}^{\infty} M_n$, where M_n denotes the closed cover of the sum $\sum_{0<|\xi-z_0|<1/n} S_{\xi}^{(D)}$, ξ belonging to C. This set is also closed and becomes empty, if and only if z_0 is an isolated boundary point.

(3) The range of values $R_{z_0}^{(D)}$. This is the product of all value-sets $\mathfrak{D}_n$ of $f(z)$ for $K_n \cdot D$. Hence, a value α belongs to $R_{z_0}^{(D)}$, if and only if, $f(z)$ takes α infinitely often near z_0 inside D. It is clear that $R_{z_0}^{(D)}$ is a G_δ set.

(4) The convergence set $\Gamma_{z_0}^{(D)}$. Suppose that z_0 is an accessible boundary point, then there is at least one path (JORDAN arc) lying inside D except the end point at z_0. $\Gamma_{z_0}^{(D)}$ is the set of all values α such that $f(z) \to \alpha$ as $z \to z_0$ along such a path. We call such a value α an asymptotic value at $z = z_0$. For convenience, we regard $\Gamma_{z_0}^{(D)} = 0$ when z_0 is non accessible.

It is obvious that $S_{z_0}^{(D)}$ includes $S_{z_0}^{(C)}$, $R_{z_0}^{(D)}$ and $\Gamma_{z_0}^{(D)}$.

IVERSEN has proved that if α does not belong to $R_{z_0}^{(D)}$, where z_0 is an isolated essential singular point, then α is an asymptotic value at z_0. The object of this paragraph is to generalize this classical theorem. *Let $f(z)$ be a uniform function, meromorphic in D, whose two cluster sets are not identical: i.e., $S_{z_0}^{(D)} - S_{z_0}^{(C)} \neq 0$. Suppose further that a value α, lying in $S_{z_0}^{(D)} - S_{z_0}^{(C)}$, does not belong to $R_{z_0}^{(D)}$.* Under these conditions we may assert that z_0 *must be accessible and α is an asymptotic value at z_0.* For proof, we have only to consider the case where z_0 is non isolated (hence, it is clear that $S_{z_0}^{(C)} \neq 0$) and α is finite. Since α does not belong to $S_{z_0}^{(C)}$, by its definition, there exists a positive number r_1 such that $\alpha \bar{\in} \overline{\sum_{0<|\xi-z_0|\leq r_1} S_{\xi}^{(D)}}$[1] where ξ varies on C. Consequently α has a positive distance ρ_1 from the set $\overline{\sum_{0<|\xi-z_0|\leq r_1} S_{\xi}^{(D)}}$.

(1) "$\alpha \in A$" and "$\alpha \bar{\in} A$" mean that "α belongs to A" and "α does not belong to A" respectively.

By the assumption $\alpha \bar{\in} R_{z_0}^{(D)}$, we may also find another positive number r_2 such that $f(z)-\alpha$ never vanishes for $|z-z_0| \leqq r_2$ inside D. Let us put $r = \min\left(\frac{r_1}{2}, \frac{r_2}{2}\right)$. Denote by K and k the circular disc $|z-z_0| < r$ and its circumference $|z-z_0| = r$ respectively. Then k is either contained entirely in D or the common part $k \cdot D$ consists of a finite or an enumerable number of cross-cuts. Consequently the common part $K \cdot D$ is a single connected domain in the former case and contains finite or an enumerable infinity of connected domains $D_i (i = 0, 1, 2, \ldots.)$ in the latter. We easily see that in either case the modulus of $f(z)-\alpha$ has a positive minimum ρ_2 on the set $k \cdot D$. Now put $\rho = \min\left(\frac{\rho_1}{2}, \frac{\rho_2}{2}\right)$. Thus we have obtained two positive numbers r and ρ such that the function $f(z)-\alpha$ does not vanish for $|z-z_0| \leqq r$ inside D and further $\overline{\sum_{0<|\xi-z_0|\leqq r} S_\xi^{(D)}}$ and $\overline{\sum_{z \in k \cdot D} f(z)}$ both lie outside the circle $(c): |w-\alpha| < \rho$. Remember that α is a cluster value of $f(z)$ at z_0. Then, there is a sequence of points z_ν inside D such that $z_\nu \to z_0$ and $f(z_\nu) \to \alpha$. Accordingly we may find a point ζ_0 in $K \cdot D$, whose image $w_0 = f(\zeta_0)$ lies within (c); we suppose ζ_0 to be contained in the component D_0 of $K \cdot D$ and denote by e_{ζ_0} the inverse functional element of $f(z)$ at w_0. Continuing analytically the element e_{ζ_0} inside the circle $(c_0): |w-\alpha| < |w_0-\alpha|$ with algebraic characters, we obtain a branch $z = \varphi_{\Delta_0}(w)$ of the inverse function of $w = f(z)$; the set of values taken by $z = \varphi_{\Delta_0}(w)$ in (c_0) is a connected domain Δ_0, ζ_0 being its boundary point. Next we investigate the relation between D_0 and Δ_0.

i) Δ_0 is contained in D_0. Suppose that there were a point $z^* = \varphi_{\Delta_0}(w^*)$ outside D_0. Then we would have a path L_z, joining the point ζ_0, lying inside D_0, to z^*, along which the analytic continuation of $f(z)$ would be possible and whose image by $w = f(z)$ would lie inside (c_0) except the end point w_0. Obviously L_z would cut the boundary of D_0, while for any point of intersection ξ, distinct from z_0, $S_\xi^{(D_0)}$ lies outside (c_0). Without difficulty, it is seen that there is a contradition.

ii) The closed cover $\bar{\Delta}_0$ of Δ_0 cannot lie inside D_0. If it did, $f(z)$ would assume α at least once in Δ_0.[1] This contradicts the fact that $f(z)-\alpha$ never vanishes for $|z-z_0| \leqq r$ inside D.

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(1) If $\bar{\Delta}_0$ were contained in D_0, $f(z)-\alpha$ would be a regular function in $\bar{\Delta}_0$, having a constant modulus: $|f(z)-\alpha| = |w_0-\alpha|$ on the boundary of Δ_0, Hence, we see that $f(z)-\alpha$ would vanish in Δ_0, applying the maximum-principle to the function $\psi(z) = (f(z)-\alpha)^{-1}$.

iii) Accordingly Δ_0 and D_0 must have at least one boundary point in common. However, it is easily seen that there is no common point other than z_0[1]. Thus, we saw that the domain Δ_0, contained in D_0, has exactly one common boundary point z_0 with D_0 and the boundary of Δ_0 consists of certain analytic contours γ_0.

Since $f(z)-a$ is regular in the closed domain $\bar{\Delta}_0$ except at z_0, satisfying the relations $0<|f(z)-a|<|w_0-a|$ inside Δ_0 and $|f(z)-a|=|w_0-a|$ on γ_0, we can assert, after IVERSEN, that there exists a continuous arc L_z contained in Δ_0, except the end point z_0, along which $f(z)$ tends to a, as z tends to z_0. First, a must belong to the cluster set $S_{z_0}^{(\Delta_0)}$. To prove this, we consider the function $\psi(z)=\dfrac{1}{f(z)-a}$, regular in Δ_0 and having a constant modulus: $|\psi(z)|=\dfrac{1}{|f(z)-a|}$ on the boundary γ_0, z_0 being excluded. Obviously $\psi(z)$ cannot be a bounded function in Δ_0, otherwise, by LINDELÖF's principle, it would hold that $|f(z)-a|\geqq|w_0-a|$. Accordingly, there is a sequence of points z'_n inside Δ_0 such that $f(z'_n)\to a$ as $z'_n\to z_0$. Denote by (c_1) the circle: $|w-a|<\dfrac{|w_0-a|}{2}$ and choose n so large that $w'_n=f(z'_n)$ falls inside (c_1). Next, let us join ζ_0 to z'_n by a simple continuous arc $L_z^{(0)}$ within Δ_0 except the end point ζ_0. Let $L_w^{(0)}$ be the image of $L_z^{(0)}$ by $w=f(z)$. Clearly $L_w^{(0)}$ is a continuous arc inside (c_0) except the end point w_0, joining w_0 to w'_n. $L_w^{(0)}$ meets at least once at the circumference of (c_1). Denote by w_1 the first point of intersection from w_0 along the curve $L_w^{(0)}$, and denote by ζ_1 the corresponding point; i.e., $w_1=f(\zeta_1)$. Thus we see that there is a continuous arc Λ_0: $\overset{\frown}{w_0 w_1}$, connecting two circumferences c_0, c_1, of (c_0), (c_1), and lying in the ring (c_0, c_1) except two end points, such that the analytic continuation of e_{ζ_0} along Λ_0 is possible and that the image of Λ_0 by this continuation is a simple arc λ_0, joining ζ_0 and ζ_1 in Δ_0. Next consider the branch $z=\varphi_{\Delta_1}(w)$ defined inside (c_1) by the element e_{ζ_1}, just obtained at w_1, and let Δ_1 be the value set of $z=\varphi_{\Delta_1}(w)$ in (c_1). Describing the circle $c_2: |w-a|=\dfrac{|w_0-a|}{4}$, whose interior is denoted by (c_2), we see that, by the same reasoning as before, there is a continuous arc

(1) If there were a common boundary point ξ, distinct from z_0, then $S_\xi^{(\Delta_0)}$ would be contained in $S_\xi^{(D_0)}$, while $S_\xi^{(\Delta_0)}$ would lie on the closed cover of (c_0) and $S_\xi^{(D_0)}$ entirely outside (c_0).

$\Lambda_1 : \widehat{w_1 w_2}$, connecting two circumferences c_1, c_2 and lying between them and having analogous properties to those of Λ_0.

Repeating the same arguments and using analogous notations, for any natural number n, we have a continuous arc $\Lambda_n : \widehat{w_n w_{n+1}}$ with the following properties:

1) Λ_n connects two circles c_n, c_{n+1} and lies between them, excluding two end points w_n, w_{n+1} where c_n is a circle: $|w-\alpha| = \frac{|w_0-\alpha|}{2^n}$;

2) Analytic continuation of e_{ζ_n} is possible along Λ_n and an element $e_{\zeta_{n+1}}$ is obtained at the end point w_{n+1}. The image of Λ_n by this continuation is a simple arc λ_n, joining ζ_n and ζ_{n+1}, inside Δ_n.

Since λ_n falls in $\Delta_n - \Delta_{n+1}$, the curve $L_z = \sum_{n=0}^{\infty} \lambda_n$ is a simple continuous curve in Δ_0, and it must converge to z_0. The last assertion comes from the fact, since we have the inequality $|f(z)-\alpha| > \delta > 0$ for $|z-z_0| \geqq \sigma > 0$ within Δ_0, all Δ_n lie in the circle $|z-z_0| < \sigma$ except a finite number of them. It is obvious that $f(z)$ tends to α as z tends to z_0 along the curve L_z.

Thus we have proved the following

Theorem 1. *Let $f(z)$ be uniform and meromorphic in an arbitrary domain D. Suppose that α is a value belonging to $S_{z_0}^{(D)} - S_{z_0}^{(C)}$ but not belonging to $R_{z_0}^{(D)}$, where z_0 is a boundary point of D. Then z_0 becomes necessarily accessible and α is an asymptotic value of $f(z)$ at z_0; that is, α belongs to $\Gamma_{z_0}^{(D)}$.*

Remark. For the method in our proof we owe much to DOOB.[1]

2. In order to apply the above result, we here propose a lemma concerning p-valent analytic functions, obtained easily by means of a method of AHLFORS. *Suppose that $f(z)$ is uniform and meromorphic in an arbitrary domain D and that $f(z)$ is at most p-valent in the neighbourhood of a frontier point z_0, which here we will assume to be non-isolated.* Denoted by K the circle: $|z-z_0| < R$, taking R so small that $f(z)$ is at most p-valent in the product $K \cdot D$. If we take the image of $K \cdot D$ by $w = f(z)$ on the RIEMANN sphere $\sum$ of radius $\frac{1}{2}$, touching the w-plane, its area is obviously at most $p\pi$. Describe a concentric circle $K_r : |z-z_0| = r < R$ and denoted by Q_r the common part of K_r and D. Q_r consists of a finite number or an enumerable

(1) He has proved this theorem in the case where D is a JORDAN domain. Our theorem is regarded as an extension of his theorem. Cf. DOOB: loc. cit.

infinity of cross-cuts, except the case in which K_r is entirely contained in D. In either case, denoting by $L(r)$ the total spherical length of the image of Q_r on the w-plane, *we may obtain the relation* $\varliminf_{r\to 0} L(r)=0$.

From
$$L(r)=\int_{Q_r}\frac{|f'(z)|}{1+|f(z)|^2}rd\theta\,,\quad (z=re^{i\theta})\,,$$

applying SCHWARZ's inequality
$$[L(r)]^2\leqq\int_{Q_r} rd\theta\int_{Q_r}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}rd\theta\,,$$
whence
$$\frac{[L(r)]^2}{2\pi r}\leqq\int_{Q_r}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}rd\theta\,.$$

If we had $\varliminf_{r\to 0} L(r)>0$, then there would exist two positive numbers r_0 and δ such that $L(r)>\delta>0$ for $r\leqq r_0$. Accordingly $\frac{\delta^2}{2\pi}\int_r^{r_0}\frac{dr}{r}\leqq\int_r^{r_0}\int_{Q_r}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}rdrd\theta$, whence we would have $\frac{\delta^2}{2\pi}\log\frac{r_0}{r}\leqq p\pi$. Thus we arrive at a contradiction, making r tend to zero.

3. An application of theorem 1 will next be treated, using the just obtained lemma. Let $f(z)$ be uniform and meromorphic in an arbitrary domain D and z_0 be a non-isolated frontier point of D.

There is a problem, not yet solved completely, to find the necessary and sufficient condition that we should have $S_{z_0}^{(D)}=S_{z_0}^{(C)}$. When D is a JORDAN domain, GROSS[1] has shown that it is sufficient that $R_{z_0}^{(D)}$ should be empty, his result being obtained directly from that of BEURLING[2] which we shall state later. Now, we show that for $S_{z_0}^{(D)}=S_{z_0}^{(C)}$ it is sufficient that $f(z)$ *should be at most p-valent near z_0 inside D, without any restriction on the domain D.* Let $f(z)$ be at most p-valent near z_0. By the lemma, we have a sequence of curves Q_{r_n} which satisfy $\lim_{n\to\infty} L(r_n)=0$, $r_n\to 0$, denoting by $L(r)$ the spherical total length of

(1) W. GROSS: loc. cit.
(2) A. BEURLING: loc. cit.

the image, by $w = f(z)$, of Qr_n on the w-plane. Any curve Qr_n is either a complete circle lying inside D or a system of cross-cuts. Suppose, first, that there are an infinite number of complete circles Qr_ν. If we had $S_{z_0}^{(D)} \neq S_{z_0}^{(C)}$, then $S_{z_0}^{(D)} - S_{z_0}^{(C)}$ would contain at least two points α and β, $\alpha \neq \beta$, since, in this case, $S_{z_0}^{(D)}$ is a continuum and z_0 is non-isolated. Consequently, by theorem 1, there would exist two paths L_α and L_β, inside D and reaching z_0, along which $f(z)$ would tend to α and β respectively, as z tends to z_0, for no value of $S_{z_0}^{(D)}$ belongs to $R_{z_0}^{(D)}$. Since L_α and L_β intersect with all circumferences Qr_ν, excluding a fininite number, α and β would coincide with each other, by the assumption $\lim\limits_{n \to \infty} L(r_n) = 0$. This is a contradiction.

Now consider the other case in which we have a subsequence $\{Qr_\nu\}$, each member being a system of cross-cuts. Again suppose that we had $S_{z_0}^{(D)} \neq S_{z_0}^{(C)}$. Then denoting by α any value of $S_{z_0}^{(D)} - S_{z_0}^{(C)}$, we would have a path L_α, along which $f(z)$ would tend to α when z tends to z_0. The curve L_α would meet at all Qr_ν except a finite number of them. Denote by z_ν the first point of intersection along L_α from z_0 and by q_ν the cross-cut, containing z_ν in its interior and by ζ'_ν and ζ''_ν the two end points of q_ν. It is obvious that the image of q_ν by $w = f(z)$ connects two cluster sets $S_{\zeta'_\nu}^{(D)}$ and $S_{\zeta''_\nu}^{(D)}$, its spherical length tending to zero as $\nu \to \infty$. Consequently the value α, written in the form $\alpha = \lim\limits_{\nu \to \infty} f(z_\nu)$, would belong to $S_{z_0}^{(C)}$, since $S_{z_0}^{(C)}$ is a closed set. This contradicts the assumption of α.

Theorem 2. *Suppose that $f(z)$ is uniform and meromorphic in an arbitrary domain D and let z_0 be a non-isolated boundary point. Suppose further that $f(z)$ is at most p-valent in the neighbourhood of z_0. Then we have the relation $S_{z_0}^{(D)} = S_{z_0}^{(C)}$.*

Remark. From this theorem follows at once, a result obtained recently by U. MINAMI[1], which is stated in the following form: Let $f(z)$ be a uniform function, holomorphic in D, such that with each boundary point ζ, excluding a point z_0, we can associate a circle, with centre ζ, in which $|f(z)| \leq m + \varepsilon$, for any positive number ε. Then we have $|f(z)| \leq m$ throughout inside D, provided that $f(z)$ is univalent near z_0, z_0 being a non-isolated boundary point.

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(1) U. MINAMI: An extension of PHRAGMÉN-LINDELÖF's theorem, Proc. Imp. Acad. Tokyo, vol. 13 (1937), pp. 241–243.

§ 2. A precision of Cartwright's theorem.

1. Recently CARTWRIGHT[1] has extended IVERSEN's theorem in a distinct way: Let $f(z)$ be uniform and meromorphic in D except a closed set E, completely contained in D, of essential singularities. Suppose that E is of logarithmic measure zero. If a value α does not belong to $R_{z_0}^{(D-E)}$, z_0 being a point on E, α is either an asymptotic value at z_0 or there is a sequence of essential singular points ζ_n tending to z_0 such that, for every n, α is an asymptotic value at ζ_n. Here we give a precise form of her theorem, in virtue of NEVANLINNA's theory of harmonic measures[2]. Indeed, *in her theorem we can suppose E to be of absolute harmonic measure zero.* It is known that, if a set E, assumed to be closed, is of logarithmic measure zero, then it is also of harmonic measure zero, its converse being not necessarily true; moreover, if E is of harmonic measure zero, its linear measure is also zero and E becomes pointwise, that is, E does not contain any continuum as its component.

To prove our assertion we enunciate two lemmas:

Lemma 1. (An extension of WEIERSTRASS' theorem.) *Let $f(z)$ be uniform and meromorphic in D except a closed set E, lying completely inside D, of essential singularities. Let z_0 be any point of E. Then the set $S_{z_0}^{(D-E)}$ contains all values, provided that E is of absolute harmonic measure zero.*

Lemma 2. (An extension of LINDELÖF's principle.) *Let D be an arbitrary domain and let E be a closed set of harmonic measure zero, consisting of boundary points. Suppose that $f(z)$ is a uniform, regular function in D which satisfies $|f(z)| \leq M$ inside D, and $\overline{\lim}\, |f(z)| \leq m$ $(m \leq M)$ for any boundary point, not belonging to E. Then we have the relation $|f(z)| \leq m$ throughout in D.*

Both lemmas may be obtained readily from NEVANLINNA's theorems[3] stated in the form of harmonic functions. For instance, we will here consider lemma 1. Obviously it is sufficient to show that $f(z)$ is not bounded near z_0. We can draw a circle $K: |z-z_0| = r$ lying entirely inside D, for E is of linear measure zero. Consider the real part $u(z)$ of $f(z)$, which is harmonic for $|z-z_0| \leq r$ inside D, and define, by POISSON's integral, another function $v(z)$ harmonic throughout

(1) M. L. CARTWRIGHT: loc. cit.
(2) R. NEVANLINNA: loc. cit. pp. 106–153.
(3) R. NEVANLINNA: loc. cit., pages 132 and 134.

in the circular domain $|z-z_0|<r$ such that $v(z)=u(z)$ on any point on K. If $f(z)$ were bounded for $|z-z_0|<r$ inside D, then we see that, using NEVANLINNA's result, there would exist the relation $v(z)=u(z)$ throughout for $|z-z_0|<r$ inside D. Consequently $f(z)$ would be identical with the analytic function defined by $v(z)$ regular for any point in the circle $|z-z_0|<r$, while z_0 is an essential singularity of $f(z)$.

2. Now we will give a proof for our assertion. Its proof is entirely similar to that of theorem 1. By hypothesis, there is a positive number r, taken to be arbitrarily small, such that the circle K_r: $|z-z_0|=r$ lies entirely inside $D'=D-E$, as E is of linear measure zero, and such that $f(z)-a$ never vanishes for $|z-z_0|\leqq r$ inside D'. Consequently we can find a positive number δ, satisfying $|f(z)-a|>\delta>0$ on K_r. By lemma 1, there exists a sequence of points z_n tending to z_0 in D' such that $f(z_n)\rightarrow a$ as $n\rightarrow\infty$. Denoting by (c) the circular disc: $|w-a|<\delta$, we may find a point ζ_0 in $|z-z_0|<r$ whose image $w_0=f(\zeta_0)$ lies inside (c). Continuing, with algebraic characters, the inverse element e_{ζ_0} at w_0 within (c_0): $|w-a|<|w_0-a|$, we have a branch $z=\varphi_{\Delta_0}(w)$ of the inverse function of $w=f(z)$. Let us denote by Δ_0 the set of values taken by $z=\varphi_{\Delta_0}(w)$ in (c_0). It is clear that Δ_0 is a connected domain, its boundary containing a point ζ_0. Denote by D'_r the part of D' lying inside K_r. By a similar reasoning in the proof of theorem 1, it is seen that i) Δ_0 is included in D'_r; ii) Δ_0 and D'_r must have at least one bounday point, belonging to E, in common, but iii) no point on K_r belongs to the boundary of Δ_0. Thus the boundary of Δ_0 consists of a finite or an enumerable number of analytic contours γ_0 lying inside D'_r and a closed subset E_0 of E. Obviously the function $f(z)-a$ has the following properties: $f(z)-a$ is regular in the closed domain $\bar{\Delta}_0$, excluding all points belonging to E_0, not vanishing inside Δ_0 and moreover statisfies that $|f(z)-a|<|w_0-a|$ in Δ_0 and $|f(z)-a|=|w_0-a|$ on the contours γ_0. Accordingly we see that the function $\psi(z)=\dfrac{1}{f(z)-a}$, regular in Δ_0, is not bounded there. In other words, a belongs to the cluster set $S_\xi^{(\Delta_0)}$, selecting a point ξ suitably in E_0. If $\psi(z)$ were bounded, then, applying NEVANLINNA's extension of LINDELÖF's principle, we would have $|\psi(z)|\leqq\left|\dfrac{1}{w_0-a}\right|$ in Δ_0, that is, $|f(z)-a|\geqq|w_0-a|$ in Δ_0. Consequently we may find a path Λ_0, joining two circum-

ferences c_0: $|w-\alpha| = |w_0-\alpha|$ and c_1: $|w-\alpha| = \frac{|w_0-\alpha|}{2}$ and lying inside the ring (c_0, c_1), excluding two end points w_0, w_1 along which the analytic continuation of e_{ζ_0} is possible, an element e_{ζ_1} being obtained at the end point w_1 on c_1. Moreover it is possible to select Λ_0 so that its image λ_0 is a simple continuous arc inside Δ_0 joining two points ζ_0 and ζ_1. Next, from the just obtained element e_{ζ_1} at w_1, we may similarly define an analytic function $z = \varphi_{\Delta_1}(w)$ inside (c_1), whose value set Δ_1 on the z-plane is contained in Δ_0, the boundaries of Δ_0 and Δ_1 having a common subset E_1 of E_0. We may also find a curve Λ_1 with an analogous property to Λ_0, joining two circumferences c_1: $|w-\alpha| = \frac{|w_0-\alpha|}{2}$ and c_2: $|w-\alpha| = \frac{|w_0-\alpha|}{2^2}$ and lying between them, excluding two end points. Repeating the same aguments (Compare the proof of theorem 1.), we finally obtain a path L_w joining w_0 and α inside (c_0), along which the continution of e_{ζ_0} is possible, except the end point α, and whose image L_z on the z-plane by this continuation $z = \varphi_{L_w}(w)$ is a simple curve inside Δ_0. We must show that L_z tends to a point of E_0. It is obvious that the cluster set $S_\alpha^{(L_w)}$(1) of $z = \varphi_{L_w}(w)$ at $w = \alpha$ is a continuum. Since the continuum $S_\alpha^{(L_w)}$ must be contained in a pointwise set E, it follows that $S_\alpha^{(L_w)}$ contains a single point z^* in E_0. In other words, the image L_z converges to z^* in E_0 and it is evident that $f(z)$ tends to α along L_z, if z tends to z^*. Our assertion has been proved completely, considering r to be taken as small as we please. Thus we have, as a precision of CARTWRIGHT's theorem,

Theorem 3. *Let $f(z)$ be uniform and meromorphic in D except a closed set E, completely contained in D, of essential singularities. Suppose that E is of harmonic measure zero. If a value α does not belong to $R_{z_0}^{(D-E)}$, where z_0 is a point on E, then α is either an asymptotic value at z_0 or there is a sequence of essential singularities ζ_n tending to z_0 such that, for every n, α is an asymptotic value at ζ_n.*

Finally we will give, without proof, a further extension of IVERSEN's theorem containing theorems 1 and 3 as its special cases.

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(1) This is the set of values β such that there is a sequence of points w_ν tending to α on the path L_w which satisfies the relation $\lim_{\nu\to\infty} \varphi_{L_w}(w_\nu) = \beta$.

Theorem 4[1]. *Let D be an arbitrary domain, C being its boundary, and let E be a closed set, included in C, which contains only accessible boundary points and whose absolute harmonic measure is equal to zero. Suppose that $f(z)$ is a uniform function, meromorphic in D, having a set E of essential singular points. Denoting by z_0 any point in E, we may define a new cluster set $S^{*(C)}_{z_0}$ by the formula $S^{*(C)}_{z_0} = \prod_{n=1}^{\infty} \overline{\sum_{0<|\xi-z_0|<1/n} S^{(D)}_{\xi}}$ where ξ belongs to the set $C-E$*[2]. *Supposing further that $S^{(D)}_{z_0}-S^{*(C)}_{z_0}$ is not a vacuum, if a value a, contained in $S^{(D)}_{z_0}-S^{*(C)}_{z_0}$, does not belong to $R^{(D)}_{z_0}$, then a is either an asymptotic value at z_0 or there exists a sequence of essential singularities ζ_n in E tending to z_0 such that a is an asymptotic value at each ζ_n*

§ 3. Some applications of Beurling's theorem.

Complements to the maximum principle.

1. Recently BEURLING[3] has obtained the following important theorem:

Theorem B_1. *Let D be a connected domain and suppose that z_0 is a boundary point being regular on the problem of DIRICHLET*[4]. *If $f(z)$ is a uniform function, holomorphic and bounded in the neighbourhood of z_0 inside D, then we get $\overline{\lim}_{z\to z_0}|f(z)| = \overline{\lim}_{\substack{z'\to z_0 \\ z'\neq z_0}}\left(\overline{\lim}_{z\to z'}|f(z)|\right)$, denoting any boundary point by z'.*

He has proved it, applying an important, somewhat complicated lemma and recently S. IRIE[5] has given a direct, elementary proof, imposing a slightly stronger restriction on z_0. The above theorem is written in the form:

Theorem B_2. *Under the same condition on D and z_0 as in theorem B_1, if a is not contained in $S^{(D)}_{z_0}$, then $\rho(S^{(D)}_{z_0}, a) = \rho(S^{(C)}_{z_0}, a)$, where ρ*

(1) The proof is somewhat long and obtained by means of both methods in theorems 1 and 3.

(2) It is clear that $S^{*(C)}_{z_0}$ is also closed and contained in $S^{(C)}_{z_0}$ (and so in $S^{(D)}_{z_0}$).

(3) BEURLING: loc. cit. See pp. 100–103.

(4) Denote by E the set of values r such that the circumference $|z-z_0|=r$ $(0\leq r\leq 1)$ contains a point, not belonging to D. If E contains a segment $[0, \rho]$, selecting ρ to be sufficiently small, then it is known that z_0 is regular concerning the problem of DIRICHLET.

(5) IRIE has assumed that z_0 is an accessible boundary point contained in a boundary-component which has at least two points Cf. S. IRIE: Sur un théorème de M. BEURLING, Proc. Imp. Acad. vol. 13 (1937), pp. 244–246.

denotes the spherical distance, supposing that $f(z)$ is uniform and meromorphic in D.

For, rotating the RIEMANN sphere $\sum$ by the formula $w^* = \dfrac{1+\bar{a}w}{w-a}$, we may replace a by ∞ in the above statement, considering $w^*(z) = \dfrac{1+\bar{a}f}{f-a}$, then theorem B_2 is identical with theorem B_1.

Next we state two elementary lemmas.

Lemma 1. *Let M and $N(M > N)$ be two closed sets, on the w-plane, such that, for any exterior point P of M, they always satisfy $\rho(M, P) = \rho(N, P)$. Then we have $B(M) < B(N)$ where $B(M)$ and $B(N)$ denote the boundaries of M and of N respectively.*

The proof is easy. Denote by Q any point in $B(M)$, then, since M is closed, there is a sequence of exterior points P_n of M (and N) such that $P_n \to Q$. Clearly $\rho(M, Q) = \lim \rho(M, P_n)$, $\rho(N, Q) = \lim \rho(N, P_n)$ and, by hypothesis, $\rho(M, P_n) = \rho(N, P_n)$ for any n. Consequently $\rho(N, Q) = \rho(M, Q) = 0$, whence follows our assertion.

Its converse holds good. That is, we have

Lemma 2. *From the assumption that $B(M) < B(N)$ and $M > N$, it follows that, for any exterior point P of M, we have $\rho(M, P) = \rho(N, P)$.*

Because we have $\rho(M, P) \leqq \rho(N, P)$ and $\rho(B(M), P) \geqq \rho(B(N), P)$, by hypothesis, while $\rho(B(M), P) = \rho(M, P)$, $\rho(B(N), P) = \rho(N, P)$, whence follows at once $\rho(M, P) = \rho(N, P)$, for any exterior point P of M.

By these lemmas, theorem B_2 is written in the following form:

Theorem B_3. *Let $f(z)$ be uniform and meromorphic in D. Then we have $B(S_{z_0}^{(D)}) < B(S_{z_0}^{(C)})$ under the same condition on D and z_0 as in theorem B_1.*

Clearly theorem B_1, B_2 and B_3 are equivalent with one other. The complementary set of $S_{z_0}^{(C)}$ with respect to the w-plane is an open set and so decomposed into a finite or an enumerable number of connected components Δ_i $(i = 1, 2, 3, \ldots)$. Now, as an application of theorem B_3, we have

Theorem 4. *Denote by Δ any one of Δ_i $(i = 1, 2, 3, \ldots)$. Then either Δ is entirely included in $S_{z_0}^{(D)}$ or Δ never contains any point in $S_{z_0}^{(D)}$.*

Suppose that Δ contains a point α in $S_{z_0}^{(D)}$. If in Δ there were a point β, not belonging to $S_{z_0}^{(D)}$, then we would find a boundary point of $S_{z_0}^{(D)}$, (of $S_{z_0}^{(C)}$ by theorem B_3), on a continuous arc L, joining two points

α and β inside $\varDelta$, since $\varDelta$ is a connected domain. Thus we arrived at a contradiction.

BEURLING's results are regarded as localizations of the maximum principle of analytic functions. Concerning the general principle of maximum modulus some analogous results may be obtained.

Let $f(z)$ be a uniform function meromorphic in an arbitrary domain D, C being its boundary. Suppose that E is a closed set, contained in C, whose absolute harmonic measure is zero. Denote by $\mathfrak{D}$ the set of values taken by $f(z)$ in D and put $S_{C-E}=\sum_{z'\in C-E} S_{z'}^{(D)}$, i.e., the sum of all $S_{z'}^{(D)}$, as z' varies in $C-E$.

Then it is obvious that their closed covers $\overline{\mathfrak{D}}$ and $\overline{S}_{C-E}$ satisfy an inequality $\overline{\mathfrak{D}}>\overline{S}_{C-E}$. Corresponding to theorem B_1, we may take NEVANLINNA's extension of LINDELÖF's principle which we have used already as lemma 2 in § 2.

Theorem N_1. *If the function $f(z)$ satisfies that $|f(z)|\leqq M$ in D and $\overline{\lim}\,|f(z)|\leqq m$ $(m\leqq M)$ for any boundary point belonging to $C-E$, then $|f(z)|\leqq m$ throughout in D.*

Theorem N_2. *Let $f(z)$ be uniform and meromorphic in D, its boundary being denoted by C, and let E be a closed set, contained in C, whose absolute harmonic measure is zero. Then, for any exterior point α of $\mathfrak{D}$, we have $\rho(\overline{\mathfrak{D}},\alpha)=\rho(\overline{S}_{C-E},\alpha)$.*

Under the same condition on $f(z)$ and E as in theorem N_2, we obtain the following three theorems:

Theorem N_3. $B(\overline{\mathfrak{D}})<B(\overline{S}_{C-E})$.

Theorem 5. *Denote by $\varDelta$ any component of the complementary set of $\overline{S}_{C-E}$ with respect to the w-plane. Then $\varDelta$ is either entirely included in $\overline{\mathfrak{D}}$ or does not contain any point in $\overline{\mathfrak{D}}$.*

As a corollary, we get a precise form of MINAMI's result [1].

Theorem 6. *If a connected domain $\varDelta^*$, containing no point of S_{C-E}, includes an exterior point α of $\mathfrak{D}$, then $\varDelta^*$ excludes all points of $\overline{\mathfrak{D}}$.*

Because $\varDelta^*$ contains no point of S_{C-E} and so $\varDelta^*$ is contained in a component of the complementary set of $\overline{S}_{C-E}$.

2. Now we suppose that $f(z)$ is uniform and meromorphic in D and z_0 is an arbitrary boundary point. Let us consider the case in which $S_{z_0}^{(D)}$ contains an interior point. Denoting by α_0 such a point,

(1) U. MINAMI: Sizyô Danwakwai, No. 116, 1936.

we may find a circle $K_0: |w-\alpha_0| < r_0$, whose closed cover $\overline{K}_0$ is contained entirely inside $S_{z_0}^{(D)}$. Let k_n be the common part of D and the circle: $|z-z_0| < \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$. Then there is a point ζ_1 inside k_1 whose image $\alpha_1 = f(\zeta_1)$ belongs to K_0, since α_0 is a cluster value of $f(z)$ at z_0. Consequently we may find a circle $K_1: |w-\alpha_1| < r_1$ such that its closed cover $\overline{K}_1$ is entirely inside K_0 all values of $\overline{K}_1$ are assumed by $f(z)$ inside k_1. Repeating the same method, we have a sequence of closed circular discs $\overline{K}_n$ with the following properties: $\overline{K}_0 > \overline{K}_1 \cdots > \overline{K}_n > \cdots$ and every value in $\overline{K}_n$ is assumed by $f(z)$ in k_n. Let β be a point in $\prod_{n=1}^{\infty} \overline{K}_n \neq 0$, then β is assumed by $f(z)$ infinitely often near z_0. Using this elementary fact, we obtain, by theorem B_3,

Theorem 7[1]. *Let $f(z)$ be uniform and meromorphic in D and let z_0 be a boundary point, regular concerning the problem of DIRICHLET. Then the set $R_{z_0}^{(D)}$ is everywhere dense in the open set $G = S_{z_0}^{(D)} - S_{z_0}^{(C)}$, provided that G is not empty. Consequently, we have $S_{z_0}^{(D)} = S_{z_0}^{(C)}$, if $R_{z_0}^{(D)}$ is a vacuum. (It is clear that $R_{z_0}^{(D)}$ is so, if $f(z)$ does not take any value infinitely often near z_0).*

Remark 1. Our theorem 2 is improved, when z_0 satisfies the condition of regularity concerning the problem of DIRICHLET. There is a difficult problem. Is it true that BEURLING's property $B(S_{z_0}^{(D)}) \subset B(S_{z_0}^{(C)})$ holds without any condition of regularity on z_0?

Remark 2. Suppose that $f(z)$ is uniform and meromorphic in a domain D, whose order of connection is finite. Then, for any non-isolated boundary point z_0, $S_{z_0}^{(D)} - S_{z_0}^{(C)}$ is an open set, consisting of a finite or enumerable number of connected domains Δ_i $(i = 1, 2, 3, \ldots)$, each being enclosed by $S_{z_0}^{(C)}$, (i.e., any boundary point of each Δ_i lies on $S_{z_0}^{(C)}$). We have the following results, which are somewhat related to BEURLING's problem[2].

(i) If z_0 is not accessible, $R_{z_0}^{(D)}$ includes every value of $S_{z_0}^{(D)} - S_{z_0}^{(C)}$.

(ii) If z_0 is simply accessible, $R_{z_0}^{(D)}$ includes every value of $S_{z_0}^{(D)} - S_{z_0}^{(C)}$, except at most two such values, and, if there are two such exceptional values, then $R_{z_0}^{(D)}$ must coincide with the w-plane excluding these two points.

(iii) If z_0 is k-ply accessible, k greater than 1, $R_{z_0}^{(D)}$ contains every value in $S_{z_0}^{(D)} - S_{z_0}^{(C)}$ except at most k such values.

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(1) This is regarded as an extension of WIERSTRASS' theorem.

(2) A. BEURLING: loc. cit. See p. 103.

(i) is already proved in theorem 1. Let C_ν ($\nu = 1, 2, \dots m$) be all components of the boundary C and suppose that z_0 lies in C_1. Suppose that near z_0 we have $k+1$ distinct exceptional values a_ν ($\nu = 1, 2, \dots, k+1$) in PICARD's sense, contained in $S_{z_0}^{(D)} - S_{z_0}^{(C)}$ then, by means of theorem 1, there exist $k+1$ paths L_ν ($\nu = 1, 2, \dots, k+1$) inside D, excluding the common end point z_0, along which $f(z)$ tends to a_ν ($\nu = 1, 2, \dots, k+1$), as z tends to z_0, respectively. Without loss of generality, we may suppose that these paths do not intersect one other except at z_0. Consequently, if we describe a small circle $K: |z - z_0| = r < \delta$, where δ is the distance of $\sum_{\nu=2}^{\nu=k} C_\nu$ from z_0, the interior of K is decomposed into $K+1$ angular domains. Since z_0 is k-ply accessible, there must exist at least one angular domain $\varDelta$, which never contains any point of continuum C_1, and so is contained entirely in D. Applying LINDELÖF-IVERSEN's theorem(1), we see that $R_{z_0}^{(D)}$ contains all values except at most two. Consequently it follows that $k+1 \leqq 2$, that is, z_0 must become simply accessible. Thus our assertions (ii) and (iii) have been proved at the same time. In particular, if D is a JORDAN domain, there holds only the case (ii), and this was also proved by GROSS and DOOB(2).

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(1) F. IVERSEN: loc. cit. See p. 29 and also cf. G. JULIA: Leçons sur les fonctions uniformes, 1924, pp. 95–97.

(2) W. GROSS: loc. cit. J. L. DOOB: loc. cit. pp. 753–757.