HOKKAIDO UNIVERSITY

Title	A REMARK TO A COVARIANT DIFFERENTIATION PROCESS
Author(s)	Michihiro, Satoshi
Citation	Journal of the Faculty of Science Hokkaido Imperial University. Ser. 1 Mathematics, 9(3), 189-192
Issue Date	1940
Doc URL	https://hdl.handle.net/2115/55955
Type	departmental bulletin paper
File Information	JFSHIU_09_N3_189-192.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

A REMARK TO A COVARIANT DIFFERENTIATION PROCESS

By

Satoshi MICHIHIRO

In one of his papers CRAIG[1] has introduced a covariant derivative

$$(1)\qquad T^{\alpha\cdots}_{\beta\cdots x(m-1)\tau} - mT^{\alpha\cdots}_{\beta\cdots x(m)\lambda}\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}$$

from a tensor $T^{\alpha}_{\beta}{}^{\cdots}_{\cdots}$ whose components depend on n coordinates x and their m derivatives x, x', x'', ..., $x^{(m)}$, that is, are of order m, where

$$\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\} \equiv x'^{\alpha}\Gamma^{\lambda}_{\tau\alpha} + \frac{1}{2}x''^{\beta}f_{\tau\delta\beta}f^{\delta\lambda},$$

which was obtained by TAYLOR[2], and partial differentiation was denoted by the subscript. Thereafter, extending this process, JOHNSON[3] has introduced the following covariant derivative

$$(2)\qquad T^{\alpha\cdots}_{\beta\cdots x(m-2)\tau} - (m-1)T^{\alpha\cdots}_{\beta\cdots x(m-1)\lambda}\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\} - \frac{m(m-1)}{2}T^{\alpha\cdots}_{\beta\cdots x(m)\lambda}\left|\begin{matrix}\lambda\\ \gamma\end{matrix}\right|,$$

where

$$\left|\begin{matrix}\lambda\\ \gamma\end{matrix}\right| \equiv Q^{\lambda}_{x^{\tau}} - Q^{\lambda}_{x'^{\alpha}}\left\{\begin{matrix}\alpha\\ \gamma\end{matrix}\right\} + Q^{\alpha}\Lambda^{\lambda}_{\alpha\tau}, \quad Q^{\alpha} \equiv x''^{\alpha} + \Gamma^{\alpha}_{\beta\tau}x'^{\beta}x'^{\tau},$$

$$\Lambda^{\lambda}_{\alpha\tau} \equiv \Gamma^{\lambda}_{\alpha\tau} - \frac{1}{2}f^{\lambda\beta}\left(f_{\tau\beta\tau}\left\{\begin{matrix}\tau\\ \alpha\end{matrix}\right\} + f_{\beta\alpha\tau}\left\{\begin{matrix}\tau\\ \gamma\end{matrix}\right\} - f_{\alpha\tau\tau}\left\{\begin{matrix}\tau\\ \beta\end{matrix}\right\}\right).$$

To obtain this result, she eliminated $\partial x''^{\beta}/\partial y^{k}$ making use of a very complicated calculation. Accordingly it seems almost impossible to extend her method further.

In the present paper it is proposed to introduce a general covariant differentiation process which involves (1) and (2) as special cases.

---

(1) H. V. CRAIG, On a covariant differentiation process, Bull. of the Amer. Math. Soc., Vol. 37 (1931), p. 731-734.

(2) J. H. TAYLOR, A generalization of LEVI-CIVITA's parallelism and the FRENET formulas, Trans. of the Amer. Math. Soc., Vol. 27 (1925), p. 246-264.

(3) M. M. JOHNSON, An extension of a covariant differentiation process, Bull. of the Amer. Math. Soc., Vol. 46 (1940), p. 269-271.

The writer expresses his heartfelt thanks to Prof. A. KAWAGUCHI for his kind advices.

In a KAWAGUCHI space(1) of order m and dimension n a covariant differentiation of a contravariant vector X^i is described in the form

$$\frac{\delta X^i}{dt} = \frac{dX^i}{dt} + \Gamma^i_j X^j .$$

The notations

$$X^{(0)i} = X^i , \quad X^{(s)i} = \frac{d^s X^i}{dt^s} \qquad (s = 1, 2, \ldots)$$

will be adopted for simplicity.

Let X^i be any contravariant vector, then its r-th covariant derivative is expressed by a linear combination of X^i and its ordinary derivatives with respect to t

$$(3) \qquad \frac{\delta^r X^i}{dt^r} = \sum_{s=0}^{r} \binom{r}{s} \overset{(s)}{\Gamma}{}^i_j X^{(r-s)j} ,$$

where

$$\overset{(0)}{\Gamma}{}^i_j = \delta^i_j , \quad \overset{(1)}{\Gamma}{}^i_j = \Gamma^i_j , \quad \overset{(s)}{\Gamma}{}^i_j = \frac{d\overset{(s-1)}{\Gamma}{}^i_j}{dt} + \Gamma^i_k \overset{(s-1)}{\Gamma}{}^k_j \quad (s = 2, 3, \ldots, r) .$$

This can be proved by mathematical induction. That is, if (3) is correct for r,

$$\begin{aligned} \frac{\delta^{r+1} X^i}{dt^{r+1}} &= \sum_{s=0}^{r} \binom{r}{s} \left(\overset{(s)}{\Gamma}{}^i_j X^{(r+1-s)j} + \frac{d\overset{(s)}{\Gamma}{}^i_j}{dt} X^{(r-s)j} + \Gamma^i_k \overset{(s)}{\Gamma}{}^k_j X^{(r-s)j} \right) \\ &= \sum_{s=0}^{r} \binom{r}{s} \left(\overset{(s)}{\Gamma}{}^i_j X^{(r+1-s)j} + \overset{(s+1)}{\Gamma}{}^i_j X^{(r-s)j} \right) \\ &= \sum_{s=0}^{r+1} \binom{r+1}{s} \overset{(s)}{\Gamma}{}^i_j X^{(r+1-s)j} . \end{aligned}$$

And (3) is primitive for $r = 1$. Therefore (3) is true for all r.

Inversely :

Let X^i be any contravariant vector, then its r-th ordinary derivative with respect to t is expressed by a linear combination of X^i and its covariant derivatives

(1) A. KAWAGUCHI, Theory of connections in a KAWAGUCHI space of higher order, Proc. of the Imp. Acad., Tokyo, Vol. 13 (1937), p. 237-240.

$$(4)\qquad X^{(r)i}=\sum_{s=0}^{r}\binom{r}{s}\overset{(s)}{\Pi}{}^{i}_{j}\frac{\delta^{r-s}X^{j}}{dt^{r-s}}\,,$$

where

$$(5)\qquad \overset{(0)}{\Pi}{}^{i}_{j}=\delta^{i}_{j}\,,\qquad \overset{(1)}{\Pi}{}^{i}_{j}=\Pi^{i}_{j}=-\Gamma^{i}_{j},$$

$$\overset{(s)}{\Pi}{}^{i}_{j}=\frac{d\overset{(s-1)}{\Pi}{}^{i}_{j}}{dt}+\overset{(s-1)}{\Pi}{}^{i}_{k}\Pi^{k}_{j}\qquad(s=2,3,\ldots,r)\,.$$

In one of his papers Prof. A. KAWAGUCHI has introduced the covariant derivation[1]

$$(6)\qquad \overset{p-\rho}{D}_{ij}(T)X^{j}=\sum_{\alpha=\rho}^{p}\binom{\alpha}{\rho}T_{i(\alpha)j}X^{(\alpha-\rho)j}\qquad(\rho=1,2,\ldots,p)$$

along a curve, T_i being any covariant vector of order p and X^i an arbitrary contravariant vector of any order. Use of (4) leads the right member of (6) to

$$\begin{aligned}\overset{p-\rho}{D}_{ij}(T)X^{j}&=\sum_{\alpha=\rho}^{p}\binom{\alpha}{\rho}T_{i(\alpha)k}\sum_{\lambda=0}^{\alpha-\rho}\binom{\alpha-\rho}{\lambda}\overset{(\lambda)}{\Pi}{}^{k}_{j}\frac{\delta^{\alpha-\rho-\lambda}X^{j}}{dt^{\alpha-\rho-\lambda}}\\&=\sum_{\alpha=\rho}^{p}\binom{\alpha}{\rho}T_{i(\alpha)k}\sum_{\lambda=\rho}^{\alpha}\binom{\alpha-\rho}{\alpha-\lambda}\overset{(\alpha-\lambda)}{\Pi}{}^{k}_{j}\frac{\delta^{\lambda-\rho}X^{j}}{dt^{\lambda-\rho}}\\&=\sum_{\lambda=\rho}^{p}\binom{\lambda}{\rho}\sum_{\alpha=\lambda}^{p}\binom{\alpha}{\lambda}T_{i(\alpha)k}\overset{(\alpha-\lambda)}{\Pi}{}^{k}_{j}\frac{\delta^{\lambda-\rho}X^{j}}{dt^{\lambda-\rho}}\,.\end{aligned}$$

Therefore it is obtained covariant tensors

$$(7)\qquad \sum_{\alpha=\lambda}^{p}\binom{\alpha}{\lambda}T_{i(\alpha)k}\overset{(\alpha-\lambda)}{\Pi}{}^{k}_{j}\qquad(\lambda=1,2,\ldots,p)\,.$$

Let any tensor $T^{l}_{i}{}^{\cdots}_{\cdots}$ *be considered, then*

$$(8)\qquad \sum_{\alpha=\lambda}^{p}\binom{\alpha}{\lambda}T^{l}_{i}{}^{\cdots}_{\cdots(\alpha)k}\overset{(\alpha-\lambda)}{\Pi}{}^{k}_{j}\qquad(\lambda=1,2,\ldots,p)$$

are tensors whose indices are one more than those of $T^{l}_{i}{}^{\cdots}_{\cdots}$, *where* $\overset{(s)}{\Pi}{}^{k}_{j}$ $(s=0,1,\ldots,p)$ *are defined by* (5).

For $p=m$, $\lambda=m-1$, one obtains

$$(9)\qquad T^{l}_{i}{}^{\cdots}_{\cdots(m-1)k}\overset{(0)}{\Pi}{}^{k}_{j}+mT^{l}_{i}{}^{\cdots}_{\cdots(m)k}\overset{(1)}{\Pi}{}^{k}_{j}\,,$$

(1) A. KAWAGUCHI, Some intrinsic derivations in a generalized space, Proc. of the Imp. Acad., Tokyo, Vol. 12 (1936), p. 149-151.

and, for $p=m$, $\lambda=m-2$, one obtains

$$(10)\quad T^{l}_{i}{}^{\cdots}{}_{(m-2)k}\overset{(0)}{\Pi}{}^{k}_{j}+(m-1)T^{l}_{i}{}^{\cdots}{}_{(m-1)k}\overset{(1)}{\Pi}{}^{k}_{j}+\frac{m(m-1)}{2}T^{l}_{i}{}^{\cdots}{}_{(m)k}\overset{(2)}{\Pi}{}^{k}_{j}$$

from (8).

CRAIG has used $\left\{ \begin{matrix} i \\ j \end{matrix} \right\}$ in place of our Γ^{i}_{j}, and (9) reduces to his covariant differentiation (1). (10) is different partly from JOHNSON's. But one can reconcile the present process to hers, as follows. Making use of

$$(11)\qquad \frac{\delta dx^{i}}{dt}=dx'^{i}+\left\{ \begin{matrix} i \\ j \end{matrix} \right\}dx^{j},$$

one has

$$df_{ij}=\left(f_{ij(0)k}-f_{ij(1)l}\left\{ \begin{matrix} l \\ k \end{matrix} \right\}\right)dx^{k}+f_{ij(1)l}\frac{\delta dx^{l}}{dt},$$

and in the same way

$$df_{jk}=\left(f_{jk(0)i}-f_{jk(1)l}\left\{ \begin{matrix} l \\ i \end{matrix} \right\}\right)dx^{i}+f_{jk(1)l}\frac{\delta dx^{l}}{dt},$$

$$df_{ki}=\left(f_{ki(0)j}-f_{ki(1)l}\left\{ \begin{matrix} l \\ j \end{matrix} \right\}\right)dx^{j}+f_{ki(1)l}\frac{\delta dx^{l}}{dt}.$$

From these it is found that Λ^{i}_{jk} are transformed just as well as the parameters of an affine connection under a coordinate transformation. Therefore one can introduce the following base connection

$$(12)\qquad \delta Q^{i}=dQ^{i}+\Lambda^{i}_{jk}Q^{j}dx^{k}=dx''^{i}+\left| \begin{matrix} \cdot & i \\ j & \end{matrix} \right|dx^{j}+Q^{i}_{x'^{j}k}\frac{\delta dx^{k}}{dt}.$$

From (11), (12) and

$$T^{l}_{i}{}^{\cdots}{}_{(m-2)j}dx^{j}+(m-1)T^{l}_{i}{}^{\cdots}{}_{(m-1)j}dx'^{j}+\frac{m(m-1)}{2}T^{l}_{i}{}^{\cdots}{}_{(m)j}dx''^{j},$$

which is a tensor obtained immediately by using the vector dx^{i} instead of X^{i} in (6), her covariant derivative (2) is obtained.

August, 1940.