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GEOMETRY IN A SPACE WITH GENERALIZED METRICS II ¹⁾

By

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Introduction

As a natural generalization of a RIEMANNIAN space, we have a FINSLER space²⁾. In a FINSLER space the distance between two consecutive points is given by $ds = L(x, dx)$ while in a RIEMANNIAN space it is given by $ds = \sqrt{g_{ij}(x)dx^i dx^j}$, where L is a positively homogeneous function of degree one in the dx^i . In a FINSLER space $g_{ij}(x, dx) = \frac{1}{2} \partial^2 L / \partial dx^i \partial dx^j$ ³⁾ is adopted as the fundamental tensor and the functions $g_{ij}(x, dx)$ are homogeneous of degree zero in the dx^i .

Other generalizations have been made by E. CARTAN⁴⁾, L. BERWALD⁵⁾, A. KAWAGUCHI⁶⁾ and S. HOKARI⁷⁾. The first two studied a theory of a generalized space (CARTAN space⁸⁾) in which geometry is ruled by an $(n-1)$ -ple integral. On the other hand the last two established geometry in an n -dimensional metric space with a connection which depends on an m -dimensional ($m < n$) surface element (x^i, p_a^i) ($i = 1, 2, \dots, n$; $a = 1, 2, \dots, m$).

The author, in his recent work⁹⁾, studied an n -dimensional metric space with an euclidean connection having the fundamental tensor g_{ij} depending on a line-element (x, x') , where g_{ij} are homogeneous functions of degree zero in the x'^i and are given *a priori*. The contracted tensor A_{0k}^i (k fixed) was decomposed into $l^i A_k$ by Postulate 5¹⁰⁾.

1) The cost of this research has been defrayed from the *Scientific Research Expenditure of the Department of Education*.

2) P. FINSLER [1]. (The numbers in the brackets after the author's name refer to the bibliography at the end of the present paper). See also E. CARTAN [1] and L. BERWALD [1].

3) See E. CARTAN [1], p. 11.

4) E. CARTAN [2].

5) L. BERWALD [2], [3], [4].

6) A. KAWAGUCHI [1], [2].

7) A. KAWAGUCHI and S. HOKARI [1], [2].

8) See BERWALD [2].

9) T. OHKUBO [1].

10) See T. OHKUBO [1], p. 50.

But, in general, it is not natural and, in this paper, it is proposed to take in place of that postulate the following: "The two vectors σ^i and $K_k^i \omega^k$ are identical".

In Chapter I an euclidean connection is determined in the space. The method employed is similar to that of L. BERWALD. But it is to be noticed that the vectors θ_i and σ^i play an important part in this theory (§ 6). In Chapter II the theory of curves is explained and in Chapter III the torsion and curvature tensors are calculated. The last chapter is devoted to the establishment of theory of hypersurfaces, which is almost analogous to that in a FINSLER space.

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Chapter I. A general metric space with an euclidean connection.

§ 1. *The space with an euclidean connection.* First it is proposed to explain the notion of a space of line-elements with an euclidean connection. A line-element means the figure formed by a point $(x^1, x^2, \dots, x^n)$ and a direction issuing from this point; this direction can be defined by n homogeneous parameters $x'^1, x'^2, \dots, x'^n$ which have vector character and only their mutual ratios are essential. The point (x) is called the element. A space of line-elements with an euclidean connection is denoted by $K_n^{(1)}$ and defined analytically by the following two postulates for the metrics and for the parallel displacement of a vector. It is desirable that the metrics is euclidean in the neighbourhood of each line-element:

(I) The square of the distance from any point (x) to its consecutive point $(x+dx)$ with respect to the line-element in (x) is represented by a positive definite quadratic differential form $g_{ij} dx^i dx^j$, the coefficients $g_{ij} = g_{ji}$ depending on only the line-element.

Postulate I means that the g_{ij} are the functions of x^i and x'^i and positively homogeneous of degree zero in the x'^i . Moreover it is assumed that they are analytic in a certain simple-connected region of the x^i and for all the values of the x'^i with exception of the value-system $(0, 0, \dots, 0)$.

The figure which determines the metrics is therefore two points infinitely near to each other $(x), (x+dx)$ and the line-element (x, x') . The figure is said to be an infinitesimal vector (dx) in the line-element (x, x') . One thus gets the fragment of an euclidean space which is

associated to each line-element and he can also arrive at finite vectors, tensors and any other geometric objects. Hereafter the components of a geometric object are always considered as functions of a line-element, that is, x^i and x'^i , which are homogeneous of degree zero in the x'^i .

For a finite vector Postulate I shows that the square of the length of a vector (X) in a line-element (x, x') is given by

$$(1) \quad g_{ij}(x, x')X^iX^j.$$

Hereafter the expression $g_{ij}x'^ix'^j$ will be denoted by g and the inverse system of g_{ij} by g^{ij} and raising and lowering of a suffix can be defined by means of g^{ij} and g_{ij} .

The unit vector along a line-element (x, x') has the contravariant and covariant components

$$(2) \quad l^i = x'^i g^{-\frac{1}{2}}, \quad l_i = g_{ij}l^j$$

respectively.

To obtain the euclidean connection, it is necessary to define the parallel displacement of a vector from a line-element to its consecutive one.

(II) Let (X) (whose components are necessarily homogeneous functions of degree zero in the x'^i) be any vector in a line-element (x, x') and ($x+dx, x'+dx'$) a consecutive line-element of (x, x'). Then the vector ($X+dX$) in the line-element ($x+dx, x'+dx'$) is said to be parallel to the vector (X) in the line-element (x, x'), if

$$(3) \quad DX^i = dX^i + C_{jk}^i X^j dx'^k + \Gamma_{jk}^i X^j dx^k = dX^i + \omega_j^i X^j = 0$$

holds good, where the C_{jk}^i , Γ_{jk}^i are given functions of x^i and x'^i , and

$$(4) \quad \omega_j^i = C_{jk}^i dx'^k + \Gamma_{jk}^i dx^k.$$

As the vector ($X+dX$) should depend on only two line-elements (x, x') and ($x+dx, x'+dx'$) it must be unaltered when $\rho(x, x')x'^i$ takes the place of the x'^i where the function $\rho(x, x')$ is positively homogeneous of degree zero in the x'^i . This leads to the following:

(a) The functions $\Gamma_{jk}^i(x, x')$ and C_{jk}^i are homogeneous of degree zero and -1 in the x'^i respectively;

$$(b) \quad C_{jk}^i x'^k = 0.$$

The functions C_{jk}^i and Γ_{jk}^i should not be absolutely arbitrary. It is desirable that parallel displacement is euclidean:

(III) The length of a vector in a line-element does not change when it is moved by parallel displacement to a consecutive line-element.

This condition gives the following relations:

$$(5) \quad dg_{ij} = g_{ik}\omega_j^k + g_{jk}\omega_i^k = \omega_{ji} + \omega_{ij},$$

$$(6) \quad g_{ij;k} = \Gamma_{ijk} + \Gamma_{jik}, \quad g_{ij;k} = C_{ijk} + C_{jik},$$

where

$$(7) \quad \Gamma_{ijk} = g_{jl}\Gamma_{ik}^l, \quad C_{ijk} = g_{jl}C_{ik}^l,$$

and the partial differentiation with respect to x^i is denoted by $,i$ and that with respect to x'^i by $;i$.

§ 2. *Determination of the functions C_{jk}^i .* In order to determine the coefficients C_{jk}^i , Γ_{jk}^i , which depend on g_{ij} and its derivatives only and define the euclidean connection, additional postulates must be introduced.

(IV) Let (X) and (Y) be two vectors in the same line-element, and $(\bar{D}X)$ and $(\bar{D}Y)$ their covariant differentials when their contravariant components X^i and Y^i remain unaltered and when their common line-element is subject to the same infinitesimal rotation about the center. Then the law of symmetry

$$\vec{X} \bar{D} \vec{Y} = \vec{Y} \bar{D} \vec{X}$$

holds good.

By the definition of the $\bar{D}\vec{X}$ one has $\bar{D}X^i = X^j C_{jk}^i dx'^k$, hence the above law of symmetry is written as follows:

$$X^i Y^k C_{kih} dx'^h = Y^i X^k C_{kih} dx'^h.$$

As the relation holds for any vectors and for any differentials of the x'^i , it follows that

$$(8) \quad C_{ikh} = C_{kih}.$$

From (6) and (8) one obtains

$$(9) \quad C_{ijk} = \frac{1}{2} g_{ij;k}.$$

The functions C_{ijk} thus obtained are homogeneous of degree -1 in the x'^i and symmetric with respect to the first two indices. When the functions C_{ijk} are multiplied by g^{jl} and summed up from 1 to n with regard to j , one gets

$$(10) \quad C_{ik}^l = \frac{1}{2} g^{jl} g_{ij;k}.$$

The functions C_{ik}^l are homogeneous of degree -1 in the x'^i .

§ 3. *The tensor A_{ijk} .* Now a general method will be shown which permits deducing a new tensor from any given tensor.

The simplest tensors in the space, which we have encountered till now, are the unit vector l^i and the fundamental tensor g_{ij} . Let $f(x, x')$ be a homogeneous function of any degree p in the x'^i , then the expressions

$$(11) \quad f_{//i} = g^{\frac{1}{2}} f_{;i}$$

are the homogeneous functions of the same degree p .

Let a notation be introduced for the contracted product of any geometric object and the unit vector l^i , that is,

$$T_{ij} l^j = T_{i0}.$$

On account of the homogeneity of $f(x, x')$ one has

$$(12) \quad f_{//0} = pf.$$

Next, consider any tensor (contravariant, covariant or mixed), for example, a covariant tensor with two indices $T_{ij}(x, x')$ whose components are necessarily homogeneous of degree zero in the x'^i . The expressions $T_{ij//k}$ are the components of the tensor with one more covariant index. Put

$$(13) \quad A_{ijk} = \frac{1}{2} g_{ij//k} = g^{\frac{1}{2}} C_{ijk},$$

which are the components of a tensor. From (13) it follows:

The RIEMANNian spaces are characterized by $A_{ijk} = 0$.

On account of the homogeneity of the g_{ij} it follows:

$$(14) \quad A_{i00} = 0.$$

The tensor A_{ijk} is symmetric for the first two indices. If they are symmetric for all indices, the considered space is FINSLERIAN. The following relations are easily verified :

$$(15) \quad \begin{cases} g_{//i} = 2g(l_i + A_{00i}), & l_{//j} = \delta_j^i - l^i(l_j + A_{00j}), \\ l_{i//j} = 2A_{i0j} + g_{ij} - l_i l_j - l_i A_{00j}. \end{cases}$$

§ 4. *The angular metrics at a point.* The angle between a line-element and its consecutive line-element with the same center is obtained as follows. Let the angle be $d\varphi$, then one has

$$(16) \quad \cos d\varphi = \frac{l_j(\bar{l}^j + \delta \bar{l}^j)}{\sqrt{(l^i + \delta l^i)(l_i + \delta l_i)}} = \frac{1}{\sqrt{1 + \delta l^i \delta l_i}},$$

because by the definition the relation $l_i l^i = 1$ is obtained from which another relation $l_i \delta \bar{l}^i = l^i \delta \bar{l}_i = 0$ is gotten by covariant differentiation, where the notation $\bar{\delta}$ shows the covariant differentiation when the center remains unaltered. After a short calculation from (16) one gets (neglecting the higher order of the infinitesimal $d\varphi$)

$$(17) \quad \begin{aligned} d\varphi^2 &= g_{ij} \delta l^i \delta l^j \\ &= g^{-1} \{ g_{ij} dx'^i dx'^j + 2A_{0ij} dx'^i dx'^j + A_{0i}^l A_{0j} dx'^i dx'^j \\ &\quad - 2l_i dx'^i A_{00j} dx'^j - (l_i dx'^i)^2 - (A_{00i} dx'^i)^2 \}. \end{aligned}$$

It is clear that the last member remains unaltered when x'^i are substituted by $\rho(x, x')x'^i$, where $\rho(x, x')$ is a homogeneous function of degree zero in the x'^i .

§ 5. *Covariant derivatives.* By definition the covariant differentials of any vector $v^i(x, x')$ and the unit vector l^i are written as follows :

$$(18) \quad Dv^i = dv^i + C_{jk}^i v^j dx'^k + \Gamma_{jk}^i v^j dx^k,$$

$$(19) \quad \omega^i = Dl^i = g^{-\frac{1}{2}}(\delta_k^i + A_{0k}^i) dx'^k + x'^i d(g^{-\frac{1}{2}}) + \Gamma_{0k}^i dx^k.$$

Now put

$$(20) \quad \delta_k^i + A_{0k}^i = H^i_k$$

and assume that the determinant $|H^i_k|$ does not vanish identically, then one obtains the inverse system K_i^k , that is,

$$(21) \quad H^i_k K_i^j = \delta_k^j.$$

When (19) is multiplied by $g^{\frac{1}{2}} K_i^l$, one has

$$(22) \quad dx'^l = g^{\frac{1}{2}} K_i^l \omega^i - g^{\frac{1}{2}} K_i^l \Gamma_{0k}^i dx^k - g^{\frac{1}{2}} x'^l d(g^{-\frac{1}{2}}),$$

because $x'^i K_i^l = x'^l$.

When (22) is substituted into (18) on account of the relation (b), one obtains

$$(18)' \quad Dv^i = dv^i + \Gamma_{jk}^{*i} v^j dx^k + T_{jk}^i v^j \omega^k,$$

where

$$(23) \quad \Gamma_{jk}^{*i} = \Gamma_{jk}^i - T_{jr}^i \Gamma_{0k}^r, \quad T_{jk}^i = A_{jl}^i K_k^l.$$

As A_{jk}^i , K_k^l , N^i and ω^k are the components of tensors or vectors, the expressions Γ_{jk}^{*i} are transformed in like manner the coefficients of the affine connection under the transformation of coördinates. Making use of (22) one can write (18) in another form:

$$(18)'' \quad Dv^i = \Gamma_h v^i dx^h + \Gamma'_h v^i \omega^h,$$

where

$$(24) \quad \Gamma_h v^i = v^i_{;h} - g^{\frac{1}{2}} K_r^k \Gamma_{0h}^r v^i_{;k} + \Gamma_{jh}^{*i} v^j,$$

$$(25) \quad \Gamma'_h v^i = g^{\frac{1}{2}} K_h^k v^i_{;k} + T_{jh}^i v^j,$$

because $v^i_{;k} x'^k = 0$. We call $\Gamma_h v^i$ and $\Gamma'_h v^i$ the covariant derivatives of the first and second kinds of v^i respectively. In particular we have for the vector l^i and the tensor g_{ij} :

$$(26) \quad \Gamma_h l^i = l^i_{;h} + l^i (l_k + A_{00k}) K_r^k \Gamma_{0h}^r = 0,$$

$$(27) \quad \Gamma_h g_{ij} = 0.$$

§ 6. The vectors θ_i and σ^i . In this section will be introduced a very important vector θ_i .

Theorem. Let $g^{-\frac{1}{2}} F_i(x, x')$ be the components of a covariant vector in the space $K_n^{(1)}$ which are the homogeneous functions of degree zero in the x'^i , then the PFAFFian forms

$$(28) \quad \theta_j = \frac{1}{2}(F_{i;j} + F_{j;i})dl^i + \frac{1}{2}\left\{-g^{\frac{1}{2}}F_{i;j;l}\Gamma_{00}^{*l} + F_{i;j,l}l^l - g^{-\frac{1}{2}}F_{i,j} + g^{-\frac{1}{2}}F_{j,i}\right\}dx^i$$

are also the components of a vector.

Proof. Under the transformation of the coördinates $x^\lambda = x^\lambda(x^i)$ one gets the relations :

$$(29) \quad (F_{i;j} + F_{j;i})dl^i = (F_{\lambda;\mu} + F_{\mu;\lambda})dl^\lambda Q_j^\mu - (F_{\lambda;\mu} + F_{\mu;\lambda})Q_{il}^\lambda l^l Q_j^\mu dx^i,$$

$$(30) \quad -g^{\frac{1}{2}}F_{i;j;l}\Gamma_{00}^{*l}dx^i = -g^{\frac{1}{2}}F_{\lambda;\mu;\nu}\Gamma_{00}^{*\nu}dx^\lambda Q_j^\mu - g^{\frac{1}{2}}F_{\lambda;\mu;\nu}dx^\lambda Q_{kl}^\nu l^k l^l Q_j^\mu,$$

$$(31) \quad F_{i;j,l}l^l dx^i = F_{\lambda;\mu;\nu}l^\nu dx^\lambda Q_j^\mu + g^{\frac{1}{2}}F_{\lambda;\mu;\nu}dx^\lambda Q_{kl}^\nu l^k l^l Q_j^\mu + F_{\lambda;\mu}Q_{il}^\lambda l^l dx^i Q_j^\mu + F_{\lambda;\mu}dx^\lambda Q_{jk}^\mu l^k,$$

$$(32) \quad g^{-\frac{1}{2}}(-F_{i,j} + F_{j,i})dx^i = g^{-\frac{1}{2}}(-F_{\lambda,\mu} + F_{\mu,\lambda})dx^\lambda Q_j^\mu - F_{\lambda;\mu}dx^\lambda Q_{jk}^\mu l^k + F_{\mu;\lambda}Q_j^\mu Q_{ik}^\lambda l^k dx^i,$$

where

$$Q_i^\lambda = \frac{\partial x^\lambda}{\partial x^i}, \quad Q_{ij}^\lambda = \frac{\partial^2 x^\lambda}{\partial x^i \partial x^j}.$$

When (29), (30), (31) and (32) are substituted into (28), one obtains

$$\theta_i(x^k, x'^k, dx^k, dx'^k) = \theta_\lambda(x^\mu, x'^\mu, dx^\mu, dx'^\mu)Q_i^\lambda,$$

as $g(x, x')$ is the scalar.

Q. E. D.

Now put $g_i(x, x') = g_{ij}x'^j$, $2G_{ij} = g_{i;j} + g_{j;i}$ and take g_i instead of the functions F_i in the above theorem. If it is assumed that the determinant $|G_{ij}|$ does not vanish identically, it is possible to define the inverse system E^{ij} , that is, $G_{ij}E^{jk} = \delta_j^k$. As G_{ij} and E^{ik} are the components of the tensors, there follows

Theorem. The expressions

$$(33) \quad \sigma^i(x, x', dx, dx') = dl^i + \frac{1}{2}E^{ij}\left\{-g^{\frac{1}{2}}g_{k;j;l}\Gamma_{00}^{*l} + g_{k;j,l}l^l - g^{-\frac{1}{2}}g_{k,j} + g^{-\frac{1}{2}}g_{j,k}\right\}dx^k$$

are the components of a vector.

Finally from (18)' one gets

$$(34) \quad K_k^i \omega^k = dl^i + \Gamma_{0k}^{*i} dx^k.$$

§ 7. *Determination of Γ_{jk}^i .* In order to determine the coefficients Γ_{jk}^i let the last two postulates be introduced :

(V)¹⁾ The two vectors σ^i and $K_k^i \omega^k$ are identical.

(VI) The coefficients Γ_{jk}^{*i} are symmetric with respect to j and k , namely :

$$(35) \quad \Gamma_{jk}^{*i} = \Gamma_{kj}^{*i}.$$

Postulate (V) gives the relations

$$(36) \quad \frac{1}{2} E^{ij} \left\{ -g^{\frac{1}{2}} g_{k;j;l} \Gamma_{00}^{*l} + g_{k;j,0} - g^{-\frac{1}{2}} g_{k,j} + g^{-\frac{1}{2}} g_{j,k} \right\} = \Gamma_{0k}^{*i}.$$

When (36) are multiplied by l^k , one gets

$$(2G_u + g^{\frac{1}{2}} g_{0;i;l}) \Gamma_{00}^{*l} = g_{0;i,0} - g^{-\frac{1}{2}} g_{0,i} + g^{-\frac{1}{2}} g_{i,0}.$$

Put $2G_u + g^{\frac{1}{2}} g_{0;i;l} = M_{il}$ and assume that the determinant $|M_{il}|$ does not vanish identically, so that N^{jk} can be defined as the inverse system of M_{il} . Therefore the above equations are solved for Γ_{00}^{*l} as follows :

$$(37) \quad \Gamma_{00}^{*i} = N^{ij} \left\{ g_{0;j,0} - g^{-\frac{1}{2}} g_{0,j} + g^{-\frac{1}{2}} g_{j,0} \right\}.$$

If (37) are substituted into (36), Γ_{0k}^{*i} is gotten from the given functions g_{ij} and their derivatives :

$$(38) \quad \Gamma_{0k}^{*i} = \frac{1}{2} E^{ij} \left\{ -g_{k;j;l} N^{rl} \left[g^{\frac{1}{2}} g_{0;r,0} - g_{0,r} + g_{r,0} \right] + g_{k;j,0} - g^{-\frac{1}{2}} g_{k,j} + g^{-\frac{1}{2}} g_{j,k} \right\}.$$

On the other hand from (23) one gets

$$(39) \quad \Gamma_{0k}^{*i} = \Gamma_{0k}^i - T_{0r}^i \Gamma_{0k}^r = (\delta_r^i - T_{0r}^i) \Gamma_{0k}^r.$$

1) In a FINSLER space where the arc length of a curve $x^i = x^i(t)$ is given by the integral $\int F\left(x, \frac{dx}{dt}\right) dt$, the coefficients of the euclidean connection Γ_{jk}^i and C_{jk}^i are determined perfectly by the postulates I, II, III, IV, VI only and the postulate V $\sigma^i = K_k^i \omega^k$ is satisfied identically (See E. CARTAN [1]).

And from (20) by multiplication of K_j^k one has the relations

$$K_j^i + A_{0k}^i K_j^k = \delta_j^i$$

from which it follows

$$(40) \quad \delta_j^i - A_{0k}^i K_j^k = K_j^i.$$

From (39), (40) and (20) is gotten

$$(41) \quad \Gamma_{0k}^i = H_{\cdot r}^i \Gamma_{0k}^{*r}.$$

On account of (6) and (23) one gets

$$(42) \quad \Gamma_{ijk}^* + \Gamma_{jik}^* = g_{ij,k} - 2T_{ijl} \Gamma_{0k}^l$$

and from (33)

$$(43) \quad \Gamma_{ijk}^* = \Gamma_{kji}^*.$$

From (42) and (43) it follows by the same calculation as in a RIEMANNIAN space ($A_{ijk} = 0$)

$$(44) \quad \Gamma_{ijk}^* = \gamma_{ijk} + T_{ikl} \Gamma_{0j}^l - T_{jkl} \Gamma_{0i}^l - T_{jil} \Gamma_{0k}^l,$$

where γ_{ijk} are the CHRISTOFFEL's symbols of the first kind with respect to g_{ij} . From (44) and (23) one gets

$$(45) \quad \Gamma_{ijk} = \gamma_{ijk} + T_{ikl} \Gamma_{0j}^l - T_{jkl} \Gamma_{0i}^l.$$

When (39) are substituted into (44) and (45) on account of (21), one obtains

$$(46) \quad \Gamma_{ijk}^* = \gamma_{ijk} + A_{ikr} \Gamma_{0j}^{*r} - A_{jkr} \Gamma_{0i}^{*r} - A_{jir} \Gamma_{0k}^{*r},$$

$$(47) \quad \Gamma_{ijk} = \gamma_{ijk} + A_{ikr} \Gamma_{0j}^{*r} - A_{jkr} \Gamma_{0i}^{*r}.$$

If the suffix j in Γ_{ijk}^* and Γ_{ijk} is raised up by multiplication of g^{jh} , one obtains

$$(48) \quad \Gamma_{ik}^{*h} = \gamma_{ik}^h + A_{ikr} \Gamma_{0j}^{*r} g^{jh} - A_{kr}^h \Gamma_{0i}^{*r} - A_{ir}^h \Gamma_{0k}^{*r},$$

$$(49) \quad \Gamma_{ik}^h = \gamma_{ik}^h + A_{ikr} \Gamma_{0j}^{*r} g^{jh} - A_{kr}^h \Gamma_{0i}^{*r},$$

where γ_{ik}^h are CHRISTOFFEL's symbols of the second kind with respect to g_{ij} .

Finally if (36) are substituted into (48) and (49), Γ_{ik}^{*h} and Γ_{ik}^h are obtained as functions of g_{ij} and their derivatives up to the second.

Thus the coefficients Γ_{jk}^i and C_{jk}^i of the connection in $K_n^{(1)}$ can be completely determined.

Chapter II. Theory of curves.

§ 1. *The geodesic line and the extremal.* In this chapter the unit tangential vector l^i to a curve is taken as line-element and the arc length s as parameter. So we have

$$(1) \quad l^i = \frac{dx^i}{ds} = \dot{x}^i,$$

$$(2) \quad g(x, \dot{x}) = 1.$$

If a contravariant vector l^i is transported in its own direction parallelly, there is obtained the line which is called a geodesic line. From Chap. I, (18)' follows its equations

$$\frac{Dl^i}{ds} = H^i_k \left(\frac{d^2 x^k}{ds^2} + \Gamma_{jk}^{*k} \frac{dx^j}{ds} \frac{dx^k}{ds} \right) = 0,$$

which can be written in the form

$$(3) \quad \frac{d^2 x^i}{ds^2} + \Gamma_{jk}^{*i} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0,$$

because the determinant of the tensor H^i_k does not vanish.

Next calculate the differential equations of the extremal in the theory of variation $\delta s = 0$. The EULER's equations of the base function $g^{\frac{1}{2}}$ are

$$(4) \quad \frac{d}{ds} (g^{\frac{1}{2}})_{;i} - (g^{\frac{1}{2}})_{;i} = 0,$$

that is,

$$(5) \quad g^{\frac{1}{2}}(g^{\frac{1}{2}})_{;i;j} \frac{d^2 x^j}{ds^2} + g^{\frac{1}{2}}(g^{\frac{1}{2}})_{;i,j} \frac{dx^j}{ds} - g^{\frac{1}{2}}(g^{\frac{1}{2}})_{;i} = 0.$$

On the other hand differentiating (2) with respect to s and then multiplying with $(g^{\frac{1}{2}})_{;i}$ one gets

$$(6) \quad (g^{\frac{1}{2}})_{;i}(g^{\frac{1}{2}})_{;j} \frac{d^2 x^j}{ds^2} + (g^{\frac{1}{2}})_{;i}(g^{\frac{1}{2}})_{;j} \frac{dx^j}{ds} = 0.$$

When (6) is added to (5) side by side, the equations of the extremal are obtained :

$$g_{;i;j} \frac{d^2 x^j}{ds^2} + g_{;i,j} \frac{dx^j}{ds} - g_{,i} = 0 ,$$

that is,

$$(2G_{ij} + g^{\frac{1}{2}} g_{0;i;j}) \frac{d^2 x^j}{ds^2} + g g_{0;i,0} - g^{\frac{1}{2}} g_{0,i} + g^{\frac{1}{2}} g_{i,0} = 0 ,$$

and according to Chapt. 1, (37) their contravariant forms are the same as (3). Therefore in our space $K_n^{(1)}$ the geodesic line is identical with the extremal in the theory of variation $\delta s = 0$ as in a FINSLER space.

§ 2. *FRENET's formulae.* In this section the space of three dimensions only is considered. As a curve is regarded as the locus of their tangential line-elements, the theory of curves is the same as in the euclidean geometry. The unit tangential vector is l^i and its covariant derivative with respect to s is

$$\frac{Dl^i}{ds} = H^i_k \left(\frac{dl^k}{ds} + \Gamma_{0k}^{*h} \frac{dx^k}{ds} \right)$$

and we put

$$(7) \quad \frac{Dl^i}{ds} = \kappa n^i ,$$

where n^i is a unit vector along $\frac{Dl^i}{ds}$ and κ is called the curvature. It is seen that the vector n^i is perpendicular to the vector l^i , because when the relation $l^i l_i = 1$ is differentiated covariantly, $l^i \frac{Dl_i}{ds} + l_i \frac{Dl^i}{ds} = 0$ is obtained, namely $l_i \frac{Dl^i}{ds} = 0$.

Now let a unit vector b^i be introduced which is perpendicular to each of the vectors l^i and n^i and has the same line-element as l^i and n^i . Then the covariant derivative $\frac{Dn^i}{ds}$ is written in the form

$$(8) \quad \frac{Dn^i}{ds} = \xi l^i + \eta n^i + \zeta b^i ,$$

but η must be zero because the vector $\frac{Dn^i}{ds}$ is perpendicular to the vector n^i . When the relation $n^i l_i = 0$ is differentiated covariantly, one has $l_i \frac{Dn^i}{ds} = -n^i \frac{Dl_i}{ds}$. From the last equation and (7) one gets

$$(9) \quad l_i \frac{Dn^i}{ds} = -\kappa.$$

When (8) are multiplied by l_i and summed up for i and substituted by (9), $\xi = -\kappa$ is obtained. So that (8) becomes

$$(10) \quad \frac{Dn^i}{ds} = -\kappa l^i + \zeta b^i.$$

Analogously one obtains the result:

$$(11) \quad \frac{Db^i}{ds} = -\zeta n^i.$$

Equations (7), (10) and (11) give the FRENET's formulae.

Chapter III. The torsion and curvature tensors.

§ 1. *The torsion tensor.* The covariant differential (18)' in Chap. I of any vector v^i is written in the form

$$(1) \quad Dv^i = dv^i + \omega_j^{*i} v^j,$$

where

$$(2) \quad \omega_j^{*i} = \Gamma_{jh}^{*i} dx^h + T_{jh}^i \omega^h = \omega_j^i.$$

If Δv^i is the covariant differential of v^i which corresponds to the increment δ , the torsion of the space is defined by

$$(3) \quad (\Delta D - D\Delta)x^i = \Omega^i,$$

where $Dx^i = dx^i$, $\Delta x^i = \delta x^i$ and d and δ are exchangeable.

From (3) one obtains

$$(4) \quad \Omega^i = [dx^k \omega_k^{*i}] = T_{kh}^i [dx^k \omega^h].$$

Therefore the tensor T_{kh}^i is the torsion tensor of the space.

§ 2. *The curvature tensors.* The curvature of the space is defined by

$$(5) \quad (\Delta D - D\Delta)v^i = \Omega_k^i v^k.$$

Then one has

$$(6) \quad \Omega_k^i = [\omega_k^{*h} \omega_h^{*i}] - (\omega_k^{*i})'$$

or

$$(6)' : \quad \Omega_{ij} = [\omega_j^{*m} \omega_{im}^*] - (\omega_{ij}^*)',$$

where

$$\Omega_{ij} = \Omega_i^l g_{jl}.$$

These forms are decomposed in the following wise:

$$(7) \quad \Omega_{ij} = \frac{1}{2} R_{ijhk} [dx^h dx^k] + P_{ijhk} [dx^h \omega^k] + \frac{1}{2} S_{ijhk} [\omega^h \omega^k].$$

But as the relation $k\omega_i = 0$ holds good this decomposition is not unique. If the conditions

$$(8) \quad P_{ijh0} = 0, \quad S_{ijh0} = 0$$

are added, the decomposition is unique. Thus there are obtained the curvature tensors R_{ijhk} , P_{ijhk} , S_{ijhk} of the space. From (2) the first term of the second member of (6)' is written in the form:

$$(9) \quad [\omega_j^{*m} \omega_{im}^*] = \frac{1}{2} (\Gamma_{jh}^{*m} \Gamma_{imk}^* - \Gamma_{jk}^{*m} \Gamma_{imh}^*) [dx^h dx^k] + T_{jk}^m \Gamma_{imh}^* [\omega^k dx^h] \\ + \Gamma_{jh}^{*m} T_{imk} [dx^h \omega^k] + \frac{1}{2} (T_{jh}^m T_{imk} - T_{jk}^m T_{imh}) [\omega^h \omega^k].$$

On the other hand, as $\omega_{ij} = \Gamma_{ijk} dx^k + C_{ijk} dx'^k$ from Chap. I, (18), the second term of the second member of (6)' is written in the form

$$(10) \quad (\omega_{ij})' = \frac{1}{2} \{ \Gamma_{ijk, h} - \Gamma_{ijh, k} - g^{\frac{1}{2}} \Gamma_{ijk, s} K_r^s \Gamma_{0h}^r + g^{\frac{1}{2}} \Gamma_{ijh, s} K_r^s \Gamma_{0k}^r \\ - g^{\frac{1}{2}} C_{ijr, h} K_s^r \Gamma_{0k}^s + g^{\frac{1}{2}} C_{ijr, k} K_s^r \Gamma_{0h}^s \} [dx^h dx^k] \\ + g^{\frac{1}{2}} \Gamma_{ijh, s} K_k^s [\omega^k dx^h] + \frac{1}{2} g^{\frac{1}{2}} (C_{ijr, h} K_k^r - C_{ijr, k} K_h^r) [\omega^h \omega^k].$$

When (9) and (10) are substituted into equations (6)', one gets from (7):

$$(11) \quad R_{ijhk} = \Gamma_{jh}^{*m} \Gamma_{imk}^* - \Gamma_{jk}^{*m} \Gamma_{imh}^* + \Gamma_{ijk, h} - \Gamma_{ijh, k} - \Gamma_{ijk||s} K_r^s K_{0h}^r + \Gamma_{ijh||s} K_r^s \Gamma_{0k}^r - g^{\frac{1}{2}} C_{ijr, h} K_s^r \Gamma_{0k}^s + g^{\frac{1}{2}} C_{ijr, k} K_s^r \Gamma_{0h}^s,$$

$$(12) \quad P_{ijhk} = g^{\frac{1}{2}} C_{ijr, h} K_k^r - \Gamma_{ijh||s} K_k^s + \Gamma_{jh}^{*m} T_{imk} - T_{jk}^m \Gamma_{imk}^*$$

and

$$(13) \quad S_{ijhk} = T_{jh}^m T_{imk} - T_{jk}^m T_{imh}.$$

The tensors R_{ijhk} and S_{ijhk} are skew-symmetric with respect to h, k and i, j respectively. For the tensors P_{ijhk} and S_{ijhk} the conditions (8) are satisfied identically. In fact, as $K_s^r l^s = l^r$ it follows that

$$\Gamma_{ijh||s} K_k^s l^k = \Gamma_{ijh; s} x'^s = 0,$$

because Γ_{ijh} are the homogeneous functions of degree zero in the x'^k . Similarly it follows $C_{ijr, h} K_k^r l^k = 0$, $T_{imk} l^k = 0$. Therefore one gets $P_{ijh0} = 0$ and $S_{ijh0} = 0$.

For our connection we have the equations $Dg_{ik} = 0$, therefore the relations

$$(14) \quad (\Delta D - D\Delta)g_{ij} = \mathcal{Q}_{ij} + \mathcal{Q}_{ji} = 0$$

hold. As any two of the three exterior products $[dx^h dx^k]$, $[dx^h \omega^k]$, $[\omega^h \omega^k]$ can be made to vanish by a suitable selection of dx^h , δx^h , while the other remains without vanishing, it follows that the first two indices of the three curvature tensors are skew-symmetric by substitution of the last equations in (7); namely

$$(15) \quad R_{ijhk} + R_{jihk} = 0, \quad P_{ijhk} + P_{jihk} = 0, \quad S_{ijhk} + S_{jihk} = 0.$$

From the skew-symmetry of the curvature tensors one gets immediately:

$$R_r^r{}_{hk} = P_r^r{}_{hk} = S_r^r{}_{hk} = 0.$$

§ 3. *BIANCHI's identities.* BIANCHI's identities are given by exterior derivation of $\mathcal{Q}^i$ and $\mathcal{Q}_j^i$, namely:

$$(16) \quad (\mathcal{Q}^i)' - [dx^k \mathcal{Q}_k^i] + [\omega_k^i \mathcal{Q}^k] = 0,$$

$$(17) \quad (\mathcal{Q}_{ij})' - [\omega_j^k \mathcal{Q}_{ik}] + [\omega_i^k \mathcal{Q}_{jk}] = 0.$$

he gets

$$(8) \quad x_i^\alpha x_i^\beta = \delta_\alpha^\beta, \quad g_{\alpha\beta} x_k^\alpha = g_{ik} x_\alpha^i, \quad g^{\alpha\beta} x_\beta^i = g^{ik} x_k^\alpha, \quad g^{\alpha\beta} = g^{ik} x_i^\alpha x_k^\beta.$$

Any tensor defined in the normal line-element $(x(u), x'(u))$ on the hypersurface is decomposed in the manner:

$$(9) \quad T_i^k = T_\alpha^\beta x_i^\alpha x_\beta^k + T_{\alpha 0} x_i^\alpha l^\beta + T^{0\alpha} l_i x_\alpha^k + T_{00} l_i l^k,$$

where $T_\beta^\alpha = T_j^i x_\beta^j x_i^\alpha$, $T_{\alpha 0} = T_{\alpha 0} x_\alpha^i$, $T_0^\alpha = T_0^\alpha x_i^\alpha$. The T_i^k are called the space components and the T_β^α the hypersurface components of the tensor T_i^k . If $T_{i0} = T_0^k = 0$, $T_{00} = 0$ and the T_i^k is the hypersurface tensor. An example of a hypersurface tensor is the tensor $g_{ij} - l_i l_j$ and its hypersurface components are $g_{\alpha\beta}$. From (7), (8) and (9) one gets:

$$(10) \quad g_{\alpha\beta} x_i^\alpha x_j^\beta = g_{ij} - l_i l_j, \quad g^{\alpha\beta} x_\alpha^i x_\beta^j = g^{ij} - l^i l^j,$$

$$(11) \quad x_\alpha^i x_j^\alpha = \delta_j^i - l_j l^i.$$

Putting

$$(12) \quad \begin{aligned} \Gamma_{\beta k}^{*\alpha} &= \Gamma_{ik}^{*h} x_\beta^i x_h^\alpha, \quad \Gamma_{\alpha 0\beta}^* = \Gamma_{ik}^{*h} x_\alpha^i x_\beta^k l_h, \\ \Gamma_{00}^{*\alpha} &= \Gamma_{ik}^{*h} x_h^\alpha l^i l^k, \end{aligned}$$

one obtains the relations:

$$(13) \quad \begin{cases} \Gamma_{00}^{*i} = \Gamma_{00}^{*\alpha} x_\alpha^i + \Gamma_{000}^* l^i, & \Gamma_{\alpha k}^{*\alpha} = \Gamma_{rk}^{*r} - \Gamma_{0k}^{*0}, \\ \Gamma_{\alpha h}^{*i} \Gamma_{jk}^{*\alpha} = \Gamma_{rh}^{*i} \Gamma_{jk}^{*r} - \Gamma_{0h}^{*i} \Gamma_{jk}^{*0}. \end{cases}$$

The second fundamental form of the hypersurface is defined as in the RIEMANNIAN space by

$$(14) \quad a_{\alpha\beta} du^\alpha du^\beta = \pi_\beta du^\beta \quad (a_{\alpha\beta} = a_{\beta\alpha}),$$

where

$$(15) \quad \pi_p = l_i D x_p^i = -\omega_i x_p^i = -\omega_p.$$

From (14) and (15) one gets

$$(16) \quad \begin{aligned} a_{\alpha\beta} &= l_i \left\{ \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma_{jk}^{*i} \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} + C_{jk}^{*i} \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x'^k}{\partial u^\beta} \right\} \\ &= \mathfrak{S}_\alpha^\alpha \left(l_i \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma_{\alpha\beta}^{*0} \right) \\ &= \mathfrak{R}_\alpha^\alpha(\quad , \quad), \end{aligned}$$

where $\mathfrak{S}^\alpha_\alpha$ is the inverse system of $\delta^\alpha_\alpha + T^\alpha_{\alpha\delta} g^{\delta\alpha}$ and $\mathfrak{R}^\alpha_\alpha = K^\alpha_{\delta\alpha} g^{\delta\alpha} g_{\delta\alpha}$.

If one puts

$$(17) \quad \pi^\alpha_{\beta\tau} = \gamma^\alpha_{\beta\tau} du^\tau = x^\alpha_i Dx^\beta_i = -x^\beta_i Dx^\alpha_i,$$

the functions $\gamma^\alpha_{\beta\tau}$ are the coefficients of the parallel displacement induced in the hypersurface. Because for a hypersurface vector ($v_i l^i = 0$) one has

$$\delta v_\alpha = (Dv_i) x^\alpha_i = D(v_i x^\alpha_i) - v_i Dx^\alpha_i = dv_\alpha - \pi^\alpha_\alpha v_\tau,$$

where the parallel displacement in the hypersurface is denoted by the symbol δ .

From (17) one gets

$$(18) \quad \gamma^\alpha_{\beta\tau} = x^\alpha_i \frac{\partial^2 x^i}{\partial u^\beta \partial u^\tau} + \Gamma^{*\alpha}_{\beta\tau} - T^\alpha_{\beta\sigma} g^{\sigma\delta} a_{\delta\tau}.$$

§ 2. *The torsion and curvature tensors of the hypersurface.* The decompositions of ω^i and Dx^α_i are

$$(19) \quad \omega^i = \omega^\alpha x^\alpha_i = -\pi^\alpha_\alpha x^\alpha_i,$$

and

$$(20) \quad Dx^\alpha_i = \pi^\alpha_\alpha u^\beta_i - \omega_\alpha l^i = \pi^\alpha_\alpha x^\beta_i + \pi_\alpha l^i = dx^\alpha_i + \omega^i_k x^\alpha_k.$$

The above two equations lead to the following:

$$(19)' \quad \frac{\partial l^i}{\partial u^\alpha} + \Gamma^{*i}_{\alpha k} \frac{\partial x^k}{\partial u^\alpha} = -a^\alpha_\alpha \frac{\partial x^k}{\partial u^\alpha} K_{k\alpha}^i$$

and

$$(20)' \quad \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma^{*i}_{\alpha\beta} = \gamma^\alpha_{\alpha\beta} x^\alpha_i + T^\alpha_{\alpha\sigma} a^\sigma_\beta + a_{\alpha\beta} l^i.$$

As $Dg_{ij} = 0$, one gets

$$(21) \quad \delta g_{\alpha\beta} = 0$$

or

$$(22) \quad \frac{\partial g_{\alpha\beta}}{\partial u^\tau} = \gamma_{\alpha\beta\tau} + \gamma_{\beta\alpha\tau} \quad (\gamma_{\alpha\beta\tau} = g_{\alpha\beta} \gamma^\alpha_{\alpha\tau}).$$

This shows that the induced connection is euclidean. The torsion tensor of the hypersurface is defined by

$$(23) \quad B^\sigma = [du^\alpha \pi_\alpha^\sigma] = \frac{1}{2} B_{\alpha\beta}^\sigma [du^\alpha du^\beta],$$

where

$$(24) \quad B_{\alpha\beta}^\sigma = \gamma_{\alpha\beta}^\sigma - \gamma_{\beta\alpha}^\sigma = a_\alpha^\varepsilon T_{\beta\varepsilon}^\sigma - a_\beta^\varepsilon T_{\alpha\varepsilon}^\sigma$$

by virtue of (18).

From (22) by suitable exchange of indices and addition are obtained, on account of (24),

$$(25) \quad \gamma_{\alpha\beta\tau} = \frac{1}{2} \left\{ \frac{\partial g_{\alpha\beta}}{\partial u^\tau} + \frac{\partial g_{\beta\tau}}{\partial u^\alpha} - \frac{\partial g_{\tau\alpha}}{\partial u^\beta} \right\} + \frac{1}{2} \{ B_{\alpha\beta\tau} + B_{\tau\alpha\beta} - B_{\beta\tau\alpha} \} \\ = \quad \quad \quad + a_\alpha^\varepsilon T_{\tau\beta\varepsilon} - a_\beta^\varepsilon T_{\tau\alpha\varepsilon}.$$

Moreover from (24) we have the relations:

$$(26) \quad B_{\alpha\beta\tau} = -B_{\tau\beta\alpha}, \quad B_{\alpha\beta\tau} + B_{\beta\tau\alpha} + B_{\tau\alpha\beta} = 0.$$

On the other hand the curvature tensor of the hypersurface is defined by

$$(27) \quad B_\alpha^\beta = [\pi_\alpha^\beta \pi_\tau^\beta] - (\pi_\alpha^\beta)' = \frac{1}{2} B_{\alpha\tau\delta}^\beta [du^\tau du^\delta],$$

where

$$(28) \quad B_{\alpha\tau\delta}^\beta = \frac{\partial \gamma_{\alpha\tau}^\beta}{\partial u^\delta} - \frac{\partial \gamma_{\alpha\delta}^\beta}{\partial u^\tau} + \gamma_{\alpha\tau}^\gamma \gamma_{\gamma\delta}^\beta - \gamma_{\alpha\delta}^\gamma \gamma_{\gamma\tau}^\beta.$$

§ 3. *BIANCHI's identities. Conditions of integrability.* BIANCHI's identities are:

$$(29) \quad (B^\alpha)' - [du^\tau B_\tau^\alpha] + [\pi_\tau^\alpha B^\tau] = 0$$

and

$$(30) \quad (B_{\alpha\beta})' - [\pi_\beta^\tau B_{\alpha\tau}] + [\pi_\alpha^\tau B_{\beta\tau}] = 0 \quad (B_{\alpha\beta} = g_{\tau\beta} B_\alpha^\tau).$$

(29) yields the identities:

$$(31) \quad B_{\alpha\beta|\tau}^\sigma + B_{\alpha\tau}^\gamma B_{\gamma\beta}^\sigma - B_{\alpha\sigma}^\gamma B_{\gamma\tau}^\beta + \text{cyclic}(\alpha\beta\gamma) = 0,$$

where $B_{\alpha\beta|\tau}^\sigma$ represents the covariant derivation in the sense of the connection on the hypersurface and +cyclic($\alpha\beta\gamma$) shows the expres-

sions made from the written expression by exchange of the indices α , β and γ cyclically. On the other hand from (30) one gets:

$$(32) \quad B_{\alpha\beta\gamma\delta|\varepsilon} + B_{\alpha\beta\delta\tau} B_{\varepsilon\tau}^{\gamma} + \text{cyclic}(\gamma\delta\varepsilon) = 0.$$

From Chap. III, (6) and the definition of covariant differential one gets

$$(33) \quad [Dx_{\alpha}^k \omega_k^i] - (Dx_{\alpha}^i)' = \Omega_k^i x_{\alpha}^k.$$

On account of (20) the first term of the first member of (33) becomes

$$(34) \quad [Dx_{\alpha}^k \omega_k^i] = [\pi_{\alpha}^{\beta} \pi_{\beta}^{\gamma}] x_{\alpha}^i + [\pi_{\alpha}^{\beta} \pi_{\beta}^{\gamma}] l^i - [\pi_{\alpha}^{\beta} dx_{\beta}^i] + [\pi_{\alpha}^{\beta} \omega_{\beta}^i] l^k.$$

On the other hand the second term becomes

$$(35) \quad (Dx_{\alpha}^i)' = [dx_{\beta}^i \pi_{\alpha}^{\beta}] + (\pi_{\alpha}^{\beta})' x_{\beta}^i + \pi_{\alpha}^{\beta} l^i - [\pi_{\alpha}^{\sigma} \pi_{\sigma}^{\gamma}] x_{\alpha}^i - [\omega_{\alpha}^i \pi_{\alpha}^{\gamma}] l^k.$$

Therefore if (35) is subtracted from (34), one obtains, taking heed of (27),

$$(36) \quad \Omega_k^i x_{\alpha}^k = B_{\alpha}^{\sigma} x_{\sigma}^i + [\pi_{\alpha}^{\sigma} \pi_{\sigma}^{\gamma}] x_{\alpha}^i + \{[\pi_{\alpha}^{\sigma} \pi_{\sigma}^{\gamma}] - \pi_{\alpha}^{\gamma}\} l^i.$$

When (36) is multiplied by x_{α}^{β} and l_i and summed up for i , then one gets

$$(37) \quad \Omega_k^i x_{\alpha}^k x_{\alpha}^{\beta} = \Omega_{\alpha}^{\beta} = D_{\alpha}^{\beta} [du^{\gamma} du^{\delta}] = B_{\alpha}^{\beta} + [\pi_{\alpha}^{\beta} \pi_{\alpha}^{\gamma}]$$

and

$$(38) \quad \Omega_k^0 x_{\alpha}^k = \Omega_{\alpha}^0 = [\pi_{\alpha}^{\beta} \pi_{\beta}^{\gamma}] - \pi_{\alpha}^{\gamma}$$

respectively, where

$$D_{\alpha}^{\beta} = R_{\alpha}^{\beta} - P_{\alpha}^{\beta} a_{\alpha}^{\sigma} + P_{\alpha}^{\beta} a_{\sigma}^{\sigma} + S_{\alpha}^{\beta} a_{\sigma}^{\sigma} a_{\alpha}^{\sigma}.$$

(37) becomes explicitly

$$(39) \quad R_{\alpha}^{\beta} - P_{\alpha}^{\beta} a_{\alpha}^{\sigma} + P_{\alpha}^{\beta} a_{\sigma}^{\sigma} + S_{\alpha}^{\beta} a_{\sigma}^{\sigma} a_{\alpha}^{\sigma} = B_{\alpha}^{\beta} + a_{\alpha}^{\beta} a_{\alpha}^{\sigma} - a_{\sigma}^{\beta} a_{\alpha}^{\sigma}.$$

Equations (39) are the GAUSS' equations for the hypersurface. From (38) it follows:

$$(40) \quad R_{\alpha}^0 - P_{\alpha}^0 a_{\alpha}^{\sigma} + P_{\alpha}^0 a_{\sigma}^{\sigma} + S_{\alpha}^0 a_{\sigma}^{\sigma} a_{\alpha}^{\sigma} = a_{\alpha\tau|\delta} - a_{\alpha\delta|\tau} + B_{\tau\delta}^{\varepsilon} a_{\alpha\varepsilon}.$$

This is the CODAZZI's equations for the hypersurface.

By exterior derivation of (37) and (38) one obtains more identities on account of (30), (27) and (37), (38) themselves:

$$(41) \quad (\mathcal{Q}_{\alpha\beta})' = [\pi_{\beta}^{\tau} \mathcal{Q}_{\alpha\tau}] - [\pi_{\alpha}^{\tau} \mathcal{Q}_{\beta\tau}] + [\pi_{\beta} \mathcal{Q}_{\alpha 0}] - [\pi_{\alpha} \mathcal{Q}_{\beta 0}],$$

$$(42) \quad (\mathcal{Q}_{\alpha 0})' = [\pi_{\alpha}^{\tau} \mathcal{Q}_{\tau 0}] - [\beta_{\alpha}^{\tau} \pi_{\tau}].$$

Bibliography.

- L. BERWALD, [1], Über FINSLERSche und verwandte Räume, C.R., 2. Congr. Math. des Pays Slaves, Prag, 1935, 1-16.
- , [2], Über die n -dimensionalen CARTANSchen Räume und eine Normalform der zweiten Variation eines $(n-1)$ -fachen Oberflächenintegrals, Acta Math., 71, 191-248.
- , [3], Über FINSLERSche und CARTANSche Geometrie I. Geometrische Erklärungen der Krümmung und des Hauptskalars eines zweidimensionalen FINSLERSchen Raumes, Mathematica, Cluj, 16 (1940).
- , [4], Über FINSLERSche und CARTANSche Geometrie II. Invarianten bei der Variation vielfacher Integrale und Parallelhyperflächen in CARTANSchen Räumen, Comp. Math., 7, Fasc. 1 (1939), 141-176.
- E. CARTAN, [1], Les espaces de FINSLER, Actualités scientifiques, 79, Paris, Hermann et Cie, 1934.
- , [2], Les espaces métriques fondés sur la notion d'aire, Actualités scientifiques, 72, Paris, Hermann et Cie, 1933.
- A. KAWAGUCHI, [1], Ein metrischer Raum, der eine Verallgemeinerung des FINSLERSchen Raumes ist, Monatsh. für Math. und Phys., 43 (1936), 289-297.
- , [2], Die Differentialgeometrie höherer Ordnung II. Über die n -dimensionalen metrischen Räume mit vom m -dimensionalen Flächenelement abhängigem Zusammenhang, Journ. Fac. of Science, Hokkaido Imp. Univ., Ser. I, 9 (1940), 153-188.
- A. KAWAGUCHI and S. HOKARI, [1], Die Grundlegung der Geometrie der fünf-dimensionalen metrischen Räume auf Grund des Begriffs des zwei-dimensionalen Flächeninhalts, Proc. of the Imp. Acad., 16 (1940), 313-319.
- , [2], Die Grundlegung der Geometrie der n -dimensionalen metrischen Räume auf Grund des Begriffs des K -dimensionalen Flächeninhalts, *ibid.*, 320-325.
- T. OHKUBO, [1], Geometry in a space with a generalized metrics, Tensor, 3 (1940), 48-55 (Japanese).