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REMARK ON THE STRUCTURE OF LIE AND JORDAN RINGS

By

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The purpose of this paper⁽¹⁾ is to investigate the form of multiplication of Lie and Jordan rings when we seek for their model in a non-commutative ring.

Suppose L is the module with operator in which a multiplication is defined and the following conditions are satisfied:

- (1) $a \times (b+c) = a \times b + a \times c, \quad (b+c) \times a = b \times a + c \times a$
- (2) $\lambda(a \times b) = (\lambda a) \times b = a \times (\lambda b), \quad a, b, c \in L, \quad \lambda \in P,$

where P is a operator domain.

Suppose that P is the field whose characteristic is m ($0 < m \leq \infty$) (in our case we say the characteristic of a field is ∞ when it is 0 in usual sense), and S is the non-commutative ring in which L is mapped as the module by an operator-homomorphism φ .

Now we assume that $a \times b$ is mapped on the polynomial with coefficients in P of degree n ($< m$) with respect to x, y which are the images of a, b by the homomorphism φ respectively:

$$\begin{aligned} \varphi(a \times b) &= \sum \xi_{j_1, \dots, j_r}^{i_1, \dots, i_r} x^{i_1} y^{j_1} \dots x^{i_r} y^{j_r} + C, \\ \xi &\in P, \quad x, y, C \in S, \quad i_1 + \dots + i_r \leq n, \quad i_k \geq 0, \\ j_1 + \dots + j_r &\leq n, \quad j_k \geq 0, \quad i_1 + \dots + i_r + j_1 + \dots + j_r \neq 0, \end{aligned}$$

where $\sum$ covers $i_1, \dots, i_r, j_1, \dots, j_r$ satisfying the above conditions.

Being $0 \times b = a \times 0 = 0 \times 0 = 0$, it follows immediately that C and the sum of all terms with $i_1 + \dots + i_r = 0$ or $j_1 + \dots + j_r = 0$ vanish. Therefore we can assume from the beginning that

$$\varphi(a \times b) = \sum \xi_{j_1, \dots, j_r}^{i_1, \dots, i_r} x^{i_1} y^{j_1} \dots x^{i_r} y^{j_r}, \quad \begin{aligned} i_1 + \dots + i_r &> 0 \\ j_1 + \dots + j_r &> 0. \end{aligned}$$

(1) This paper has been completed by the encouragement of Prof. M. MORIYA and the helps of my colleagues.

Now for an arbitrary element λ of P it holds from (2)

$$\varphi(a \times (\lambda b)) = \sum \xi_{j_1, \dots, j_r}^{i_1, \dots, i_r} x^{i_1} (\lambda y)^{j_1} \dots x^{i_r} (\lambda y)^{j_r},$$

$$\varphi(\lambda(a \times b)) = \lambda \sum \xi_{j_1, \dots, j_r}^{i_1, \dots, i_r} x^{i_1} y^{j_1} \dots x^{i_r} y^{j_r},$$

whence

$$\sum \lambda^{j_1 + \dots + j_r} \xi_{j_1, \dots, j_r}^{i_1, \dots, i_r} x^{i_1} y^{j_1} \dots x^{i_r} y^{j_r} = \sum \lambda \xi_{j_1, \dots, j_r}^{i_1, \dots, i_r} x^{i_1} y^{j_1} \dots x^{i_r} y^{j_r}.$$

Arranging the expansion with respect to λ , we have $\sum_{t=1}^n \lambda^t A_t = 0$ where A_t is a polynomial in x, y not containing λ . If $\lambda \neq 0$ it follows:

$$(3) \quad \sum_{t=0}^{n-1} \lambda^t A_{t+1} = 0.$$

Since there exist at least n elements different from 0 in P , we have from (3):

$$A_1 + \lambda_1 A_2 + \dots + \lambda_1^{n-1} A_n = 0$$

$$\dots \dots \dots \lambda_i \neq 0 \quad (i = 1, 2, \dots, n)$$

$$A_1 + \lambda_n A_2 + \dots + \lambda_n^{n-1} A_n = 0.$$

Since

$$\begin{vmatrix} 1, & \lambda_1, & \dots, & \lambda_1^{n-1} \\ 1, & \lambda_2, & \dots, & \lambda_2^{n-1} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 1, & \lambda_n, & \dots, & \lambda_n^{n-1} \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_3) (\lambda_1 - \lambda_4) \dots (\lambda_{n-1} - \lambda_n) \neq 0,$$

then $A_t = 0$ ($i=1, \dots, n$). Hence $j_1 + \dots + j_r = 1$. Similarly we have $i_1 + \dots + i_r = 1$.

Therefore changing the coefficients, we can put

$$(4) \quad \varphi(a \times b) = \xi xy + \xi' yx.$$

Now if we add to L the next condition (5) besides (1), (2):

$$(5) \quad a \times b = b \times a,$$

it follows:

$$\xi xy + \xi' yx = \xi yx + \xi' xy, \quad (\xi - \xi') (xy - yx) = 0.$$

Since S is non-commutative there exist at least two elements x, y such that $xy \neq yx$, therefore $\xi = \xi'$, i.e., (4) will be written as $\varphi(a \times b) = \xi(xy + yx)$, and the module L is represented as a Jordan ring.

When there exists an element x in S such that $x^2 \neq 0$ and if we adopt besides (1), (2) the next condition (6) instead of (5):

$$(6) \quad a \times a = 0,$$

we have from (4) $(\xi + \xi')x^2 = 0$. As $x^2 \neq 0$, we have $\xi' = -\xi$, i.e. (4) will be written as $\varphi(a \times b) = \xi(xy - yx)$, and in this case the polynomial $a \times (b \times c) + b \times (c \times a) + c \times (a \times b)$ is mapped on 0, i.e. the module L is represented as a Lie ring.

When for all elements x of S $x^2 = 0$, then

$$0 = (x+y)^2 = x^2 + xy + yx + y^2 = xy + yx, \quad yx = -xy,$$

therefore we can describe (4) as $\varphi(a \times b) = \lambda xy$ and the module L is represented as an associative ring.