HOKKAIDO UNIVERSITY

Title	PARTIALLY ORDERED ABELIAN SEMIGROUPS I. ON THE EXTENSION OF THE STRONG PARTIAL ORDER DEFINED ON ABELIAN SEMIGROUPS
Author(s)	Nakada, Osamu
Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 11(4), 181-189
Issue Date	1951
Doc URL	https://hdl.handle.net/2115/55970
Type	departmental bulletin paper
File Information	JFSHIU_11_N4_181-189.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

PARTIALLY ORDERED ABELIAN SEMIGROUPS

I. ON THE EXTENSION OF THE STRONG PARTIAL ORDER DEFINED ON ABELIAN SEMIGROUPS

By

Osamu NAKADA

Definition 1. A set S is said to be a *partially ordered abelian semigroup* (p. o. semigroup), when in S are satisfied the following conditions:

I) S is an abelian semigroup under the multiplication, that is:

1) A single-valued product ab is defined in S for any pair a, b of S,

2) $ab = ba$ for any a,b of S,

3) $(ab)c = a(bc)$ for any a, b, c of S.

II) S is a partially ordered set under the relation $\geqq$, that is:

1) $a \geqq a$,

2) $a \geqq b$, $b \geqq a$ imply $a = b$,

3) $a \geqq b$, $b \geqq c$ imply $a \geqq c$.

III) Homogeneity: $a \geqq b$ implies $ac \geqq bc$ for any c of S.

A partial order which satisfies the condition III) is called a *partial order defined on an abelian semigroup.*

If S is an abelian group, then S is said to be a *partially ordered abelian group* (p. o. group).

Moreover, if a partial order defined on an abelian semigroup (group) S is a linear order, then S is said to be a *linearly ordered abelian semigroup* (*group*) (l. o. semigroup (l. o. group)).

We write $a > b$ for $a \geqq b$ and $a \neq b$.

Definition 2. A partial order defined on an abelian semigroup S (or a p. o. semigroup S) is called *strong*, when the following condition is satisfied: $ac \geqq bc$ implies $a \geqq b$.

Theorem 1. A partial order defined on an abelian group G is always strong.

Proof. Since G is a group, there exists an inverse element c^{-1} of c. By the homogeneity $ac \geqq bc$ implies $(ac)c^{-1} \geqq (bc)c^{-1}$. Therefore $a \geqq b$.

Theorem 2. In the strong p. o. semigroup S the following properties are held:

1) $ac = bc$ implies $a = b$ (product cancellation law).

2) $ac > bc$ implies $a > b$ (order cancellation law).

3) $a > b$ implies $ac > bc$ for any c of S.

Proof. 1): If $ac = bc$, or, what is the same, if $ac \geqq bc$ and $bc \geqq ac$, then $a \geqq b$ and $b \geqq a$, that is, $a = b$.

2): If $ac > bc$ implies $a = b$, then $ac = bc$, which is absurd.

3): If $a > b$ implies $ac = bc$ for some c of S, then by 1) we have $a = b$ which contradicts the hypothesis $a > b$.

Theorem 3. In the l. o. semigroup S the following properties are held:

1) $ac > bc$ implies $a > b$,

2) $a^n > b^n$ for some positive integer n implies $a > b$.

Proof. 1): If, under the hypothesis $ac > bc$, $a \not> b$, then by the linearity of S, $b \geqq a$. By the homogeneity we have $bc \geqq ac$, this contradicts the hypothesis. 2): Similarly,if $a^n > b^n$ implies $a \not> b$, then we have $b^n \geqq a^n$.

Theorem 4. In the l. o. semigroup S the following conditions are equivalent to each other:

1) $ac \geqq bc$ implies $a \geqq b$ (strong),

2) $ac = bc$ implies $a = b$,

3) $a > b$ implies $ac > bc$ for all c of S.

Proof. 1) $\rightarrow$ 2): See Theorem 2, 1). 2) $\rightarrow$ 3): Suppose that $a > b$ implies $ac = bc$ for some c of S. By 2) we have $a = b$. 3) $\rightarrow$ 1): Suppose that $ac \geqq bc$ implies $a \ngeq b$. By the linearity we have $b > a$, therefore we have $bc > ac$ by 3).

Definition 3. Two p. o. semigroups S and S' will be called *order-isomorphic* if there exists an algebraic isomorphism $x \longleftrightarrow x'$ between them which preserves order: if $a \longleftrightarrow a'$, $b \longleftrightarrow b'$, then $a \geqq b$ if and only if $a' \geqq b'$.

A p. o. semigroup S will be said to be *order-embedded* in a p. o. semigroup S', if there exists an order-isomorphism of S into S'.

Theorem 5. A p. o. semigroup S can be order-embedded in a p. o. group if and only if S is strong.

Proof. Necessity: By Theorem 1.

Sufficiency: By Theorem 2, the product cancellation law is held in

S. Let G be the set of all symbols (a, a'), $a, a' \in S$. We introduce the equality of the elements of G as follows: (a, a') is equal to (b, b') if and only if $ab' = a'b$. As we can then prove, the above-defined equality fulfils the equivalence relation. In particular $(ax, a'x) = (a, a')$ for any x of S. Next, we define the multiplication of the elements in G as follows: $(a, a')(b, b') = (ab, a'b')$. If $(a, a') = (c, c')$ and $(b, b') = (d, d')$, then $(ab, a'b') = (cd, c'd')$. One can easily verify the commutative and associative laws of multiplication. Moreover, (x, x) is the unit element of G and (a', a) is an inverse element of (a, a'). Therefore G is an abelian group under the multiplication introduced above.

Now let us define an order in G as follows: $(a, a') \geqq (b, b')$ if and only if $ab' \geqq a'b$ in S. By the strongness of S it follows immediately that if $(a, a') = (c, c')$, $(b, b') = (d, d')$ and $(a, a') \geqq (b, b')$, then $(c, c') \geqq (d, d')$. Moreover, it is easy to see that the above-defined order $\geqq$ fulfils the conditions II) 1), 2), 3) and III). Therefore G becomes a p. o. group. The correspondence $a \longleftrightarrow (ax, x)$ is the order-isomorphism of S into G.

Such an obtained group $G = Q(S)$, which is the minimal p. o. group containing S and uniquely determined by S apart from its order-isomorphism, will be called the *quotient group* of the p. o. semigroup S.

Corollary. A l. o. semigroup S can be order-embedded in a l. o. group if and only if S is strong.

Theorem 6. Let S be a p. o. semigroup with the unit element e. $e \geqq a$ for any a of S if and only if $a \geqq ab$ for any a, b of S.

Proof. Necessity: $e \geqq b$ for any b of S implies $ae = a \geqq ab$ for any a, b of S.

Sufficiency: If $a \geqq ab$ for any a, b of S, then we put $a = e$. Thus we have $e \geqq b$ for any b of S. Moreover, if S has the zero element, i. e., the element 0 such that $0a = 0$ for any a of S, then $a \geqq 0$ for any a of S.

Corollary. Let S be a p. o. semigroup order-embedded in a p. o. group G. $e \geqq a$ for any a of S, where e is the unit element of G, if and only if $a \geqq ab$ for any a, b of S.

Theorem 7. Let S be a strong p. o. semigroup, G be the quotient group of S and e the unit element of G. $e \geqq a$ for any a of S and $e > a$ $(a \in G)$ implies $a \in S$ if and only if $a \geqq ab$ for any a, b of S and if $a > b$, then there exists an element c of S such that $b = ac$.

Proof. Necessity: By Corollary of Theorem 6, $a \geqq ab$ for any a, b of S. If $a > b$, then $e > a^{-1}b$, and hence $a^{-1}b = c \in S$. Therefore $b = ac$.

Sufficiency: It is clear that $e \geqq a$ for any a of S. Moreover, let x be any element of G such that $e > x$. We can put $x = a^{-1}b$, $a, b \in S$. Thus we obtain $a > b$. Hence there exists an element c of S such that $b = ac$, therefore $x = a^{-1}b = c \in S$.

Definiton 4. Let S be a p.o.semigroup. An element a of S is called *positive* or *negative*, when $a^2 \geqq a$ or $a \geqq a^2$ respectively. In a p. o. group these coincide with the usual definition.

A partial order defined on S is called *directed*, when to any a, b of S there exists an element c of S such that $a \geqq c$ and $b \geqq c$.

Theorem 8. Let G be a p. o. group and S be the p. o. semigroup of all negative elements of G. Then $G = Q(S)$ if and only if G is directed.(1)

Proof. Necessity: By Theorem 7, $a \geqq ab$ for any a, b of S. Therefore S is directed. Let x, y be any elements of G. One can write $x = ac^{-1}$, $y = bc^{-1}$, $a, b, c \in S$. Since S is directed, there exists an element d of S such that $a \geqq d$ and $b \geqq d$. And hence if we put $z = dc^{-1}$, we have $x \geqq z$ and $y \geqq z$. Therefore G is directed.

Sufficiency: Let x be any element of G. If a be chosen such that $x \geqq a$ and $e \geqq a$ (e is the unit element of G), then

$$x = a(ax^{-1})^{-1}, \quad e \geqq a, \quad e \geqq ax^{-1}.$$

Definition 5. An element of a semigroup S is said to be of *infinite order* if all its powers are different. If there exists a positive integer n such that $a^i \neq a^j$ for $1 \leqq i < j \leqq n$ and $a^n = a^k$ for all integers $k \geqq n$, then a is called *quasi-idempotent* and such positive integer n is called the *length* of a. If the length of a is 1 then a is idempotent in the usual sense.

Theorem 9. An element of a l. o. semigroup S is of infinite order or quasi-idempotent.

Proof. Let a be not of infinite order. There exist positive integers n, m such that $a^n = a^m$, $m > n$, and n is the least. Since S is a l. o. semigroup,

$$a > a^2 > \cdots > a^{n-1} > a^n \geqq a^{n+1} \geqq \cdots \geqq a^m = a^n \text{ (or its dual).}$$

Therefore $a^n = a^k$ for all $k \geqq n$.

(1) Cf. A. H. Clifford: Partially ordered abelian groups, Ann. Math., vol. 41 (1940), pp. 465–473.

Theorem 10. Let S be a strong l. o. semigroup. Then $a^n = b^n$ implies $a = b$. And if there exists a quasi-idempotent element e, then e is the unit element.

Proof. Since S is strong, $a > b$ implies $a^2 > ab > b^2$. Hence for all positive integers n, $a^n > b^n$. Next, the length of e must be 1. Hence $e^2 = e$. For every x of S, $ex = e^2x$ and hence $x = ex$, that is, e is the unit element. Therefore S has at most one quasi-idempotent element.

Definition 6. A partial order defined on an abelian semigroup S is called *normal*, when the following condition is satisfied :[2]

$$a^n \geqq b^n \text{ for some positive integer } n \text{ implies } a \geqq b.$$

Theorem 11. A strong l. o. semigroup S is always normal.

Proof. Suppose that $a \ngeqq b$. Then we have, by the linearity of S, $b > a$, which implies $b^n > a^n$ for every positive integer n.

Corollary. A l. o. group G is always normal.

Theorem 12. In the normal p. o. semigroup the following properties are held: 1) $a^n > b^n$ implies $a > b$, 2) $a^n = b^n$ implies $a = b$.

Proof. 1): By the normality, $a^n > b^n$ implies $a \geqq b$. If $a = b$, then we have $a^n = b^n$. 2): The normality means that if $a^n = b^n$, or what is the same $a^n \geqq b^n$ and $b^n \geqq a^n$, then $a \geqq b$ as well as $b \geqq a$, that is, $a = b$.

Corollary. An element of a normal p. o. group has an infinite order, except the unit element.

Definition 7. Suppose that two partial orders P and Q are defined on the same semigroup S and that the relation $a > b$ in P implies $a > b$ in Q; then Q will be called an *extension* of P. An extension which defines a linear order on S will be called a *linear extension.*

In the set $\mathfrak{P}$ of all partial orders defined on the same semigroup S, we put $Q \succ P$ if and only if Q is an extension of P. Then $\mathfrak{P}$ is a partially ordered set under this relation $\succ$.

Theorem 13. Let P be a strong partial order defined on an abelian semigroup S and x and y are any two elements non-comparable in P. Then there exists an extension Q, which is strong, of P such that $x > y$ in Q if and only if P is normal.[3]

Proof. Sufficiency: Let P be a normal strong partial order defined

(2) Cf. L. Fuchs: On the extension of the partial order of groups, Amer. Journ. Math., vol. 72 (1950), pp. 191–194.

(3) Cf. L. Fuchs: l. c.

on S and the elements x and y are not comparable in P. Let us define a relation Q as follows:

We put $a > b$ in Q if and only if $a \neq b$ and there are two non-negative integers n, m, such that not both zero and

$$(\S) \qquad a^n y^m \geqq b^n x^m \quad \text{in} \quad P,$$

where if $m = 0$ or $n = 0$ (§) means that $a^n \geqq b^n$ or $y^m \geqq x^m$ in P respectively.

First, we note that n is never zero, for otherwise we should have $y^m \geqq x^m$ in P, whence (by the normality) we have $y \geqq x$ in P against the hypothesis.

i) We begin with verifying that $a > b$ and $b > a$ in Q are contradictory. Suppose that $a > b$ and $b > a$, namely $a^n y^m \geqq b^n x^m$ and $b^t y^j \geqq a^i x^j$ in P for some non-negative integers n, m, i, j. By multiplying i times the first, n times the second inequality, one obtains $(ab)^{ni} y^{mi+nj} \geqq (ab)^{ni} x^{mi+nj}$ in P. By the strongness of P we have $y^{mi+nj} \geqq x^{mi+nj}$ in P. If $mi + nj$ does not vanish, by the normality we have $y \geqq x$, this contradicts the hypothesis. On the other hand, if $mi + nj$ is zero, i. e., both m and j vanish, then $a^n \geqq b^n$ and $b^i \geqq a^i$ in P. Therefore we have $a \geqq b$ and $b \geqq a$ in P, that is, $a = b$ which is absurd.

ii) We show the transitivity of Q. Assume that $a > b$ and $b > c$ in Q, i. e., for some non-negative integers n, m, i, j, $a^n y^m \geqq b^n x^m$ and $b^i y^j \geqq c^i x^j$ in P. By multiplying as in i) we get $a^{ni} y^{mi+nj} \geqq c^{ni} x^{mi+nj}$ in P. Here ni is not zero, and $a = c$ is by i) impossible, so that $a > c$ in Q.

iii) We prove next the homogeneity of Q. $a \neq b$ implies $ac \neq bc$ for any c of S, since P is strong. Hence if $a > b$ in Q, namely, if $a \neq b$ and $a^n y^m \geqq b^n x^m$ in P for some n, m, then $ac \neq bc$ and $(ac)^n y^m \geqq (bc)^n x^m$ in P. Therefore $a > b$ implies $ac > bc$ in Q for any c of S.

iv) Q is an extension of P, for if $a > b$ in P, then $ay^0 > bx^0$ in P, therefore $a > b$ in Q.

v) It is clear that $x > y$ in Q. In fact, $xy \geqq yx$ in P.

vi) We may prove the normality and the strongness of Q. Indeed, supposing $a^n > b^n$ in Q for some positive integer n, i.e., $(a^n)^t y^j \geqq (b^n)^t x^j$ in P, we see at once that $a > b$ in Q. Suppose that $ac > bc$ in Q, i.e., $(ac)^n y^m \geqq (bc)^n x^m$ in P for some n, m. Then by the strongness of P we are led to the result $a > b$ in Q.

Necessity: Let us assume that there exist elements a and b such that $a^n \geqq b^n$ and $a \ngeqq b$ in P. Then a and b can not be comparable in P by the strongness of P. And hence there exists a strong extension

Q of P in which $b > a$. This is however absurd, since by the strongness of Q this would imply $b^n > a^n$ in Q, contrary to the hypothesis $a^n \geqq b^n$ in P.

Definition 8. If $P_1, P_2, \cdots, P_\alpha, \cdots$ is a well-ordered chain of partial orders defined on the same abelian semigroup S such that each of them is some extension of the preceding ones, then the *union* of the chain may be defined to be a partial order P defined on S such that $a \geqq b$ in P if and only if $a \geqq b$ in P_α holds for some one and hence for all subsequent subscripts α.

It is easy to see that P is normal or strong if all P_α are normal or strong respectively.

Theorem 14. For every normal strong partial order P defined on an abelian semigroup S and every two elements x, y non-comparable in P, there exists a normal strong linear extension L_{xy} with the property that $x > y$ in L_{xy}.

Proof. By Theorem 13 there exists a normal strong extension Q of P such that $x > y$ in Q. Let $\mathfrak{P}'$ be a set of all normal strong partial orders defined on S which are extensions of Q. $\mathfrak{P}'$ is a partially ordered set as a subset of $\mathfrak{P}$ in Definition 7. By ZORN's lemma there exists a maximal linearly ordered subset $\mathfrak{P}^*$ of $\mathfrak{P}'$. Let L_{xy} be an union of $\mathfrak{P}^*$. Then L_{xy} is a maximal order, that is, order which has no proper extension. By Theorem 13 this can happen only in case any two elements are comparable in L_{xy}, that is to say, L_{xy} is linear. Moreover, L_{xy} is strong and normal, and $x > y$ in L_{xy}.

Theorem 15. A strong linear order may be defined on an abelian semigroup S if and only if in S are satisfied the following conditons: 1) $ax = bx$ implies $a = b$, 2) $a^n = b^n$ for some positive integer n implies $a = b$.

Proof. The necessity is obvious by Theorems 2 and 12. If we consider a vacuous partial order P of S in the sense of TUKEY, then P is the partial order defined on S. And conditions 1) and 2) say that P is strong and normal. Therefore, by Theorem 14 for any x, y of S there exists a strong linear extension L_{xy} of P in which $x > y$.

Corollary. A linear order may be defined on an abelian group if and only if all its elements, except the unit element, are of infinite order.[4]

(4) F. LEVI: Arithmetische Gesetze im Gebiete diskreter Gruppen, Rendiconti Palermo, vol. 35 pp. (1913), 225–236.
G. BIRKHOFF: Lattice Theory, second edition, Theorem 14, p. 224.

Definition 9. Let $\mathfrak{S} = \{P_\alpha\}$ be any set of partial orders, each defined on the same abelian semigroup S. We define the new partial order P on S as follows: For any two elements a, b we put $a \geqq b$ in P if and only if $a \geqq b$ in every P_α of the set $\mathfrak{S}$. Indeed, P is again a partial order defined on S, moreover P is normal or strong if all P_α of $\mathfrak{S}$ are normal or strong respectively. The partial order P is said to be the *product* of the P_α or to be *realized* by the set $\mathfrak{S}$ of partial orders, written $P = \Pi P_\alpha$.

Let $\mathfrak{G} = \{G_\alpha\}$ be a set of l. o. groups and G the (restricted or complete) direct product of G_α. Then one can introduce a partial order defined on G as usual, so that G becomes a p. o. group. We shall call G a *vector-group*. It is clear that a vector-group is always strong and normal.

Theorem 16. A strong partial order P defined on an abelian semigroup S may be realized by a certain set of strong linear orders if and only if P is normal.

Proof. The necessity is obvious, since by Theorem 11 a strong linear order, and hence every product of strong linear orders, is normal. On the other hand, if P is not linear, then there exist to any pair of elements x, y non-comparable in P the corresponding linear extensions L_{xy} and L_{yx} described in Theorem 14. It is easy to see that these linear orders realize P.

Theorem 17. A p. o. semigroup S can be order-embedded in a vector-group if and only if S is normal and strong.

Proof. Let P be a partial order defined on S. If P is strong and normal, then by Theorem 16 P is realized by a certain set of strong linear orders, which are extensions of P, defined on the semigroup S; $P = \Pi P_\alpha$. Let S_α be the strong l. o. semigroup when we consider that P_α is the strong linear order defined on S. And let G_α be the quotient group of S_α. G_α is a l. o. group. Then S is order-embedded in the direct product G of G_α. The necessity is obvious.

Corollary. A p. o. group G can be order-embedded in a vector-group if and only if G is normal[5].

Theorem 18. Let $\mathfrak{T} = \{S_\alpha\}$ be a set of strong l. o. semigroups and S the (restricted or complete) direct product of S_α. Then one can introduce a linear order defined on S, so that S becomes a strong l. o.

(5) A. H. Clifford: l.c., Theorem 1.

semigroup.[6]

Proof. We may consider that the S_α are well-ordered. Elements of S are then given by their components: $x = \{x_\alpha\}$, $x_\alpha \in S_\alpha$.

Let us define a relation P in S as follows:

We put $x > y$ in P if and only if $x \neq y$ and

$$x_\alpha = y_\alpha \quad \text{for all} \quad \alpha < \beta \quad \text{and} \quad x_\beta > y_\beta .$$

We see readily that P is a strong linear order defined on S.

Mathematical Institute,
Hokkaido University.

(6) Cf. K. IWASAWA: On linearly ordered groups, Journ. Math. Soc. Japan, vol. 1 (1948), pp. 1–9.