A GENERALIZATION OF THE FRENET'S FORMULAE

By

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§ 1. **Introduction.** P. DIENES considered the mixed tensors of rank ω $A^{\cdot i \nu}_{\cdot j \nu}\colon (x, \dot{x}, \ddot{x}, \cdots)$, whose componentes are functions of a line-element, and applies them to many theorems in his papers[1]. The purpose of the present paper is to show that the FRENET's formulae may take the same form as those found in the RIEMANNian geometry. At first let us give a brief explanation of notations and contractions according to P. DIENES. The absolute derivatives of a contravariant vector A^i or of a covariant vector B_i along a curve, be expressed by the following forms

$$\frac{\delta A^i}{\delta t} = \frac{dA^i}{dt} + f^i{}_\rho A^\rho\,, \qquad \frac{\delta B_i}{\delta t} = \frac{dB_i}{dt} + f_i{}^\rho B_\rho\,,$$

where the connection parameters $f^i{}_\rho$ and $f_i{}^\rho$ are functions of the parameter of the curve and have no relation among themselves. Then the absolute derivative of the tensor $A^{\cdot i \nu}_{\cdot j \nu}\colon$ along the curve may by expressed in the form

$$\text{(1)} \qquad \frac{\delta A^{\cdot i \nu}_{\cdot j \nu}\colon}{\delta t} = \frac{dA^{\cdot i \nu}_{\cdot j \nu}\colon}{dt} + \sum_{\rho=1}^{\omega} A^{\cdot i \nu \cdot \binom{i_\rho = \alpha}{}}_{\cdot j \nu \cdot \binom{}{j_\rho = \beta}} f^{i_\rho}{}_\alpha f_{j_\rho}{}^\beta\,,$$

where only one of the two functions $f^{i_\rho}{}_\alpha$ and $f_{j_\rho}{}^\beta$ in every term is equal to 1.

The space in which absolute derivatives of tensors are of the form (1) is called a *monodromic* space. In the monodromic space, the absolute differentiation of the sum or the product of two tensors A and B follows to the ordinarily established formal rules,

$$\frac{\delta(A+B)}{\delta t} = \frac{\delta A}{\delta t} + \frac{\delta B}{\delta t}\,, \qquad \frac{\delta AB}{\delta t} = \frac{\delta A}{\delta t} B + A \frac{\delta B}{\delta t}\,.$$

There are two kinds of metric tensors, one of which is the covariant tensor g_{ij} and the other the contravariant g^{ij}. These tensors are related to each other as $g_{ij} g^{ik} = E^k_j$, where $E^k_j = 0$ if $k \neq j$ and $= 1$ if $k = j$. If $A^i{}_j$ are the components of a tensor A, the length of the tensor may be defined and written as

$$g_{ik} g^{jl} A^i{}_j A^k{}_l = \vdash g_{ij} g^{kl} A^\alpha{}_\beta A^\gamma{}_\delta \dashv_{(i\alpha)(j\gamma)(k\beta)(l\delta)} = \vdash AA \dashv = |A|^2\,.$$

Similarly, if $A^i{}_j$ and $B^i{}_j$ are components of two tensors A and B, whose lengths are $|A|$ and $|B|$ respectively, θ being the angle between these tensors, we have

$$|A|\,|B| \cos\theta = \vdash AB \dashv .$$

A prominent property of this space is that there is the following relation between an absolute differentiation and a contraction:

$$\text{(2)} \qquad \left\vdash \frac{\delta A^{:i\nu:}_{:j\nu:}}{\delta t} \right\dashv_{(i\rho, j\sigma)} = \frac{\delta \vdash A^{:i\nu:}_{:j\nu:} \dashv_{(i\rho, j\sigma)}}{\delta t} + A^{:i\nu:[i\rho=\alpha]}_{:j\nu:[j\sigma=\beta]} C^\beta{}_\alpha ,$$

where $C^\beta{}_\alpha = f^\beta{}_\alpha + f_\alpha{}^\beta$, and we notice that $\dfrac{\delta E^\alpha_\beta}{\delta t} = C^\alpha{}_\beta$, therefore the $C^\alpha{}_\beta$ is a tensor.

§ 2. **Isotropic space.** Let us consider an arbitrarily given continuous, differentiable field of a normalized orthogonal n-nuple $\overset{(i)}{A}{}^\alpha$ along a curve. Since the absolute derivative of a vector at any point is also a vector at the point, there must be the functions P^{ik} which satisfy the relations

$$\text{(3)} \qquad \frac{\delta \overset{(i)}{A}{}^\alpha}{\delta t} = P^{ik} \overset{(k)}{A}{}^\alpha \qquad (i, k = 1, 2, \cdots\cdots, n).$$

Applying the relation (2) twice we have

$$\left\vdash \frac{\delta(\overset{(i)}{A}{}^\alpha \overset{(j)}{A}{}^\beta g_{\gamma\delta})}{\delta t} \right\dashv_{(\alpha,\gamma)(\beta,\delta)} = \left\vdash \frac{\delta(\overset{(i)}{A}{}^\alpha \overset{(j)}{A}{}^\beta g_{\gamma\beta})}{\delta t} \right\dashv_{(\alpha,\gamma)} + \overset{(i)}{A}{}^\alpha \overset{(j)}{A}{}^\beta g_{\alpha\delta} C^\delta{}_\beta$$
$$= \overset{(i)}{A}{}^\alpha \overset{(j)}{A}{}^\beta (g_{\gamma\beta} C^\gamma{}_\alpha + g_{\alpha\delta} C^\delta{}_\beta).$$

On the other hand, (3) gives

$$\left\vdash \frac{\delta(\overset{(i)}{A}{}^\alpha \overset{(j)}{A}{}^\beta g_{\gamma\delta})}{\delta t} \right\dashv_{(\alpha,\gamma)(\beta,\delta)} = \left\vdash \frac{\delta \overset{(i)}{A}{}^\alpha}{\delta t} \overset{(j)}{A}{}^\beta g_{\gamma\delta} + \overset{(i)}{A}{}^\alpha \frac{\delta \overset{(j)}{A}{}^\beta}{\delta t} g_{\gamma\delta} + \overset{(i)}{A}{}^\alpha \overset{(j)}{A}{}^\beta \frac{\delta g_{\gamma\delta}}{\delta t} \right\dashv_{(\alpha,\gamma)(\beta,\delta)}$$
$$= P^{ij} + P^{ji} + \overset{(i)}{A}{}^\alpha \overset{(j)}{A}{}^\beta \frac{\delta g_{\alpha\beta}}{\delta t} .$$

Comparing the two above relations, we get

$$\text{(4)} \qquad P^{ij} + P^{ji} + \overset{(i)}{A}{}^\alpha \overset{(j)}{A}{}^\beta \frac{\delta g_{\alpha\beta}}{\delta t} = \overset{(i)}{A}{}^\alpha \overset{(j)}{A}{}^\beta (g_{\gamma\beta} C^\gamma{}_\alpha + g_{\alpha\delta} C^\delta{}_\beta).$$

Particularly, if there is the relation

$$\text{(5)} \qquad \frac{\delta g_{\alpha\beta}}{\delta t} = g_{\gamma\beta} C^\gamma{}_\alpha + g_{\alpha\delta} C^\delta{}_\beta$$

at any point of space, (4) may be replaced by the more simple relation

$$(6) \qquad P^{ij} + P^{ji} = 0\,.$$

Similarly, for a normalized orthogonal covariant n–nuple $\overset{(k)}{A}_\alpha$, there are some functions P_{ik} along the curve, satisfying the following equations:

$$(7) \qquad \frac{\delta \overset{(i)}{A}_\alpha}{\delta t} = P_{ik} \overset{(k)}{A}_\alpha\,,$$

$$(8) \qquad \overset{(i)}{A}_\alpha \overset{(j)}{A}_\beta (g^{\alpha\rho} C^\beta{}_\rho + g^{\beta\rho} C^\alpha{}_\rho) = P_{ij} + P_{ji} + \overset{(i)}{A}_\alpha \overset{(j)}{A}_\beta \frac{\delta g^{\alpha\beta}}{\delta t}\,,$$

particularly, if we have the relation

$$(9) \qquad \frac{\delta g^{\alpha\beta}}{\delta t} = g^{\alpha\rho} C^\beta{}_\rho + g^{\beta\rho} C^\alpha{}_\rho$$

at any point of the space, the equation (8) may be replaced by the relation

$$(10) \qquad P_{ij} + P_{ji} = 0\,.$$

More generally, consider a normalized orthogonal system of tensors $\overset{(i)}{A}{}^{\cdot i_\nu \cdot}_{\cdot j_\nu \cdot}$ along the curve, then any tensor of the same kind may be expressed by a linear form of the tensors $\overset{(i)}{A}{}^{\cdot i_\nu \cdot}_{\cdot j_\nu \cdot}$. Then there exist some functions $P^{\alpha\beta}$ on the curve, which satisfy the equation

$$(11) \qquad \frac{\delta \overset{(\alpha)}{A}{}^{\cdot i_\nu \cdot}_{\cdot j_\nu \cdot}}{\delta t} = P^{\alpha\beta} \overset{(\beta)}{A}{}^{\cdot i_\nu \cdot}_{\cdot j_\nu \cdot}\,, \qquad (\alpha, \beta = 1, 2, \cdots\cdots, \omega)$$

and we get the following equation:

$$(12) \qquad P^{\alpha\beta} + P^{\beta\alpha} + \overset{(\alpha)}{A}{}^{\cdot i_\nu \cdot}_{\cdot j_\nu \cdot} \overset{(\beta)}{A}{}^{\cdot k_\nu \cdot}_{\cdot l_\nu \cdot} \left[\sum_{\rho=1}^{\omega} \left(\frac{\delta g_{i_\rho k_\rho}}{\delta t} g^{\cdot j_\nu l_\nu \cdot}_{\cdot i_\nu k_\nu}{}_{[i_\rho, k_\rho]} + \frac{\delta g^{j_\rho l_\rho}}{\delta t} g^{\cdot j_\nu l_\nu \cdot [j_\rho, l_\rho]}_{\cdot i_\nu k_\nu \cdot} \right) \right]$$
$$= \overset{(\alpha)}{A}{}^{\cdot i_\nu \cdot}_{\cdot j_\nu \cdot} \overset{(\beta)}{A}{}^{\cdot k_\nu \cdot}_{\cdot l_\nu \cdot} \left[\sum_{\rho=1}^{\omega} (C_{i_\rho, k_\rho} + C_{k_\rho, i_\rho})\, g^{\cdot j_\nu l_\nu \cdot}_{\cdot i_\nu k_\nu}{}_{[i_\rho, k_\rho]} \right.$$
$$\left. + (C^{l_\rho j_\rho} + C^{j_\rho l_\rho})\, g^{\cdot j_\nu l_\nu \cdot [j_\rho, l_\rho]}_{\cdot i_\nu k_\nu \cdot} \right],$$

where $\qquad C^{ij} = g^{i\rho} C^j{}_\rho\,, \quad C_{ij} = g_{i\rho} C^\rho{}_j\,.$

If the relations (5) and (9) are satisfied simultaneously in the space, then the relation (12) is simply expressed as in (6) and (10) by the form

$$(10)' \qquad P^{\alpha\beta} + P^{\beta\alpha} = 0\,.$$

The relations (6) and (8) give some dependence among $f_\rho{}^i$, $f^i{}_\rho$ and g_{ij} or g^{ij}. We call therefore the space in which (6) and (8) are satisfied, an *isotropic* space. We shall examine in detail the isotropic space.

From (5) we have

$$\left|-\frac{\delta g^{ij} g_{\rho k}}{\delta t}-\right|_{(j,\rho)} = \frac{\delta g^{i\rho}}{\delta t} g_{\rho k} + C^{i}{}_{k} + g^{i\rho} g_{k\sigma} C^{\sigma}{}_{\rho}\,,$$

on the other hand from (2) the following equation is obtained:

$$\left|-\frac{\delta g^{ij} g_{\rho k}}{\delta t}-\right|_{(j,\rho)} = \frac{\delta E^{i}_{k}}{\delta t} + g^{i\sigma} g_{\rho k} C^{\rho}{}_{\sigma} = C^{i}{}_{k} + g^{i\sigma} g_{\rho k} C^{\rho}{}_{\sigma}\,.$$

Comparing the above two equations we have

$$(5)' \qquad \frac{\delta g^{i\rho}}{\delta t} = 0\,.$$

Conversely, the relation (5) can be deduced from (5)′. Similarly it will be seen that the relation (9) is equivalent to the form

$$(9)' \qquad \frac{\delta g_{\alpha\beta}}{\delta t} = 0\,.$$

In the isotropic space we get the relations $\frac{\delta g_{\alpha\beta}}{\delta t} = \frac{\delta g^{\alpha\beta}}{\delta t} = 0$, which are equivalent to the equation $g^{\alpha\rho} C^{\beta}{}_{\rho} + g^{\beta\rho} C^{\alpha}{}_{\rho} = 0$, but do not necessitate the condition $C^{\beta}{}_{\rho} = 0$, namely $f^{\beta}{}_{\rho} + f_{\rho}{}^{\beta} = 0$ as in the RIEMANNian geometry.

Since these relations are satisfied both in the FINSLER[2] space and the CARTAN space, the two spaces are isotropic spaces.

§ 3. **The FRENET's formulae and the curvature.** Consider an arbitrarily given continuous differentiable tensor field $\overset{(1)}{B}{}^{\cdot i\nu\cdot}_{\cdot j\nu\cdot}$ along a curve and linearly independent tensors $\overset{(i)}{B}$ defined by the following equations:

$$\frac{\delta \overset{(1)}{B}{}^{\cdot i\nu\cdot}_{\cdot j\nu\cdot}}{\delta t} = \overset{(2)}{B}{}^{\cdot i\nu\cdot}_{\cdot j\nu\cdot}\,, \;\cdots,\; \frac{\delta \overset{(i)}{B}{}^{\cdot i\nu\cdot}_{\cdot j\nu\cdot}}{\delta t} = \overset{(i+1)}{B}{}^{\cdot i\nu\cdot}_{\cdot j\nu\cdot}\,, \cdots,\; \frac{\delta \overset{(\omega-1)}{B}{}^{\cdot i\nu\cdot}_{\cdot j\nu\cdot}}{\delta t} = \overset{(\omega)}{B}{}^{\cdot i\nu\cdot}_{\cdot j\nu\cdot}\,.$$

By the SCHMIDT's method[3] as expounded in the RIEMANNian geometry, we have then a normalized orthogonal system of tensor fields $\overset{(i)}{A}{}^{\cdot i\nu\cdot}_{\cdot j\nu\cdot}$ along a curve

$$\overset{(i)}{A} = \frac{\begin{vmatrix} (1,1), \cdots (1, i-1)\, \overset{(1)}{B} \\ \cdots\cdots\cdots\cdots\cdots \\ (i,1), \cdots (i, i-1), \overset{(i)}{B} \end{vmatrix}}{\sqrt{D_{i-1} D_i}}\,, \quad \text{where } D_i = \begin{vmatrix} (1,1), \cdots (1,i) \\ \cdots\cdots\cdots\cdots \\ (i,1), \cdots (i,i) \end{vmatrix}\,, \quad (i,j) = |\text{-}\overset{i}{B}\overset{j}{B}\text{-}|\,.$$

Since $\overset{(i)}{A}$ are homogeneous linear forms of $\overset{(1)}{B}, \overset{(2)}{B}, \cdots, \overset{(i)}{B}$, we have

$$\frac{\delta \overset{(i)}{A}}{\delta t} = a_1 \overset{(1)}{B} + a_2 \overset{(2)}{B} + \cdots + a_i \overset{(i)}{B} + \frac{D_{i-1}}{\sqrt{D_{i-1} D_i}} \overset{(i+1)}{B} .$$

But $\overset{(i)}{B}$ is a homogeneous linear form of $\overset{(1)}{A}, \overset{(2)}{A}, \cdots, \overset{(i)}{A}$. Hence

$$\frac{\delta \overset{(i)}{A}}{\delta t} = \sum_{j=1}^{i+1} P^{ij} \overset{(j)}{A} \quad \text{i.e.} \quad P^{i,i+k} = 0 , \quad \text{when} \quad k > 1 .$$

It follows by (10)′ and (11) that in an isotropic space we have also

$$P^{i+k\, i} = P^{i\, i+k} = 0 \ \text{ and } \ P^{ii} = 0 , \quad \text{when} \quad k > 1 .$$

On the other hand

$$P^{i\, i+1} = |\text{-} \overset{(i+1)}{A} \frac{\delta \overset{(i)}{A}}{\delta t} \text{-}| = |\text{-} \overset{(i+1)}{A} \overset{(i+1)}{B} \text{-}| \frac{D_{i-1}}{\sqrt{D_{i-1} D_i}} = \frac{\sqrt{D_{i-1} D_{i+1}}}{D_i} .$$

Let θ be the extremal angle between a tensor $\overset{(i+1)}{B}$ and a plane F which is determined by the tensor $\overset{(1)}{B}, \overset{(2)}{B}, \cdots, \overset{(i)}{B}$. P. Dienes used the relation[1]

$$\cos\theta = \frac{\sum_{\rho=1}^{i} \overset{(\rho)}{D_i} |\text{-} \overset{(\rho)}{B} \overset{(i+1)}{B} \text{-}|}{\sqrt{\sum_{j,k=1}^{i} \overset{(j)}{D_i} \overset{(k)}{D_i} |\text{-} \overset{(i)}{B} \overset{(k)}{B} \text{-}| |\text{-} \overset{i+1}{B} \overset{i+1}{B} \text{-}|}} ,$$

$$\overset{(k)}{D_i} = \begin{vmatrix} (1,1), \cdots, (1,k-1), (1,i+1), (1,k+1), \cdots, (1,i) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (i,1), \cdots, (i,k-1), (i,i+1), (i,k+1), \cdots, (i,i) \end{vmatrix} .$$

Denote by α the angle between F and $P\ (C : t \| t^\circ)$. The limit of the quotient $\frac{\alpha}{\Delta t}$ or that of $\frac{\sin\alpha}{\Delta t}$, as $\Delta t \to 0$, is called the i^{th} curvature of the tensor field. He had following relations[1]:

$$\frac{1}{\gamma_i} = \lim_{\Delta t \to 0} \frac{\alpha}{\Delta t} = \frac{\sqrt{D_{i-1} D_{i+1}}}{D_i} .$$

Hence we have the Frenet's formulae, as in the Riemannian geometry;

$$\frac{\delta \overset{(i)}{A}{}^{\cdot i \nu \cdot}_{\cdot j \nu \cdot}}{\delta t} = \frac{1}{\gamma_i} \overset{(i+1)}{A}{}^{\cdot i \nu \cdot}_{\cdot j \nu \cdot} - \frac{1}{\gamma_{i-1}} \overset{(i-1)}{A}{}^{\cdot i \nu \cdot}_{\cdot j \nu \cdot} \qquad (i = 2, \cdots, \omega - 1) ,$$

$$\frac{\delta \overset{(1)}{A}{}^{\cdot i \nu \cdot}_{\cdot j \nu \cdot}}{\delta t} = \frac{1}{\gamma_1} \overset{(2)}{A}{}^{\cdot i \nu \cdot}_{\cdot j \nu \cdot} , \qquad \frac{\delta \overset{(\omega)}{A}{}^{\cdot i \nu \cdot}_{\cdot j \nu \cdot}}{\delta t} = - \frac{1}{\gamma_{\omega-1}} \overset{(\omega-1)}{A}{}^{\cdot i \nu \cdot}_{\cdot j \nu \cdot} .$$

Consider a parallel polyhedron composed of tensors $\overset{(1)}{B}, \overset{(2)}{B}, \cdots, \overset{(i)}{B}$, then its volume $\sqrt{V_i}$ is expressed in the form

$$\sqrt{V_i} = \sqrt{\begin{vmatrix} (1,1), \cdots, (1,i) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (i,1), \cdots, (i,i) \end{vmatrix}} \, .$$

Since

$$0 = \begin{vmatrix} (1,1), \cdots, (1,i), (1,i+1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (i,1), \cdots, (i,i), (i,i+1) \\ (j,1), \cdots, (j,i), (j,i+1) \end{vmatrix} = (j, i+1)\, V_i - \sum_{k=1}^{i} (i,k) \overset{(k)}{D}_i \, , \quad j \leqq i \, ,$$

$$V_{i+1} = \begin{vmatrix} (1,1) \cdots\cdots\cdots\cdots\cdots\cdots (1,i+1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (i,1) \cdots\cdots\cdots\cdots (i,i), \ (i,i+1) \\ (i+1,1) \cdots (i+1,i), (i+1,i+1) \end{vmatrix} = (i+1,i+1)\, V_i - \sum_{k=1}^{i} \overset{(k)}{D}_i (k,i+1),$$

$$\frac{\sum_{j=1}^{i} \overset{(j)}{D}_i (j, i+1)}{\sqrt{(i+1,i+1)} \sqrt{\sum_{j=1}^{i} \overset{(j)}{D}_i (j, i+1)\, V_i}} = \frac{\sqrt{\sum_{j=1}^{i} \overset{(j)}{D}_i (j, i+1)}}{\sqrt{(i+1, i+1)\, V_i}} \, ,$$

we have at last

$$\sqrt{V_{i+1}} = \sqrt{(i+1, i+1)} \, \sin\theta \, \sqrt{V_i} \, , \qquad \gamma_i^2 = \frac{V_{i-1}\, V_{i+1}}{V_i^2} \, .$$

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