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ON INTRINSIC THEORIES IN THE MANIFOLD OF SURFACE-ELEMENTS OF HIGHER ORDER

By

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Introduction. It is well known that the space in which a measure of a hypersurface: $x^i = x^i(u^1, u^2, \dots, u^{n-1})$, $i = 1, 2, \dots, n$ is given by the $(n-1)$ -ple integral: $\int_{(n-1)} F(x^i, \partial x^i / \partial u^a) du^1 \dots du^{n-1}$ is called a CARTAN space. As it is shown by CARTAN, this space is to be regarded as a manifold of hyperplane-elements $(x^i, \partial x^i / \partial u^a)$. The geometry of CARTAN space were discussed by E. CARTAN [1]⁽¹⁾ and L. BERWALD [6][7] at large. Thereafter, T. OHKUBO [9] and the present author [10][11] extended this theory to the $(n-1)$ -ple integral of higher order of special forms. Recently, the present author [12] have established a geometry of an $(n-1)$ -ple integral of the second order in general form, but the space in which the theories are discussed was regarded as a manifold of hypersurface-elements of the third order. On the other hand the theory of K -spreads in an n -dimensional manifold which are concerned with a system of partial differential equations of the second order was studied at first by J. DOUGLAS, and the theory was treated in the manifold of all K -dimensional surface-elements of order 1. Thereafter A. KAWAGUCHI and H. HOMBU [5] studied the theory of K -spreads of the m -th order ($m \geq 2$), and the manifold of all K -dimensional surface-elements of the $(m-1)$ -th order was based in this case. In this paper we aim to establish the foundation of differential geometries in the manifold of K -dimensional surface-elements of higher order under the transformation group of the surface-elements which is deduced from the groups of arbitrary transformations of coordinates and parameters, and treat of the geometry of multiple integral of higher order in detail.

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§ 1. The manifold $F_n^{(m)}$ and notations. In an n -dimensional space X_n with point coordinates $x^1, x^2, \dots, x^n$ a K -dimensional surface is defined analytically by the parametric equations

(1) Numbers in brackets refer to the references at the end of the paper.

$$x^i = x^i(u^a), \quad a = 1, 2, \dots, K,$$

where u^a are K essential parameters for the K -dimensional surface.

At every point on this K -dimensional surface a K -dimensional surface element of the m -th order can be determined by

$$x^i = x^i(u^a), p_{a(1)}^i \equiv p_{a_1}^i = \frac{\partial x^i}{\partial u^{a_1}}, \quad p_{a(2)}^i \equiv p_{a_1 a_2}^i = \frac{\partial^2 x^i}{\partial u^{a_1} \partial u^{a_2}},$$

$$\dots\dots\dots, p_{a(m)}^i \equiv p_{a_1 a_2 \dots a_m}^i = \frac{\partial^m x^i}{\partial u^{a_1} \partial u^{a_2} \dots \partial u^{a_m}}.$$

Now, adjoining arbitrary system of values $x^i, p_{a(1)}^i, \dots, p_{a(m)}^i$ to every point in X_n , we have the $n \binom{K+m}{K}$ -dimensional manifold $F_n^{(m)}$. We shall name the quantity which is transformed according to the tensor law under the transformation groups of coordinates and parameters:

$$(1.1) \quad x^{i'} = x^{i'}(x^1, x^2, \dots, x^n),$$

$$(1.2) \quad u^{a'} = u^{a'}(u^1, u^2, \dots, u^K)$$

the intrinsic quantity according to E. BORTOLOTTI.

We can speak of x -transformations or u -transformations alone, and of x -tensors or u -tensors accordingly. Tensor will mean, unless otherwise mentioned, a geometrical object which has the proper law of transformation for both sorts of indices.

Throughout this paper we shall use the notations

$$X_{i'}^i = \frac{\partial x^i}{\partial x^{i'}}, \quad X_{i'}^{i'} = \frac{\partial x^{i'}}{\partial x^i}, \quad X_{i'(2)}^i = \frac{\partial^2 x^i}{\partial x^{i'_1} \partial x^{i'_2}}, \quad \dots\dots\dots,$$

$$U_{a'}^a = \frac{\partial u^a}{\partial u^{a'}}, \quad U_{a'}^{a'} = \frac{\partial u^{a'}}{\partial u^a}, \quad U_{a'(2)}^a = \frac{\partial^2 u^a}{\partial u^{a'_1} \partial u^{a'_2}}, \quad \dots\dots\dots.$$

and

$$F_{;a} = \frac{\partial F}{\partial u^a}, \quad F_{;i} = \frac{\partial F}{\partial x^i}, \quad F_{;i}^{a(s)} = \frac{\partial}{\partial p_{a(s)}^i} F = \frac{l_1! l_2! \dots l_s!}{s!} \frac{\partial}{\partial p_{a(s)}^i} F,$$

when the indices $a_1, a_2, \dots, a_s$ consist of $l_1, l_2, \dots, l_s$ blocks of the same indices.

Moreover we shall often use the notations for indices in the form

$$I_{(a_1 a_2 \dots a_r)} J_{a_{r+1} \dots a_t} \equiv I_{(a(r)} J_{a(t-r))}.$$

§ 2. Transformation laws of the various quantities. Under the transformation (1.1) the surface-elements in $F_n^{(m)}$ has the laws of transformation as the forms:

$$\begin{aligned} x^i &= x^i(x^{i'}), & p_a^i &= X_{i'}^i p_{a'}^{i'}, \\ p_{a(2)}^i &= X_{i'}^i p_{a(2)}^{i'} + X_{j'k'}^i p_{a'}^{j'} p_{a_2}^{k'}, \\ &\dots\dots\dots, \\ p_{a(m)}^i &= X_{i'}^i p_{a(m)}^{i'} + R_{a(m)}^i(x', p_{(1)}^{i'}, p_{(2)}^{i'}, \dots\dots, p_{(m-1)}^{i'}), \end{aligned}$$

so that $dp_{a(t)}^i = \sum_{s=0}^t \frac{\partial p_{a(t)}^i}{\partial p_{a(s)}^{i'}} dp_{a(s)}^{i'}$, putting $dp_{a(0)}^i = dx^i$.

It is well known that

$$\begin{aligned} (2.1) \quad \frac{\partial p_{a(t)}^i}{\partial p_{a(s)}^{i'}} &= \frac{t!(s-r)!}{s!(t-r)!} \delta_{a(r)}^{\beta(r)} \frac{\partial p_{a(t-r)}^i}{\partial p_{a(s-r)}^{i'}} \\ &= \binom{t}{s} \delta_{a_1}^{\beta_1} \dots \delta_{a_s}^{\beta_s} \frac{\partial p_{a_{s+1} \dots a_t}^i}{\partial x^{i'}} = \binom{t}{s} \delta_{a(s)}^{\beta(s)} (X_{i'}^i)_{a(t-s)} [5] \\ &\quad (t \geq s \geq r), \end{aligned}$$

putting $\delta_{a(r)}^{\beta(r)} = \delta_{a_1}^{\beta_1} \delta_{a_2}^{\beta_2} \dots \delta_{a_r}^{\beta_r}$.

On the other hand, by the transformations (1.2) the partial derivatives $f_{;a(r)} = \frac{\partial^r f}{\partial u^{a_1} \partial u^{a_2} \dots \partial u^{a_r}}$ ($r=1, 2, \dots, m$) are transformed in the manners

$$f_{;a'} = U_{a'}^a f_{;a}, \quad f_{;a'(2)} = U_{a'_1}^{a_1} U_{a'_2}^{a_2} f_{;a(2)} + U_{a'_1 a'_2}^{\beta_1 \beta_2} f_{;\beta}, \dots\dots\dots$$

and in general

$$(2.2) \quad f_{;a'(s)} = \sum_{t=0}^s A_{a'(s)}^{a(t)} f_{;a(t)},$$

so that, we have

$$(2.3) \quad p_{a'(s)}^{i'} = \sum_{t=0}^s A_{a'(s)}^{a(t)} p_{a(t)}^i \quad (s=0, 1, \dots, m)$$

and consequently

$$dp_{a'(s)}^{i'} = \sum_{t=0}^s A_{a'(s)}^{a(t)} dp_{a(t)}^i \quad (s=0, 1, \dots, m),$$

putting $A_{a'(0)}^{a(0)} = 1$, $A_{a'(1)}^{a(0)} = A_{a'(2)}^{a(0)} = \dots = 0$.

It is easily seen that the quantities $A_{a'(s)}^{a(t)}$ are polynomials of the derivatives $U_{a'_1}^{\beta_1}, U_{a'_1 a'_2}^{\beta_1 \beta_2}, \dots, U_{a'_1 a'_2 \dots a'_s}^{\beta_1 \beta_2 \dots \beta_s}$, and determined from the recurring formulae

$$(2.4) \quad A_{a'(s)}^{a(t)} = \begin{cases} U_{a'(s)}^{a(s)} = U_{a'_1}^{a_1} U_{a'_2}^{a_2} \dots U_{a'_s}^{a_s} & t = s, \\ A_{a'(s-1)}^{a(t-1)} U_{a'_s}^{a_s} + A_{a'(s-1); a'_s}^{a(t)} & s > t > 1, \\ U_{a'_1 a'_2 \dots a'_s}^{a_1 a_2 \dots a_s} & t = 1. \end{cases}$$

Now, we shall prove the formula

$$(2.5) \quad \binom{r}{t} A_{a'(s)}^{a(r)} = \sum_{u=t}^{s-r+t} \binom{s}{u} A_{a'(u)}^{a(t)} A_{a'(s-u)}^{a(r-t)}.$$

Let ϕ and ψ be any two functions of the parameters u^a 's, then it follows that

$$(\phi \cdot \psi)_{a'(s)} = \sum_{u=0}^s \binom{s}{u} (\phi_{a'(u)} \psi_{a'(s-u)}) \quad [5]$$

which is known as the generalized LEIBNIZ formula. Therefore, by the transformation (1.2) we have

$$\sum_{r=0}^s (\phi \cdot \psi)_{a'(s)} A_{a'(s)}^{a(r)} = \sum_{u=0}^s \binom{s}{u} \left(\sum_{v=0}^u \sum_{l=0}^{s-u} \phi_{a'(v)} \psi_{a'(l)} A_{a'(u)}^{a(v)} A_{a'(s-u)}^{a(l)} \right)$$

or

$$\begin{aligned} \sum_{r=0}^s \sum_{t=0}^r \binom{r}{t} \phi_{a'(t)} \psi_{a'(r-t)} A_{a'(s)}^{a(r)} \\ = \sum_{u=v}^s \sum_{v=0}^s \sum_{l=0}^{s-u} \binom{s}{u} \phi_{a'(v)} \psi_{a'(l)} A_{a'(u)}^{a(v)} A_{a'(s-u)}^{a(l)}. \end{aligned}$$

Comparing the coefficients of $\phi_{a'(t)} \psi_{a'(r-t)}$ on both sides, we have the formula (2.5).

Let us consider an operator $P^{a(l)}$ applied to any quantity of the manifold $F_n^{(m)}$, that is

$$P^{a(l)}(L) = \sum_{t=l}^m \binom{t}{l} L_{; i^{(l)} \beta_i^{(t-l)}} dp_{\beta(t-l)}^i \quad [5],$$

then we have

Theorem 1. Under the transformation group (1.1) and (1.2) the operator $P^{a(l)}$ has the law of transformation :

$$(2.6) \quad P^{a(l)}(L) = \sum_{u=l}^m A_{a'(u)}^{a(l)} P^{a'(u)}(L).$$

Proof. If we effect the x -transformations alone, it follows that

$$\begin{aligned} P^{a(l)}(L) &= \sum_{t=l}^m \binom{t}{l} \sum_{s=t}^m L_{; i^{(s)} \gamma_{i'}^{(s)}} \frac{\bar{\partial} p_{\gamma(s)}^{i'}}{\bar{\partial} p_{a'(l) \beta(t-l)}^i} dp_{\beta(t-l)}^i \\ &= \sum_{t=l}^m \binom{t}{l} \sum_{s=t}^m L_{; i^{(s)} \gamma_{i'}^{(s)}} \frac{s! (t-l)!}{t! (s-l)!} \delta_{(r(l))}^{a(l)} \frac{\bar{\partial} p_{\gamma(s-l)}^{i'}}{\bar{\partial} p_{\beta(t-l)}^i} dp_{\beta(t-l)}^i \\ &= \sum_{s=l}^m \binom{s}{l} L_{; i^{(s)} \gamma_{i'}^{(s)}} \sum_{t=l}^s \delta_{(r(l))}^{a(l)} \frac{\bar{\partial} p_{\gamma(s-l)}^{i'}}{\bar{\partial} p_{\beta(t-l)}^i} dp_{\beta(t-l)}^i \\ &= \sum_{s=l}^m \binom{s}{l} L_{; i^{(l)} \gamma_{i'}^{(s-l)}} dp_{\gamma(s-l)}^{i'} = P^{a(l)}(L). \end{aligned}$$

By effecting the u -transformations we have

$$\begin{aligned}
 P^{\alpha(l)}(L) &= \sum_{t=l}^m \binom{t}{l} \sum_{s=t}^m L_{; \tau'(s)} \frac{\bar{\partial} p_{\tau'(s)}^{i'}}{\bar{\partial} p_{\alpha(l)\beta(t-l)}^{j'}} dp_{\beta(t-l)}^{j'} \\
 &= \sum_{s=t}^m \sum_{t=l}^m \binom{t}{l} L_{; \tau'(s)} A^{\alpha(l)\beta(t-l)}_{\tau'(s)} dp_{\beta(t-l)}^{i'} \\
 &= \sum_{s=t}^m \sum_{t=l}^m L_{; \tau'(s)} \sum_{u=l}^{s-t+l} \binom{s}{u} A^{\alpha(l)}_{\tau'(u)} A^{\beta(t-l)}_{\tau'(s-u)} dp_{\beta(t-l)}^{i'} \\
 &= \sum_{s=l}^m \sum_{r=0}^{s-l} \sum_{u=l}^{s-r} \binom{s}{u} L_{; \tau'(s)} A^{\beta(r)}_{\tau'(s-u)} dp_{\beta(r)}^{i'} A^{\alpha(l)}_{\tau'(u)} \\
 &= \sum_{s=l}^m \sum_{u=l}^s \binom{s}{u} L_{; \tau'(s)} \sum_{r=0}^{s-u} A^{\beta(r)}_{\tau'(s-u)} dp_{\beta(r)}^{i'} A^{\alpha(l)}_{\tau'(u)} \\
 &= \sum_{u=l}^m \left(\sum_{s=u}^m \binom{s}{u} L_{; \tau'(s)} dp_{\tau'(s-u)}^{i'} \right) A^{\alpha(l)}_{\tau'(u)} \\
 &= \sum_{u=l}^m P^{\tau'(u)}(L) A^{\alpha(l)}_{\tau'(u)}.
 \end{aligned}$$

Theorem 2. If T^A be an intrinsic quantity of $F_n^{(m)}$ whose transformation law under the transformations (1.1) and (1.2) is $T^A = \mathcal{Q}^A_A T^{A'}$, then $P^{\alpha(l)}(T^A)$ obeys the transformation law

$$(2.7) \quad P^{\alpha(l)}(T^A) = \sum_{u=l}^m A^{\alpha(l)}_{\alpha'(u)} \mathcal{Q}^A_{A'} P^{\alpha'(u)}(T^{A'}).$$

Theorem 3. If w^i be any vector of $F_n^{(m)}$ and L be a scalar of $F_n^{(m)}$, the quantity

$$(2.8) \quad D_j(L)w^j = \sum_{r=0}^m L_{; \beta(r)} w^j_{|\beta(r)} \quad [5]$$

is invariant under the transformations (1.1) and (1.2), where $|\beta$ denotes the total differentiation with respect to u^β , that is

$$w^j_{|\beta} = w^j_{;\beta} + \sum_{s=0}^m w^j_{; \alpha(s)} p_{\alpha(s)\beta}^i.$$

Proof. If we effect the x -transformations alone, it follows that

$$\begin{aligned}
 \sum_{r=0}^m L_{; \beta(r)} w^j_{|\beta(r)} &= \sum_{r=0}^m \sum_{s=r}^m L_{; \alpha(s)} \frac{\bar{\partial} p_{\alpha(s)}^{i'}}{\bar{\partial} p_{\beta(r)}^j} w^j_{|\beta(r)} \\
 &= \sum_{r=0}^m \sum_{s=r}^m L_{; \alpha(s)} \binom{s}{r} \delta^{\beta(r)}_{\alpha(s)} \left(\frac{\partial x^{i'}}{\partial x^j} \right)_{|\alpha(s-r)} w^j_{|\beta(r)} \\
 &= \sum_{r=0}^s \sum_{s=0}^m L_{; \alpha(s)} \binom{s}{r} \left(\frac{\partial x^{i'}}{\partial x^j} \right)_{|\alpha(s-r)} w^j_{|\alpha(r)} \\
 &= \sum_{s=0}^m L_{; \alpha(s)} \left(\frac{\partial x^{i'}}{\partial x^j} w^j \right)_{|\alpha(s)} = \sum_{s=0}^m L_{; \alpha(s)} w^{i'}_{|\alpha(s)}.
 \end{aligned}$$

By effecting the u -transformations we have

$$\begin{aligned} \sum_{s=0}^m L_{; \ell'}^{(s)} w_{| \alpha(s)}^{\ell'} &= \sum_{s=0}^m \sum_{r=s}^m L_{; \ell'}^{(s)} \frac{\bar{\partial} p_{\beta'(r)}^{\ell'}}{\bar{\partial} p_{\alpha(s)}^{j'}} w_{| \alpha(s)}^{j'} \\ &= \sum_{s=0}^m \sum_{r=s}^m L_{; \ell'}^{(s)} A_{\beta'(r)}^{\alpha(s)} w_{| \alpha(s)}^{\ell'} \\ &= \sum_{r=0}^m \sum_{s=0}^r L_{; \ell'}^{(r)} A_{\beta'(r)}^{\alpha(s)} w_{| \alpha(s)}^{\ell'} = \sum_{r=0}^m L_{; \ell'}^{(r)} w_{| \beta'(r)}^{\ell'}. \end{aligned}$$

Theorem 4. Let T^A be an intrinsic quantity of $F_n^{(m)}$ whose transformations law under the transformations (1.1) and (1.2) is $T^A = \mathfrak{A}_A^A T^A$, and v^i be any vector of $F_n^{(m)}$, then the quantities

$$D_j^{\alpha(u)}(T^A) v^j = \sum_{r=u}^m \binom{r}{u} T^A_{; \alpha(u) \beta(r-u)} v^j_{| \beta(r-u)} \quad (u = 1, 2, \dots, m)$$

are transformed by the transformations (1.1) and (1.2) in the manners

$$(2.9) \quad D_j^{\alpha(u)}(T^A) v^j = \mathfrak{A}_{A'}^A \sum_{t=u}^m A_{\lambda'(t)}^{\alpha(u)} D_j^{\lambda'(t)}(T^{A'}) v^{j'}.$$

Proof. When we put $L = T^A$ and $w^j = \phi v^j$ into (2.8), ϕ being any scalar of $F_n^{(m)}$, one obtains the intrinsic quantity

$$\begin{aligned} D_j(T^A) \phi v^j &= \sum_{r=0}^m T^A_{; \beta(r)} (\phi v^j)_{| \beta(r)} \\ &= \sum_{r=0}^m T^A_{; \beta(r)} \sum_{u=0}^r \binom{r}{u} \phi_{| \beta(u)} v^j_{| \beta(r-u)} \\ &= \sum_{u=0}^m \left(\sum_{r=u}^m \binom{r}{u} T^A_{; \beta(r)} v^j_{| \beta(r-u)} \right) \phi_{| \beta(u)}, \end{aligned}$$

so that we can conclude (2.9) by virtue of (2.2).

§ 3. Intrinsic operators and intrinsic PFAFFian form. Let $f(x^i, p_{\alpha(1)}^i, \dots, p_{\alpha(m)}^i)$ be any scalar of $F_n^{(m)}$, then it is seen from theorem 2 and the above mentioned theorem that the quantities

$$(3.1a) \quad \sum_{l=1}^m f_{| \alpha(l)} P_j^{\alpha(l)}(T^A),$$

$$(3.1b) \quad \sum_{l=1}^m f_{| \alpha(l)} D_j^{\alpha(l)}(T^A) v^j$$

are tensors of the same kind with T^A .

Suppose now that we have the quantity $G_{\beta r}^{\alpha}$ whose transformation law under the transformations (1.1) and (1.2) is the same as that of coefficient of the affine connection of u -tensor, that is

$$(3.2) \quad U_{\beta\tau}^{\alpha'} = G_{\beta\tau}^{\alpha} U_{\alpha'}^{\alpha} - G_{\beta'}^{\alpha'} U_{\beta}^{\beta'} U_{\tau}^{\tau'},$$

then we can derive from $f_{/\alpha(1)}, f_{/\alpha(2)}, \dots, f_{/\alpha(t)}$ the intrinsic quantities $f_{\beta(s)} (s=1, 2, \dots, t)$ in the following way.

First of all we see that $f_{\beta(1)} = f_{/\alpha_1} \delta_{\beta_1}^{\alpha_1}$ is an intrinsic quantity. Assume that there are the quantities $'K_{\beta(s-1)}^{\alpha(l)}$ ($l=1, 2, \dots, s-1$) such that $f_{\beta(s-1)} = \sum_{l=1}^{s-1} f_{/\alpha(l)} 'K_{\beta(s-1)}^{\alpha(l)}$ is intrinsic, namely

$$(3.3) \quad \sum_{l=1}^{s-1} f_{/\alpha(l)} 'K_{\beta(s-1)}^{\alpha(l)} = \left(\sum_{l=1}^{s-1} f_{/\alpha'(l)} 'K_{\beta'(s-1)}^{\alpha'(l)} \right) U_{\beta'(s-1)}^{\beta'(s-1)}.$$

Differentiating the above equation with respect to u^{β} and symmetrizing with respect to the indices $\beta_1, \beta_2, \dots, \beta_s$ one gets

$$(3.4) \quad \begin{aligned} & \sum_{l=1}^{s-1} f_{/\alpha(l)(\beta_s)} 'K_{\beta(s-1)}^{\alpha(l)} + \sum_{l=1}^{s-1} f_{/\alpha(l)} 'K_{(\beta(s-1)/\beta_s)}^{\alpha(l)} \\ &= \left(\sum_{l=1}^{s-1} f_{/\alpha'(l)(\beta'_s)} 'K_{\beta'(s-1)}^{\alpha'(l)} + \sum_{l=1}^{s-1} f_{/\alpha'(l)} 'K_{(\beta'(s-1)/\beta'_s)}^{\alpha'(l)} \right) U_{\beta'(s)}^{\beta'(s)} \\ &+ (s-1) \left(\sum_{l=1}^{s-1} f_{/\alpha'(l)} 'K_{\beta'(s-1)}^{\alpha'(l)} \right) U_{(\beta'_1, \dots, \beta'_{s-2}, \beta'_{s-1}, \beta_s)}^{\beta'_1, \dots, \beta'_{s-2}, \beta'_{s-1}, \beta_s}. \end{aligned}$$

Eliminating $U_{\beta'_1, \dots, \beta'_{s-1}, \beta_s}^{\beta'_1, \dots, \beta'_{s-1}, \beta_s}$ from the above equation and (3.2) putting $\alpha' = \beta'_{s-1}$, $\beta = \beta_{s-1}$, $\tau = \beta_s$, we have under the consideration of (3.3)

$$\begin{aligned} & \sum_{l=1}^{s-1} f_{/\alpha(l)(\beta_s)} 'K_{\beta(s-1)}^{\alpha(l)} + \sum_{l=1}^{s-1} f_{/\alpha(l)} 'K_{(\beta(s-1)/\beta_s)}^{\alpha(l)} \\ & - (s-1) \sum_{l=1}^{s-1} f_{/\alpha(l)} 'K_{(\beta(s-2)/\tau)}^{\alpha(l)} G_{\beta_{s-1}\beta_s}^{\tau} \\ &= \left(\sum_{l=1}^{s-1} f_{/\alpha'(l)(\beta'_s)} 'K_{\beta'(s-1)}^{\alpha'(l)} + \sum_{l=1}^{s-1} f_{/\alpha'(l)} 'K_{(\beta'(s-1)/\beta'_s)}^{\alpha'(l)} \right. \\ & \left. - (s-1) \sum_{l=1}^{s-1} f_{/\alpha'(l)} 'K_{(\beta'(s-2)/\tau')}^{\alpha'(l)} G_{\beta'_{s-1}\beta'_s}^{\tau'} \right) U_{\beta'(s)}^{\beta'(s)}. \end{aligned}$$

Therefore, if we put

$$(3.5) \quad \delta_{(\beta_s)}^{\alpha_s} K_{\beta(s-1)}^{\alpha(l)} + 'K_{(\beta(s-1)/\beta_s)}^{\alpha(l)} - (s-1) 'K_{(\beta(s-2)/\tau)}^{\alpha(l)} G_{\beta_{s-1}\beta_s}^{\tau} = 'K_{\beta(s)}^{\alpha(l)},$$

the quantity $f_{\beta(s)} = \sum_{l=1}^s f_{/\alpha(l)} 'K_{\beta(s)}^{\alpha(l)}$ is also an intrinsic quantity. Thus we can see that there exist the intrinsic quantities $f_{\beta(s)} = \sum_{l=1}^s f_{/\alpha(l)} 'K_{\beta(s)}^{\alpha(l)}$ ($s=1, 2, \dots, m$) whose coefficients are determined from the recurring formula (3.5) putting $'K_{\beta(1)}^{\alpha(1)} = \delta_{\beta_1}^{\alpha_1}$.

It is easily seen from (3.5) that

$$(3.6) \quad 'K_{\beta(s)}^{\alpha(s)} = \delta_{\beta(s)}^{\alpha(s)}.$$

Moreover we can find the quantities $K_{r(l)}^{\beta(s)}$ ($1 \leq s \leq t \leq m$) such that the relations

$$(3.7) \quad \sum_{s=l}^t {}'K_{\beta(s)}^{\alpha(l)} K_{r(l)}^{\beta(s)} = \delta_{r(l)}^{\alpha(l)} \quad (1 \leq l \leq t \leq m)$$

hold. Specially, we have from (3.5), (3.6) and the above relations

$$(3.8) \quad K_{a(l)}^{r(l)} = \delta_{a(l)}^{r(l)}, \quad K_{a(l)}^{r(l-1)} = -{}'K_{a(l)}^{r(l-1)} = (\text{linear form of } G_{a\beta}^r).$$

Therefore, from (3.1a) and (3.1b) we have the following theorems:

Theorem 5. *The operators $\mathfrak{P}^{\beta(s)}$ defined by*

$$\mathfrak{P}^{\beta(s)} = \sum_{t=s}^m K_{r(t)}^{\beta(s)} P^{r(t)} \quad (s = 1, 2, \dots, m)$$

are intrinsic operators, that is

$$(3.9) \quad \mathfrak{P}^{\beta(s)}(L) = U_{\beta'(s)}^{\beta(s)} \mathfrak{P}^{\beta'(s)}(L),$$

and if T^A be an intrinsic quantity of $F_n^{(m)}$ whose transformation law is $T^A = \mathfrak{P}_A^{A'} T^{A'}$, then $\mathfrak{P}^{\beta(s)}(T^A)$ ($s = 1, 2, \dots, m$) are also intrinsic quantities whose transformation laws are

$$(3.10) \quad \mathfrak{P}^{\beta(s)}(T^A) = \mathfrak{P}_A^{A'} U_{\beta'(s)}^{\beta(s)} \mathfrak{P}^{\beta'(s)}(T^{A'}).$$

Theorem 6. *Let T^A be an intrinsic quantity of $F_n^{(m)}$ and v^i be any vector, then*

$$\mathfrak{D}^{\beta(l)}(T^A) v^i = \sum_{r=l}^m K_{a(r)}^{\beta(l)} D_i^{a(r)}(T^A) v^i \quad (l = 1, 2, \dots, m)$$

are also intrinsic quantities.

Moreover we have

Theorem 7. *If the PFAFFian form $\sum_{r=0}^m P^J a_i^{(r)} dp_{a(r)}^i$ defined in the manifold $F_n^{(m)}$ has the tensor character with respect to the index J , and v^i be any vector of $F_n^{(m)}$,*

$$P^J a_i^{(m)} dv^i + \sum_{r=0}^m P^J \beta_i^{(r)} a_j^{(m)} v^j dp_{\beta(r)}^i$$

has also the tensor character with respect to the index J and $\alpha_{(m)}$. [5].

Suppose now that we have the quantity $\Gamma_{j\beta}^i$ whose transformation law under the transformations (1.1) and (1.2) is

$$\Gamma_{j\beta}^i = \Gamma_{j'\beta'}^{i'} X_{i'}^i X_j^{j'} U_{\beta'}^{\beta} - X_{j',k'}^{i'} X_j^{j'} p_{\beta'}^{k'} U_{\beta'}^{\beta},$$

then we have

Theorem 8. *If $\omega_{\beta(s)}^i(d) = \sum_{r=0}^s P_{\beta(s)}^i a_k^{(r)} dp_{a(r)}^k$ be an intrinsic PFAFFian*

form, then

$$\begin{aligned} \omega_{\beta(s+1)}^i(d) &= \sum_{r=0}^s P_{(\beta(s) | k | \beta_{s+1})}^i{}^{a(r)} dp_{a(r) | \beta_{s+1}}^k + \sum_{r=0}^s (P_{(\beta(s) | k | \beta_{s+1})}^i{}^{a(r)} \\ &\quad + \Gamma_{j(\beta_{s+1})}^i P_{\beta(s)}^j{}^{a(r)} - s G_{(\beta_1 \beta_2 \dots \beta_{s+1})}^r{}^{a(r)} \Gamma_{k}^{a(r)}) dp_{a(r)}^k \end{aligned}$$

is also an intrinsic PFAFFian form.

Proof. It has been proved in the work [5] of A. KAWAGUCHI and H. HOMBU that this theorem is true under the x -transformations alone. We shall now prove it under the u -transformations (x fixed). Since

$$P_{\beta(s)}^i{}^{a(r)} = U_{\beta(s)}^{\beta'(s)} \sum_{t=r}^s P_{\beta'(s)}^i{}^{a'(t)} \frac{\partial p_{a'(t)}^j}{\partial p_{a(r)}^k},$$

we have

$$\begin{aligned} \sum_{r=0}^s P_{\beta(s)}^i{}^{a(r)} dp_{a(r) | \beta_{s+1}}^k &= U_{\beta(s)}^{\beta'(s)} \sum_{r=0}^s \sum_{t=r}^s P_{\beta'(s)}^i{}^{a'(t)} \frac{\partial p_{a'(t)}^j}{\partial p_{a(r)}^k} dp_{a(r) | \beta_{s+1}}^k, \\ \sum_{r=0}^s P_{\beta(s)}^i{}^{a(r)} / \beta_{s+1} dp_{a(r)}^k &= \sum_{r=0}^s \sum_{t=r}^s \left(U_{\beta(s)}^{\beta'(s)} P_{\beta'(s)}^i{}^{a'(t)} \frac{\partial p_{a'(t)}^j}{\partial p_{a(r)}^k} \right) / \beta_{s+1} dp_{a(r)}^k \\ &= U_{\beta(s) | \beta_{s+1}}^{\beta'(s)} \sum_{r=0}^s \sum_{t=r}^s P_{\beta'(s)}^i{}^{a'(t)} \frac{\partial p_{a'(t)}^j}{\partial p_{a(r)}^k} dp_{a(r)}^k \\ &\quad + U_{\beta(s)}^{\beta'(s)} \sum_{r=0}^s \sum_{t=r}^s P_{\beta'(s)}^i{}^{a'(t)} / \beta'_{s+1} U_{\beta'_{s+1}}^{\beta'_{s+1}} \frac{\partial p_{a'(t)}^j}{\partial p_{a(r)}^k} dp_{a(r)}^k \\ &\quad + U_{\beta(s)}^{\beta'(s)} \sum_{r=0}^s \sum_{t=r}^s P_{\beta'(s)}^i{}^{a'(t)} \left(\frac{\partial p_{a'(t)}^j}{\partial p_{\beta(r)}^k} \right) / \beta_{s+1} dp_{a(r)}^k \\ &= U_{\beta(s) | \beta_{s+1}}^{\beta'(s)} \sum_{t=0}^s P_{\beta'(s)}^i{}^{a'(t)} dp_{a'(t)}^j \\ &\quad + U_{\beta(s)}^{\beta'(s)} U_{\beta'_{s+1}}^{\beta'_{s+1}} \sum_{t=0}^s P_{\beta'(s)}^i{}^{a'(t)} / \beta'_{s+1} dp_{a'(t)}^j \\ &\quad + U_{\beta(s)}^{\beta'(s)} \sum_{r=0}^s \sum_{t=r}^s P_{\beta'(s)}^i{}^{a'(t)} \left(\frac{\partial p_{a'(t)}^j}{\partial p_{a(r)}^k} \right) / \beta_{s+1} dp_{a(r)}^k, \\ \Gamma_{j \beta_{s+1}}^i P_{\beta(s)}^j{}^{a(r)} dp_{a(r)}^k &= U_{\beta(s)}^{\beta'(s)} U_{\beta'_{s+1}}^{\beta'_{s+1}} \Gamma_{j \beta'_{s+1}}^i P_{\beta'(s)}^j{}^{a'(r)} dp_{a'(r)}^k \end{aligned}$$

and

$$\begin{aligned} \sum_{r=0}^s s G_{(\beta_1 \beta_2 \dots \beta_{s+1})}^r{}^{a(r)} dp_{a(r)}^k &= s (G_{\beta'_1 \beta'_2}^r{}^{a(r)} U_{\beta'_1}^r U_{\beta'_2}^{\beta'_1} U_{\beta'_3}^{\beta'_2} + U_{(\beta'_1 \beta'_2 | r')} U_{\beta'_3}^{\beta'_1} \dots U_{\beta'_{s+1}}^{\beta'_s} U_{r'}^{\beta'_{s+1}}) \\ &= s (G_{\beta'_1 \beta'_2}^r{}^{a(r)} U_{\beta'_1}^r U_{\beta'_2}^{\beta'_1} U_{\beta'_3}^{\beta'_2} + U_{(\beta'_1 \beta'_2 | r')} U_{\beta'_3}^{\beta'_1} \dots U_{\beta'_{s+1}}^{\beta'_s} U_{r'}^{\beta'_{s+1}}) \end{aligned}$$

$$\begin{aligned}
& \times \sum_{r=0}^s \sum_{t=r}^s P_{\beta'_3, \dots, \beta'_{s+1}}^i \nu^{a'(t)} \frac{\partial p_{a'(t)}^j}{\partial p_{a(r)}^k} dp_{a(r)}^k \\
& = s U_{\beta(s+1)}^{\beta'(s+1)} \sum_{t=0}^s G_{\beta'_1 \beta'_2}^{\tau'} P_{\beta'_3, \dots, \beta'_{s+1}}^i \nu^{a'(t)} dp_{a'(t)}^j \\
& + U_{(\beta(s)/\beta_{s+1})}^{\beta'(s)} \sum_{t=0}^s P_{\beta'(s)}^i \nu^{a'(t)} dp_{a'(t)}^j.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
& U_{\beta(s)}^{\beta'(s)} \sum_{r=0}^s \sum_{t=r}^s P_{\beta'(s)}^i \nu^{a'(t)} \left(\frac{\partial p_{a'(t)}^j}{\partial p_{a(r)}^k} dp_{a(r)\beta_{s+1}}^k + \left(\frac{\partial p_{a'(t)}^j}{\partial p_{a(r)}^k} \right)_{/\beta_{s+1}} dp_{a(r)}^k \right) \\
& = U_{\beta(s)}^{\beta'(s)} \sum_{r=0}^s \sum_{t=r}^s P_{\beta'(s)}^i \nu^{a'(t)} \left(A_{a'(t)}^{a(r)} \sum_{l=1}^{r+1} A_{a(r)\beta_{s+1}}^{\tau'(l)} dp_{\tau'(l)}^j \right. \\
& \quad \left. + A_{a'(t)/\beta_{s+1}}^{a(r)} \sum_{l=1}^r A_{a(r)}^{\tau'(l)} dp_{\tau'(l)}^j \right) \\
& = U_{\beta(s)}^{\beta'(s)} \sum_{r=0}^s \sum_{t=r}^s P_{\beta'(s)}^i \nu^{a'(t)} \left(A_{a'(t)}^{a(r)} \sum_{l=1}^{r+2} A_{a(r)}^{\tau'(l-1)} U_{\beta_{s+1}}^{\tau' l} dp_{\tau'(l)}^j \right. \\
& \quad \left. + A_{a'(t)}^{a(r)} \sum_{l=1}^r A_{a(r)/\beta_{s+1}}^{\tau'(l)} dp_{\tau'(l)}^j + A_{a'(t)/\beta_{s+1}}^{a(r)} \sum_{l=1}^r A_{a(r)}^{\tau'(l)} dp_{\tau'(l)}^j \right) \\
& = U_{\beta(s)}^{\beta'(s)} \sum_{l=1}^{s+1} \sum_{t=r}^s \sum_{r=l-1}^s P_{\beta'(s)}^i \nu^{a'(t)} A_{a'(t)}^{a(r)} A_{a(r)}^{\tau'(l-1)} U_{\beta_{s+1}}^{\tau' l} dp_{\tau'(l)}^j \\
& \quad + U_{\beta(s)}^{\beta'(s)} \sum_{l=1}^s \sum_{t=r}^s \sum_{r=l}^s P_{\beta'(s)}^i \nu^{a'(t)} (A_{a'(t)}^{a(r)} A_{a(r)/\beta_{s+1}}^{\tau'(l)} + A_{a'(t)/\beta_{s+1}}^{a(r)} A_{a(r)}^{\tau'(l)}) dp_{\tau'(l)}^j \\
& = U_{\beta(s)}^{\beta'(s)} \sum_{l=1}^{s+1} \sum_{t=l-1}^s \sum_{r=l-1}^t P_{\beta'(s)}^i \nu^{a'(t)} A_{a'(t)}^{a(r)} A_{a(r)}^{\tau'(l-1)} U_{\beta_{s+1}}^{\tau' l} dp_{\tau'(l)}^j \\
& = U_{\beta(s)}^{\beta'(s)} \sum_{l=1}^{s+1} \sum_{t=l-1}^s P_{\beta'(s)}^i \nu^{a'(t)} \delta_{a'(t)}^{\tau'(l-1)} U_{\beta_{s+1}}^{\tau' l} dp_{\tau'(l)}^j \\
& = U_{\beta(s)}^{\beta'(s)} U_{\beta_{s+1}}^{\beta(s+1)} \sum_{l=0}^s P_{\beta'(s)}^i \nu^{a'(l)} dp_{\tau'(l)\beta_{s+1}}^j,
\end{aligned}$$

consequently, we have the intrinsic PFAFFIAN form

$$\omega_{\beta(s+1)}^i(d) = \sum_{r=0}^{s+1} P_{\beta(s+1)}^i \nu^{a(r)} dp_{a(r)}^k,$$

putting

$$\begin{aligned}
(3.11) \quad P_{\beta(s+1)}^i \nu^{a(s+1)}_k &= P_{\beta(s)}^i \nu^{a(s)}_k \delta_{\beta_{s+1}}^{a_{s+1}}, \\
P_{\beta(s+1)}^i \nu^{a(r)}_k &= P_{\beta(s)}^i \nu^{a(r-1)}_{|k|} \delta_{\beta_{s+1}}^{a_r} + P_{i(\beta(s))|k|/\beta_{s+1}}^{a(r)} \\
&\quad + \Gamma_{j(\beta_{s+1})}^i P_{\beta(s)}^j \nu^{a(r)}_k - s G_{(\beta_{s+1}\beta_s)}^{\tau} P_{\beta_{s-1}, \dots, \beta_1}^i \nu^{a(r)}_k.
\end{aligned}$$

§ 4. *K*-ple integral of the *m*-th order. We shall now proceed to discuss the geometry of the *K*-ple integral of the *m*-th order:

$$(4.1) \quad \int_{(K)} F(x^i, p_{a(1)}^i, \dots, p_{a(m)}^i) du^1 \dots du^K \quad (m > 1)$$

by using of the results obtained in the preceding paragraphs, where the function $F(x^i, p_{a(1)}^i, \dots, p_{a(m)}^i)$ is differentiable to sufficient order with respect to its arguments.

If the integral (4.1) be regarded as defining a measure of K -dimensional surface in an n -dimensional manifold, it is adequate to suppose that the integral (4.1) is invariant under any parameter transformations. In order this it is necessary and sufficient that the function F is transformed under the parameter transformations (1.2) in the manner

$$(4.2) \quad F(x^i, p_{a'(1)}^i, \dots, p_{a'(m)}^i) = \Delta F(x^i, p_{a(1)}^i, \dots, p_{a(m)}^i), \quad \Delta = |U_{a'}^a|,$$

From this one has the well known relations [5]

$$(4.3a) \quad \sum_{s=1}^m s p_{r\beta(s-1)}^i F_{;i}^{\alpha\beta(s-1)} = \delta_r^\alpha F,$$

$$(4.3b) \quad \sum_{s=t}^m \binom{s}{t} p_{r\beta(s-t)}^i F_{;i}^{\alpha(t)\beta(s-t)} = 0 \quad (m \geq t > 1).$$

When $t = m$, (4.3b) becomes

$$F_{;i}^{\alpha(m)} p_r^i = 0 \quad (r = 1, 2, \dots, K).$$

Differentiating with respect to $p_{\lambda(m)}^j$ we have

$$F_{;i}^{\alpha(m)} p_r^i = 0 \quad (r = 1, 2, \dots, K),$$

so that

$$(4.4) \quad F_{;[\lambda_1}^{\alpha(m)} F_{;j_1}^{\lambda(m)} F_{;i_2}^{\beta(m)} F_{;j_2}^{\mu(m)} \dots F_{;i_{n-K}}^{\tau(m)} F_{;j_{n-K}}^{\nu(m)} p_r^i = 0 \quad (r = 1, 2, \dots, K),$$

$$(4.5) \quad F_{;[\lambda_1}^{\alpha(m)} F_{;j_1}^{\lambda(m)} F_{;i_2}^{\beta(m)} F_{;j_2}^{\mu(m)} \dots F_{;i_{n-K}}^{\tau(m)} F_{;j_{n-K}}^{\nu(m)} p_r^j = 0 \quad (r = 1, 2, \dots, K).$$

On the other hand it is evident that

$$(4.6) \quad \varepsilon_{i_1 i_2 \dots i_{n-K} i_{n-K+1} \dots i_n} p_{i_1}^{\ell_{n-K+1}} \dots p_{i_{n-K}}^{\ell_n} p_r^i = 0 \quad (r = 1, 2, \dots, K),$$

where $\varepsilon_{i_1 i_2 \dots i_n} = n! \delta_{[i_1}^1 \dots \delta_{i_n]}^n$.

We can see from (4.4), (4.5) and (4.6) that there is one system of the quantities $\rho^{(\alpha_1 \alpha_2 \dots \alpha_N) (\lambda_1 \lambda_2 \dots \lambda_N)}$ such that

$$F_{;[\lambda_1}^{\alpha(m)} F_{;j_1}^{\lambda(m)} \dots F_{;i_{n-K}}^{\alpha(m)} F_{;j_{n-K}}^{\nu(m)} = \varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} \times p_{i_1}^{\ell_{n-K+1}} \dots p_{i_{n-K}}^{\ell_n} p_{j_1}^{\ell_{n-K+1}} \dots p_{j_{n-K}}^{\ell_n} \rho^{(\alpha_1 \dots \alpha_N) (\lambda_1 \dots \lambda_N)},$$

where $(\alpha_1 \alpha_2 \dots \alpha_N)$ and $(\lambda_1 \lambda_2 \dots \lambda_N)$ represent $(\alpha_1 \dots \alpha_m \beta_1 \dots \beta_m \dots \gamma_1 \dots \gamma_m)$ and $(\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m \dots \nu_1 \dots \nu_m)$ respectively, and consequently $N = m(n-K)$.

It is easily seen from the definition that under the transformations (1.1) and (1.2) the quantity $\rho^{a(N)\lambda(N)} = \rho^{(\alpha_1, \dots, \alpha_N) (\lambda_1, \dots, \lambda_N)}$ obeys the transformation law

$$\rho^{a'(N)\lambda'(N)} = \Delta^{(m-K)-2} D^2 \rho^{a(N)\lambda(N)} U_{a(N)}^{a'(N)} U_{\lambda(N)}^{\lambda'(N)}$$

or

$$\rho^{a'(N)\lambda'(N)} = \Delta^{(n-K)-2} D^2 \sum_{a(N), \lambda(N)} \rho^{a(N), \lambda(N)} U_{a(N)}^{a'(N)} U_{\lambda(N)}^{\lambda'(N)}$$

where $\sum_{a(N), \lambda(N)}$ denotes the summation for the all different combinations of N indices $\alpha_1, \alpha_2, \dots, \alpha_N$ and the all different combinations of N indices $\lambda_1, \lambda_2, \dots, \lambda_N$, and $'U_{a(N)}^{a'(N)}$ denotes $\frac{N!}{l_1! l_2! \dots l_t!} U_{a(N)}^{a'(N)}$, when $\alpha_1, \alpha_2, \dots, \alpha_N$ consist of $l_1, l_2, \dots, l_t$ blocks of the same indices.

Consequently, the $\binom{K+N-1}{N}$ -rowed determinant $\rho = |\rho^{a(N)\lambda(N)}|$ is transformed by the transformations (1.1) and (1.2) as follows:

$$\rho' = \Delta^x D^y |'U_{a(N)}^{a'(N)}|^2 \rho,$$

where $x = \binom{K+N-1}{N} (n-K-2)$, $y = 2 \binom{K+N-1}{N}$ and $|'U_{a(N)}^{a'(N)}|$ represents the $\binom{K+N-1}{N}$ -rowed determinant.

In the same manner it should be obtained that

$$\rho = \Delta^{-x} D^{-y} |'U_{a(N)}^{a'(N)}| \rho',$$

so that $|'U_{a(N)}^{a'(N)}|^2 |'U_{a(N)}^{a'(N)}| = 1$. Hence we can conclude that $|'U_{a(N)}^{a'(N)}|$ is a power of Δ multiplying a suitable constant, since $|'U_{a(N)}^{a'(N)}|$ is a homogeneous function of $U_a^{a'}$'s. Under some consideration we see that

$$|'U_{a(N)}^{a'(N)}| = \Delta^{\binom{K+N-1}{N-1}}$$

and consequently we have

$$\rho' = \Delta^r D^y \rho,$$

putting $r = \binom{K+N-1}{N} (n-K-2) - 2 \binom{K+N-1}{N-1}$.

When we put

$$P^{a(N)\lambda(N)} = F^{-\frac{2}{K}} \rho^{\frac{-2}{y}} \rho^{a(N)\lambda(N)}$$

and denote by $P_{a(N)\lambda(N)}$ the inverse of $P^{a(N)\lambda(N)}$, that is $P^{a(N)\lambda(N)} P_{a(N)\mu(N)} = \delta_{\mu(N)}^{\lambda(N)}$, assuming that the $\binom{K+N-1}{N}$ -rowed determinant $P = |P^{a(N)\lambda(N)}|$ does not vanish, these are transformed by the transformations (I.1) and (1.2) in the manners

$$\begin{aligned} P^{a'(N)\lambda'(N)} &= P^{a(N)\lambda(N)} U_{a(N)}^{a'(N)} U_{\lambda(N)}^{\lambda'(N)}, \\ P_{a'(N)\lambda'(N)} &= P_{a(N)\lambda(N)} U_{a'(N)}^{a(N)} U_{\lambda'(N)}^{\lambda(N)}. \end{aligned}$$

§ 5. Determination of the quantities $G_{\beta\tau}^a$ and Γ_{jr}^i . We shall now attempt to determine the quantity $G_{\beta\tau}^a$ as mentioned in § 3 in use of $P^{a(N)\lambda(N)}$ and $P_{a(N)\mu(N)}$. If we put

$$Q_{\mu(N)\nu}^{\lambda(N)} = P^{\lambda(N)a(N)} (P_{\mu(N)a(N)/\nu}),$$

under the transformation (1.1) and (1.2) it follows that

$$\begin{aligned} Q_{\mu'(N)\nu'}^{\lambda'(N)} &= P^{\lambda'(N)a'(N)} (P_{\mu'(N)a'(N)} U_{\mu'(N)}^{\mu(N)} U_{a'(N)}^{a(N)})_{/\nu'} \\ &= Q_{\mu(N)\nu}^{\lambda(N)} U_{\lambda(N)}^{\lambda'(N)} U_{\mu'(N)}^{\mu(N)} U_{\nu'}^{\nu} + P^{\lambda'(N)a'(N)} (U_{\mu'(N)/\nu'}^{\mu(N)} P_{\mu(N)a(N)} U_{a'(N)}^{a(N)} \\ &\quad + U_{a'(N)/\nu'}^{a(N)} P_{\mu(N)a(N)} U_{\mu'(N)}^{\mu(N)}). \end{aligned}$$

Putting $\lambda'_1 = \mu'_1$, $\lambda'_2 = \mu'_2$, $\dots$, $\lambda'_{N-1} = \mu'_{N-1}$, $\lambda'_N = \lambda'$, $\mu'_N = \mu'$ and contracting over the indices $\mu'_1, \mu'_2, \dots, \mu'_{N-1}$ we have

$$\begin{aligned} Q_{\mu'\nu'}^{\lambda'} &= Q_{\mu\nu}^{\lambda} U_{\lambda}^{\lambda'} U_{\mu}^{\mu'} U_{\nu'}^{\nu} + NP^{(\mu'_1, \dots, \mu'_{N-1}, \sigma')}_{(\alpha'_1, \dots, \alpha'_{N-1}, \lambda')} \\ &\quad \times P_{(\mu'_1, \dots, \mu'_{N-1}, \mu')}_{(\alpha'_1, \dots, \alpha'_{N-1}, \rho')} U_{\alpha'}^{\rho'} U_{\sigma'}^{\alpha}, \\ &\quad + U_{\mu'_1}^{(\mu'_1)} U_{\mu'_2}^{(\mu'_2)} \dots U_{\mu'_{N-1}}^{(\mu'_{N-1})} U_{\mu}^{\lambda'} (U_{\mu'_1}^{\mu_1} \dots U_{\mu'_{N-1}}^{\mu_{N-1}} U_{\mu}^{\mu'}), \end{aligned}$$

or

$$\begin{aligned} (5.1) \quad Q_{\mu'\nu'}^{\lambda'} &= Q_{\mu\nu}^{\lambda} U_{\lambda}^{\lambda'} U_{\mu}^{\mu'} U_{\nu'}^{\nu} + K_{\mu'\rho'}^{\sigma'\lambda'} U_{\alpha'}^{\rho'} U_{\sigma'}^{\alpha} \\ &\quad + p U_{\mu}^{\lambda'} U_{\mu'\nu'}^{\mu} + q \delta_{\mu'}^{\lambda'} \partial_{\nu'} \log \Delta, \end{aligned}$$

where we put $Q_{\mu'\nu'}^{\lambda'} = Q_{\mu'(N-1)\mu'\nu'}^{\lambda'}$,

$$\begin{aligned} K_{\mu'\rho'}^{\sigma'\lambda'} &= NP^{(\mu'_1, \dots, \mu'_{N-1}, \sigma')}_{(\alpha'_1, \dots, \alpha'_{N-1}, \lambda')} P_{(\mu'_1, \dots, \mu'_{N-1}, \mu')}_{(\alpha'_1, \dots, \alpha'_{N-1}, \rho')} \\ p &= \frac{(K+2) \dots (K+N)}{N!} \quad \text{and} \quad q = \frac{(N-1)(K+2) \dots (K+N-1)}{N!} (N \neq 2), \end{aligned}$$

$$q = \frac{1}{2} \quad (N=2).$$

Moreover from (5.1) we have

$$(5.2) \quad Q_{\mu'} = Q_{\mu} U_{\mu}^{\mu'} + K_{\mu'\rho'}^{\sigma'\lambda'} U_{\alpha'}^{\rho'} U_{\sigma'}^{\alpha} + (p+q) \partial_{\mu'} \log \Delta,$$

putting $Q_{\mu'} = Q_{\mu'}^{\lambda'}$.

By eliminating $\partial_{\nu'} \log \Delta$ from (5.1) and (5.2) we get

$$(5.3) \quad (p+q) Q_{\mu'}^{\lambda'} - q \delta_{\mu'}^{\lambda'} Q_{\nu'} = [(p+q) Q_{\mu'}^{\lambda'} - q \delta_{\mu'}^{\lambda'} Q_{\nu'}] U_{\lambda'}^{\lambda'} U_{\mu'}^{\mu'} U_{\nu'}^{\nu'} \\ + [(p+q) \delta_{\nu'}^{\omega'} K_{\mu'}^{\sigma'} + p(p+q) \delta_{\rho'}^{\lambda'} \delta_{\mu'}^{\sigma'} \delta_{\nu'}^{\omega'} - q \delta_{\mu'}^{\lambda'} K_{\nu'}^{\sigma' \omega'}] U_{\sigma'}^{\sigma'} U_{\omega'}^{\omega'}.$$

Now we put

$$(p+q) K_{\mu'}^{\sigma'} \delta_{\nu'}^{\omega'} + p(p+q) \delta_{\rho'}^{\lambda'} \delta_{\mu'}^{\sigma'} \delta_{\nu'}^{\omega'} - q \delta_{\mu'}^{\lambda'} K_{\nu'}^{\sigma' \omega'} = N_{\mu' \nu'}^{\lambda' \sigma' \omega'}.$$

and assume that the K^3 -rowed determinant $|N_{\mu' \nu'}^{\lambda' \sigma' \omega'}|$ is different from zero, then we can obtain the quantities $n_{\lambda' \nu'}^{\mu' \sigma' \omega'}$ such that

$$N_{\mu' \nu'}^{\lambda' \sigma' \omega'} n_{\lambda' \nu'}^{\mu' \sigma' \omega'} = \delta_{\rho'}^{\lambda'} \delta_{\beta'}^{\sigma'} \delta_{\gamma'}^{\omega'}.$$

Since $K_{\mu'}^{\sigma'}$ has the tensor character with respect to its indices under the transformations (1.1) and (1.2), the quantities $N_{\mu' \nu'}^{\lambda' \sigma' \omega'}$ and $n_{\lambda' \nu'}^{\mu' \sigma' \omega'}$ are tensors. Hence, if we put

$$n_{\lambda' \nu'}^{\mu' \sigma' \omega'} ((p+q) Q_{\mu'}^{\lambda'} - q \delta_{\mu'}^{\lambda'} Q_{\nu'}) = G_{\beta' \gamma'}^{\alpha'},$$

it is easily seen from (5.3) that the quantity $G_{\beta' \gamma'}^{\alpha'}$ obeys the law of transformation:

$$G_{\beta' \gamma'}^{\alpha'} = G_{\beta \gamma}^{\alpha} U_{\alpha'}^{\alpha} U_{\beta'}^{\beta} U_{\gamma'}^{\gamma} + U_{\alpha'}^{\alpha} U_{\beta' \gamma'}^{\alpha}.$$

Thus obtained $G_{\beta' \gamma'}^{\alpha'}$ will play an important rôle in our theories.

Let us next consider the EULER vector which is concerned with the first variation of the integral (4.1):

$$(5.4) \quad E_i(F) = \sum_{r=0}^m (-1)^r (F; \frac{\alpha^{(r)}}{i})_{| \alpha^{(r)}},$$

It is seen from the first variation of (4.1) that the EULER vector E_i is transformed by the parameter transformations as follows:

$$(5.5) \quad E_i(F') = \Delta E_i(F).$$

If in (5.4) F be substituted by $F^* = F\phi$, ϕ being any function of u 's, we have

$$E_i(F\phi) = \sum_{r=0}^m E_i^{\alpha^{(r)}} \phi_{| \alpha^{(r)}},$$

where $E_i^{\alpha^{(r)}} (r = 0, 1, \dots, m)$ are vectors of the form

$$E_i^{\alpha^{(r)}} = \sum_{s=r}^m (-1)^s \binom{s}{r} (F; \frac{\alpha^{(r)} \beta^{(s-r)}}{i})_{| \beta^{(s-r)}} \quad (r = 0, 1, \dots, m),$$

and called the SYNGE vectors. If we effect the parameter transformations, it follows that

$$\begin{aligned} E_i(F'\phi') &= \sum_{s=0}^m E_i^{a'(s)} \phi_{|a'(s)} = \sum_{s=0}^m \sum_{r=0}^s E_i^{a(s)} A_{a'(s)}^{a(r)} \phi_{|a(r)} \\ &= \sum_{r=0}^m \sum_{s=r}^m E_i^{a'(s)} A_{a'(s)}^{a(r)} \phi_{|a(r)}. \end{aligned}$$

On the other hand by (5.5) we have

$$E_i(F'\phi') = \Delta E_i(F\phi) = \Delta \sum_{r=0}^m E_i^{a(r)} \phi_{|a(r)}.$$

Hence, the SYNGE vectors are transformed by the parameter transformations in the manners

$$E_i^{a(r)} = \Delta^{-1} \sum_{s=r}^m E_i^{a'(s)} A_{a'(s)}^{a(r)} \quad (r = 0, 1, \dots, m),$$

so that we can derive from the SYNGE vectors a system of the intrinsic vectors $\mathfrak{S}_i^{\beta(l)}$ ($l = 0, 1, \dots, m$) in similar manner as theorem 5, that is

$$\mathfrak{S}_i^{\beta(l)} = -\frac{1}{F} \sum_{r=l}^m E_i^{a(r)} K_{a(r)}^{\beta(l)} \quad (l = 1, 2, \dots, m).$$

By the definitions of $E_i^{a(r)}$ and $K_{a(r)}^{\beta(l)}$ it is seen that $\mathfrak{S}_i^{\beta(l)}$ is a quantity of $F^{(2m-1)}_n$ and the relations

$$\begin{aligned} (5.6) \quad \mathfrak{S}_i^{\beta} p_r^i &= \delta_r^{\beta}, \quad \mathfrak{S}_i^{\beta(l)} p_r^i = 0 \quad (l = 2, 3, \dots, m) [5], \\ (\delta_j^i - p_a^i \mathfrak{S}_j^a) p_r^j &= 0, \quad (\delta_j^i - p_a^i \mathfrak{S}_j^a) E_i^r = 0, \\ (\delta_j^i - p_a^i \mathfrak{S}_j^a) (\delta_k^j - p_\beta^j \mathfrak{S}_k^\beta) &= (\delta_k^i - p_a^i \mathfrak{S}_k^a) \end{aligned}$$

hold.

We shall now go on to derive the quantity Γ_{jr}^i which obeys the transformation law as mentioned in § 3. When we put $T^A = F; \tau_j^{(m)}$ and $l = m - 1$ in theorem 6, one gets the intrinsic quantity

$$\begin{aligned} \frac{1}{F} (m K_{a(m-1)}^{\beta(m-1)} F; \tau_j^{(m)}; a(m-1)\beta v^i_{| \beta} + K_{a(m-1)}^{\beta(m-1)} F; \tau_j^{(m)}; a(m-1) v^i \\ + K_{a(m)}^{\beta(m-1)} F; \tau_j^{(m)}; a(m) v^i), \end{aligned}$$

or by virtue of (3.8) we have

$$\begin{aligned} (5.7) \quad \frac{m}{F} F; \tau_j^{(m)}; \beta(m-1)\beta v^i_{| \beta} + \frac{1}{F} F; \tau_j^{(m)}; \beta(m-1) \\ + \frac{m(m-1)}{2} F; \tau_j^{(m)}; \omega_1 \omega_2 (\beta_1 \dots \beta_{m-2} G_{\omega_1 \omega_2}^{\beta(m-1)}) v^i, \end{aligned}$$

where v^i is a vector of $F^{(m)}_n$. The $n \binom{K+m-1}{m}$ -rowed matrix $(F; \tau_j^{(m)}; \beta(m))$

has the rank $\binom{K+m-1}{m} (n-K)$ at most because of $F; \tau_j^{(m)}; \beta_i^{(m)} p_r^i = 0$ ($r = 1, 2, \dots, K$). Suppose that the matrix $(F; \tau_j^{(m)}; \beta_i^{(m)})$ is of rank $\binom{K+m-1}{m} (n-K)$, then by virtue of (5.6) it is seen that we can find the quantities $G_{a(m)\tau(m)}^k$ such that

$$(5.8) \quad G_{a(m)\tau(m)}^k F; \tau_j^{(m)}; \beta_i^{(m)} = F(\delta_i^k - p_a^k \mathfrak{S}_i^a) \delta_{a(m)}^{\beta(m)}.$$

We see that thus obtained $G_{a(m)\tau(m)}^k$ obey the transformation law of the form

$$(5.9) \quad G_{a(m)\tau(m)}^k = G_{a'(m)\tau'(m)}^k X_{k'}^k X_j^j U_{a(m)}^{a'(m)} U_{\tau(m)}^{\tau'(m)} + R^a p_a^j.$$

We may write all system of the solutions of (5.8) in the forms

$$(5.10) \quad G_{a(m)\tau(m)}^k = g_{a(m)\tau(m)}^k + \varphi_{a(m)\tau(m)}^i \mathfrak{S}_i^a p_a^k + \psi_{a(m)\tau(m)}^a p_a^j,$$

where $g_{a(m)\tau(m)}^k$ and $\varphi_{a(m)\tau(m)}^i$ are quantities of $F_n^{(m)}$ and $\psi_{a(m)\tau(m)}^a$ are any quantities. Accordingly, it is known that the quantity $G_{a(m)\tau(m)}^k T_j^A$ is intrinsic and is the same for all system of the solutions of (5.8), when T_j^A is an intrinsic quantity satisfying the relations $p_r^j T_j^A = 0$ ($r = 1, 2, \dots, K$), that is to say, the intrinsic quantity $G_{a(m)\tau(m)}^k T_j^A$ is uniquely determined by the equations (5.8).

If (5.7) be multiplied by $G_{a(m)\tau(m)}^k$ and summed for $\tau_1, \tau_2, \dots, \tau_m$ and j , we have the intrinsic quantity

$$\begin{aligned} m(\delta_i^k - p_a^k \mathfrak{S}_i^a) \delta_{a(m)}^{\beta(m-1)\beta} v_{i\beta}^i + \frac{1}{F} G_{a(m)\tau(m)}^k (F; \tau_j^{(m)}; \beta_i^{(m-1)}) \\ + \frac{m(m-1)}{2} F; \tau_j^{(m)}; \omega_1 \omega_2 (\beta_i^{(m-1)} \dots \beta_{m-2} G_{\omega_1 \omega_2}^{\beta(m-1)}) v^i. \end{aligned}$$

Putting $\beta_1 = \alpha_1, \beta_2 = \alpha_2, \dots, \beta_{m-1} = \alpha_{m-1}$ and contracting over these indices, we get

$$\begin{aligned} (\delta_i^k - p_a^k \mathfrak{S}_i^a) \frac{(K+1) \dots (K+m-1)}{(m-1)!} v_{i\alpha_m}^i + \frac{1}{F} G_{a(m)\tau(m)}^k (F; \tau_j^{(m)}; \alpha_i^{(m-1)}) \\ + \frac{m(m-1)}{2} F; \tau_j^{(m)}; \omega_1 \omega_2 (\alpha_i^{(m-1)} \dots \alpha_{m-2} G_{\omega_1 \omega_2}^{\alpha(m-1)}) v^i. \end{aligned}$$

Accordingly, if we put

$$\begin{aligned} A_{ia}^k = \frac{(m-1)!}{(K+1) \dots (K+m-1)} \frac{1}{F} G_{a(m-1)\tau(m)}^k (F; \tau_j^{(m)}; \alpha_i^{(m-1)}) \\ + \frac{m(m-1)}{2} F; \tau_j^{(m)}; \omega_1 \omega_2 (\alpha_i^{(m-1)} \dots \alpha_{m-2} G_{\omega_1 \omega_2}^{\alpha(m-1)}) \quad (m > 2), \end{aligned}$$

$$\begin{aligned} \Lambda_{ia}^k = & \frac{1}{K+1} \frac{1}{F} G_{a,1}^k F_{;j}^{(2)} F_{;i}^{(2)} + (F_{;j}^{(2)} \omega_1 \omega_2 G_{\omega_1 \omega_2}^{a_1} \\ & + 2 \mathfrak{G}_{[i}^{(r_1)} F_{;j]}^{(r_2) a_1} - 2 \mathfrak{G}_{[i}^{a_1} F_{;j]}^{(r_1) r_2}) \end{aligned} \quad (m=2),$$

we have the intrinsic quantity

$$(5.11) \quad \Delta_a v^k = (\delta_i^k - p_\beta^k \mathfrak{G}_i^\beta) v^i_{/a} + \Lambda_{ia}^k v^i$$

which is the same for all systems of the solutions of (5.8). If w_i be any covariant vector of $F_{(n)}^{(m)}$ such that $w_i p_r^i = 0$ ($r = 1, 2, \dots, K$), one obtains from (5.11) the intrinsic derivative of w_i :

$$\Delta_a w_i = w_{i/a} - \Lambda_{ia}^k w_k$$

by means of $\Delta_a (v^i w_i) = (v^i w_i)_{/a}$. Accordingly, it is easily seen that the quantity

$$(5.12) \quad \begin{aligned} \Delta_a F_{;i}^{a(m)} = & \frac{1}{F} (F_{;i}^{a(m)}_{/a} - \Lambda_{ia}^k F_{;k}^{a(m)} + m G_{\beta a}^{(a_1} F_{;i}^{a_2 \dots a_m) \beta} \\ & - G_{\beta a}^\beta F_{;i}^{a(m)}) \end{aligned}$$

is intrinsic. Moreover, we see from (5.10) and the definition of Λ_{ia}^k that $\Lambda_{ia}^k F_{;k}^{a(m)}$ does not depend on $\mathfrak{G}_i^a$, so that (5.12) can be written in the form

$$(5.13) \quad \Delta_{(a} F_{;i}^{a(m)}) = H_{a i}^{a(m) \beta(m+1)} p_{\beta(m+1)}^j + P_{a i}^{a(m)} (x^j, p_{\beta(1)}^j, \dots, p_{\beta(m)}^j).$$

When $m > 2$, we have from (4.3b) an identity

$$\sum_{s=m-1}^m \binom{s}{m-1} p_{a r(s-m+1)}^i F_{;i}^{\beta(m-1) r(s-m+1)} = 0.$$

Differentiating this with respect to $p_{a(m)}^j$ one obtains

$$F_{;j}^{a(m) \beta(m-1)} p_a^i = -m F_{;j}^{a(m) \beta(m-1) r} p_{a r}^i,$$

and consequently it follows that

$$\begin{aligned} p_r^i \Delta_a F_{;i}^{a(m)} &= p_r^i (F_{;i}^{a(m)}_{/a} - \Lambda_{ia}^k F_{;k}^{a(m)}) \\ &= -F_{;i}^{a(m)} p_{a r}^i - \frac{(m-1)!}{(K+1) \dots (K+m-1)} \frac{1}{F} G_{\beta(m-1) a}^k F_{;j}^{(m)} F_{;i}^{(m-1) \beta} F_{;k}^{a(m)} p_r^i \\ &= -F_{;i}^{a(m)} p_{a r}^i + \frac{m!}{(K+1) \dots (K+m-1)} \frac{1}{F} G_{\beta(m-1) a}^k F_{;j}^{(m)} F_{;i}^{(m-1) \beta} p_\beta^i r F_{;k}^{a(m)} \\ &= -F_{;i}^{a(m)} p_{a r}^i + \frac{m!}{(K+1) \dots (K+m-1)} (\delta_i^k - p_\lambda^k E_i^\lambda) \delta_{\beta(m-1) a}^\beta p_\beta^i r F_{;k}^{a(m)} \\ &= -F_{;i}^{a(m)} p_{a r}^i + F_{;i}^{a(m)} p_{a r}^i = 0. \end{aligned}$$

When $m = 2$, it follows that

$$\begin{aligned}
p_r^i \Delta_a F;_i^{a(2)} &= p_r^i (F;_i^{a(2)}|_a - \Lambda_{ia}^k F;_k^{a(2)}) \\
&= -F;_i^{a(2)} p_{ar}^i - \frac{1}{K+1} \frac{1}{F} G_{\beta, a}^k{}^j{}_{r(2)} (F;_j^{\lambda(2)};_i^{\beta} p_r^i + 2\delta_r^{\lambda(2)} F;_j^{\beta}{}_{\lambda(2)} \\
&\quad - \delta_r^{\beta} F;_j^{\lambda(2)}{}_{\lambda(2)} F;_k^{a(2)}).
\end{aligned}$$

On the other hand, differentiating the identity

$$F;_i^{\beta} p_r^i + 2F;_i^{\beta}{}_{\lambda(2)} p_{\beta r}^i = \delta_r^{\beta} F$$

with respect to $p_{r(2)}^j$, we have

$$F;_j^{\lambda(2)};_i^{\beta} p_r^i + 2F;_j^{\beta}{}_{\lambda(2)} \delta_r^{\lambda(2)} + 2F;_j^{\lambda(2)};_i^{\beta} p_{\beta r}^i - \delta_r^{\beta} F;_j^{\lambda(2)} = 0$$

and consequently

$$\begin{aligned}
p_r^i \Delta_a F;_i^{a(2)} &= -F;_i^{a(2)} p_{ar}^i + \frac{2}{K+1} \frac{1}{F} G_{\beta, a}^k{}^j{}_{r(2)} F;_j^{\lambda(2)};_i^{\beta} p_{\beta r}^i F;_k^{a(2)} \\
&= -F;_i^{a(2)} p_{ar}^i + \frac{2}{K+1} \frac{1}{F} (\delta_i^k - p_{\lambda}^k E_i^{\lambda}) \delta_{\beta}^{\lambda} \delta_a^{\beta} p_{ar}^i F;_k^{a(2)} \\
&= 0.
\end{aligned}$$

Hence, we have from (5.13) the relations

$$(5.14) \quad p_r^i P_a^{a(m)}{}^i = 0 \quad (r = 1, 2, \dots, K).$$

Suppose now that the rank of the $n \binom{K+m}{m+1}$ -rowed matrix $(H_a^{a(m)}{}^i{}_{\beta(m+1)}{}_j)$ is $(n-K) \binom{K+m}{m+1}$, then we can find the quantities $'H_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_k$ such that

$$(5.15) \quad 'H_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_k H_a^{a(m)}{}^i{}_{\beta(m+1)}{}_j = (\delta_j^k - p_{\lambda}^k \mathfrak{E}_j^{\lambda}) \delta_{\beta(m+1)}^{\beta}.$$

We may write all systems of the solutions of (5.15) in the form

$$(5.16) \quad 'H_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_k = h_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_k + \varphi_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_j \mathfrak{E}_j^{\lambda} p_{\lambda}^k + \phi_{a(m)}^{a(m)}{}^{\beta}{}_{\beta(m+1)}{}_k p_{\beta}^i,$$

where $h_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_k$ and $\varphi_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_j$ are quantities of $F_n^{(m)}$, and $\phi_{a(m)}^{a(m)}{}^{\beta}{}_{\beta(m+1)}{}_k$ are any quantities.

If (5.13) be multiplied by $'H_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_k$ and contracted over the indices $i, \alpha, \alpha_1, \dots, \alpha_m$, it is obtained the intrinsic quantity

$$(5.17) \quad (\delta_j^k - p_{\lambda}^k \mathfrak{E}_j^{\lambda}) p_{r(m+1)}^j + 'H_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_k P_a^{a(m)}{}^i{}_k$$

and by (5.14) it is seen that the quantity $'H_{a(m)}^{a(m)}{}^i{}_{\beta(m+1)}{}_k P_a^{a(m)}{}^i{}_k$ is the same for all solutions of (5.15).

If (5.17) be multiplied by $(\delta_k^i - p_{\mu}^i \mathfrak{E}_k^{\mu})$ and summed for k , we have, in consequence of (5.6) and (5.16),

$$(\delta_j^i - p_\lambda^i \mathfrak{G}_j^\lambda) [p_{r(m+1)}^j + H_{r(m+1)}^j (x^i, p_{a(1)}^i, \dots, p_{a(m)}^i)]$$

or

$$(5.18) \quad T_{r(m+1)}^i = p_{r(m+1)}^i + H_{r(m+1)}^i - p_\lambda^i \chi_{r(m+1)}^\lambda,$$

putting $H_{r(m+1)}^k = h_{a(m)}^{a(i)} p_{r(m+1)}^k P_a^{a(m)} i$, $\chi_{r(m+1)}^\lambda = \mathfrak{G}_j^\lambda (p_{r(m+1)}^j + H_{r(m+1)}^j)$. It should be observed that $H_{r(m+1)}^i$ is a quantity of $F_n^{(m)}$.

Let us now observe the following facts. If Q^A be any quantity of $F_n^{(m)}$ which is transformed by arbitrary transformations of parameters in the manner

$$Q^{A'} (x^i, p_{a'(1)}^i, \dots, p_{a'(m)}^i) = R^{A'} (Q^A, U_{a'(1)}^a, \dots, U_{a'(s)}^a),$$

$R^{A'}$ being a rational function of its arguments, then the relations

$$Q^A;_i^{a(m)} p_r^i = 0 \quad (r = 1, 2, \dots, K)$$

hold when $s < m$. On the other hand, since $A_{a'(s)}^{a(t)} (t \leq s)$ are polynomials of $U_{a'(1)}^a, U_{a'(2)}^a, \dots, U_{a'(s-t+1)}^a$, the equation of transformation of the quantity

$$F;_i^{a(r)} p_{a(s)}^{b(s-r)} /_{B(s-r-1)} \quad (1 \leq r < s \leq m)$$

appearing in the course of formation of the SYNGE vector contains the derivatives of u^a 's not more than the $(m-r)$ -th order. And the quantity $K_{r(t)}^{\beta(s)} (1 \leq s \leq t \leq m-1)$ determined from (3.5) and (3.7) has the transformation law:

$$K_{a'(t)}^{\beta'(s)} = U_{\beta'(s)}^{\beta(s)} \sum_{r=s}^t A_{a'(t)}^{r(r)} K_{r(r)}^{\beta(s)}$$

in which $U_{a'(t-s+1)}^r$'s are contained as the highest derivatives of u^r .

Let us now define the differential operator D_r applied to the quantity Q^A as follows:

$$D_r Q^A = Q^A;_r + \sum_{s=0}^{m-1} Q^A;_i^{a(s)} p_{a(s)r}^i - Q^A;_i^{a(m)} (H_{a(m)r}^i - p_\lambda^i \chi_{a(m)r}^\lambda)$$

or

$$D_r Q^A = Q^A;_r + \sum_{s=0}^{m-1} Q^A;_i^{a(s)} p_{a(s)r}^i - Q^A;_i^{a(m)} H_{a(m)r}^i,$$

then it is also a quantity of $F_n^{(m)}$ and its transformation law under (1.1) and (1.2) is the same as that of $Q^A;_r$. Consequently, if the quantities $K_{r(t)}^{\beta(s)}$ and $\mathfrak{G}_i^{a(r)}$ in which the operator D_r is applied instead of the operator $/r$ are denoted by $\tilde{K}_{r(t)}^{\beta(s)}$ and $\tilde{\mathfrak{G}}_i^{a(r)}$ respectively, these are quantities of $F_n^{(m)}$, and the transformation laws of $\tilde{K}_{r(t)}^{\beta(s)}$ and $\tilde{\mathfrak{G}}_i^{a(r)}$ are the

same as that of $K_{r(i)}^{\beta(s)}$ and $\mathfrak{G}_i^{\alpha(r)}$ respectively. Moreover the same relations as (5.6) hold in this case.

We can now find one and only one system of the quantities $\mathring{G}_{a(m)r(m)}^k$ satisfying the equations

$$\begin{aligned} \mathring{G}_{a(m)r(m)}^k F_j^{\tau(m); \beta(m)} &= F(\delta_i^k - p_a^k \mathfrak{G}_i^a) \delta_{a(m)}^{\beta(m)}, \\ \mathring{G}_{a(m)r(m)}^k \mathfrak{G}_j^a &= 0 \quad (\alpha = 1, 2, \dots, K), \end{aligned}$$

since we have supposed that the rank of the matrix $(F_j^{\tau(m); \beta(m)})$ is $(n-K) \binom{K+m-1}{m}$. And it is easily seen that thus obtained $\mathring{G}_{a(m)r(m)}^k$ is intrinsic and symmetric with respect to the rows $\alpha(m)$ and $r(m)$, so that

$$(5.19) \quad \mathring{G}_{a(m)r(m)}^k \mathfrak{G}_k^a = 0 \quad (\alpha = 1, 2, \dots, K).$$

If we put

$$\begin{aligned} \mathring{A}_{ia}^k &= \frac{(m-1)}{(K+1) \dots (K+m-1)} \frac{1}{F} \mathring{G}_{a(m-1)a}^k F_j^{\tau(m); \alpha(m-1)} \\ &\quad + \frac{m(m-1)}{2} F_j^{\tau(m); \omega_1, \omega_2(\alpha, \dots, \alpha_{m-1})} \mathring{G}_{\omega_1 \omega_2}^{\alpha(m-1)} \quad (m \geq 2), \end{aligned}$$

then

$$(\delta_i^k - p_\beta^k \mathfrak{G}_i^\beta) v_{i/a}^k + \mathring{A}_{ia}^k v^i$$

is an intrinsic quantity as it is seen from (5.11).

When we put $v^i = p_r^i v^r$, by virtue of (5.6) it follows that $\mathfrak{G}_i^r v^i = v^r$, and consequently we have

$$\begin{aligned} (5.20) \quad (\delta_i^k - p_\beta^k \mathfrak{G}_i^\beta) v_{i/a}^k + \mathring{A}_{ia}^k v^i &= v_{i/a}^k - p_\beta^k (v_{i/a}^\beta + \mathring{G}_{a\tau}^\beta v^\tau) \\ &\quad + (p_\beta^k \mathring{G}_{a\tau}^\beta \mathfrak{G}_i^\tau + p_\beta^k \mathfrak{G}_i^\beta v_{i/a} + \mathring{A}_{ia}^k) v^i \end{aligned}$$

from which we see that

$$(5.21) \quad \Delta_a v^k = v_{i/a}^k + \Gamma_{ia}^k v^i$$

defines an intrinsic derivative of the vector v^k , where we put

$$\Gamma_{ia}^k = \mathring{A}_{ia}^k + p_\beta^k \mathfrak{G}_{i/a}^\beta + \mathring{G}_{a\tau}^\beta p_\beta^k \mathfrak{G}_i^\tau.$$

From (5.5), (5.19) and (5.20) we have

$$\mathfrak{G}_k^\beta (v_{i/a}^k + \Gamma_{ia}^k v^i) = v_{i/a}^\beta + \mathring{G}_{a\tau}^\beta v^\tau,$$

Consequently, if one defines the intrinsic derivative along the K -dimensional surface $x^i = x^i(u^a)$ by (5.21), putting

$$p_{a(s)}^i = \frac{\partial^s x^i}{\partial u^{a_1} \dots \partial u^{a_s}} \quad (s = 0, 1, \dots, m), \text{ then}$$

$$\Delta_a v^\beta \equiv \mathring{\mathbb{G}}_k^\beta \Delta_a v^k = v^\beta_{,a} + \mathring{G}_{a\tau}^\beta v^\tau$$

may be regarded as the intrinsic derivative induced on the K -dimensional surface. Therefore, the covariant differential of the vector v^k along a K -dimensional surface is given by

$$\delta_s v^k = dv^k + \Gamma_{j\tau}^k v^j du^\tau$$

and the induced covariant differential is given by

$$\delta_s v^\beta = du^\beta + \mathring{G}_{a\tau}^\beta v^a du^\tau,$$

$$\text{when we put } p_{a(s)}^i = \frac{\partial^s x^i}{\partial u^{a_1} \dots \partial u^{a_s}} \quad (s = 0, 1, \dots, m).$$

In order to determine the base connections and the connections in $F_n^{(m)}$ we put

$$\mathring{\Gamma}_{j\tau}^i = \mathring{A}_{j\tau}^i + p_{\beta}^i D_{\tau} \mathring{\mathbb{G}}_j^{\beta} + \mathring{G}_{a\tau}^{\beta} p_{\beta}^i \mathring{\mathbb{G}}_j^a \quad [14][15],$$

then it is a quantity of $F_n^{(m)}$ and has the same transformation law as that of $\Gamma_{j\tau}^i$ under the transformations (1.1) and (1.2), that is,

$$(5.22) \quad \mathring{\Gamma}_{j\tau}^i = \mathring{\Gamma}_{j'\tau'}^{i'} X_{i'}^i X_{j'}^{j'} U_{\tau'}^{\tau} - X_{j,k}^i X_{j'}^{j'} p_{\tau}^{k'} U_{\tau'}^{\tau},$$

as it is seen from the tensor character of (5.21).

§ 6. Base connections in $F_n^{(m)}$ and covariant differentials. In order to define the base connections in $F_n^{(m)}$, we shall introduce the intrinsic PFAFFIAN forms by means of theorem 8.

Since dx^i may be regarded as an intrinsic PFAFFIAN form,

$$\omega_{\beta(1)}^i(d) = P_{\beta(1)}^i{}^{a(1)} dp_{a(1)}^k + P_{\beta(1)k}^i dx^k$$

is an intrinsic PFAFFIAN form, where the coefficients $P_{\beta(1)}^i{}^{a(1)}$ and $P_{\beta(1)k}^i$ are determined from the recurring formulae (3.11) as follows:

$$P_{\beta(1)}^i{}^{a(1)} = \delta_k^i \delta_{\beta_1}^{a_1}, \quad P_{\beta(1)k}^i = \mathring{\Gamma}_{k\beta_1}^i,$$

so that

$$\omega_{\beta(1)}^i(d) = dp_{\beta(1)}^i + \mathring{\Gamma}_{k\beta_1}^i dx^k$$

is an intrinsic PFAFFIAN form of $F_n^{(m)}$. In general, we obtain the intrinsic PFAFFIAN forms

$$(6.1) \quad \omega_{\beta(s)}^i(d) = \sum_{r=0}^s P_{\beta(s)}^i{}^{a(r)} dp_{a(r)}^k \quad (s = 0, 1, \dots, m),$$

where the coefficients $P_{\beta(s)}^i \alpha_k^{(r)}$ are determined from the recurring formulae

$$\begin{aligned} P_{\beta(s)}^i \alpha_k^{(r)} = & P_{(\beta(s-1))}^i \alpha_k^{(r-1)} \delta_{\beta(s)}^{\alpha_r} + D_{(\beta,s)} P_{\beta(s-1)}^i \alpha_k^{(r)} \\ & + \Gamma_{j(\beta,s)}^i P_{\beta(s-1)}^j \alpha_k^{(r)} - (s-1) G_{(\beta,s,\beta_{s-1})}^{\tau} P_{\beta_{s-2} \dots \beta_1}^i \alpha_k^{(r)} \\ & (0 \leq r \leq s \leq m), \end{aligned}$$

putting $P_{\beta(0)}^i \alpha_k^{(0)} = \delta_k^i$.

It is evident from the above recurring formulae that the coefficients of the differential $dp_{\alpha(s)}^k$ in the PFAFFIAN form $\omega_{\beta(s)}^i(d)$, say $P_{\beta(s)}^i \alpha_k^{(s)}$, is of the form $\delta_k^i \delta_{\beta(s)}^{\alpha(s)}$, so that (6.1) becomes

$$(6.2) \quad \omega_{\beta(s)}^i(d) = dp_{\beta(s)}^i + \sum_{r=0}^{s-1} P_{\beta(s)}^i \alpha_k^{(r)} dp_{\alpha(s-r)}^k \quad (s = 0, 1, \dots, m).$$

Hence, we can define the base connetctions in $F_n^{(m)}$ by the equations

$$\omega_{\beta(s)}^i(d) = 0 \quad (s = 0, 1, \dots, m).$$

We shall next introduce a covariant differential of vector of $F_n^{(m)}$ by means of theorem 5.

We have seen in theorem 5 that the transformation law of the quantity $\mathfrak{P}^\beta(L)$ defined by

$$\mathfrak{P}^\beta(L) = \sum_{t=1}^m K_{\tau(t)}^\beta \sum_{s=t}^m \binom{s}{t} L_{\tau(t)}^{\alpha(s-t)} dp_{\alpha(s-t)}^k$$

is

$$\mathfrak{P}^\beta(L) = U_\beta^\beta \mathfrak{P}^{\beta'}(L),$$

when L is a quantity of $F_n^{(m)}$. If we put $L = \Gamma_{j\beta}^i$ into $\mathfrak{P}^\beta(L)$ and contract over the index β , it follows from (5.22) that the quantity $\mathfrak{P}^\beta(\Gamma_{j\beta}^i)$ is transformed by the transformations (1.1) and (1.2) in the manner

$$\mathfrak{P}^\beta(\Gamma_{j\beta}^i) = \mathfrak{P}^{\beta'}(\Gamma_{j'\beta'}^{i'}) X_{i'}^i X_{j'}^{j'} - K X_{j'k}^i X_{j'}^{j'} dx^{k'},$$

so that the PFAFFIAN form

$$(6.3) \quad \Gamma_j^i = \frac{1}{K} \mathfrak{P}^\beta(\Gamma_{j\beta}^i)$$

obeys the transformation law

$$\Gamma_j^i = X_{i'}^i X_{j'}^{j'} \Gamma_{j'}^{i'} - X_{j'k}^i X_{j'}^{j'} dx^{k'}.$$

Consequently, if v^i be a vector of $F_n^{(m)}$,

$$(6.4) \quad \delta v^i = dv^i + \Gamma_j^i v^j$$

defines a covariant differential of the vector v^i .

We may write (6.4) in the form

$$(6.5) \quad \delta v^i = dv^i + \sum_{s=0}^{m-1} C_j^{i\beta(s)} v^j dp_{\beta(s)}^k,$$

when we put

$$C_j^{i\beta(s)} = \frac{1}{K} \sum_{t=1}^{m-s} \binom{t+s}{t} K_{\tau(t)}^{\beta} \Gamma_{j\beta}^{i; \tau(t)\beta(s)} \quad (s = 0, 1, \dots, m-1).$$

On the other hand, we may also introduce another intrinsic differential of vector by means of theorem 7.

From (6.2) we have the intrinsic PFAFFIAN form

$$\omega_{\beta(m)}^j(d) = dp_{\beta(m)}^j + \sum_{s=0}^m P_{\beta(m)}^j \alpha_i^{(s)} dp_{\alpha(s)}^i.$$

Applying theorem 7 one obtains the intrinsic quantity

$$\delta_i^j \delta_{\beta(m)}^{\alpha(m)} dv^i + \sum_{s=0}^{m-1} P_{\beta(m)}^j \tau_i^{(s)} \alpha_k^{(m)} v^k dp_{\tau(s)}^i$$

from which we obtain the intrinsic differential of the vector v^i :

$$\delta v^j = dv^j + \frac{m!}{K(K+1)\dots(K+m-1)} \sum_{s=0}^{m-1} P_{\beta(m)}^j \tau_i^{(s)} \beta_k^{(m)} v^k dp_{\tau(s)}^i.$$

§ 7. Covariant derivatives, torsion tensors and curvature tensors.

When we put

$$\delta v^i = \sum_{s=0}^m (\nabla_k^{\beta(s)} v^i) \omega_{\beta(s)}^k(d),$$

it is obtained from (6.2) and (6.5) the recurring formulae for the covariant derivatives $\nabla_k^{\beta(s)} v^i$ ($s = 0, 1, \dots, m$), that is

$$(7.1) \quad \begin{aligned} \nabla_k^{\beta(m)} v^i &= v^i; \beta_k^{(m)}, \\ \nabla_j^{\alpha(t)} v^i &= v^i; \alpha_j^{(t)} + C_k^i \alpha_j^{(t)} v^k - \sum_{s=t+1}^m P_{\beta(s)}^k \alpha_j^{(t)} \nabla_k^{\beta(s)} v^i. \end{aligned}$$

We shall next determine the torsion tensors and the curvature tensors of $F_n^{(m)}$.

If δ_1 and δ_2 denote the intrinsic differential operators corresponding to the increments d_1 and d_2 respectively, we can find the torsion tensors $A_{\alpha(r)}^i \beta_j^{(p)} \tau_k^{(q)}$ and $A_j^i \omega_{\beta(s)}^k$ by means of the equations

$$\begin{aligned} \delta_1 \omega_{\alpha(r)}^i(d_2) - \delta_2 \omega_{\alpha(r)}^i(d_1) &= \sum_{p=0}^r \sum_{q=0}^r A_{\alpha(r)}^i \beta_j^{(p)} \tau_k^{(q)} \omega_{\beta(p)}^j(d_2) \omega_{\tau(q)}^k(d_1) \\ &+ \sum_{p=0}^r \sum_{q=r+1}^m A_{\alpha(r)}^i \beta_j^{(p)} \tau_k^{(q)} [\omega_{\beta(p)}^j(d_2) \omega_{\tau(q)}^k(d_1)]^* \quad (r = 0, 1, \dots, m). \end{aligned}$$

Indeed, we have

*The symbol $[\omega_{\beta(p)}^j(d_2) \omega_{\tau(q)}^k(d_1)]$ means $\omega_{\beta(p)}^j(d_2) \omega_{\tau(q)}^k(d_1) - \omega_{\beta(p)}^j(d_1) \omega_{\tau(q)}^k(d_2)$.

$$A_{a(r)}^i \beta_j^{(r)} \gamma_k^{(q)} = C_{j k}^i \gamma_{a(r)}^{(q)} \delta_{a(r)}^{\beta(r)} - C_{k j}^i \beta_{a(r)}^{(r)} \delta_{a(r)}^{\gamma(q)} - \sum_{s=q+1}^{m-1} A_{a(r)}^i \beta_j^{(r)} \omega_s^{(s)} P_{\omega(s)}^i \gamma_k^{(q)} \\ (r=m, m-1, \dots, 0; q=m-1, \dots, 0),$$

$$A_{a(r)}^i \beta_j^{(p)} \gamma_k^{(q)} = P_{a(r)}^i \beta_j^{(p)} \gamma_k^{(q)} - P_{a(r)}^i \gamma_k^{(q)} \beta_j^{(p)} \\ + C_{i k}^j P_{a(r)}^i \beta_j^{(p)} - C_{i j}^k P_{a(r)}^i \gamma_k^{(q)} \\ - \sum_{t=p+1}^{r-1} A_{a(r)}^i \delta_{i k}^{(t)} \gamma_k^{(q)} P_{\delta(t)}^i \beta_j^{(p)} - \sum_{s=q+1}^{m-1} A_{a(r)}^i \beta_j^{(p)} \omega_s^{(s)} P_{\omega(s)}^i \gamma_k^{(q)} \\ - \sum_{t=p+1}^{r-1} \sum_{s=q+1}^{m-1} A_{a(r)}^i \delta_{i h}^{(t)} \omega_s^{(s)} P_{\delta(t)}^i \beta_j^{(p)} P_{\omega(s)}^i \gamma_k^{(q)} \\ - A_{i k}^j P_{a(r)}^i \beta_j^{(p)} - \sum_{s=q+1}^{m-1} A_{i h}^j \omega_s^{(s)} P_{a(r)}^i \beta_j^{(p)} P_{\omega(s)}^i \gamma_k^{(q)} \\ (r=m, \dots, 1; p=r-1, \dots, 0; q=m-1, \dots, 0),$$

where we put $P_{\omega(s)}^i \gamma_k^{(q)} = 0$ ($s \leq q$) and $A_{i k}^j = A_{a(0)}^i \beta_{i l}^{(0)} \gamma_k^{(q)}$.

Next, by means of the equation

$$[\delta_1 \delta_2 - \delta_2 \delta_1] v^i = \sum_{t=0}^{m-1} R_{i h}^{t a(t) \beta(m)} [\omega_{\beta(m)}^k (d_1) \omega_{a(t)}^h (d_2)] v^i \\ + \sum_{s=0}^{m-1} \sum_{t=0}^{m-1} R_{i h}^{t a(t) \beta(s)} \omega_{\beta(s)}^k (d_1) \omega_{a(t)}^h (d_2) v^i,$$

we have the recurring formulae for the curvature tensors:

$$R_{i h}^{t a(m-1) \beta(m)} = C_{i h}^{t a(m-1) \beta(m)}, \\ R_{i h}^{t a(t) \beta(m)} = C_{i h}^{t a(t) \beta(m)} - \sum_{s=t+1}^{m-1} R_{i g}^{t a(s) \beta(m)} P_{i(s)}^g \alpha_{h}^{a(t)} \\ (t=m-1, m-2, \dots, 1, 0), \\ R_{i h}^{\beta(t) a(s)} = C_{j h}^i \beta_{a(s)}^{(t)} - C_{j f}^i \alpha_{a(s)}^{(t)} + C_{j f}^i C_{i h}^j \beta_{a(s)}^{(t)} \\ - C_{j h}^i C_{i f}^j \alpha_{a(s)}^{(t)} + R_{i f}^{\beta(t) a(s)} P_{\gamma(m)}^k \beta_{h}^{\beta(t)} \\ - R_{i h}^{\beta(t) \gamma(m)} P_{\gamma(m)}^k \alpha_{f}^{a(s)} + \sum_{r=s+1}^{m-1} R_{i g}^{\beta(r) \mu(m)} P_{\mu(m)}^k \beta_{h}^{\beta(t)} P_{\delta(r)}^g \alpha_{f}^{a(s)} \\ - \sum_{r=t+1}^{m-1} R_{i g}^{\beta(r) \mu(m)} P_{\mu(m)}^k \alpha_{f}^{a(s)} P_{\delta(r)}^g \beta_{h}^{\beta(t)} \\ + \sum_{r=s+1}^{m-1} R_{i h}^{\beta(t) \delta(r)} P_{\delta(r)}^k \alpha_{f}^{a(s)} - \sum_{r=t+1}^{m-1} R_{i k}^{\beta(t) a(s)} P_{\delta(r)}^k \beta_{h}^{\beta(t)} \\ - \sum_{r=s+1}^{m-1} \sum_{p=t+1}^{m-1} R_{i g}^{\beta(p) \delta(r)} P_{\delta(r)}^k \alpha_{f}^{a(s)} P_{\gamma(p)}^g \beta_{h}^{\beta(t)} \\ (t, s=m-1, m-2, \dots, 1, 0).$$

§ 8. Method of A. KAWAGUCHI. In the case of one parameter

Prof. A. KAWAGUCHI [3] has introduced a base connection in the manifold of line-elements of higher order. We shall generalize this method to the manifold of surface-elements of higher order.

By theorem 5 the quantities

$$\frac{1}{F} \mathfrak{P}^{\tau(s)}(F; \alpha_i^{(m)}) = \frac{1}{F} \sum_{l=s}^m K_{\beta(l)}^{\tau(s)} \sum_{t=l}^m \binom{t}{l} F; \alpha_i^{(m)}; \beta(l) \mu_j^{(t-l)} dp_{\mu(t-l)}^j \quad (s = 1, 2, \dots, m)$$

are intrinsic. Putting $t-l=r$, we have the intrinsic quantities

$$(8.1) \quad \frac{1}{F} \binom{m}{s} F; \alpha_i^{(m)}; \tau(s) \mu_j^{(m-s)} dp_{\mu(m-s)}^j + \frac{1}{F} \sum_{r=0}^{m-s-1} M_{\alpha_i^{(m)} \tau(s) \mu_j^{(r)}}^{\alpha_i^{(m)} \tau(s) \mu_j^{(r)}} dp_{\mu(r)}^j \quad (s = 1, 2, \dots, m),$$

where

$$M_{\alpha_i^{(m)} \tau(s) \mu_j^{(r)}}^{\alpha_i^{(m)} \tau(s) \mu_j^{(r)}} = \sum_{l=s}^{m-r} K_{\beta(l)}^{\tau(s)} \binom{l+r}{l} F; \alpha_i^{(m)}; \beta(l) \mu_j^{(r)}.$$

Hence, we can derive from (8.1) the intrinsic PFAFFIAN forms

$$(8.2) \quad \delta p_{r(m-s)}^k = (\delta_i^k - p_a^k \mathfrak{G}_j^a) dp_{r(m-s)}^j + \sum_{r=0}^{m-s-1} N_{\tau(m-s)}^k \beta_j^{(r)} dp_{\beta(r)}^j \quad (s = 1, 2, \dots, m),$$

where

$$N_{\tau(m-s)}^k \beta_j^{(r)} = \frac{1}{F} \binom{m+K-1}{s}^{-1} \mathfrak{G}_{\tau(m-s) \nu(s) \alpha_i^{(m)}}^k M_{\alpha_i^{(m)} \nu(s) \beta_j^{(r)}}^{\alpha_i^{(m)} \nu(s) \beta_j^{(r)}}.$$

Moreover we have the intrinsic quantity

$$\frac{1}{F} \delta F; \alpha_i^{(m)} = \frac{1}{F} (dF; \alpha_i^{(m)} - \Gamma_i^j F; \alpha_j^{(m)})$$

from which one gets the intrinsic PFAFFIAN form

$$(8.3) \quad \delta p_{r(m)}^k = (\delta_j^k - p_a^k \mathfrak{G}_j^a) dp_{r(m)}^j + \sum_{t=0}^{m-1} N_{\tau(m)}^k \beta_j^{(t)} dp_{\beta(t)}^j,$$

where

$$N_{\tau(m)}^k \beta_j^{(t)} = \frac{1}{F} G_{\alpha_i^{(m)} \tau(m)}^k (F; \alpha_i^{(m)}; \beta_j^{(t)} - C_{\alpha_i^{(m)} \tau(m)}^{\beta_j^{(t)}} F; \alpha_i^{(m)}).$$

We may define the base connections in $F_n^{(m)}$ by the equations

$$\delta p_{r(t)}^k = 0 \quad (t = 0, 1, \dots, m).$$

Using of (6.5), (8.2) and (8.3) we can determine the covariant derivatives $\nabla_j^{\alpha_i^{(t)}} v^i$ under the conditions $\nabla_j^{\alpha_i^{(t)}} v^i p_r^j = 0$ ($i = 1, 2, \dots, K$), that is,

$$\nabla_j^{a(m)} v^i = v^i_{;j}{}^{a(m)},$$

$$\nabla_j^{a(i)} v^i = v^i_{;j}{}^{a(i)} + C_j^{ia(i)} v^k - \sum_{s=i+1}^m P_{\beta(s)}^k{}^{a(i)} \nabla_k^{\beta(s)} v^i.$$

§ 9. Method of D. D. KOSAMBI. D. D. KOSAMBI has introduced a system of covariant derivatives in his work [2] on the path space of higher order by using of the special method. We shall now generalize this method in our metric space.

If v^j be a vector of $F_n^{(m)}$, the quantity $\nabla_i^{a(m)} v^j = v^j_{;i}{}^{a(m)}$ is an intrinsic derivative.

Now we see that when $m > 1$,

$$\begin{aligned} \nabla_i^{a(m-1)} v^j &= \frac{m}{K+m-1} (\nabla_i^{a(m-1)\sigma} \Delta_\sigma v^j - \Delta_\sigma \nabla_i^{a(m-1)\sigma} v^j) \\ &\quad - \frac{m}{K+m-1} (\nabla_i^{a(m-1)\sigma} \Gamma_{i\sigma}^j) v^l \end{aligned}$$

is an intrinsic quantity, because $\nabla_i^{a(m-1)\sigma} \Gamma_{i\sigma}^j$ is a tensor. Moreover we see that $\nabla_i^{a(m-1)} v^j$ may be written in the form

$$(9.1) \quad \nabla_i^{a(m-1)} v^j = v^j_{;i}{}^{a(m-1)} + S_{i\beta(m)}^{ja(m-1)} \nabla_i^{\beta(m)} v^j \quad (m > 1),$$

where

$$(9.2) \quad \begin{aligned} S_{i\beta(m)}^{ja(m-1)} &= \frac{m}{K+m-1} (\Gamma_{i\beta m}^j \delta_{\beta(m-1)}^{a(m-1)} - G_{\beta m} \delta_{\beta(m-1)}^{a(m-1)} \delta_i^j) \\ &\quad - (m-1) G_{\beta m \beta_{m-1}}^{a(m-1)} \delta_{\beta(m-2)}^{a(m-2)} \delta_i^j, \end{aligned}$$

putting $G_{\beta m} = G_{\beta m \lambda}^\lambda$.

We shall prove that in general the representations

$$\nabla_i^{a(r)} v^j = v^j_{;i}{}^{a(r)} + \sum_{s=r+1}^m S_{i\beta(s)}^{ja(r)} \nabla_i^{\beta(s)} v^j \quad (r = 1, 2, \dots, m-1)$$

and

$$\nabla_i v^j = v^j_{;i} + \sum_{s=1}^m S_{i\beta(s)}^j \nabla_i^{\beta(s)} v^j + \frac{1}{K} \nabla_i^{\beta} \Gamma_{i\beta}^j v^l$$

is true when we put

$$\begin{aligned} \nabla_i^{a(r)} v^j &= \frac{r+1}{K+r} (\nabla_i^{a(r)\beta} \Delta_\beta - \Delta_\beta \nabla_i^{a(r)\beta}) v^j \\ &\quad - (1 - \delta_r^j) \frac{r+1}{K+r} (\nabla_i^{a(r)\beta} \Gamma_{i\beta}^j) v^l \quad (r = 0, 1, \dots, m-1). \end{aligned}$$

First of all we have

$$\begin{aligned} & \bar{\nabla}_i^{\alpha(m)} (D_r v^j) - D_r (\bar{\nabla}_i^{\alpha(m)} v^j) \\ &= v^j_{;i} \delta_r^{\alpha(m-1)} \delta_r^{\alpha(m)} - H_{\beta(m)r}^i \delta_i^{\alpha(m)} \bar{\nabla}_i^{\beta(m)} v^j \end{aligned}$$

or by (9.1)

$$\begin{aligned} (9.3) \quad & \bar{\nabla}_i^{\alpha(m)} (D_r v^j) - D_r (\bar{\nabla}_i^{\alpha(m)} v^j) \\ &= \delta_r^{\alpha(m)} \bar{\nabla}_i^{\alpha(m-1)} v^j + U_{\beta(m)r}^i \delta_i^{\alpha(m)} \bar{\nabla}_i^{\beta(m)} v^j, \end{aligned}$$

where

$$(9.4) \quad U_{\beta(m)r}^i \delta_i^{\alpha(m)} = -S_{i\beta(m)}^{\alpha(m-1)} \delta_r^{\alpha(m)} - H_{\beta(m)r}^i \delta_i^{\alpha(m)},$$

$H_{\beta(m)r}^i$ being that of § 5.

Let us now assume that the representations of two kinds:

$$(9.5) \quad \bar{\nabla}_i^{\alpha(r)} v^j = v^j_{;i} \delta_i^{\alpha(r)} + \sum_{s=r+1}^m S_{i\beta(s)}^{\alpha(r)} \bar{\nabla}_i^{\beta(s)} v^j \quad (r=m, m-1, \dots, t)$$

and

$$\begin{aligned} (9.6) \quad & \bar{\nabla}_i^{\alpha(r+1)} D_r v^j - D_r \bar{\nabla}_i^{\alpha(r+1)} v^j \\ &= \delta_r^{\alpha(r+1)} \bar{\nabla}_i^{\alpha(r)} v^j + \sum_{s=r+1}^m U_{\beta(s)r}^i \delta_i^{\alpha(r+1)} \bar{\nabla}_i^{\beta(s)} v^j \\ & \quad (r=m-1, m-2, \dots, t) \end{aligned}$$

are true, then, after some calculation we see that the equalities

$$\bar{\nabla}_i^{\alpha(r-1)} v^j = v^j_{;i} \delta_i^{\alpha(r-1)} + \sum_{s=r}^m S_{i\beta(s)}^{\alpha(r-1)} \bar{\nabla}_i^{\beta(s)} v^j \quad (r > 1),$$

$$\bar{\nabla}_i^{\alpha} v^j = v^j_{;i} + \sum_{s=1}^m S_{i\beta(s)}^{\alpha} \bar{\nabla}_i^{\beta(s)} v^j + \frac{1}{K} (\bar{\nabla}_i^{\beta} \Gamma_{i\beta}^j) v^j$$

and

$$\bar{\nabla}_i^{\alpha(r)} D_r v^j - D_r \bar{\nabla}_i^{\alpha(r)} v^j = \delta_r^{\alpha(r)} \bar{\nabla}_i^{\alpha(r-1)} v^j + \sum_{s=r}^m U_{\beta(s)r}^i \delta_i^{\alpha(r)} \bar{\nabla}_i^{\beta(s)} v^j$$

hold good, where the coefficients $S_{i\beta(s)}^{\alpha(r-1)}$, $S_{i\beta(s)}^k$ and $U_{\beta(s)r}^i \delta_i^{\alpha(r)}$ are determined from (9.2), (9.4) and the recurring formulae

$$\begin{aligned} S_{i\beta(m)}^{\alpha(r-1)} &= \frac{r+1}{K+r} \left(\sum_{t=r+1}^m S_{i\omega(t)}^{\alpha(r-1)r} U_{\beta(m)r}^i \omega_j^{\omega(t)} \right. \\ & \quad \left. - D_r S_{i\beta(m)}^{\alpha(r-1)r} - H_{\beta(m)r}^i \delta_i^{\alpha(r-1)r} \right) \quad (r < m), \end{aligned}$$

$$S_{i\beta(r)}^{\alpha(r-1)} = \frac{r+1}{K+r} (S_{i\beta(r)}^{\alpha(r-1)r} + \Gamma_{i\beta r}^i \delta_{\beta(r-1)}^{\alpha(r-1)})$$

$$\begin{aligned}
& - (r-1) G_{\beta_r \beta_{r-1}}^{(a) r-1} \delta_{\beta(r-2)}^{\alpha(r-2)} \delta_i^{\alpha(r-1)} - G_{\beta_r} \delta_{\beta(r-1)}^{\alpha(r-1)} \delta_i^{\alpha(r)}, \\
S_{i\beta(s)}^{(a) r-1} &= \frac{r+1}{K+r} (S_{i\beta(s)}^{(a) r-1} \delta_r^{\alpha(r-1)} - D_r S_{i\beta(s)}^{(a) r-1} + \sum_{t=r+1}^s S_{i\omega(t)}^{(a) r-1} U_{\beta(s)r}^{\omega(t)} U_j^{\omega(t)}) \\
& \quad (s=r+1, r+2, \dots, m-1)
\end{aligned}$$

and

$$\begin{aligned}
U_{\beta(m)r}^{\alpha(r)} &= \sum_{t=r+1}^m \delta_{i\omega(t)}^{(a) r} U_{\beta(m)r}^{\omega(t)} - S_{i\beta(m)}^{(a) r-1} \delta_r^{\alpha(r)} \\
& \quad - H_{\beta(m)r}^{(a) r} - D_r S_{i\beta(m-1)}^{(a) r}, \\
U_{\beta(r)r}^{\alpha(r)} &= F_{i\beta(r)}^{\alpha(r)} - r G_{\beta_r \beta_{r-1}}^{(a) r-1} \delta_{\beta(r-1)}^{\alpha(r-1)} \delta_i^{\alpha(r)} + S_{i\beta(r)}^{(a) r} \delta_r^{\alpha(r)} - S_{i\beta(r)}^{(a) r-1} \delta_r^{\alpha(r)}, \\
U_{\beta(s)r}^{\alpha(r)} &= \sum_{t=r+1}^s S_{i\omega(t)}^{(a) r} U_{\beta(s)r}^{\omega(t)} - S_{i\beta(s)}^{(a) r-1} \delta_r^{\alpha(r)} \\
& \quad + S_{i\beta(s)}^{(a) r} \delta_r^{\alpha(r)} - D_r S_{i\beta(s)}^{(a) r} \quad (s=r+1, \dots, m-1).
\end{aligned}$$

Consequently, we have the intrinsic PFAFFIAN forms of the second kind

$$\delta_{\beta(s)}^K = dp_{\beta(s)}^i - \sum_{r=0}^{s-1} S_{i\alpha(s)}^{(a) r} dp_{\beta(r)}^{\alpha(r)} \quad (s=0, 1, \dots, m).$$

§ 10. Metric tensors and metric connection. If we put

$$G_{\alpha(m-1)\alpha\beta(m-1)\beta}^{(a) m-1} \mathfrak{G}_i^{\alpha(m-1)} \mathfrak{G}_j^{\beta(m-1)} = G_{\alpha\beta}$$

it is easily seen from (5.6), (5.9) and (5.10) that when $m > 2$, $G_{\alpha\beta}$ is an intrinsic quantity of $F_n^{(m)}$ and is the same for all the solutions $G_{\alpha(m-1)\alpha\beta(m-1)\beta}^{(a) m-1}$ of the equations (5.8). When $m=2$, we put

$$\frac{1}{F} G_{\alpha(2)\alpha\beta(2)\beta}^{(a) 2} F_i^{\alpha(2)} F_j^{\beta(2)} = f$$

and derive the intrinsic vectors $\mathring{H}_i^{\alpha}$ from the scalar f as if we derive the intrinsic vectors $\mathring{E}_i^{\alpha}$ from the scalar F . If we put

$$(\delta_a^i - p_\lambda^i \mathfrak{G}_a^\lambda) (\delta_b^j - p_\mu^j \mathfrak{G}_b^\mu) G_{\alpha_1 \alpha \beta_1 \beta}^{\alpha_1 \beta_1} \mathring{H}_i^{\alpha_1} \mathring{H}_j^{\beta_1} = G_{\alpha\beta},$$

this is an intrinsic quantity of $F_n^{(m)}$ and is the same for all the solutions $G_{\alpha(2)\alpha\beta(2)\beta}^{(a) 2}$.

Moreover, if we put $F^{\frac{2}{K}} G^{-\frac{1}{K}} G_{\alpha\beta} = g_{\alpha\beta}$ assuming that $G = |G_{\alpha\beta}| \neq 0$, then the measure of K -dimensional surface is given by

$$\int_{(K)} |g_{\alpha\beta}|^{\frac{1}{2}} du^1 \dots du^K.$$

Hence, it is adequate to take $g_{\alpha\beta}$ as the metric tensor on the K -dim-

ensional surface, when we put $p_{\alpha(s)}^i = \partial^s x^i / \partial u^{\alpha_1} \dots \partial u^{\alpha_s}$ ($s=0, \dots, m$).

Now we put

$$\frac{1}{F} g_{\alpha_1 \beta_1} g_{\alpha_2 \beta_2} \dots g_{\alpha_m \beta_m} F_{; i}^{\alpha(m); \beta(m)} + g_{\alpha \beta} \mathring{G}_i^\alpha \mathring{G}_j^\beta = g_{ij},$$

and assume that the determinant $|g_{ij}|$ does not vanish, then g_{ij} is a tensor of $F_n^{(m)}$ and the relation

$$g_{ij} p_\alpha^i p_\beta^j = g_{\alpha \beta}$$

holds good, so that we may take g_{ij} as the metric tensor of $F_n^{(m)}$. If $g^{\alpha \beta}$ and g^{ij} be the inverses of $g_{\alpha \beta}$ and g_{ij} respectively, it is easily seen that

$$g^{ij} g_{\alpha \beta} \mathring{G}_j^\beta = p_\alpha^i, \quad g^{ij} \mathring{G}_i^\alpha \mathring{G}_j^\beta = g^{\alpha \beta}.$$

By the method of Prof. A. KAWAGUCHI [4] we obtain the metric connection:

$$dv^i + \frac{1}{2} (\Gamma_j^i - g^{ik} \Gamma_k^l g_{lj} + g^{ik} dg_{jk}) v^j = 0.$$

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