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Author(s)	Nakano, Hidegorô
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LINEAR TOPOLOGIES ON SEMI-ORDERED LINEAR SPACES

By

Hidegorô NAKANO

Normed semi-order linear spaces are considered first by L. KANTOROVITCH.⁽¹⁾ In this paper we shall consider linear topologies on semi-ordered linear spaces.

Let R be a linear space. A manifold $V \subset R$ is called a *vicinity*, if for any $a \in R$ we can find $\varepsilon > 0$ such that $\xi a \in V$ for $|\xi| \leq \varepsilon$. A collection of vicinities $\mathfrak{B}$ is said to be a *linear topology* on R , if

- 1) $U \subset V \in \mathfrak{B}$ implies $U \in \mathfrak{B}$,
- 2) $U, V \in \mathfrak{B}$ implies $UV \in \mathfrak{B}$,
- 3) $V \in \mathfrak{B}$ implies $\xi V \in \mathfrak{B}$ for every real number ξ ,
- 4) for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $\xi U \subset V$ for $|\xi| \leq 1$,
- 5) for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $U \times U \subset V$,

adopting the notations:

$$\xi U = \{\xi x : x \in U\}, \quad U \times V = \{x + y : x \in U, y \in V\}.$$

A subset $\mathfrak{B} \subset \mathfrak{B}$ is called a *basis* of a linear topology $\mathfrak{B}$, if for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ and $\varepsilon > 0$ such that $\varepsilon U \subset V$.

Let R be now a semi-ordered linear space and universally continuous, that is, for any system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$) there exists $\bigcap_{\lambda \in \Lambda} a_\lambda$. In this paper we shall consider only such linear topologies $\mathfrak{B}$ on R that $\mathfrak{B}$ have a basis composed only of vicinities V subject to the conditions:

- 6) $a \in V, |x| \leq a$ implies $x \in V$,
- 7) $0 \leq a_\lambda \in V$ ($\lambda \in \Lambda$), $a_\lambda \uparrow_{\lambda \in \Lambda} a$ implies $a \in V$.

Here $a_\lambda \uparrow_{\lambda \in \Lambda} a$ means that for any two $\lambda_1, \lambda_2 \in \Lambda$ we can find $\lambda \in \Lambda$ such that

$$a_\lambda \geq a_{\lambda_1} \cup a_{\lambda_2}, \quad \text{and} \quad a = \bigcup_{\lambda \in \Lambda} a_\lambda.$$

For such a linear topology, we shall prove as a principal result that the manifold $\{x : a \leq x \leq b\}$ is complete as a uniform space in WEIL's

(1) L. KANTOROVITCH: Lineare halbgeordnete Räume, Math. Sbornik, 2 (44), (1937), 121-168.

sense.⁽²⁾

For a vicinity V subject to the conditions 6), 7), putting

$$\|x\|_V = \inf_{\xi \in V} \frac{1}{|\xi|},$$

we obtain a *pseudo-norm* on R . A manifold $A \subset R$ is said to be *topologically bounded*, by a linear topology $\mathfrak{B}$, if $\sup_{x \in A} \|x\|_V < +\infty$ for every such vicinity $V \in \mathfrak{B}$. A linear topology $\mathfrak{B}$ on R is said to be *monotone complete*, if for any topologically bounded system $0 \leq a_\lambda \in R$ ($\lambda \in \Lambda$) such that $a_\lambda \uparrow_{\lambda \in \Lambda}$, we can find $a \in R$ for which $a_\lambda \uparrow_{\lambda \in \Lambda} a$. With this definition, we can prove that if a linear topology $\mathfrak{B}$ is monotone complete, then R is complete by $\mathfrak{B}$ in WEIL's sense. This result may be considered as a generalization of the famous RIESZ-FISCHER's theorem about L_p -spaces.

A vicinity V is said to be *convex*, if $V \times V \subset 2V$. A linear topology $\mathfrak{B}$ is said to be *convex*, if $\mathfrak{B}$ has a basis composed only of convex vicinities. There exists a linear topology $\mathfrak{B}$ on R of which the totality of convex vicinities subject to the conditions 6), 7) is a basis. This linear topology $\mathfrak{B}$ is called the *strong topology* of R . A linear topology $\mathfrak{B}$ is said to be *sequential*, if $\mathfrak{B}$ has a basis composed of at most countable vicinities. We shall prove that if a linear topology $\mathfrak{B}$ is sequential, convex, complete, and $\bigcap_{V \in \mathfrak{B}} V = \{0\}$, then $\mathfrak{B}$ is the strong topology of R .

Let R be now reflexive and $\bar{R}$ its conjugate space.⁽³⁾ The so-called weak linear topology of R by $\bar{R}$ is not a linear topology in our sense. However there exists the weakest linear topology $\mathfrak{W}$ among our linear topologies by which every $\bar{a} \in \bar{R}$ is topologically continuous. This linear topology $\mathfrak{W}$ is called the *absolute weak topology* of R , as the system of vicinities $\{x: \bar{a}(|x|) \leq 1\}$ for all positive $\bar{a} \in \bar{R}$ is a basis of $\mathfrak{W}$. We can prove that the absolute weak topology $\mathfrak{W}$ of R is weaker than the strong topology $\mathfrak{S}$ of R , i.e., $\mathfrak{W} \subset \mathfrak{S}$, but $\mathfrak{W}$ is equivalent to $\mathfrak{S}$, i.e., a manifold $A \subset R$ is topologically bounded by $\mathfrak{W}$, if and only if A is so by $\mathfrak{S}$.

A pseudo-norm $\|x\|$ on R is said to be *reflexive*, if for

$$\bar{A} = \{\bar{x}: \sup_{\|x\| \leq 1} |\bar{x}(x)| \leq 1\},$$

we have $\|x\| = \sup_{\bar{x} \in \bar{A}} |\bar{x}(x)|$. A linear topology $\mathfrak{B}$ on R is said to be *reflexive*, if $\mathfrak{B}$ has a basis $\mathfrak{B}$ such that the pseudo-norm $\|x\|_V$ is reflexive

2) A. WEIL: Sur les espaces à structure uniforme, Actual. Sci. et Industr. Paris, (1938).

3) H. NAKANO: Modularized semi-ordered linear spaces, Tokyo Math. Book Series I (1950),

§22. This book will be denoted by MSLS in this paper.

for every $V \in \mathfrak{B}$. The absolute weak topology of R is reflexive. We shall prove that if the strong topology of R is sequential, then it is reflexive. This result is a generalization of the theorem: if there is a complete norm on R , then there exists a complete reflexive norm on R .

We shall make use of notations in MSLS and the following notations:

$$A^+ = \{x^+ : x \in A\}, \quad A^- = \{x^- : x \in A\}, \quad |A| = \{|x| : x \in A\},$$

$$A \cup B = \{x \cup y : x \in A, y \in B\}, \quad A \cap B = \{x \cap y : x \in A, y \in B\}.$$

$$A \times B = \{x + y : x \in A, y \in B\}$$

for manifolds A, B of R .

§ 1. Linear topologies

Let R be a universally continuous semi-ordered linear space. A set of positive elements V is said to be a *positive vicinity*, if

- 1) for any $a \geq 0$ we can find $\varepsilon > 0$ such that $\varepsilon a \in V$,
- 2) $0 \leq b \leq a \in V$ implies $b \in V$,
- 3) $V \ni a_\lambda \uparrow_{\lambda \in A} a$ implies $a \in V$.

A positive vicinity V is said to be *convex*, if $x, y \in V, \lambda + \mu = 1, \lambda, \mu \geq 0$ implies $\lambda x + \mu y \in V$.

With this definition, we see easily that if V is a positive vicinity (convex), then ξV also is a positive vicinity (convex) for $\xi > 0$, and for two positive vicinity U, V (convex), both UV and $U \times V$ are positive vicinities (convex).

A collection $\mathfrak{B}$ of positive vicinities is called a *linear topology*, if

- 1') $U \subset V \in \mathfrak{B}$ implies $U \in \mathfrak{B}$,
- 2') $U, V \in \mathfrak{B}$ implies $UV \in \mathfrak{B}$,
- 3') $V \in \mathfrak{B}$ implies $\xi V \in \mathfrak{B}$ for every $\xi > 0$,
- 4') for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $U \times U \subset V$.

For a linear topology $\mathfrak{B}$ on R , a subset $\mathfrak{B} \subset \mathfrak{B}$ is called a *basis* of $\mathfrak{B}$, if for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ and $\alpha > 0$ such that $\alpha U \subset V$. With this definition, we can prove easily

Theorem 1.1 *If a collection of positive vicinities $\mathfrak{B}$ satisfies*

- 1'') for any $U, V \in \mathfrak{B}$ we can find $W \in \mathfrak{B}$ and $\alpha > 0$ such that $\alpha W \subset UV$,
- 2'') for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ and $\alpha > 0$ such that $U \times U \subset \alpha V$,

then there exists uniquely a linear topology $\mathfrak{B}$ of which $\mathfrak{B}$ is a basis.

A linear topology $\mathfrak{B}$ is said to be *convex*, if $\mathfrak{B}$ has a basis composed

only of convex positive vicinities. A linear topology $\mathfrak{B}$ is said to be *sequential*, if $\mathfrak{B}$ has a basis composed of at most countable positive vicinities. A sequence of positive vicinities V_ν ($\nu = 1, 2, \dots$) is said to be *decreasing*, if

$$V_\nu \supset V_{\nu+1} \times V_{\nu+1} \quad \text{for every } \nu = 1, 2, \dots$$

If a linear topology $\mathfrak{B}$ is sequential, then we can find obviously by definition a decreasing sequence $V_\nu \in \mathfrak{B}$ ($\nu = 1, 2, \dots$) as a basis of $\mathfrak{B}$. Such a basis is called a *decreasing basis* of $\mathfrak{B}$. If $V_\nu \in \mathfrak{B}$ ($\nu = 1, 2, \dots$) is a decreasing basis of $\mathfrak{B}$, then for any $V \in \mathfrak{B}$ we can find ν such that $V_\nu \subset V$. Because we can find by definition μ and $\varepsilon > 0$ such that $\varepsilon V_\mu \subset V$. For such $\varepsilon > 0$, we can find $\nu > \mu$ such that $\frac{1}{2^{\nu-\mu}} < \varepsilon$, and then we have

$$V_\nu \subset \frac{1}{2^{\nu-\mu}} V_\mu \subset \varepsilon V_\mu \subset V,$$

because we have $V_\nu \supset 2V_{\nu+1}$ for every $\nu = 1, 2, \dots$.

A decreasing basis $V_\nu \in \mathfrak{B}$ ($\nu = 1, 2, \dots$) is said to be *convex*, if every V_ν ($\nu = 1, 2, \dots$) is convex. With this definition, we see at once by definition

Theorem 1.2. If a linear topology $\mathfrak{B}$ is sequential and convex, then $\mathfrak{B}$ has a convex decreasing basis.

A linear topology $\mathfrak{B}$ is said to be of *single vicinity* if $\mathfrak{B}$ has a basis composed only of a single positive vicinity. With this definition we have obviously

Theorem 1.3. If a linear topology $\mathfrak{B}$ is of single vicinity and convex, then there is a convex positive vicinity which is a basis of $\mathfrak{B}$.

§ 2. Pseudo-norms

A functional $\|x\|$ ($x \in R$) on R is said to be a *pseudo-norm* on R , if

- 1) $0 \leq \|x\| < +\infty$ for every $x \in R$,
- 2) $|x| \leq |y|$ implies $\|x\| \leq \|y\|$,
- 3) $\|\xi x\| = |\xi| \|x\|$ for every real number ξ ,
- 4) $0 \leq x_\lambda \uparrow_{\lambda \in A} x$ implies $\|x\| = \sup_{\lambda \in A} \|x_\lambda\|$.

A pseudo-norm $\|x\|$ ($x \in R$) is said to be *convex*, if

$$\|x+y\| \leq \|x\| + \|y\| \quad \text{for every } x, y \in R.$$

For a pseudo-norm $\|x\|$ ($x \in R$), putting

$$V = \{x : \|x\| \leq 1, x \geq 0\},$$

we see easily that V is a positive vicinity. Furthermore, if $\|x\| (x \in R)$ is convex, then this positive vicinity V is convex. .

Conversely, for a positive vicinity V , putting

$$(1) \quad \|x\|_V = \inf_{\xi | x| \in V} \frac{1}{\xi},$$

we obtain a pseudo-norm $\|x\|_V (x \in R)$, which will be called the *pseudo-norm* of V . With this definition, we see easily

$$(2) \quad V = \{x : \|x\|_V \leq 1, x \geq 0\}.$$

Furthermore we can prove easily

$$(3) \quad \|x\|_{\xi V} = \frac{1}{\xi} \|x\|_V \quad \text{for } \xi > 0,$$

$$(4) \quad V \subset U \text{ implies } \|x\|_V \geq \|x\|_U \quad \text{for every } x \in R,$$

$$(5) \quad V \times V \subset U \text{ implies } \|x+y\|_U \leq \text{Max}\{\|x\|_V, \|y\|_V\}.$$

By virtue of Theorem 1.1, we can prove easily

Theorem 2.1. *For a system of pseudo-norms $\|x\|_\lambda (\lambda \in \Lambda)$ on R , if for any $\lambda \in \Lambda$ we can find $\sigma \in \Lambda$ such that*

$$\|x+y\|_\lambda \leq \|x\|_\sigma + \|y\|_\sigma \quad \text{for every } x, y \in R,$$

then there exists uniquely a linear topology $\mathfrak{B}$ on R such that the totality of

$$V_{\lambda_1, \lambda_2, \dots, \lambda_\kappa} = \{x : \|x\|_{\lambda_\nu} \leq 1 (\nu=1, 2, \dots, \kappa), x \geq 0\}$$

for every finite number of elements $\lambda_\nu \in \Lambda (\nu=1, 2, \dots, \kappa)$ is a basis of $\mathfrak{B}$.

A pseudo-norm $\|x\| (x \in R)$ is said to be *proper*, if $\|x\|=0$ implies $x=0$. A pseudo-norm is called a *norm*, if it is convex and proper.

Theorem 2.2. *For a convex pseudo-norm $\|x\| (x \in R)$ there exists uniquely a normal manifold N of R such that $\|x\| (x \in N)$ is proper in N and $\|x\|=0$ for every $x \in N^\perp$.*

Proof. Putting $N = \{x : \|x\|=0\}$, we see easily that N is a normal manifold of R . For such N , it is evident that $\|x\|=0$ for every $x \in N$. Conversely, if $\|x\|=0$, then we have naturally $x \in N$, and hence $[N^\perp]x=0$. Thus $\|x\|$ is proper in $N^\perp$. If $\|x\|$ is proper in a normal manifold M and $\|x\|=0$ for every $x \in M^\perp$, then it is evident that $M^\perp = N$.

A system of pseudo-norms $\|x\|_\lambda (\lambda \in \Lambda)$ is said to be *proper*, if $\|x\|_\lambda = 0$ for all $\lambda \in \Lambda$ implies $x=0$. With this definition, we have

Theorem 2.3. *For a system of pseudo-norms $\|x\|_\lambda (\lambda \in \Lambda)$ on R , if for any $\lambda \in \Lambda$ we can find $\sigma \in \Lambda$ such that*

$$\|x+y\|_\lambda \leq \|x\|_\sigma + \|y\|_\sigma \quad \text{for every } x, y \in R,$$

then there exists uniquely a normal manifold N of R such that the system $\|x\|_\lambda$ ($\lambda \in \Lambda$) is proper in N and $\|x\|_\lambda = 0$ for every $\lambda \in \Lambda$ and $x \in N^\perp$.

Proof. Putting $M = \{x : \|x\|_\lambda = 0 \text{ for all } \lambda \in \Lambda\}$, we see easily that M is a normal manifold of R and $M^\perp$ satisfies our requirement. Furthermore the uniqueness is obvious.

We shall say that R is separated by a linear topology $\mathfrak{B}$, or that $\mathfrak{B}$ is separative if $\bigcap_{V \in \mathfrak{B}} V = \{0\}$. With this definition, we see at once

Theorem 2.4. A linear topology $\mathfrak{B}$ is separative, if and only if for a basis $\mathfrak{B}$ of $\mathfrak{B}$, the system of pseudo-norms $\|x\|_V$ ($V \in \mathfrak{B}$) is proper.

§ 3. Completeness

Let $\mathfrak{B}$ be a linear topology on R . A system of manifolds A_λ ($\lambda \in \Lambda$) is said to be a CAUCHY system by $\mathfrak{B}$, if $\prod_{\nu=1}^k A_{\lambda_\nu} \neq \emptyset$ for every finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \dots, k$), and for any $V \in \mathfrak{B}$ we can find $\lambda \in \Lambda$ such that

$$|x - y| \in V \quad \text{for every } x, y \in A_\lambda.$$

A CAUCHY system A_λ ($\lambda \in \Lambda$) is said to be convergent to a limit $a \in R$, if for any $V \in \mathfrak{B}$ we can find $\lambda \in \Lambda$ such that

$$|x - a| \in V \quad \text{for every } x \in A_\lambda.$$

If $\mathfrak{B}$ is separative, then we see easily that the limit of a CAUCHY system is uniquely determined, if it is convergent.

We see easily by definition that for a basis $\mathfrak{B}$ of $\mathfrak{B}$, a system of manifolds A_λ ($\lambda \in \Lambda$) is a CAUCHY system by $\mathfrak{B}$, if and only if $\prod_{\lambda=1}^k A_{\lambda_\nu} \neq \emptyset$ for every finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \dots, k$) and for any $V \in \mathfrak{B}$ and $\varepsilon > 0$ we can find $\lambda \in \Lambda$ such that

$$\|x - y\|_V \leq \varepsilon \quad \text{for every } x, y \in A_\lambda.$$

Furthermore we see that a CAUCHY system A_λ ($\lambda \in \Lambda$) is convergent to a limit $a \in R$, if and only if for any $V \in \mathfrak{B}$ and $\varepsilon > 0$ we can find $\lambda \in \Lambda$ such that

$$\|x - a\|_V \leq \varepsilon \quad \text{for every } x \in A_\lambda.$$

By virtue of the formula §2 (5), we can prove easily

Theorem 3.1. For two CAUCHY system A_λ and B_λ ($\lambda \in \Lambda$), all $A_\lambda \cup B_\lambda$, $A_\lambda \cap B_\lambda$, and $A_\lambda \times B_\lambda$ ($\lambda \in \Lambda$) are CAUCHY systems, furthermore, if A_λ and

B_λ ($\lambda \in \Lambda$) are convergent respectively to limits a and b , then $A_\lambda \cup B_\lambda$, $A_\lambda \cap B_\lambda$, and $A_\lambda \times B_\lambda$ ($\lambda \in \Lambda$) are convergent to $a \cup b$, $a \cap b$, and $a + b$ respectively.

We see further easily

Theorem 3.2. For a CAUCHY system A_λ ($\lambda \in \Lambda$), all A_λ^+ , A_λ^- , $|A_\lambda|$, αA_λ , and $[N]A_\lambda$ ($\lambda \in \Lambda$) are CAUCHY systems for every real number α and projection operator $[N]$. If a CAUCHY system A_λ ($\lambda \in \Lambda$) is convergent to a limit a , then A_λ^+ , A_λ^- , $|A_\lambda|$, αA_λ , and $[N]A_\lambda$ ($\lambda \in \Lambda$) are convergent to a^+ , a^- , $|a|$, αa , and $[N]a$ respectively.

A manifold A of R is said to be complete by a linear topology $\mathfrak{B}$, if every CAUCHY system $A_\lambda \subset A$ ($\lambda \in \Lambda$) is convergent to a limit $a \in A$. With this definition we have

Theorem 3.3. For every positive element $a \in R$, $\{x : |x| \leq a\}$ is complete by $\mathfrak{B}$.

Proof. We shall consider firstly the case where $\mathfrak{B}$ is sequential and separative. Let $V_\nu \in \mathfrak{B}$ ($\nu = 1, 2, \dots$) be a decreasing basis of $\mathfrak{B}$. We set

$$A = \{x : |x| \leq a\}$$

and assume that $A_\lambda \subset A$ ($\lambda \in \Lambda$) is a CAUCHY system by $\mathfrak{B}$. Then we can find $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \dots$) such that

$$\sup_{x, y \in A_{\lambda_\nu}} \|x - y\|_{V_\nu} \leq \frac{1}{\nu} \quad (\nu = 1, 2, \dots).$$

For such $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \dots$) we can find

$$a_\mu \in \prod_{\nu=1}^{\mu} A_{\lambda_\nu} \quad (\mu = 1, 2, \dots).$$

As $V_{\nu+1} \times V_{\nu+1} \subset V_\nu$, we conclude by the formula §2(5)

$$\left\| \left(\sum_{\nu=\mu}^{\sigma} |a_{\nu+1} - a_\nu| \right) \right\|_{V_{\mu-1}} \leq \max_{\mu \leq \nu \leq \sigma} \|a_{\nu+1} - a_\nu\|_{V_\nu} \leq \frac{1}{\mu}.$$

On the other hand we have

$$\bigcup_{\nu=\mu}^{\sigma} a_\nu - a_\mu = \bigcup_{\nu=\mu}^{\sigma} (a_\nu - a_\mu) \leq \sum_{\nu=\mu}^{\sigma} |a_{\nu+1} - a_\nu|,$$

and hence $\left\| \bigcup_{\nu=\mu}^{\sigma} a_\nu - a_\mu \right\|_{V_{\mu-1}} \leq \frac{1}{\mu}$. This relation yields by 4) in §2

$$\left\| \bigcup_{\nu=\mu}^{\infty} a_\nu - a_\mu \right\|_{V_{\mu-1}} \leq \frac{1}{\mu} \quad (\mu = 2, 3, \dots).$$

We obtain likewise

$$\left\| a_\mu - \bigcap_{\nu=\mu}^{\infty} a_\nu \right\|_{V_{\mu-1}} \leq \frac{1}{\mu} \quad (\mu = 2, 3, \dots).$$

Consequently we have by the formula §2 (5)

$$\left\| \bigcup_{\nu=\mu}^{\infty} a_{\nu} - \bigcap_{\nu=\mu}^{\infty} a_{\nu} \right\|_{V_{\mu-2}} \leq \frac{1}{\mu} \quad (\mu=3, 4, \dots).$$

Thus, putting $l_{\mu} = \bigcup_{\nu=\mu}^{\infty} a_{\nu} - \bigcap_{\nu=\mu}^{\infty} a_{\nu}$, $l = \bigcap_{\mu=1}^{\infty} l_{\mu}$, we obtain $\|l\|_{V_{\mu-2}} \leq \frac{1}{\mu}$ for every $\mu=3, 4, \dots$. As $\|x\|_{V_1} \leq \|x\|_{V_2} \leq \dots$ by §2 (4), we conclude hence $\|l\|_{V_{\mu}} = 0$ for every $\mu=1, 2, \dots$, and hence $l=0$, as $\mathfrak{B}$ is separative by assumption. Therefore there exists $a \in R$ such that $\lim_{\nu \rightarrow \infty} a_{\nu} = a$, and naturally $a \in A$. Furthermore we have

$$\|a - a_{\mu}\|_{V_{\mu-2}} \leq \frac{1}{\mu} \quad \text{for every } \mu=3, 4, \dots,$$

because $\bigcup_{\nu=\mu}^{\infty} a_{\nu} \geq a \geq \bigcap_{\nu=\mu}^{\infty} a_{\nu}$. This relation shows that A_{λ} ($\lambda \in \Lambda$) is convergent to a by $\mathfrak{B}$.

Now we consider the general case. Let $A_{\lambda} \subset A$ ($\lambda \in \Lambda$) be an arbitrary CAUCHY system by $\mathfrak{B}$ and $V_{\nu} \in \mathfrak{B}$ ($\nu=1, 2, \dots$) an arbitrary decreasing sequence. By virtue of Theorem 2.3, we can find a normal manifold $N_{V_1, V_2, \dots}$ of R such that the system $\|x\|_{V_{\nu}}$ ($\nu=1, 2, \dots$) is proper in $N_{V_1, V_2, \dots}$ and $\|x\|_{V_{\nu}} = 0$ for every $x \in N_{V_1, V_2, \dots}^{\perp}$ and $\nu=1, 2, \dots$. Recalling Theorem 2.1, we can find then a linear topology $\mathfrak{B}_{V_1, V_2, \dots}$ on $N_{V_1, V_2, \dots}$ such that $[N_{V_1, V_2, \dots}]V_{\nu}$ ($\nu=1, 2, \dots$) is a basis of $\mathfrak{B}_{V_1, V_2, \dots}$. This linear topology $\mathfrak{B}_{V_1, V_2, \dots}$ is obviously sequential and separative by Theorem 2.4. Furthermore, as $[N_{V_1, V_2, \dots}]A_{\lambda}$ ($\lambda \in \Lambda$) is a CAUCHY system by $\mathfrak{B}_{V_1, V_2, \dots}$, there exists uniquely a limit $a \in [N_{V_1, V_2, \dots}]A$ of $[N_{V_1, V_2, \dots}]A_{\lambda}$ ($\lambda \in \Lambda$), as proved just above.

Corresponding to every decreasing sequence $V_{\nu} \in \mathfrak{B}$ ($\nu=1, 2, \dots$), we obtain thus uniquely a normal manifold $N_{V_1, V_2, \dots}$ and a limit $a_{V_1, V_2, \dots} \in [N_{V_1, V_2, \dots}]A$ of $[N_{V_1, V_2, \dots}]A_{\lambda}$ ($\lambda \in \Lambda$). We see further by Theorem 3.2 that for every two decreasing sequences V_{ν} and $U_{\nu} \in \mathfrak{B}$ ($\nu=1, 2, \dots$), we have

$$[N_{V_1, V_2, \dots}][N_{U_1, U_2, \dots}]a_{V_1, V_2, \dots} = [N_{V_1, V_2, \dots}][N_{U_1, U_2, \dots}]a_{U_1, U_2, \dots}.$$

Therefore we can find $a \in A$ such that

$$[N_{V_1, V_2, \dots}]a = a_{V_1, V_2, \dots}$$

for every decreasing sequence $V_{\nu} \in \mathfrak{B}$ ($\nu=1, 2, \dots$). Such $a \in A$ is a limit of A_{λ} ($\lambda \in \Lambda$). Because, for any $V \in \mathfrak{B}$ we can find a decreasing sequence $V_{\nu} \in \mathfrak{B}$ ($\nu=1, 2, \dots$) such that $V \supset V_1 \times V_1$, and $\lambda \in \Lambda$ such that

$$\sup_{x \in [N_{V_1, V_2, \dots}]A_{\lambda}} \|x - a_{V_1, V_2, \dots}\|_{V_1} \leq 1,$$

and hence $\sup_{x \in A_\lambda} \|[N_{V_1, V_2, \dots}](x-a)\|_{V_1} \leq 1$. As

$$\|[N_{V_1, V_2, \dots}^\perp](x-a)\|_{V_1} = 0,$$

we obtain by §2(5)

$$\sup_{x \in A_\lambda} \|x-a\|_V \leq 1,$$

that is, $|x-a| \in V$ for every $x \in A_\lambda$. Therefore A is complete by $\mathfrak{B}$.

Theorem 3.4. $\{x : a \leq x \leq b\}$ is complete by every linear topology $\mathfrak{B}$ for every two elements $a \leq b$.

Proof. Putting $A = \{x : |x| \leq |a| + |b|\}$, $B = \{x : a \leq x \leq b\}$, we have obviously $B \subset A$ and A is complete by $\mathfrak{B}$ on account of Theorem 3.3. For a CAUCHY system $A_\lambda \subset B$ ($\lambda \in \Lambda$) there exists hence a limit $c \in A$ of A_λ ($\lambda \in \Lambda$), and then we obtain by Theorem 3.1 that $(c \smile a) \frown b$ is a limit of

$$(A_\lambda \smile a) \frown b = A_\lambda \quad (\lambda \in \Lambda),$$

and it is evident that $(c \smile a) \frown b \in B$. Therefore B is complete by $\mathfrak{B}$.

§ 4. Topologically bounded manifolds

A manifold A of R is said to be *topologically bound* by a linear topology $\mathfrak{B}$, if

$$\sup_{x \in A} \|x\|_V < +\infty \quad \text{for every } V \in \mathfrak{B}.$$

With this definition, it is obvious by the formula §2(4) that a manifold A is topologically bounded by a linear topology $\mathfrak{B}$, if and only if for a basis $\mathfrak{B}$ of $\mathfrak{B}$ we have

$$\sup_{x \in A} \|x\|_V < +\infty \quad \text{for every } V \in \mathfrak{B}.$$

We can prove easily by definition

Theorem 4.1. If a manifold A is topologically bounded by a linear topology $\mathfrak{B}$, then all A^+ , A^- , $|A|$, αA , $[N]A$ are topologically bound by $\mathfrak{B}$ for every real number α and projection operator $[N]$. If both manifolds A and B are topologically bounded by $\mathfrak{B}$, then all $A \smile B$, $A \frown B$, and $A \times B$ are topologically bounded by $\mathfrak{B}$.

A manifold A of R is said to be *order bound* or *merely bounded*, if we can find a positive element $a \in R$ such that $|x| \leq a$ for every $x \in A$. Every bounded manifold is obviously topologically bounded by every linear topology.

A linear topology $\mathfrak{B}$ on R is said to be *monotone complete*, if for any

topologically bounded manifold of positive elements $a_\lambda \uparrow_{\lambda \in \Lambda} a$, we can find $a \in R$ such that $a_\lambda \uparrow_{\lambda \in \Lambda} a$.

Theorem 4.2. *If a linear topology $\mathfrak{B}$ on R is monotone complete, then R is complete by $\mathfrak{B}$.*

Proof. Let A_λ ($\lambda \in \Lambda$) be a CAUCHY system by $\mathfrak{B}$. We suppose firstly that $\mathfrak{B}$ is separative. As A_λ^+ ($\lambda \in \Lambda$) also is by Theorem 3.2 a CAUCHY system, corresponding to every $x \geq 0$, we obtain uniquely by Theorem 3.3 a limit a_x of a CAUCHY system $A_\lambda^+ \cap x$ ($\lambda \in \Lambda$). For this limit a_x , we have obviously by Theorem 3.1 $0 \leq a_x \uparrow_{x \geq 0}$. Furthermore the system a_x ($x \geq 0$) is topologically bounded by $\mathfrak{B}$. Because for any $V \in \mathfrak{B}$ we can find by definition $U \in \mathfrak{B}$ such that $U \times U \times U \times U \subset V$, and $\lambda_1 \in \Lambda$ such that $\|y - z\|_U \leq 1$ for every $y, z \in A_{\lambda_1}^+$, and hence by § 2(5) $\sup_{y, z \in A_{\lambda_1}^+} \|y\|_{U \times U} < +\infty$.

For any $x \geq 0$ we can find by definition, $\lambda_2 \in \Lambda$ such that

$$\|a_x - z\|_{U \times U} \leq 1 \quad \text{for every } z \in A_{\lambda_2}^+ \cap x.$$

For an element $b \in A_{\lambda_1}^+ A_{\lambda_2}^+$, we have then by § 2(5)

$$\|a_x\|_V \leq \text{Max}\{1, \|b \cap x\|_{U \times U}\} \leq \text{Max}\{1, \|b\|_{U \times U}\},$$

and hence $\|a_x\|_V \leq \text{Max}\{1, \sup_{y \in A_{\lambda_1}^+} \|y\|_{U \times U}\}$ for every $x \geq 0$.

Therefore there exists by assumption $a \in R$ such that $a_x \uparrow_{x \geq 0} a$. As we have by Theorem 3.1

$$a_x \cap y = a_{x \cap y} \quad \text{for every } x, y \geq 0,$$

we obtain $a \cap x = a_x$ for every $x \geq 0$. For any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $U \times U \subset V$, and further $\lambda_0 \in \Lambda$ such that

$$\sup_{y, z \in A_{\lambda_0}^+} \|y - z\|_U \leq 1.$$

Thus, for any $y \in A_{\lambda_0}^+$, putting $x = y \cap a$, we can find $\lambda_1 \in \Lambda$ such that

$$\sup_{z \in A_{\lambda_1}^+} \|z \cap x - a\|_U = \sup_{z \in A_{\lambda_1}^+} \|z \cap x - a_x\|_U \leq 1,$$

and for $z \in A_{\lambda_0}^+ A_{\lambda_1}^+$, we have

$$\|y - z \cap x\|_U = \|y \cap x - z \cap x\|_U \leq \|y - z\|_U \leq 1.$$

Consequently we obtain by § 2(5)

$$\|y - a\|_V \leq 1 \quad \text{for every } y \in A_{\lambda_0}^+.$$

Therefore a is a limit of A_λ^+ ($\lambda \in \Lambda$). We obtain likewise a limit b of A_λ^-

($\lambda \in A$). Thus we see by Theorem 3.1 that $a - b$ is a limit of A_λ ($\lambda \in A$).

In general, we can find by Theorem 2.3 a normal manifold N of R , such that the system of pseudo-norms $\|x\|_V$ ($V \in \mathfrak{B}$) is proper in N and $\|x\|_V = 0$ for every $x \in N^\perp$ and $V \in \mathfrak{B}$. Then there exists a limit $a \in N$ of $[N]A_\lambda$ ($\lambda \in A$), as proved just above. This limit a also is a limit of A_λ ($\lambda \in A$), because for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $U \times U \subset V$, and we have by §2 (5) for every $x \in R$

$$\|x - a\|_V \leq \| [N]x - a \|_V.$$

A linear topology $\mathfrak{B}$ on R is said to be *complete*, if R is complete by $\mathfrak{B}$. We can state then by Theorem 4.2 that every monotone complete linear topology is complete.

Theorem 4.3. If a linear topology $\mathfrak{B}$ on R is separative, convex, and complete, and a manifold A of R is topologically bounded by $\mathfrak{B}$, then we have for every positive vicinity W

$$\sup_{x \in A} \|x\|_W < +\infty.$$

Proof. If $\sup_{x \in A} \|x\|_W = +\infty$, then we can find $x_\nu \in A$ ($\nu = 1, 2, \dots$) such that $\|x_\nu\|_W \geq \nu 2^\nu$ for every $\nu = 1, 2, \dots$. As A is by assumption topologically bounded by $\mathfrak{B}$, we have obviously $\sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \|x_\nu\|_V < +\infty$ for every $V \in \mathfrak{B}$. As $\mathfrak{B}$ is convex and complete by assumption, we can find $a \in R$ such that

$$\lim_{\mu \rightarrow \infty} \left\| \sum_{\nu=1}^{\mu} \frac{1}{2^\nu} |x_\nu| - a \right\|_V = 0 \quad \text{for every } V \in \mathfrak{B}.$$

As $\mathfrak{B}$ is separative by assumption, we conclude easily that $a = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} |x_\nu|$, and hence we have

$$\|a\|_W \geq \frac{1}{2^\nu} \|x_\nu\|_W \geq \nu \quad \text{for every } \nu = 1, 2, \dots,$$

contradicting $\|a\|_W < +\infty$.

§ 5. Equivalence

A linear topology $\mathfrak{B}$ on R is said to be *equivalent* to a linear topology $\mathfrak{U}$ on R , if $\mathfrak{B}$ has the same topologically bounded manifolds with $\mathfrak{U}$, that is, a manifold A is topologically bounded by $\mathfrak{B}$ if and only if A is so by $\mathfrak{U}$. With this definition, we have obviously

Theorem 5.1. If a linear topology $\mathfrak{B}$ is monotone complete, then every

linear topology equivalent to $\mathfrak{B}$ is also monotone complete.

We shall say that a linear topology $\mathfrak{B}$ on R is *stronger* than a linear topology $\mathfrak{U}$ on R , or that $\mathfrak{U}$ is *weaker* than $\mathfrak{B}$, if $\mathfrak{B} \supset \mathfrak{U}$. With this definition we have obviously by Theorem 4.3.

Theorem 5.2. *If a linear topology $\mathfrak{B}$ is separative, convex, and complete, then every linear topology stronger than $\mathfrak{B}$ is equivalent to $\mathfrak{B}$.*

By virtue of Theorem 1.1, we see easily that there exists uniquely a linear topology $\mathfrak{B}$ of which the totality of convex vicinity in R is a basis. This linear topology $\mathfrak{B}$ is called the *strong topology* of R . With this definition, we have obviously that the strong topology of R is the strongest convex linear topology on R , that is, the strong topology of R is stronger than every other convex linear topology on R .

Recalling Theorem 5.2, we obtain at once

Theorem 5.3. *If a linear topology $\mathfrak{B}$ on R is separative, convex, and complete, then $\mathfrak{B}$ is equivalent to the strong topology of R .*

Theorem 5.4. *If a linear topology $\mathfrak{B}$ on R is sequential and equivalent to a linear topology $\mathfrak{U}$ on R , then $\mathfrak{B}$ is stronger than $\mathfrak{U}$.*

Proof. Let $V_\nu \in \mathfrak{B} (\nu=1, 2, \dots)$ be a decreasing basis of $\mathfrak{B}$. If $\mathfrak{B}$ is not stronger than $\mathfrak{U}$, then we can find $U \in \mathfrak{U}$ such that $U \notin \mathfrak{B}$. For such U , there is a sequence $a_\nu \in R (\nu=1, 2, \dots)$ such that

$$\nu U \ni a_\nu \in V_\nu \quad \text{for every } \nu=1, 2, \dots,$$

and hence we have by the formulas (2), (3) in §2

$$\|a_\nu\|_{V_\nu} \leq 1, \quad \|a_\nu\|_U \geq \nu \quad \text{for every } \nu=1, 2, \dots.$$

Then $\{a_1, a_2, \dots\}$ is a bounded by $\mathfrak{B}$ but not by $\mathfrak{U}$; contradicting assumption.

On account of this Theorem 5.4, we conclude by Theorem 5.3

Theorem 5.5. *If a linear topology $\mathfrak{B}$ on R is sequential, separative, convex, and complete, then $\mathfrak{B}$ is the strong topology of R .*

§6. Continuous linear topologies

A pseudo-norm $\|x\|$ on R is said to be *continuous*, if $R \ni x_\nu \downarrow_{\nu \rightarrow \infty} 0$ implies $\lim_{\nu \rightarrow \infty} \|x_\nu\| = 0$. A linear topology $\mathfrak{B}$ on R is said to be *continuous*, if the pseudo-norm $\|x\|_V$ is continuous for every $V \in \mathfrak{B}$. With this definition, we see at once by the formulas (3), (4) in §2 that $\mathfrak{B}$ is continuous if and only if for a basis $\mathfrak{B}$ of $\mathfrak{B}$, the pseudo-norm $\|x\|_V$ is continuous for every $V \in \mathfrak{B}$.

Theorem 6.1. *If a linear topology $\mathfrak{B}$ on R is sequential, separative and*

continuous, then R is superuniversally continuous, that is, for any system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$) we can find countable $\lambda_\nu \in \Lambda$ ($\nu=1, 2, \dots$) such that

$$\bigcap_{\nu=1}^{\infty} a_{\lambda_\nu} = \bigcap_{\lambda \in \Lambda} a_\lambda.$$

Proof. Let $V_\nu \in \mathfrak{B}$ ($\nu=1, 2, \dots$) be a decreasing basis of $\mathfrak{B}$. $0 \leq x_\lambda \downarrow_{\lambda \in \Lambda}$ implies then

$$\inf_{\lambda \in \Lambda} \{ \sup_{x_\sigma \leq x_\lambda} \|x_\lambda - x_\sigma\|_{V_\nu} \} = 0 \quad \text{for every } \nu=1, 2, \dots.$$

Because, if $0 \leq x_\lambda \downarrow_{\lambda \in \Lambda}$ and

$$\inf_{\lambda \in \Lambda} \{ \sup_{x_\sigma \leq x_\lambda} \|x_\lambda - x_\sigma\|_{V_\nu} \} \geq \varepsilon > 0$$

for some ν , then we can find $\lambda_\mu \in \Lambda$ ($\mu=1, 2, \dots$) such that

$$x_{\lambda_1} \geq x_{\lambda_2} \geq \dots, \quad \|x_{\lambda_\mu} - x_{\lambda_{\mu+1}}\|_{V_\nu} \geq \varepsilon \quad (\mu=1, 2, \dots).$$

Then, putting $x_0 = \bigcap_{\mu=1}^{\infty} x_{\lambda_\mu}$, we have $x_{\lambda_\mu} - x_0 \downarrow_{\mu=1}^{\infty} 0$, but

$$\|x_{\lambda_\mu} - x_0\|_{V_\nu} \geq \|x_{\lambda_\mu} - x_{\lambda_{\mu+1}}\|_{V_\nu} \geq \varepsilon$$

for every $\mu=1, 2, \dots$, contradicting the assumption that $\mathfrak{B}$ is continuous.

Therefore for $0 \leq x_\lambda \downarrow_{\lambda \in \Lambda}$ we can find $\lambda_\nu \in \Lambda$ ($\nu=1, 2, \dots$) such that $x_{\lambda_\nu} \downarrow_{\nu=1}^{\infty} 0$ and

$$\sup_{x_\sigma \leq x_{\lambda_\nu}} \|x_{\lambda_\nu} - x_\sigma\|_{V_\nu} \leq \frac{1}{2^\nu} \quad \text{for every } \nu=1, 2, \dots.$$

Then, putting $x_0 = \bigcap_{\nu=1}^{\infty} x_{\lambda_\nu}$, we have for every $\sigma \in \Lambda$

$$\|x_{\lambda_\nu} - x_0 \wedge x_\sigma\|_{V_\nu} \leq \frac{1}{2^\nu} \quad (\nu=1, 2, \dots),$$

because $x_{\lambda_\nu} - x_{\lambda_\mu} \wedge x_\sigma \uparrow_{\mu=1}^{\infty} x_{\lambda_\nu} - x_0 \wedge x_\sigma$, $\|x_{\lambda_\nu} - x_{\lambda_\mu} \wedge x_\sigma\|_{V_\nu} \leq \frac{1}{2^\nu}$ for $\mu \geq \nu$.

Thus we obtain naturally for every $\sigma \in \Lambda$

$$\|x_0 - x_0 \wedge x_\sigma\|_{V_\nu} \leq \frac{1}{2^\nu} \quad (\nu=1, 2, \dots).$$

As $\mathfrak{B}$ is separative by assumption, we obtain hence $x_0 - x_0 \wedge x_\sigma = 0$, and consequently $x_0 \leq x_\sigma$ for every $\sigma \in \Lambda$. Therefore $x_\lambda \downarrow_{\lambda \in \Lambda} x_0$.

Theorem 6.2. *If a linear topology $\mathfrak{B}$ on R is continuous, then $a_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies $\inf_{\lambda \in \Lambda} \|a_\lambda\|_V = 0$ for every $V \in \mathfrak{B}$.*

Proof. For any $V \in \mathfrak{B}$ we can find a decreasing sequence $V_\nu \in \mathfrak{B}$ ($\nu=1, 2, \dots$) such that $V_1 \times V_1 \subset V$. For such $V_\nu \in \mathfrak{B}$ ($\nu=1, 2, \dots$), we can

find by Theorem 2.3 a normal manifold N of R such that the system of pseudo-norms $\|x\|_{V_\nu}$ ($\nu=1, 2, \dots$) is proper in N and $\|x\|_{V_\nu}=0$ for every $x \in N^\perp$ and $\nu=1, 2, \dots$. Then the linear topology on N , of which $\{x: \|x\|_{V_\nu} \leq 1, 0 \leq x \in N\}$ ($\nu=1, 2, \dots$) is a basis, is obviously sequential, separative, and continuous. Thus N is superuniversally continuous by Theorem 6.1. Therefore, if $R \ni a_\lambda \downarrow_{\lambda \in A} 0$, then we can find $\lambda_\mu \in A$ ($\mu=1, 2, \dots$) such that

$$[N]a_{\lambda_\mu} \downarrow_{\mu=1}^\infty 0,$$

and hence $\lim_{\mu \rightarrow \infty} \|[N]a_{\lambda_\mu}\|_{V_1} = 0$, because $\mathfrak{B}$ is continuous by assumption. As $\|[N^\perp]a_{\lambda_\mu}\|_{V_1} = 0$, we obtain hence by §2(5)

$$\|a_{\lambda_\mu}\|_V \leq \|[N]a_{\lambda_\mu}\|_{V_1} \quad \text{for every } \mu=1, 2, \dots$$

Consequently we have $\lim_{\mu \rightarrow \infty} \|a_{\lambda_\mu}\|_V = 0$. Thus we have naturally

$$\inf_{\lambda \in A} \|a_\lambda\|_V = 0.$$

Theorem 6.3. *If a linear topology $\mathfrak{B}$ on R is sequential, separative, continuous, and complete, then R is regularly complete, that is, for any double sequence $a_{\nu, \mu} \downarrow_{\nu=1}^\infty 0$ ($\mu=1, 2, \dots$), we can find ν_μ ($\mu=1, 2, \dots$) such that $\sum_{\mu=1}^\infty a_{\nu_\mu, \mu}$ is convergent.*

Proof. Let $V_\nu \in \mathfrak{B}$ ($\nu=1, 2, \dots$) be a decreasing basis of $\mathfrak{B}$. If $a_{\nu, \mu} \downarrow_{\nu=1}^\infty 0$ ($\mu=1, 2, \dots$), then we have

$$\lim_{\nu \rightarrow \infty} \|a_{\nu, \mu}\|_{V_\mu} = 0 \quad \text{for every } \mu=1, 2, \dots,$$

because $\mathfrak{B}$ is continuous by assumption. Thus we can find ν_μ ($\mu=1, 2, \dots$) such that $a_{\nu_\mu, \mu} \in V_{\sigma^{-1}}$. Then we have obviously

$$\sum_{\mu=\sigma}^{\rho} a_{\nu_\mu, \mu} \in V_{\sigma^{-1}} \quad \text{for } \rho > \sigma.$$

As $\mathfrak{B}$ is complete and separative, we see easily that $\sum_{\mu=1}^\infty a_{\nu_\mu, \mu}$ is convergent. Therefore R is regularly complete.

§7. Linear functionals

Let $\mathfrak{B}$ be a linear topology on R . A linear functional φ on R is said to be *topologically bounded* by $\mathfrak{B}$, if $\sup_{x \in A} |\varphi(x)| < +\infty$ for every topologically bounded manifold A .

For any positive element $a \in R$, $\{x: 0 \leq x \leq a\}$ is obviously topologically bounded by $\mathfrak{B}$. Thus we have

Theorem 7.1. *If a linear functional φ on R is topologically bounded by $\mathfrak{B}$, then φ is bounded, that is,*

$$\sup_{0 \leq x \leq a} |\varphi(x)| < +\infty \quad \text{for every } a \geq 0.$$

Conversely we have

Theorem 7.2. *If a linear topology $\mathfrak{B}$ on R is separative, convex, and complete, then every bounded linear functional φ on R is topologically bounded by $\mathfrak{B}$.*

Proof. Let φ be a positive linear functional on R . If φ is not topologically bounded by $\mathfrak{B}$, then we can find a sequence $a_\nu \geq 0$ ($\nu = 1, 2, \dots$) such that $\{a_\nu\}$ is topologically bounded, but

$$\varphi(a_\nu) \geq \nu 2^\nu \quad (\nu = 1, 2, \dots).$$

Then we have obviously $\sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \|a_\nu\|_\nu < +\infty$ for every $V \in \mathfrak{B}$. As $\mathfrak{B}$ is separative, convex, and complete by assumption, we obtain hence that $\sum_{\nu=1}^{\infty} \frac{1}{2^\nu} a_\nu$ is convergent, and putting $a = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} a_\nu$, we have that $\varphi(a) \geq \varphi\left(\frac{1}{2^\nu} a_\nu\right) \geq \nu$ for every $\nu = 1, 2, \dots$, contradicting $\varphi(a) < +\infty$.

A linear functional φ on R is said to be *topologically continuous* by a linear topology $\mathfrak{B}$, if we can find $V \in \mathfrak{B}$ such that

$$|\varphi(x)| \leq \|x\|_V \quad \text{for every } x \in R.$$

With this definition, we see at once by the formulas (3), (4) in §2 that a linear functional φ on R is topologically continuous by $\mathfrak{B}$, if and only if for a basis $\mathfrak{B}$ of $\mathfrak{B}$ we can find $V \in \mathfrak{B}$ and $\alpha > 0$ such that

$$|\varphi(x)| \leq \alpha \|x\|_V \quad \text{for every } x \in R.$$

If a linear functional φ on R is topologically continuous by $\mathfrak{B}$, then φ is obviously by definition topologically bounded by $\mathfrak{B}$.

If a linear functional φ on R is *universally continuous*, that is, if $x_\lambda \downarrow_{\lambda \in A} 0$ implies $\inf_{\lambda \in A} |\varphi(x_\lambda)| = 0$, then, putting

$$V = \{x : \sup_{|y| \leq x} |\varphi(y)| \leq 1, x \geq 0\},$$

we see easily that V is a convex positive vicinity. Thus we have

Theorem 7.3. *If a linear functional φ on R is universally continuous, then φ is topologically continuous by the strong topology of R .*

Recalling Theorem 6.2, we obtain immediately

Theorem 7.4. If a linear topology $\mathfrak{B}$ on R is continuous, then every topologically continuous linear functional on R is universally continuous.

If a convex pseudo-norm $\|x\|$ on R is not continuous, then we can find a linear functional φ on R such that

$$\sup_{|x| \leq 1} |\varphi(x)| < +\infty,$$

but there is a sequence $a_\nu \downarrow_{\nu \rightarrow \infty} 0$ for which we have $\lim_{\nu \rightarrow \infty} \varphi(a_\nu) > 0$. (c.f. MSLS Theorem 31.10). Therefore we have

Theorem 7.5. For a convex linear topology $\mathfrak{B}$ on R , if every topologically continuous linear functional on R is continuous, then $\mathfrak{B}$ is continuous.

§ 8. Reflexive linear topologies

Let R be a reflexive semi-ordered linear space and $\bar{R}$ the conjugate space of R . For any positive $\bar{a} \in \bar{R}$, putting

$$(1) \quad V_{\bar{a}} = \{x : \bar{a}(x) \leq 1, x \geq 0\},$$

we obtain obviously a convex positive vicinity $V_{\bar{a}}$. For this $V_{\bar{a}}$ we have obviously

$$(2) \quad \|x\|_{V_{\bar{a}}} = \bar{a}(|x|) \quad \text{for every } x \in R,$$

because $\|x\|_{V_{\bar{a}}} = \inf_{\xi |x| \in V_{\bar{a}}} \frac{1}{\xi} = \inf_{\bar{a}(\xi |x|) \leq 1} \frac{1}{\xi} = \bar{a}(|x|)$.

Recalling Theorem 1.1, we see easily that there exists uniquely a linear topology $\mathfrak{B}$ on R such that the system $V_{\bar{a}}$ ($0 \leq \bar{a} \in \bar{R}$) is a basis of $\mathfrak{B}$. This linear topology $\mathfrak{B}$ is called the *absolute weak topology* of R . With this definition we have

Theorem 8.1. The absolute weak topology $\mathfrak{B}$ of R is separative, convex, continuous, and monotone complete.

Proof. It is evident by definition that $\mathfrak{B}$ is separative, convex, and continuous. If a system of positive elements $x_\lambda \uparrow_{\lambda \in A}$ is topologically bounded by $\mathfrak{B}$, then we have by the formula (2)

$$\sup_{\lambda \in A} \bar{a}(x_\lambda) = \sup_{\lambda \in A} \|x_\lambda\|_{V_{\bar{a}}} < +\infty$$

for every positive $\bar{a} \in \bar{R}$. Therefore there exists $a \in R$ such that $x_\lambda \uparrow_{\lambda \in A} a$. (c.f. MSLS. Theorem 24.4)

Theorem 8.2. A manifold A of R is topologically bounded by the absolute weak topology $\mathfrak{B}$ if and only if A is weakly bounded, that is,

$$\sup_{x \in A} |\bar{x}(x)| < +\infty \quad \text{for every } \bar{x} \in \bar{R}.$$

Proof. If A is weakly bounded, then we have

$$\sup_{x \in A} \bar{a}(|x|) < +\infty \quad \text{for } 0 \leq \bar{a} \in \bar{R}$$

(MSLS. Theorem 24.15). Thus we obtain by (2)

$$\sup_{x \in A} \|x\|_{V_{\bar{a}}} < +\infty \quad \text{for } 0 \leq \bar{a} \in \bar{R},$$

and hence A is topologically bounded by $\mathfrak{B}$. Conversely, if A is topologically bounded by $\mathfrak{B}$, then we have by (2)

$$\sup_{x \in A} |\bar{a}(x)| \leq \sup_{x \in A} |\bar{a}|(|x|) = \sup_{x \in A} \|x\|_{V_{|\bar{a}|}} < +\infty,$$

and hence A is weakly bounded.

Recalling Theorem 5.3, we obtain by Theorem 8.1

Theorem 8.3. The strong topology of R is separative and equivalent to the absolute weak topology of R .

A pseudo-norm $\|x\|$ on R is said to be *reflexive*, if for

$$\bar{A} = \{\bar{x} : \sup_{|x| \leq 1} |\bar{x}(x)| \leq 1\},$$

we have $\|x\| = \sup_{\bar{x} \in \bar{A}} |\bar{x}(x)|$ for every $x \in R$. With this definition, we see at once that every reflexive pseudo-norm is convex.

Let $\mathfrak{B}$ be the absolute weak topology of the conjugate space $\bar{R}$. For every topologically bounded manifold $\bar{A}$ of $\bar{R}$ by $\mathfrak{B}$, putting

$$V = \{x : |\bar{x}|(x) \leq 1 \text{ for every } \bar{x} \in \bar{A}, x \geq 0\},$$

we see easily that V is a positive vicinity in R and the pseudo-norm $\|x\|_V$ is reflexive.

Theorem 8.4. If a pseudo-norm $\|x\|$ ($x \in R$) is convex and continuous, then it is reflexive.

Proof. By virtue of BANACH's extension theorem, for any $a \in R$ we can find a linear functional φ on R such that

$$\varphi(a) = \|a\|, \quad |\varphi(x)| \leq \|x\| \quad \text{for every } x \in R.$$

As $\|x\|$ ($x \in R$) is convex and continuous by assumption, we see by Theorem 6.2 that φ is universally continuous, and hence $\varphi \in \bar{R}$. Furthermore, putting

$$\bar{A} = \{\bar{x} : \sup_{\|x\| \leq 1} |\bar{x}(x)| \leq 1\},$$

we have obviously $\varphi \in \bar{A}$, and hence

$$\sup_{\bar{x} \in \bar{A}} |\bar{x}(a)| \geq \varphi(a) = \|a\|.$$

On the other hand, it is evident that $\|a\| \geq \sup_{\bar{x} \in \bar{A}} |\bar{x}(a)|$. Thus we conclude $\|a\| = \sup_{\bar{x} \in \bar{A}} |\bar{x}(a)|$ for every $a \in R$, that is, the pseudo-norm $\|x\|$ ($x \in R$) is reflexive by definition.

A linear topology $\mathfrak{B}$ on R is said to be *reflexive*, if there is a basis $\mathfrak{B}$ of $\mathfrak{B}$ such that $\|x\|_V$ is reflexive for every $V \in \mathfrak{B}$. With this definition, we have obviously by Theorem 8.4

Theorem 8.5. *If a linear topology $\mathfrak{B}$ on R is convex and continuous, then $\mathfrak{B}$ is reflexive.*

Consequently we obtain by Theorem 8.1

Theorem 8.6. *The absolute weak topology of R is reflexive.*

Theorem 8.7. *If the strong topology of R is sequential, then it is reflexive.*

Proof. Let V_ν ($\nu=1, 2, \dots$) be the convex decreasing basis of the strong topology of R . Putting

$$\bar{A}_\nu = \{\bar{x} : \sup_{x \in V_\nu} \bar{x}(x) \leq 1, \quad 0 \leq \bar{x} \in \bar{R}\},$$

we see easily that every $\bar{A}_\nu$ ($\nu=1, 2, \dots$) is topologically bounded by the absolute weak topology $\bar{\mathfrak{B}}$ of $\bar{R}$. Thus, putting

$$U_\nu = \{x : \sup_{x \in \bar{A}_\nu} \bar{x}(x) \leq 1, \quad 0 \leq x \in R\},$$

we obtain a convex positive vicinity U_ν in R such that $\|x\|_{U_\nu}$ is reflexive. For any positive $\bar{a} \in \bar{R}$, putting

$$V_{\bar{a}} = \{x : \bar{a}(x) \leq 1, \quad 0 \leq x \in R\},$$

we obtain a convex vicinity $V_{\bar{a}}$ and hence we can find ν such that $V_{\bar{a}} \supset V_\nu$, because V_ν ($\nu=1, 2, \dots$) is a basis of the strong topology of R . For such ν , we have obviously $\bar{a} \in \bar{A}_\nu$, and consequently $U_\nu \subset V_{\bar{a}}$. Therefore the convex linear topology $\mathfrak{B}$, of which U_ν ($\nu=1, 2, \dots$) is a basis, is stronger than the absolute weak topology of R . Recalling Theorem 5.2, we see that $\mathfrak{B}$ is monotone complete, and hence $\mathfrak{B}$ coincides by Theorem 7.5 with the strong topology of R . Furthermore $\mathfrak{B}$ is obviously reflexive. Consequently the strong topology of R is reflexive.

If a norm $\|x\|$ on R is complete, that is, if the linear topology $\mathfrak{B}$, of which $\{x : \|x\| \leq 1, \quad 0 \leq x \in R\}$ is a basis, is complete, then $\mathfrak{B}$ is by Theorem 5.5 the strong topology of R , and hence reflexive by Theorem 8.7. Therefore we obtain.

Theorem 8.8. *If there is a complete norm on R , then there exists a complete reflexive norm on R .*