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Author(s)	Nakano, Hidegorô
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ON TRANSCENDENTAL POINTS IN PROPER SPACES OF DISCRETE SEMI-ORDERED LINEAR SPACES

By

Hidegorô NAKANO

A cardinal number $\mathfrak{f}$ is said to be *singular*, if

- 1) $\mathfrak{f} >$ the countable density $\aleph_0$,
- 2) $\mathfrak{f} > c$ implies $\mathfrak{f} > 2^c$,
- 3) for any system of cardinal numbers $c_\lambda < \mathfrak{f} (\lambda \in A)$ with a density $< \mathfrak{f}$ we have $\sum_{\lambda \in A} c_\lambda < \mathfrak{f}$.

The existence of singular cardinal numbers is not known yet. It will be extraordinarily great, if exists.

A cardinal number c is said to be *regular*, if there is no singular cardinal number $\leq c$. The countable density $\aleph_0$ is naturally regular. If a cardinal number c is regular, then 2^c also is regular. For a system of regular cardinal numbers $c_\lambda (\lambda \in A)$, if the density of A is regular, then $\sum_{\lambda \in A} c_\lambda$ also is regular.

Let S be a set and R the totality of real functions on S . R is then obviously a discrete semi-ordered linear space.⁽¹⁾ The purpose of this paper is to prove: *If the density of S is regular, then for any positive linear functional ϕ on R , we can find a finite number of elements $s_\nu \in S$ and positive numbers $\alpha_\nu (\nu = 1, 2, \dots, \kappa)$ such that*

$$\phi(\varphi) = \sum_{\nu=1}^{\kappa} \alpha_\nu \varphi(s_\nu) \quad \text{for every } \varphi \in R.$$

Let R be now an arbitrary linear space. We have defined a *strongest convex linear topology*⁽²⁾ on R , of which the totality of convex vicinities in R is a basis. By virtue of the fact stated just above, we see easily that if the density of R is regular, then R is regular⁽³⁾ (reflexive) by the strongest convex linear topology.

(1) I. HALPERIN and H. NAKANO: Discrete semi-ordered linear spaces, Canadian Jour. Math., 3 (1951), 293-298.

(2) H. NAKANO: Topology and topological spaces, Tokyo Math. Book Series III, (1951), §70.

(3) c. f. 2).

§ 1. Transcendental ideals of sets.

Let R be a set. A collection $\mathfrak{p}$ of subsets from R is said to be an ideal, if

- 1) $O \in \mathfrak{p}$,
- 2) $X \supset Y \in \mathfrak{p}$ implies $X \in \mathfrak{p}$,
- 3) $X, Y \in \mathfrak{p}$ implies $XY \in \mathfrak{p}$.

An ideal $\mathfrak{p}$ is said to be *maximal*, if there is no other ideal including $\mathfrak{p}$. For a maximal ideal $\mathfrak{p}$, we see easily that for any set $X \notin \mathfrak{p}$ we can find $Y \in \mathfrak{p}$ such that $XY = O$.

Theorem 1.1. Let $\mathfrak{p}$ be a maximal ideal, and A a set with a density c . If $X_\lambda \in \mathfrak{p} (\lambda \in A)$ implies $\prod_{\lambda \in A} X_\lambda \in \mathfrak{p}$, then for a set Γ with the density 2^c we also have that $X_\gamma \in \mathfrak{p} (\gamma \in \Gamma)$ implies $\prod_{\gamma \in \Gamma} X_\gamma \in \mathfrak{p}$.

Proof. The collection of systems $(\varepsilon_\lambda)_{\lambda \in A}$ for $\varepsilon_\lambda = 0, 1$ has by definition the density 2^c . Let $A_{(\varepsilon_\lambda)_{\lambda \in A}}$ for all $(\varepsilon_\lambda)_{\lambda \in A}$ be a partition of R , that is,

$$R = \sum_{(\varepsilon_\lambda)_{\lambda \in A}} A_{(\varepsilon_\lambda)_{\lambda \in A}},$$

$$A_{(\varepsilon_\lambda)_{\lambda \in A}} A_{(\delta_\lambda)_{\lambda \in A}} = O \quad \text{for } (\varepsilon_\lambda)_{\lambda \in A} \neq (\delta_\lambda)_{\lambda \in A}.$$

For every finite number of elements $\lambda_\nu \in A$ ($\nu = 1, 2, \dots, \kappa$), putting

$$Y_{\delta_{\lambda_1}, \delta_{\lambda_2}, \dots, \delta_{\lambda_\kappa}} = \sum_{\varepsilon_{\lambda_\nu} = \delta_{\lambda_\nu} (\nu=1,2,\dots,\kappa)} A_{(\varepsilon_\lambda)_{\lambda \in A}}$$

for $\delta_{\lambda_\nu} = 0, 1 (\nu = 1, 2, \dots, \kappa)$, we have obviously

$$R = \sum_{\varepsilon_{\lambda_\nu} = 0,1} Y_{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, \dots, \varepsilon_{\lambda_\kappa}} \quad \text{for every } \lambda_1, \lambda_2, \dots, \lambda_\kappa \in A,$$

and

$$Y_{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, \dots, \varepsilon_{\lambda_\kappa}} Y_{\varepsilon'_{\lambda_1}, \varepsilon'_{\lambda_2}, \dots, \varepsilon'_{\lambda_\kappa}} = O,$$

if $\varepsilon_{\lambda_\nu} \neq \varepsilon'_{\lambda_\nu}$ for some ν . Thus for each finite number of elements $\lambda_\nu \in A$ ($\nu = 1, 2, \dots, \kappa$) we can find uniquely $\delta_{\lambda_\nu} = 0, 1 (\nu = 1, 2, \dots, \kappa)$ such that

$$Y_{\delta_{\lambda_1}, \delta_{\lambda_2}, \dots, \delta_{\lambda_\kappa}} \in \mathfrak{p}.$$

As $Y_{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, \dots, \varepsilon_{\lambda_\kappa}} \supset Y_{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, \dots, \varepsilon_{\lambda_\kappa}, \varepsilon_{\lambda_{\kappa+1}}}$, we see easily further that there exists uniquely $(\delta_\lambda)_{\lambda \in A}$ such that

$$Y_{\delta_{\lambda_1}, \delta_{\lambda_2}, \dots, \delta_{\lambda_\kappa}} \in \mathfrak{p} \quad \text{for every } \lambda_1, \lambda_2, \dots, \lambda_\kappa \in A,$$

Then, as the totality of systems $\lambda_1, \lambda_2, \dots, \lambda_\kappa \in A$ also has the density c ,

we have by assumption

$$A_{(\delta_\lambda)_{\lambda \in \Lambda}} = \prod_{\lambda \in \Lambda} Y_{\delta_{\lambda_1}, \delta_{\lambda_2}, \dots, \delta_{\lambda_k}} \in \mathfrak{p}.$$

Therefore, for a set Γ with the density 2^c , if $\sum_{r \in \Gamma} A_r = R$, $A_r A_{r'} = 0$ for $r \neq r'$, then there exists uniquely $r \in \Gamma$ such that $A_r \in \mathfrak{p}$. If $\sum_{r \in \Gamma} B_r = R$, then we can find by the transfinite induction subsets $A_r \subset B_r$ ($r \in \Gamma$) such that

$$\sum_{r \in \Gamma} A_r = R, \quad A_r A_{r'} = 0 \quad \text{for } r \neq r',$$

and hence there exists $r \in \Gamma$ such that $B_r \in \mathfrak{p}$. If $X_r \in \mathfrak{p}$ ($r \in \Gamma$), then we have obviously

$$R = \sum_{r \in \Gamma} (R - X_r) + \prod_{r \in \Gamma} X_r, \quad R - X_r \notin \mathfrak{p} \quad \text{for every } r \in \Gamma,$$

and consequently $\prod_{r \in \Gamma} X_r \in \mathfrak{p}$, as proved just above.

Theorem 1.2. Let $\mathfrak{p}$ be a maximal ideal and Γ a set for which $X_r \in \mathfrak{p}$ ($r \in \Gamma$) implies $\prod_{r \in \Gamma} X_r \in \mathfrak{p}$. For a system of sets Λ_r ($r \in \Gamma$) if $X_\lambda \in \mathfrak{p}$ ($\lambda \in \Lambda_r$) implies $\prod_{\lambda \in \Lambda_r} X_\lambda \in \mathfrak{p}$ for every $r \in \Gamma$, then $X_\lambda \in \mathfrak{p}$ ($\lambda \in \sum_{r \in \Gamma} \Lambda_r$) implies $\prod_{\lambda \in \sum_{r \in \Gamma} \Lambda_r} X_\lambda \in \mathfrak{p}$.

Proof. We have obviously by assumption that $X_\lambda \in \mathfrak{p}$ ($\lambda \in \sum_{r \in \Gamma} \Lambda_r$) implies $\prod_{\lambda \in \Lambda_r} X_\lambda \in \mathfrak{p}$ for every $r \in \Gamma$, and hence

$$\prod_{\lambda \in \sum_{r \in \Gamma} \Lambda_r} X_\lambda = \prod_{r \in \Gamma} \left(\prod_{\lambda \in \Lambda_r} X_\lambda \right) \in \mathfrak{p}.$$

A maximal ideal $\mathfrak{p}$ is said to be *transcendental*, if $X_\nu \in \mathfrak{p}$ ($\nu = 1, 2, \dots$) implies $\prod_{\nu=1}^{\infty} X_\nu \in \mathfrak{p}$. With this definition, we conclude immediately by Theorems 1.1 and 1.2.

Theorem 1.3. If the density of R is regular, then for every transcendental maximal ideal $\mathfrak{p}$, we have $\prod_{X \in \mathfrak{p}} X \in \mathfrak{p}$, and hence $\prod X$ is composed only of a single element.

§ 2. Transcendental points of discrete semi-ordered linear spaces

Let R be a discrete semi-ordered linear space⁽⁴⁾ and $a_\lambda \in R$ ($\lambda \in \Lambda$) a basis of R , i. e., $a_\lambda \wedge a_\rho = 0$ for $\lambda \neq \rho$ and for each positive element $x \in R$ we can find uniquely a system of real numbers $\xi_\lambda \geq 0$ ($\lambda \in \Lambda$) such that

(4) c. f. 1).

$$x = \bigcup_{\lambda \in \Lambda} \xi_\lambda a_\lambda.$$

For a positive element $x = \bigcup_{\lambda \in \Lambda} \xi_\lambda a_\lambda$, putting

$$\Lambda_x = \{\lambda : \xi_\lambda \neq 0\},$$

we see easily:

$$[a]x = \bigcup_{\lambda \in \Lambda_a} \xi_\lambda a_\lambda \quad \text{for every positive element } x = \bigcup_{\lambda \in \Lambda} \xi_\lambda a_\lambda;$$

$$\Lambda_a \subset \Lambda_b \quad \text{if and only if } [a] \leq [b];$$

$$\Lambda_a \cup \Lambda_b = \Lambda_{a+b}, \quad \Lambda_a \cap \Lambda_b = \Lambda_{ab}.$$

Thus every projector $[a]$ may be represented by the set Λ_a . Therefore every point of the proper space of R may be considered as a maximal ideal of subsets from Λ . Furthermore, for a maximal ideal $\mathfrak{p}$ of subsets from Λ , if $\mathfrak{p} \ni \Lambda_a$ for a positive element $a \in R$, then $\mathfrak{p}$ is a point of the proper space of R and $\mathfrak{p} \in U_{[a]}$.

A point $\mathfrak{p}$ of the proper space of R is said to be *transcendental*⁽⁵⁾, if $\mathfrak{p} \in U_{[a_\nu]}$ ($\nu=1, 2, \dots$) implies $\mathfrak{p} \in U_{[b]} \subset \bigcap_{\nu=1}^{\infty} U_{[a_\nu]}$ for some $b \in R$. With this definition, it is evident that a point $\mathfrak{p}$ is transcendental if and only if $\mathfrak{p}$ is a transcendental maximal ideal.

For a positive element $a \in R$, the density of Λ_a is called the *dimension* of a .

Theorem 2.1. *If the dimension of every positive element of R is regular, then the proper space of R has no transcendental point up to isolated points.*

Proof. For a transcendental point $\mathfrak{p}$ of the proper space of R , we can find obviously a positive element $a \in R$ such that $\mathfrak{p} \in U_{[a]}$. Then we can consider $\mathfrak{p}$ as a transcendental maximal ideal of subsets from Λ_a . Therefore we can find by Theorem 1.3 $\lambda \in \Lambda$ such that $U_{[a_\lambda]}$ is composed only of the single point $\mathfrak{p}$, and hence $\mathfrak{p}$ is an isolated point.

From this Theorem 2.1 we conclude immediately

Theorem 2.2. *If the density of R is regular, then the proper space of R has no transcendental point up to isolated points.*

§ 3. Universally complete discrete semi-ordered linear spaces

Let a discrete semi-ordered linear space R be *universally complete*⁽⁶⁾,

(5) H. NAKANO: Ueber ein lineares Funktional auf dem teilweise geordneten Modul, Proc. Tokyo Acad., 18 (1942), 548-552.

(6) H. NAKANO: Modern spectral theory, Tokyo Math. B. S., II (1950), § 34.

i. e., for every orthogonal system of positive elements $x_\gamma \in R$ ($\gamma \in \Gamma$) there exists $\bigcup_{\gamma \in \Gamma} x_\gamma$. R is then obviously *totally unbounded*⁽⁷⁾, i. e., for an orthogonal sequence of positive elements $x_\nu \in R$ ($\nu=1, 2, \dots$), if $\bigcup_{\nu=1}^{\infty} x_\nu$ exists, then we can find a sequence of positive numbers $\alpha_\nu \uparrow_{\nu=1}^{\infty} +\infty$ for which $\bigcup_{\nu=1}^{\infty} \alpha_\nu x_\nu$ exists. Therefore every positive linear functional on R is continuous.⁽⁸⁾

Theorem 3.1. *If a discrete semi-ordered linear space R is universally complete and the density of R is regular, then for every positive linear functional Φ on R we can find a finite number of discrete positive elements $a_\nu \in R$ ($\nu=1, 2, \dots, \kappa$) such that*

$$\Phi(x) = \sum_{\nu=1}^{\kappa} \Phi([a_\nu]x) \quad \text{for every } x \in R.$$

Proof. Let $\mathfrak{S}_\Phi$ be the characteristic set⁽⁹⁾ of Φ . If $\mathfrak{S}_\Phi$ contains infinite points, then we can find a sequence of positive elements $a_\nu \in R$ ($\nu=1, 2, \dots$) such that $[a_\nu][a_\mu]=0$ for $\nu \neq \mu$ and $U_{[a_\nu]} \mathfrak{S}_\Phi \neq 0$ for every $\nu=1, 2, \dots$. Then we have $\Phi(a_\nu) > 0$ for every $\nu=1, 2, \dots$, and hence, putting

$$a = \bigcup_{\nu=1}^{\infty} \frac{\nu}{\Phi(a_\nu)} a_\nu,$$

we have

$$\Phi(a) \geq \Phi\left(\frac{\nu}{\Phi(a_\nu)} a_\nu\right) = \nu \quad \text{for every } \nu=1, 2, \dots,$$

contradicting $\Phi(a) < +\infty$. Thus $\mathfrak{S}_\Phi$ is composed only of a finite number of points. Furthermore every point of $\mathfrak{S}_\Phi$ is transcendental. Because, if a point $\mathfrak{p} \in \mathfrak{S}_\Phi$ is not transcendental, then we can find by definition a sequence of projectors $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$, such that $U_{[p_\nu]} \ni \mathfrak{p}$, but $U_{[p_\nu]}$ does not contain any other point of $\mathfrak{S}_\Phi$ for every $\nu=1, 2, \dots$. As $\mathfrak{p} \in \mathfrak{S}_\Phi$, we can find a positive element $a \in R$ such that $\Phi([p]a) > 0$ for $U_{[p]} \in \mathfrak{p}$. Then we have

$$\Phi([p_\nu]a) = \Phi([p_1]a) - \Phi([p_1] - [p_\nu])a = \Phi([p_1]a),$$

because $U_{[p_1] - [p_\nu]} \mathfrak{S}_\Phi = 0$. As Φ is continuous, we have hence

$$\Phi([p_1]a) = \lim_{\nu \rightarrow \infty} \Phi([p_\nu]a) = 0,$$

contradicting $\Phi([p_1]a) > 0$. Therefore $\mathfrak{S}_\Phi$ is composed only of a finite number of transcendental points. As the density of R is regular by

(7) H. NAKANO: *Modulated semi-ordered linear spaces*, Tokyo Math. B. S., I (1950), §17.

(8) c. f. 7) Theorem 19.8.

(9) c. f. 7) §20.

assumption, we see by Theorem 2.2 that $\mathfrak{S}_\phi$ is composed only of a finite number of isolated points. Thus we can find a finite number of discrete positive elements $a_\nu \in R$ ($\nu=1, 2, \dots, \kappa$) such that

$$\phi(x) = \sum_{\nu=1}^{\kappa} \phi([a_\nu]x) \quad \text{for every } x \in R.$$

Recalling a theorem in an earlier paper,⁽¹⁰⁾ we obtain immediately by this Theorem 3.1

Theorem 3.2. Let R be a universally complete, discrete semi-ordered linear space with a regular density. For a positive linear functional ϕ on R , if

$$\text{Min } \{\phi(x), \phi(y)\} = 0 \quad \text{for } x \wedge y = 0,$$

then there exists a positive discrete element $a \in R$ such that

$$\phi(x) = \phi([a]x) \quad \text{for every } x \in R. \quad (11)$$

(10) c. f. 5) Satz 8.

(11) The same problem was considered by E. HEWITT, but he could not succeed to prove. E. HEWITT: Linear functionals on spaces of continuous functions, Fund. Math. 37 (1950), 161-189.