ON THE INTRINSIC DERIVATIVE IN THE NON-HOLONOMIC EXSURFACE

By

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Introduction. H. V. CRAIG has shown that M-th order intrinsic derivative of an absolute tensor can be expressed in a form of contraction of extensors [1][(1)]. The principal purpose of the present paper is its generalization, that is, to find the structure of M-th order intrinsic derivative of a contravariant vector in the space called the non-holonomic exsurface which is defined by the n independent Pfaffian equations

$$(0.1) \qquad \sum_{\alpha=0}^{M} \lambda_{\alpha i}^{a'}(x, x', \cdots, x^{(M)})\, dx^{(\alpha) i} \overset{(2)}{=} 0 \qquad (a' = 1, \cdots, n)$$

in a space $K_n^{(M)}$ of line-elements of M-th order. A non-holonomic space [2] and an exsurface [3] introduced by CRAIG are both special ones of this space.

In the present paper we use certain of ideas, notations and results given in the previous paper [4] without explanation. The present author wishes to express to Prof. A. KAWAGUCHI her sincere thanks for his criticisms.

§1. The non-holonomic exsurface and the intrinsic derivative. Let us consider the Pfaffian equations defined by (0.1), then under "non-holonomic exsurface" which is denoted by N'_n we understand a set of the fields of n mutually independent excovariant extensors $\lambda_{\alpha i}^{a'}$ associated with each expoint $x^{(\alpha) i}$ of the space $K_n^{(M)}$.

We now define the displacement $ds^{a'}$ of an ideal point in N'_n corresponding to the displacement $dx^{(\alpha) i}$ of an expoint $x^{(\alpha) i}$ in $K_n^{(M)}$ as follows:

$$(1.1) \qquad ds^{a'} = \sum_{\alpha=0}^{M} \lambda_{\alpha i}^{a'}(x, x', \cdots, x^{(M)})\, dx^{(\alpha) i}.$$

Then for two infinitesimal displacements d_1 and d_2, there exists the relation

(1) Numbers in brackets refer to the references at the end of the paper.

(2) Numbers in parentheses denote times of differentiation with respect to t.

$$d_1 d_2 s^{a'} - d_2 d_1 s^{a'} = \omega_{\alpha i \beta j}^{a'} d_1 x^{(\alpha)i} d_2 x^{(\beta)j} ,$$

where

$$\omega_{\alpha i \beta j}^{a'} = \frac{\partial \lambda_{\alpha i}^{a'}}{\partial x^{(\beta)j}} - \frac{\partial \lambda_{\beta j}^{a'}}{\partial x^{(\alpha)i}} . \tag{1.2}$$

Especially, if $\omega_{\alpha i \beta j}^{a'} \equiv 0$, we have the relation*

$$\frac{\partial \lambda_{\alpha i}^{a'}}{\partial x^{(\beta)j}} - \frac{\partial \lambda_{\beta j}^{a'}}{\partial x^{(\alpha)i}} \equiv 0$$

which shows complete integrability of the system of partial differential equations $\dfrac{\partial s^{a'}}{\partial x^{(\alpha)i}} = \lambda_{\alpha i}^{a'}(x, x', \cdots, x^{(M)})$ whose solutions

$$s^{a'} = f^{a'}(x, x', \cdots, x^{(M)}, s_0^{a'}) \tag{1.3}$$

may be considered as the equations of exsurface in an n-dimensional space. Consequently we can see the following theorem:

Theorem 1.1. *If* $\omega_{\alpha i \beta j}^{a'} \equiv 0$ *and* $s^{a'}$ *are assumed to be rectangular cartesian coordinates of a point in an n-dimensional Euclidean space E, then the non-holonomic exsurface becomes the exsurface defined by* CRAIG ([3] p. 792).

The quantities $v^{a'}$ determined by an excontravariant extensor $v^{\alpha i}$ of range $R\,(R \leqq M)$ in $K_n^{(M)}$ such that

$$\binom{M}{R} v^{a'} \equiv \sum_{\alpha = M-R}^{M} \binom{\alpha}{M-R} \lambda_{\alpha i}^{a'} v^{\alpha - M + R i}$$

are called the *vector components* in N'_n corresponding to $v^{\alpha i}$. Similarly, the tensor components in N'_n corresponding to higher order excontravariant extensors can be defined. If $v^{(\alpha)i}$, $\alpha = 0, 1, \cdots, R$, are the components of the extensor obtained by differentiating the components of a vector v^i in $K_n^{(M)}$ α times along a parameterized arc, then the vector components in N'_n of $v^{(\alpha)i}$ are given by

$$\binom{M}{R} v_R^{a'} = \sum_{\alpha = M-R}^{M} \binom{\alpha}{M-R} \lambda_{\alpha i}^{a'} v^{(\alpha - M + R)i} ,$$

we call such the quantities $v_R^{a'}$ *R-th order intrinsic derivatives of the components* $v^{a'}$ in N'_n corresponding to v^i, *i.e.*,

$$v^{a'} \equiv \lambda_{Mi}^{a'} v^i ,$$

and put

$$\frac{\delta^R v^{a'}}{dt^R} \equiv v_R^{a'} .$$

Next we shall discuss such the intrinsic derivative of $v^{a'}$.

In the case that $R=1$, we have

$$(1.4)\qquad M\frac{\delta v^{a'}}{dt}\equiv v_1^{a'}=\sum_{\alpha=M-1}^{M}\binom{\alpha}{M-1}\lambda_{\alpha i}^{a'}v^{(\alpha-M+1)i}$$
$$=M\lambda_{Mi}^{a'}v^{(1)i}+\lambda_{M-1\,j}^{a'}v^j .$$

If we consider the reciprocal contravariant vectors $\lambda_{a'}^j$ of the covariant vectors $\lambda_{Mi}^{a'}$, which satisfy the equation

$$\lambda_{Mi}^{a'}\lambda_{a'}^j=\delta_i^j \qquad (\delta_i^j\text{: KRONECKER delta})$$

under the assumption that $|\lambda_{Mi}^{a'}|\neq 0$. Then multiplying the last result of (1.4) by $\frac{1}{M}\lambda_{a'}^j$ and summing with respect to a', we have

$$\frac{\delta v^{a'}}{dt}\lambda_{a'}^j=v^{(1)j}+\frac{1}{M}\lambda_{M-1\,k}^{a'}\lambda_{a'}^j v^k ,$$

on putting

$$(1.5)\qquad \Gamma_k^j=\frac{1}{M}\lambda_{M-1\,k}^{a'}\lambda_{a'}^j ,$$

the above equation is rewritten in the form

$$\frac{\delta v^{a'}}{dt}\lambda_{a'}^j=v^{(1)j}+\Gamma_k^j v^k .$$

Consequently, $K_n^{(M)}$ is the space with the affine connection Γ_k^j, Γ_k^j being the function of $x^{(\alpha)i}$, $\alpha=0,1,\cdots,M$. For the structure of $\frac{\delta v^{a'}}{dt}$ referred to the components of the non-holonomic exsurface, we can state the following theorem.

Theorem 1.2. *The relation*

$$\frac{\delta v^{a'}}{dt}=\frac{dv^{a'}}{dt}+\Gamma_{b'}^{a'}v^{b'} .$$

holds good, where

$$\Gamma_{b'}^{a'}=\Gamma_k^j\lambda_{Mj}^{a'}\lambda_{b'}^k-\sum_{\alpha=0}^{M}\frac{\partial\lambda_{Mk}^{a'}}{\partial x^{(\alpha)j}}x^{(\alpha+1)j}\lambda_{b'}^k$$
$$=\Gamma_k^j\lambda_{Mj}^{a'}\lambda_{b'}^k-\lambda_{Mk}^{a'(1)}\lambda_{b'}^k$$
$$=\frac{1}{M}\lambda_{M-1\,j}^{a'}\lambda_{b'}^j-\lambda_{Mk}^{a'(1)}\lambda_{b'}^k .$$

Proof. Differentiating the expression $v^i=\lambda_{a'}^i v^{a'}$ deduced from $v^{a'}=\lambda_{Mi}^{a'}v^i$ with respect to t, we have

$$v^{(1)i}=\lambda^i_{a'}v^{(1)a'}+\frac{\partial\lambda^i_{a'}}{\partial x^{(\alpha)j}}x^{(\alpha+1)j}v^{a'},$$

let replace $v^{(1)i}$ in the last result of (1.4) with the right-hand member of the above equation, then it follows that

$$\begin{aligned}\frac{\delta v^{a'}}{dt}&=\lambda^{a'}_{Mi}\left(\lambda^i_{b'}v^{(1)b'}+\frac{\partial\lambda^i_{b'}}{\partial x^{(\alpha)j}}x^{(\alpha+1)j}v^{b'}\right)+\frac{1}{M}\lambda^{a'}_{M-1j}\lambda^j_{b'}v^{b'}\\&=v^{(1)a'}+\left(\frac{1}{M}\lambda^{a'}_{M-1j}\lambda^j_{b'}+\lambda^{a'}_{Mi}\frac{\partial\lambda^i_{b'}}{\partial x^{(\alpha)j}}x^{(\alpha+1)j}\right)v^{b'}\\&=v^{(1)a'}+\Gamma^{a'}_{b'}v^{b'},\end{aligned}$$

on making use of $\dfrac{\partial\lambda^i_{b'}}{\partial x^{(\alpha)j}}\lambda^{a'}_{Mi}=-\lambda^i_{b'}\dfrac{\partial\lambda^{a'}_{Mi}}{\partial x^{(\alpha)j}}$.

Theorem 1.3. *There exists the relation*

$$M\frac{\delta v^{a'}}{dt}=\sum_{\beta'=M-1}^{M}\binom{\beta'}{M-1}N^{a'}_{\beta'b'}v^{b'(\beta'-M+1)},\tag{1.6}$$

where

$$N^{a'}_{\beta'b'}=\sum_{\alpha=\beta'}^{M}\binom{\alpha}{\beta'}\lambda^{a'}_{\alpha j}\lambda^{j(\alpha-\beta')}_{b'}.$$

Proof. Calculating the right-hand member of (1.6), it follows that

$$\begin{aligned}\sum_{\beta'=M-1}^{M}\binom{\beta'}{M-1}N^{a'}_{\beta'b'}v^{b'(\beta'-M+1)}&=N^{a'}_{M-1b'}v^{b'}+MN^{a'}_{Mb'}v^{b'(1)}\\&=\left\{\sum_{\alpha=M-1}^{M}\binom{\alpha}{M-1}\lambda^{a'}_{\alpha j}\lambda^{j(\alpha-M+1)}_{b'}\right\}v^{b'}+M\lambda^{a'}_{Mj}\lambda^j_{b'}v^{b'(1)}\\&=\left\{\lambda^{a'}_{M-1j}\lambda^j_{b'}+M\lambda^{a'}_{Mj}\lambda^{j(1)}_{b'}\right\}v^{b'}+M\delta^{a'}_{b'}v^{b'(1)}\\&=M\left\{v^{a'(1)}+\left(\lambda^{a'}_{Mj}\lambda^{j(1)}_{b'}+\frac{1}{M}\lambda^{a'}_{M-1j}\lambda^j_{b'}\right)v^{b'}\right\}\\&=M\left\{v^{a'(1)}+\left(\frac{1}{M}\lambda^{a'}_{M-1j}\lambda^j_{b'}-\lambda^{a'(1)}_{Mj}\lambda^j_{b'}\right)v^{b'}\right\}\\&=M\left(v^{a'(1)}+\Gamma^{a'}_{b'}v^{b'}\right)=M\frac{\delta v^{a'}}{dt}.\end{aligned}$$

In general, for the structure of the higher order intrinsic derivative of $v^{a'}$, we have the

Theorem 1.4. *It follows that*

$$\binom{M}{R}\frac{\delta^R v^{a'}}{dt^R}=\sum_{\beta'=M-R}^{M}\binom{\beta'}{M-R}N^{a'}_{\beta'b'}v^{b'(\beta'-M+R)},$$

where

$$N_{\beta' b'}^{a'} = \sum_{\alpha=\beta'}^{M} \binom{\alpha}{\beta'} \lambda_{\alpha j}^{a'} \lambda_{b'}^{j(\alpha-\beta')} .$$

Such the quantities $N_{\beta' b'}^{a'}$ are called as *intrinsic derivation coefficients* of the non-holonomic exsurface.

Proof. By virture of the definition, we can observe

$$\binom{M}{R} \frac{\delta^R v^{a'}}{dt^R} = \sum_{\alpha=M-R}^{M} \binom{\alpha}{M-R} \lambda_{\alpha j}^{a'} v^{(\alpha-M+R)j} .$$

Differentiating the equation $v^j = \lambda_{b'}^j v^{b'}$ $(\alpha - M+R)$ times by LEIBNITZ's rule, the following expression is obtained

$$v^{j(\alpha-M+R)} = \sum_{\beta=0}^{\alpha-M+R} \binom{\alpha-M+R}{\beta} \lambda_{a'}^{j(\alpha-M+R-\beta)} v^{a'(\beta)} .$$

Accordingly, it follows that

$$\begin{aligned}
\binom{M}{R} \frac{\delta^R v^{a'}}{dt^R} &= \sum_{\alpha=M-R}^{M} \binom{\alpha}{M-R} \lambda_{\alpha j}^{a'} \sum_{\beta=0}^{\alpha-M+R} \binom{\alpha-M+R}{\beta} \lambda_{b'}^{j(\alpha-M+R-\beta)} v^{b'(\beta)} \\
&= \sum_{\beta=0}^{R} \sum_{\alpha=\beta+M-R}^{M} \binom{\alpha}{M-R+\beta} \binom{M-R+\beta}{\beta} \lambda_{\alpha j}^{a'} \lambda_{b'}^{j(\alpha-M+R-\beta)} v^{b'(\beta)} \\
&= \sum_{\beta=0}^{R} \binom{M-R+\beta}{M-R} \sum_{\alpha=\beta+M-R}^{M} \binom{\alpha}{M-R+\beta} \lambda_{\alpha j}^{a'} \lambda_{b'}^{j(\alpha-M+R-\beta)} v^{b'(\beta)} \\
&= \sum_{\beta=0}^{R} \binom{M-R+\beta}{M-R} N_{M-R+\beta\, b'}^{a'} v^{b'(\beta)} \\
&= \sum_{\beta'=M-R}^{M} \binom{\beta'}{M-R} N_{\beta' b'}^{a'} v^{b'(\beta'-M+R)} \quad \text{(putting: } \beta' = M-R+\beta).
\end{aligned}$$

§ **2. Specialization.** The corresponding statements for the exsurface $x^{a'} = f^{a'}(x, x', \cdots, x^{(M)})$ are obtained by replacing $\lambda_{\alpha i}^{a'}$ in the preceding statements with $\dfrac{\partial f^{a'}}{\partial x^{(\alpha)i}}$.

Let us consider a non-holonomic space $\overline{N}_n$ which is given by n independent Pfaffian equations

$$\lambda_i^{a'}(x)\, dx^i = 0 \qquad\qquad (a' = 1, \cdots, n)$$

corresponding to an n-dimensional space K_n with an affine connection $\Gamma_j^i(x)$ referred to the coordinate system x^i, then if we take the quantities $\lambda_i^{a'} T_{\beta j}^{\ i}$ in place of $\lambda_{\beta j}^{a'}$ in the preceding statements, $T_{\beta j}^{\ i}$ being the higher order extensive derivative of δ_j^i introduced by CRAIG, that is, $D^M | \delta_{\beta j}^{\ i}$([1] p. 338), the corresponding results for the higher order intrinsic derivatives of tensors in $\overline{N}_n$ are observed. Also if $\overline{N}_n$ coincides with the original n-dimensional space K_n, then in use of the quantities $T^{i}_{\alpha j}$ corresponding to $\lambda^{a'}_{\alpha j}$, the above statements will give the results obtained by CRAIG [1].

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