PRODUCT SPACES OF SEMI-ORDERED LINEAR SPACES

By

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Let R and S be two universally continuous semi-ordered linear spaces.[1] A bilinear functional $\varphi(x, y)$ $(x \in R, y \in S)$ is said to be *bounded*, if for any $0 \leqq a \in R$, $0 \leqq b \in S$ we have

$$\sup \sum_{\nu, \mu} |\varphi(x_\nu, y_\mu)| < +\infty$$

for $\sum_{\nu=1}^{\kappa} x_\nu = a$, $\sum_{\mu=1}^{\rho} y_\mu = b$, $0 \leqq x_\nu \in R$, $0 \leqq y_\mu \in S$ $(\nu = 1, 2, \cdots, \kappa;\ \mu = 1, 2, \cdots, \rho)$. The totality of bounded bilinear functionals φ on R and S constitutes a universally continuous semi-ordered linear space if we define $\varphi \geqq \psi$ to mean $\varphi(x, y) \geqq \psi(x, y)$ for every $0 \leqq x \in R$, $0 \leqq y \in S$. For the conjugate spaces[2] $\bar{R}$ and $\bar{S}$ of R and S respectively, putting

$$\bar{x}\bar{y}(x, y) = \bar{x}(x)\bar{y}(y) \quad \text{for } x \in R,\ y \in S,\ \bar{x} \in \bar{R},\ \bar{y} \in \bar{S},$$

we obtain bounded bilinear functionals $\bar{x}\bar{y}$ on R and S for every $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$. A bilinear functional φ on R and S is said to be *singular*, if φ is orthogonal to every $\bar{x}\bar{y}$ for $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$, that is, if $|\varphi| \frown \bar{x}\bar{y} = 0$ for every $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$. A bilinear functional φ on R and S is said to be *totally continuous*, if φ is orthogonal to all singular bilinear functionals.

A bilinear functional φ on R and S is said to be *universally continuous*, if φ is bounded and $\varphi(x, y)$ is universally continuous[3] as a linear functional on R for every fixed $y \in S$ and on S for every fixed $x \in R$. We shall prove: *a universally continuous bilinear functional φ on R and S is totally continuous if and only if $0 \leqq a_\nu \leqq a$ $(\nu = 1, 2, \cdots)$, s-$\lim a_\nu = 0$*[4] *in R*

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1) A semi-ordered linear space R is said to be *universally continuous*, if for any system of positive elements $a_\lambda \in R$ $(\lambda \in \Lambda)$ there exists $\bigcap_{\lambda \in \Lambda} a_\lambda$.

2) Let R be a universally continuous semi-ordered linear space. A linear functional φ on R is said to be *universally continuous*, if $R \ni x_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies $\inf_{\lambda \in \Lambda} |\varphi(x_\lambda)| = 0$. The totality of universally continuous linear functional on R is called the *conjugate space* of R and denoted by $\bar{R}$.

3) c. f. 2).

4) s-$\lim_{\nu \to \infty} a_\nu = 0$ means that every subsequence from a_ν $(\nu = 1, 2, \cdots)$ contains a subsequence a_{ν_μ} $(\mu = 1, 2, \cdots)$ such that $\lim_{\mu \to \infty} a_{\nu_\mu} = 0$.

implies $\lim_{\nu\to\infty} \varphi(a_\nu, y)=0$, considering $\varphi(a,y)$ as elements of the conjugate space $\bar{S}$.

For a universally continuous semi-ordered linear space W, if there exists a corresponpence: $(xy)^W \in W$ for every $x \in R$ and $y \in S$, such that $(xy)^W$ is bilinear for $x \in R$ and $y \in S$, universally continuous for $x \in R$ and $y \in S$, and every order closed manifold of W containing all $\sum_{\nu=1}^{\kappa} \alpha_\nu (x_\nu y_\nu)^W$ for every finite number of $0 \leqq x_\nu \in R$, $0 \leqq y_\nu \in S$, $\alpha_\nu \geqq 0$ $(\nu=1,2,\cdots,\kappa)$ includes all positive elements of W, then W is said to be a *product space* of R and S by the *product correspondence*: $(xy)^W \in W(x \in R, y \in S)$.

A product space W of R and S by a product correspondence $(xy)^W \in W(x \in R, y \in S)$ is said to be *normal*, if the conjugate space $\bar{W}$ of W is a product space of the conjugate spaces $\bar{R}$ and $\bar{S}$ by a product correspondence $(\bar{x}\bar{y})^{\bar{W}} \in \bar{W}$ $(\bar{x} \in \bar{R}, \bar{y} \in \bar{S})$ such that

$$(\bar{x}\bar{y})^{\bar{W}}((xy)^W) = \bar{x}(x)\bar{y}(y).$$

If both R and S are semi-regular[5], then the totality of totally continuous bilinear functionals on $\bar{R}$ and $\bar{S}$ is a normal product space of R and S by a product correspondence:

$$xy(\bar{x},\bar{y}) = \bar{x}(x)\bar{y}(y) \qquad \text{for } \bar{x} \in \bar{R},\ \bar{y} \in \bar{S},\ x \in R,\ y \in S.$$

This product space is called the *upper product space* of R and S and denoted by RS. The totality of elements z of RS for which we can find $0 \leqq x \in R$, $0 \leqq y \in S$ such that $|z| \leqq xy$, also constitutes a normal product space of R and S. This product space is called the *lower product space* of R and S and denoted by $R \times S$. We shall prove: a *product space W of R and S is normal if and only if W is isomorphic to a semi-normal manifold*[6] *of the upper product space RS such that $(xy)^W$ corresponds to $xy \in RS$.*

For a normal product space W of R and S, a linear operator $F_{\bar{b}}$ on W into R is said to be a *Fubini operator* by $\bar{b} \in \bar{S}$, if $F_{\bar{b}}$ is universally continuous and

$$F_{\bar{b}}(xy)^W = \bar{b}(y)x \qquad \text{for every } x \in R,\ y \in S.$$

If both R and S are reflexive[7], then for every $\bar{b} \in \bar{S}$ there exists the

5) A univerrally continuous semi-ordered linear space R is said to be *semi-regular*, if for every $0 \neq a \in R$ there exists a universally continuous linear functional φ on R such that $\varphi(a) \neq 0$.

6) A linear manifold S of a lattice ordered linear space R is said to be *semi-normal*, if $a \in S$, $|b| \leqq |a|$ implies $b \in S$.

7) A universally continuous semi-ordered linear space R is said to be *reflexive*, if R may be considered as the conjugate space $\bar{\bar{R}}$ of $\bar{R}$.

Fubini operator $F_{\bar{b}}$ on W into R, and for the Fibini operator $F_{\bar{a}}$ on W into S by $\bar{a} \in \bar{R}$ we have

$$\bar{a}(F_{\bar{b}}z) = \bar{b}(F_{\bar{a}}z) \qquad \text{for every} \quad z \in W.$$

Furthermore we shall prove that *the bilinear functional on W and $\bar{R}$*

$$\varphi(z, \bar{x}) = \bar{x}(F_{\bar{b}}z) \qquad \text{for} \quad z \in W, \ \bar{x} \in \bar{R},$$

is singular for the Fubini operator $F_{\bar{b}}$ by every $\bar{b} \in \bar{S}$.

Let R and S be normed semi-ordered linear spaces. A norm on a product space W of R and S is said to be a *product norm*, if

$$\|(xy)^W\| = \|x\|\,\|y\| \qquad \text{for} \quad x \in R, \ y \in S.$$

For the upper product space RS, we obtain a product norm

$$\|z\|_u = \sup_{\bar{x} \in \bar{R}, \bar{y} \in \bar{S}} |z(\bar{x}, \bar{y})| \qquad \text{for} \quad \bar{x} \in \bar{R}, \ \bar{y} \in \bar{S},$$

which will be called the *upper norm*. For a product norm on a normal product space W, its conjugate norm on the conjugate space $\bar{W}$ is a product norm from $\bar{R}$ and $\bar{S}$, if and only if $\|z\| \geqq \|z\|_u$ for $z \in W$. For a normal product space W, if there exists a complete reflexive[8] norm on W, then there exists a complete seflexive product norm on W, which is equivalent to the former.

Let R and S be modulared semi-ordered linear spaces. A modular m on a normal product space W is said to be a product *modular*, if

$$m((xy)^W) \leqq \frac{1}{2}\{m(x) + m(y)\}^2 \qquad \text{for} \quad x \in R, \ y \in S.$$

For the conjugate modulars $\bar{m}$ on $\bar{R}$ and $\bar{S}$, putting

$$m_u(z) = \sup_{\bar{x} \in \bar{R}, \bar{y} \in \bar{S}} \{z(\bar{x}, \bar{y}) - \frac{1}{2}\{\bar{m}(\bar{x}) + \bar{m}(\bar{y})\}^2\},$$

we obtain a product modular on the upper product space RS. This product modular is called the *upper modular*. For the upper modular of the upper product space $\bar{R}\bar{S}$ of the conjugate spaces $\bar{R}$ and $\bar{S}$, as its conjugate modular

$$m_1(z) = \sup_{\bar{z} \in \bar{R}\bar{S}} \{\bar{z}(z) - m_u(\bar{z})\} \qquad \text{for} \quad z \in R \times S,$$

8) A norm $\|x\|$ on a universally continuous semi-ordered linear space R is said to be *reflexive*, if for the conjugate space $\bar{R}$ of R and the *conjugate norm*

$$\|\bar{x}\| = \sup_{\|x\| \leqq 1} |\bar{x}(x)| \qquad \text{for} \quad \bar{x} \in \bar{R}, \ x \in R,$$

we have $\|x\| = \sup_{\|\bar{x}\| \leqq 1} |\bar{x}(x)|$ for every $x \in R$.

we obtain a product modular on the lower product space $R\times S$, which will be call ed the *lower modular*. *A modular m on a normal product sopace W is a product modular, if and only if $m(z)\leqq m_l(z)$ for $z\in R\times S$, and its conjugate modular is a product modular from $\bar{R}$ and $\bar{S}$, if and only if $m(z)\geqq m_u(z)$ for $z\in W$.*

In this paper we shall make use of the notations in the book: H. Nakano, Modulared semi-ordered linear spaces, Tokyo (1950). This book will be denoted by MSLS.

§ 1. Bounded bilinear functionals

Let R and S be two lattice ordered linear spaces. A functional $\varphi(x, y)$ $(x\in R, y\in S)$ is said to be *bilinear*, if

$$\varphi(x_1+x_2, y) = \varphi(x_1, y)+\varphi(x_2, y),$$
$$\varphi(x, y_1+y_2) = \varphi(x, y_1)+\varphi(x, y_2),$$
$$\varphi(\alpha x, y) = \varphi(x, \alpha y) = \alpha\varphi(x, y)$$

for every real number α.

A bilinear functional φ on R and S is said to be *bounded*, if for every $0\leqq a\in R$, $0\leqq b\in S$ we have

$$\sup\sum_{\nu,\mu}|\varphi(x_\nu, y_\mu)| < +\infty$$

for $\sum_{\nu=1}^{\kappa} x_\nu = a$, $0\leqq x_\nu\in R$, $\sum_{\mu=1}^{\rho} y_\mu = b$, $0\leqq y_\mu\in S$, $\kappa, \rho = 1, 2, \cdots$.

The totality of bounded bilinear functionals on R and S is called the *biassociated space* of R and S, and denoted by $\widetilde{R,S}$. The biassociated space $\widetilde{R,S}$ constitutes obviously a linear space by the relation:

$$(\alpha\varphi+\beta\psi)(x, y) = \alpha\varphi(x, y)+\beta\psi(x, y)$$

for $\varphi, \psi\in\widetilde{R,S}$, $x\in R$, $y\in S$. For two bilinear functionals $\varphi, \psi\in\widetilde{R,S}$ we define $\varphi\geqq\psi$ to mean

$$\varphi(x, y)\geqq\psi(x, y) \qquad \text{for } 0\leqq x\in R,\ 0\leqq y\in S.$$

With this definition we have

Theorem 1.1. *The biassociated space $\widetilde{R,S}$ is a universally continuous semi-ordered linear space, and for every $\varphi, \psi\in\widetilde{R,S}$, $0\leqq x\in R$, $0\leqq y\in S$ we have*

$$\varphi\smile\psi(x, y) = \sup\sum_{\nu,\mu}\operatorname{Max}\{\varphi(x_\nu, y_\mu), \psi(x_\nu, y_\mu)\},$$
$$\varphi\frown\psi(x, y) = \inf\sum_{\nu,\mu}\operatorname{Min}\{\varphi(x_\nu, y_\mu), \psi(x_\nu, y_\mu)\}.$$

for $x = \sum_{\nu=1}^{\kappa} x_\nu$, $y = \sum_{\mu=1}^{\rho} y_\mu$, $0 \leqq x_\nu \in R$, $0 \leqq y_\mu \in S$, $\kappa, \rho = 1, 2, \cdots$.

$\widetilde{R,S} \ni \varphi_\lambda \uparrow_{\lambda \in \Lambda} \varphi$ *implies*

$$\varphi(x, y) = \sup_{\lambda \in \Lambda} \varphi_\lambda(x, y) \qquad \textit{for} \quad 0 \leqq x \in R, \ 0 \leqq y \in S,$$

and $\widetilde{R,S} \ni \varphi_\lambda \downarrow_{\lambda \in \Lambda} \varphi$ *implies*

$$\varphi(x, y) = \inf_{\lambda \in \Lambda} \varphi_\lambda(x, y) \qquad \textit{for} \quad 0 \leqq x \in R, \ 0 \leqq y \in S.$$

Proof. For $0 \leqq x \in R$, $0 \leqq y \in S$, putting

$$\omega(x, y) = \sup \sum_{\nu, \mu} \mathrm{Max}\, \{\varphi(x_\nu, y_\mu), \ \psi(x_\nu, y_\mu)\}$$

for $x = \sum_{\nu=1}^{\kappa} x_\nu$, $y = \sum_{\mu=1}^{\rho} y_\mu$, $0 \leqq x_\nu \in R$, $0 \leqq y_\mu \in S$, $\kappa, \rho = 1, 2, \cdots$, we have obviously for every $\alpha \geqq 0$

$$\omega(\alpha x, y) = \omega(x, \alpha y) = \alpha \omega(x, y).$$

For $0 \leqq a, b \in R$, if $a + b = \sum_{\nu=1}^{\kappa} c_\nu$, $0 \leqq c_\nu \in R$, then we can find $0 \leqq a_\nu$, $b_\nu \in R$ $(\nu = 1, 2, \cdots, \kappa)$ such that

$$a = \sum_{\nu=1}^{\kappa} a_\nu, \quad b = \sum_{\nu=1}^{\kappa} b_\nu, \quad c_\nu = a_\nu + b_\nu \text{ [9]}$$

and for $y = \sum_{\mu=1}^{\rho} y_\mu$, $0 \leqq y_\mu \in S$ we have

$$\begin{aligned} &\sum_{\nu, \mu} \mathrm{Max}\, \{\varphi(c_\nu, y_\mu), \ \psi(c_\nu, y_\mu)\} \\ &\leqq \sum_{\nu, \mu} \mathrm{Max}\, \{\varphi(a_\nu, y_\mu), \ \psi(a_\nu, y_\mu)\} + \sum_{\nu, \mu} \mathrm{Max}\, \{\varphi(b_\nu, y_\mu), \ \psi(b_\nu, y_\mu)\} \\ &\leqq \omega(a, y) + \omega(b, y). \end{aligned}$$

Thus we obtain $\omega(a + b, y) \leqq \omega(a, y) + \omega(b, y)$. On the other hand, if

$$a = \sum_{\nu=1}^{\kappa} a_\nu, \quad b = \sum_{\lambda=1}^{\kappa'} b_\lambda, \quad 0 \leqq a_\nu, \ b_\lambda \in R,$$

$$y = \sum_{\mu=1}^{\rho} y_\mu = \sum_{\eta=1}^{\rho'} y'_\eta, \quad 0 \leqq y_\mu, \ y'_\eta \in S,$$

then we can find $0 \leqq z_{\mu, \eta} \in S$ such that

$$y_\mu = \sum_{\eta=1}^{\rho'} z_{\mu, \eta}, \quad y'_\eta = \sum_{\mu=1}^{\rho} z_{\mu, \eta},$$

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9) Putting $a_1 = a \frown c_1$, $a_\mu = a \frown \left(\sum_{\nu=1}^{\mu} c_\nu \right) - a \frown \left(\sum_{\nu=1}^{\mu-1} c_\nu \right)$ $(\mu = 2, 3, \cdots, \kappa)$, $b_\nu = c_\nu - a_\nu$, we have obviously $\sum_{\mu=1}^{\kappa} a_\mu = a \frown \left(\sum_{\nu=1}^{\kappa} c_\nu \right) = a$, $\sum_{\nu=1}^{\kappa} b_\nu = b$.

and we have

$$\sum_{\nu,\mu}\mathrm{Max}\,\{\varphi(a_\nu,y_\mu),\ \psi(a_\nu,y_\mu)\}+\sum_{\lambda,\gamma}\mathrm{Max}\,\{\varphi(b_\lambda,y'_\gamma),\ \psi(b_\lambda,y'_\gamma)\}$$
$$\leqq\sum_{\nu,\mu,\gamma}\mathrm{Max}\,\{\varphi(a_\nu,z_{\mu,\gamma}),\ \psi(a_\nu,z_{\mu,\gamma})\}$$
$$+\sum_{\lambda,\gamma,\mu}\mathrm{Max}\,\{\varphi(b_\lambda,z_{\mu,\gamma}),\ \psi(b_\lambda,z_{\mu,\gamma})\}\leqq\omega(a+b,y)\ .$$

Therefore we obtain $\omega(a,y)+\omega(b,y)\leqq\omega(a+b,y)$. Consequently we have

$$\omega(a,y)+\omega(b,y)=\omega(a+b,y)\ .$$

for $0\leqq a,b\in R$, $0\leqq y\in S$. We also can prove likewise

$$\omega(x,a)+\omega(x,b)=\omega(x,a+b)\quad .$$

for $0\leqq x\in R$, $0\leqq a,b\in S$.

For arbitrary elements $x\in R$, $y\in S$, putting

$$\omega(x,y)=\omega(x^+,y^+)+\omega(x^-,y^-)-\omega(x^+,y^-)-\omega(x^-,y^+)\ ,$$

we see easily that ω is a bilinear functional on R and S. Furthermore it is evident that $\omega\geqq\varphi$ and $\omega\geqq\psi$. For a bilinear functional $\pi\in\widetilde{R,S}$, if $\pi\geqq\varphi$ and $\pi\geqq\psi$, then for

$$x=\sum_{\nu=1}^{\kappa}x_\nu,\ y=\sum_{\mu=1}^{\rho}y_\mu,\quad 0\leqq x_\nu\in R,\ 0\leqq y_\mu\in S$$

we have obviously

$$\sum_{\nu,\mu}\mathrm{Max}\,\{\varphi(x_\nu,y_\mu),\ \psi(x_\nu,y_\mu)\}\leqq\sum_{\nu,\mu}\pi(x_\nu,y_\mu)=\pi(x,y)\ ,$$

and hence $\omega(x,y)\leqq\pi(x,y)$ for $0\leqq x\in R$, $0\leqq y\in S$. Thus we conclude $\omega=\varphi\smile\psi$. Therefore the biassociated space $\widetilde{R,S}$ is a lattice ordered linear space. Since

$$\varphi\frown\psi=-\{(-\varphi)\smile(-\psi)\}\ ,$$

we obtain further

$$\varphi\frown\psi(x,y)=\inf\sum_{\nu,\mu}\mathrm{Min}\,\{\varphi(x_\nu,y_\mu),\ \psi(x_\nu,y_\mu)\}$$

for $x=\sum_{\nu=1}^{\kappa}x_\nu$, $y=\sum_{\mu=1}^{\rho}y_\mu$, $0\leqq x_\nu\in R$, $0\leqq y_\mu\in S$.

If $\widetilde{R,S}\ni\varphi_\lambda\uparrow_{\lambda\in\Lambda}$ and $\sup_{\lambda\in\Lambda}\varphi_\lambda(x,y)<+\infty$ for every $x\in R$, $y\in S$, then, putting

$$\psi(x,y)=\sup_{\lambda\in\Lambda}\varphi_\lambda(x,y)\qquad\text{for }0\leqq x\in R,\ 0\leqq y\in S,$$

we see easily that for $\alpha\geqq0$ we have

$$\psi(\alpha x, y) = \psi(x, \alpha y) = \alpha\psi(x, y).$$

Furthermore, as $\varphi_\lambda \uparrow_{\lambda \in \Lambda}$, we have for $0 \leqq a, b \in R$, $0 \leqq y \in S$

$$\psi(a+b, y) = \sup_{\lambda \in \Lambda} \varphi_\lambda(a+b, y)$$
$$= \sup_{\lambda \in \Lambda} \varphi_\lambda(a, y) + \sup_{\lambda \in \Lambda} \varphi_\lambda(b, y) = \psi(a, y) + \psi(b, y).$$

We also obtain likewise for $0 \leqq x \in R$, $0 \leqq a, b \in S$

$$\psi(x, a+b) = \psi(x, a) + \psi(x, b).$$

Thus, putting

$$\psi(x,y) = \psi(x^+, y^+) + \psi(x^-, y^-) - \psi(x^+, y^-) - \psi(x^-, y^+)$$

for arbitrary $x \in R$ and $y \in S$, we see easily that ψ is a bilinear functional on R and S. Furthermore, if $\varphi_\lambda \geqq 0$ for every $\lambda \in \Lambda$, then we have obviously $\psi \geqq 0$ and hence $\psi \in \widetilde{R,S}$. Therefore $\widetilde{R,S}$ is universally continuous, and $\varphi_\lambda \uparrow_{\lambda \in \Lambda} \varphi$ implies $\varphi = \psi$. Since $\varphi_\lambda \downarrow_{\lambda \in \Lambda} \varphi$ is equivalent to $-\varphi_\lambda \uparrow_{\lambda \in \Lambda} -\varphi$, we conclude that $\varphi_\lambda \downarrow_{\lambda \in \Lambda} \varphi$ implies

$$\varphi(x, y) = \inf_{\lambda \in \Lambda} \varphi_\lambda(x, y) \qquad \textit{for} \quad 0 \leqq x \in R, \ 0 \leqq y \in S.$$

Therefore we also obtain

Theorem 1.2. *If* $0 \leqq \varphi_\lambda \uparrow_{\lambda \in \Lambda}$, $\varphi_\lambda \in \widetilde{R,S}$ $(\lambda \in \Lambda)$, *and*

$$\sup_{\lambda \in \Lambda} \varphi_\lambda(x, y) < +\infty \qquad \textit{for} \quad 0 \leqq x \in R, \ 0 \leqq y \in S,$$

then there exists $\varphi \in \widetilde{R,S}$ *such that* $\varphi_\lambda \uparrow_{\lambda \in \Lambda} \varphi$.

Let M be a normal manifold of R. For each $\varphi \in \widetilde{R,S}$ we define $\varphi[M]$ to mean

$$\varphi[M](x,y) = \varphi([M]x, y) \quad \text{for} \quad x \in R, \ y \in S.$$

Then we see easily that $\varphi[M] \in \widetilde{R,S}$ for every $\varphi \in \widetilde{R,S}$.
Recalling Theorem 1.1 we obtain immediately by definition

Theorem 1.3. *For a normal manifold M of R we have*

$$(\varphi \smile \psi)[M] = \varphi[M] \smile \psi[M],$$
$$(\varphi \frown \psi)[M] = \varphi[M] \frown \psi[M]$$

for every $\varphi, \psi \in \widetilde{R,S}$.

Theorem 1.4. *For* $0 \leqq \varphi$, $\psi \in \widetilde{R,S}$ *we have*

$$\varphi[M] \frown \psi = \varphi \frown \psi[M] = (\varphi \frown \psi)[M].$$

Proof. By virtue of Theorem 1.1, we have for $0 \leqq x \in R$, $0 \leqq y \in S$
$(\varphi[M]_\frown \psi(1-[M]))(x, y)$

$$\leqq \mathrm{Min}\,\{\varphi[M]([M]x, y),\ \psi(1-[M])([M]x, y)\}$$
$$+\ \mathrm{Min}\,\{\varphi[M]((1-[M])x, y),\ \psi(1-[M])((1-[M])x, y)\} = 0\,,$$

because $\varphi, \psi \geqq 0$ by assumption. Thus we have

$$\varphi[M]_\frown \psi(1-[M]) = 0\,.$$

As $\psi = \psi[M] + \psi(1-[M])$, we conclude hence by Theorem 1.3

$$\varphi[M]_\frown \psi = \varphi[M]_\frown \psi[M] = (\varphi_\frown \psi)[M]\,.$$

For an arbitrary subset $\tilde{K} \subset \widetilde{R,S}$, we make use of the notation:

$$\tilde{K}[M] = \{\varphi[M]:\ \varphi \in \tilde{K}\}$$

for a normal manifold M of R. With this notation we have[10].

Theorem 1.5. *For every subset* $\tilde{K} \subset \widetilde{R,S}$, *we have*

$$[\tilde{K}](\varphi[M]) = ([\tilde{K}]\varphi)[M] = [\tilde{K}[M]]\varphi$$

for every $\varphi \in \widetilde{R,S}$ *and for every normal manifold* M *of* R.

For a normal manifold N of S, we define likewise $\varphi[N]$ to mean

$$\varphi[N](x, y) = \varphi(x, [N]y) \qquad \text{for} \quad x \in R,\ y \in S,$$

and it is evident that the similar results hold for $\varphi[N]$.

Let $\bar{R}$ and $\bar{S}$ be the associated spaces of R and S respectively. For every $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$, putting

$$\varphi(x, y) = \bar{x}(x)\bar{y}(y) \qquad \text{for} \quad x \in R,\ y \in S,$$

we obtain obviously $\varphi \in \widetilde{R,S}$. This φ will be denoted by $\bar{x}\bar{y}$. Then we have obviously for every real number α

$$(\alpha\bar{x})\bar{y} = \bar{x}(\alpha\bar{y}) = \alpha\bar{x}\bar{y}\,.$$

We see easily further

$$(\bar{x}_1 + \bar{x}_2)\bar{y} = \bar{x}_1\bar{y} + \bar{x}_2\bar{y}\,,$$
$$\bar{x}(\bar{y}_1 + \bar{y}_2) = \bar{x}\bar{y}_1 + \bar{x}\bar{y}_2\,,$$

and $0 \leqq \bar{x} \in \bar{R}$, $0 \leqq \bar{y} \in \bar{S}$ implies $\bar{x}\bar{y} \geqq 0$.

Theorem 1.6. $\bar{x}_1{}_\frown \bar{x}_2 = 0$, $\bar{y}_1, \bar{y}_2 \geqq 0$ *implies*

$$\bar{x}_1\bar{y}_1{}_\frown \bar{x}_2\bar{y}_2 = 0\,,$$

10) c. f. MSLS §18 (16).

$\tilde{x}_1, \tilde{x}_2 \geqq 0$, $\tilde{y}_1 \frown \tilde{y}_2 = 0$ *implies*

$$\tilde{x}_1\tilde{y}_1 \frown \tilde{x}_2\tilde{y}_2 = 0 .$$

Proof. If $\tilde{x}_1 \frown \tilde{x}_2 = 0$, $\tilde{y}_1, \tilde{y}_2 \geqq 0$, then we have by Theorem 1.1 for $0 \leqq a \in R$, $0 \leqq b \in S$.

$$\begin{aligned} &(\tilde{x}_1\tilde{y}_1 \frown \tilde{x}_2\tilde{y}_2)(a, b) \\ \leqq &\inf_{0 \leqq x \leqq a} \{\tilde{x}_1(x)\tilde{y}_1(b) + \tilde{x}_2(a-x)\tilde{y}_2(b)\} \\ \leqq &(\tilde{y}_1(b) + \tilde{y}_2(b)) \inf_{0 \leqq x \leqq a} \{\tilde{x}_1(x) + \tilde{x}_2(a-x)\} = 0 , \end{aligned}$$

because $(\tilde{x}_1 \frown \tilde{x}_2)(a) = \inf_{0 \leqq x \leqq a} \{\tilde{x}_1(x) + \tilde{x}_2(a-x)\}$. Thus we obtain $\tilde{x}_1\tilde{y}_1 \frown \tilde{x}_2\tilde{y}_2 = 0$. We also can prove likewise the other assertion.

Theorem 1.7. *For* $\tilde{x} \in \tilde{R}$, $\tilde{y} \in \tilde{S}$, *we have*

$$\begin{aligned} (\tilde{x}\tilde{y})^+ &= \tilde{x}^+\tilde{y}^+ + \tilde{x}^-\tilde{y}^- , \\ (\tilde{x}\tilde{y})^- &= \tilde{x}^+\tilde{y}^- + \tilde{x}^-\tilde{y}^+ , \\ |\tilde{x}\tilde{y}| &= |\tilde{x}|\,|\tilde{y}| . \end{aligned}$$

Proof. We have obviously

$$\tilde{x}\tilde{y} = \tilde{x}^+\tilde{y}^+ + \tilde{x}^-\tilde{y}^- - (\tilde{x}^+\tilde{y}^- + \tilde{x}^-\tilde{y}^+) ,$$

and we conclude by Theorem 1.6

$$(\tilde{x}^+\tilde{y}^+ + \tilde{x}^-\tilde{y}^-) \frown (\tilde{x}^+\tilde{y}^- + \tilde{x}^-\tilde{y}^+) = 0 .$$

Thus we obtain

$$(\tilde{x}\tilde{y})^+ = \tilde{x}^+\tilde{y}^+ + \tilde{x}^-\tilde{y}^-, \quad (\tilde{x}\tilde{y})^- = \tilde{x}^+\tilde{y}^- + \tilde{x}^-\tilde{y}^+ ,$$

and hence

$$|\tilde{x}\tilde{y}| = \tilde{x}^+\tilde{y}^+ + \tilde{x}^-\tilde{y}^- + \tilde{x}^+\tilde{y}^- + \tilde{x}^-\tilde{y}^+ = |\tilde{x}|\,|\tilde{y}| .$$

Theorem 1.8. *For* $\tilde{x}_\lambda \in \tilde{R}$ $(\lambda \in \Lambda)$, $0 \leqq \tilde{y} \in \tilde{S}$ *we have*

$$\begin{aligned} \bigcup_{\lambda \in \Lambda} \tilde{x}_\lambda \tilde{y} &= (\bigcup_{\lambda \in \Lambda} \tilde{x}_\lambda)\tilde{y} , \\ \bigcap_{\lambda \in \Lambda} \tilde{x}_\lambda \tilde{y} &= (\bigcap_{\lambda \in \Lambda} \tilde{x}_\lambda)\tilde{y} , \end{aligned}$$

if the right side has any sense. For $0 \leqq \tilde{x} \in \tilde{R}$, $\tilde{y}_\lambda \in \tilde{S}$ $(\lambda \in \Lambda)$ *we have*

$$\begin{aligned} \bigcup_{\lambda \in \Lambda} \tilde{x}\tilde{y}_\lambda &= \tilde{x}(\bigcup_{\lambda \in \Lambda} \tilde{y}_\lambda) , \\ \bigcap_{\lambda \in \Lambda} \tilde{x}\tilde{y}_\lambda &= \tilde{x}(\bigcap_{\lambda \in \Lambda} \tilde{y}_\lambda) , \end{aligned}$$

if the right side has any sense.

Proof. For $\tilde{x}_1, \tilde{x}_2 \in \tilde{R}$, $0 \leqq \tilde{y} \in \tilde{S}$, we have by Theorem 1.7

$$\bar{x}_1\bar{y}\smile\bar{x}_2\bar{y} = (\bar{x}_1\bar{y}-\bar{x}_2\bar{y})^+ + \bar{x}_2\bar{y}$$
$$= (\bar{x}_1-\bar{x}_2)^+\bar{y} + \bar{x}_2\bar{y} = (\bar{x}_1\smile\bar{x}_2)\,\bar{y}\,.$$

If $\bar{R}\ni\bar{x}_\lambda\uparrow_{\lambda\in\Lambda}\bar{x}$, then we have

$$\bar{x}(x) = \sup_{\lambda\in\Lambda}\bar{x}_\lambda(x) \qquad \text{for} \quad 0\leqq x\in R,$$

and hence by definition for $0\leqq\bar{y}\in\tilde{S}$

$$\bar{x}\bar{y}(x, y) = \sup_{\lambda\in\Lambda}\bar{x}_\lambda\bar{y}(x, y) \qquad \text{for} \quad 0\leqq x\in R,\ 0\leqq y\in S.$$

Thus we have $\bar{x}\bar{y} = \bigcup_{\lambda\in\Lambda}\bar{x}_\lambda\bar{y}$. Therefore we conclude the first assertion. The second will be proved likewise.

Theorem 1.9. $[\bar{x}_1]\leqq[\bar{x}_2]$, $[\bar{y}_1]\leqq[\bar{y}_2]$ *implies*

$$[\bar{x}_1\bar{y}_1]\leqq[\bar{x}_2\bar{y}_2]\,.$$

Proof, If $[\bar{x}_1]\leqq[\bar{x}_2]$, then we have

$$|\bar{x}_1| = [\bar{x}_2]\,|\bar{x}_1| = \bigcup_{\nu=1}^{\infty}(|\bar{x}_1|\frown\nu|\bar{x}_2|)\,.$$

Thus we obtain by Theorems 1.7 and 1.8

$$[\bar{x}_2\bar{y}_1]\,|\bar{x}_1\bar{y}_1| = \bigcup_{\nu=1}^{\infty}(|\bar{x}_1\bar{y}_1|\frown\nu|\bar{x}_2\bar{y}_1|)$$
$$= (\bigcup_{\nu=1}^{\infty}(|\bar{x}_1|\frown\nu|\bar{x}_2|))\,|\bar{y}_1| = |\bar{x}_1\bar{y}_1|\,,$$

and hence $[\bar{x}_1\bar{y}_1]\leqq[\bar{x}_2\bar{y}_1]$. Furthermore we also obtain likewise $[\bar{x}_2\bar{y}_1]\leqq[\bar{x}_2\bar{y}_2]$.

Theorem 1.10. *For* $\bar{x}_1, \bar{x}_2\in\bar{R}$; $\bar{y}_1, \bar{y}_2\in\tilde{S}$ *we have*

$$[\bar{x}_1\bar{y}_1]\,\bar{x}_2\bar{y}_2 = ([\bar{x}_1]x_2)\,([\bar{y}_1]\,\bar{y}_2)\,.$$

Proof. We have obviously

$$\bar{x}_2\bar{y}_2 = ([\bar{x}_1]\bar{x}_2)\,([\bar{y}_1]\bar{y}_2) + ((1-[\bar{x}_1])\bar{x}_2)\,([\bar{y}_1]\,\bar{y}_2) + \bar{x}_2((1-[\bar{y}_1])\bar{y}_2)\,,$$

and by Theorems 1.6 and 1.7

$$|\bar{x}_1\bar{y}_1|\frown|((1-[\bar{x}_1])\,\bar{x}_2)\,([\bar{y}_1]\,\bar{y}_2)| = 0\,,$$
$$|\bar{x}_1\bar{y}_1|\frown|\bar{x}_2((1-[\bar{y}_1])\bar{y}_2)| = 0\,.$$

Thus we obtain

$$[\bar{x}_1\bar{y}_1]\,\bar{x}_2\bar{y}_2 = [\bar{x}_1\bar{y}_1]([\bar{x}_1]\bar{x}_2)\,([\bar{y}_1]\bar{y}_2)\,.$$

On the other hand we have by Theorem 1.9

$$[([\bar{x}_1]x_2)\,([\bar{y}_1]y_2)]\leqq[\bar{x}_1\bar{y}_1]\,,$$

and hence

$$[\bar{x}_1\bar{y}_1]([\bar{x}_1]\bar{x}_2)([\bar{y}_1]\bar{y}_2) = ([\bar{x}_1]\bar{x}_2)([\bar{y}_1]\bar{y}_2) .$$

§ 2. Bounded linear operators

A mapping T of R into S is called an *operator* on R into S. An operator T on R is said to be *linear*, if

$$T(\alpha x+\beta y) = \alpha Tx+\beta Ty$$

for every $x, y \in R$ and real number α, β. In the sequel, we suppose that the semi-ordered linear space S is universally continuous.

A linear operator T on R into S is said to be *bounded*, if for every $0 \leqq a \in R$, $Tx\ (0 \leqq x \leqq a)$ is order bounded, that is, there exists

$$\bigcup_{0 \leqq x \leqq a} |Tx| .$$

For two linear operators T_1 and T_2 on R into S and real numbers α, β, putting

$$(\alpha T_1+\beta T_2)x = \alpha T_1 x+\beta T_2 x \qquad \text{for every } x \in R,$$

we obtain a linear operator $\alpha T_1+\beta T_2$ on R into S. Furthermore, if both T_1 and T_2 are bounded, then $\alpha T_1+\beta T_2$ also is bounded for every real numbers α, β.

For two linear operators T_1, T_2 on R into S, we define $T_1 \geqq T_2$ to mean $T_1 x \geqq T_2 x$ for every $0 \leqq x \in R$. With this definition we have

Theorem 2.1. *The totality of bounded linear operators on R into S constitutes a universally continuous semi-ordered linear space, and we have for* $0 \leqq a \in R$

$$(T_1 \smile T_2)\, a = \bigcup_{0 \leqq x \leqq a} (T_1 x+T_2(a-x)),$$

$$(T_1 \frown T_2)\, a = \bigcap_{0 \leqq x \leqq a} (T_1 x+T_2(a-x)) ,$$

$$|T|\, a = \bigcup_{0 \leqq x \leqq a} |Tx| ,$$

$$T_\lambda \uparrow_{\lambda \in \Lambda} T \quad \textit{implies} \quad Tx = \bigcup_{\lambda \in \Lambda} T_\lambda x \qquad \textit{for} \quad 0 \leqq x \in R,$$

$$T_\lambda \downarrow_{\lambda \in \Lambda} T \quad \textit{implies} \quad Tx = \bigcap_{\lambda \in \Lambda} T_\lambda x \qquad \textit{for} \quad 0 \leqq x \in R.$$

Proof. It is evident that the totality of bounded linear operators on R into S constitutes a linear space. Putting

$$T_0 a = \bigcup_{0 \leqq x \leqq a} (T_1 a+T_2 (a-x)) \qquad \text{for} \quad 0 \leqq a \in R,$$

we have for every real number $\alpha \geqq 0$

$$T_0\alpha a = \bigcup_{0 \leqq x \leqq \alpha a} (T_1 x + T_2(\alpha a - x)) = \alpha T_0 a\,,$$

and for $0 \leqq a, b \in R$

$$\begin{aligned} T_0(a+b) &= \bigcup_{0 \leqq x \leqq a+b} (T_1 x + T_2(a+b-x)) \\ &= \bigcup_{0 \leqq x \leqq a, 0 \leqq y \leqq b} (T_1(x+y) + T_2(a+b-x-y)) \\ &= \bigcup_{0 \leqq x \leqq a} (T_1 x + T_2(a-x)) + \bigcup_{0 \leqq y \leqq b} (T_1 y + T_2(b-y)) \\ &= T_0 a + T_0 b\,. \end{aligned}$$

Thus, putting

$$T_0 x = T_0 x^+ - T_0 x^-$$

for an arbitrary $x \in R$, we see easily that T_0 is a linear operator on R into S. Furthermore T_0 is bounded, because for $0 \leqq x \leqq a$ we have

$$|T_0 x| \leqq \bigcup_{0 \leqq x \leqq a} |T_1 x| + \bigcup_{0 \leqq x \leqq a} |T_2 x|\,.$$

It is evident that $T_0 \geqq T_1, T_2$. If $T \geqq T_1, T_2$, then we have for $0 \leqq x \leqq a$

$$T_1 x + T_2(a-x) \leqq Tx + T(a-x) = Ta\,,$$

and hence $T_0 a \leqq Ta$ for $0 \leqq a \in R$. Therefore we have $T_0 = T_1 \smile T_2$. Since

$$T_1 \frown T_2 = -((-T_1) \smile (-T_2))',$$

we obtain further for $0 \leqq a \in R$

$$\begin{aligned} T_1 \frown T_2 a &= \bigcap_{0 \leqq x \leqq a} (T_1 x + T_2(a-x))\,, \\ |T|\,a &= \bigcup_{0 \leqq x \leqq a} T(2x-a) = \bigcup_{-a \leqq x \leqq a} Tx \\ &= \bigcup_{0 \leqq x \leqq a} (Tx \smile (-Tx)) = \bigcup_{0 \leqq x \leqq a} |Tx|\,. \end{aligned}$$

If $T_\lambda \uparrow_{\lambda \in \Lambda}$ and $T_\lambda x$ $(\lambda \in \Lambda)$ is bounded for every $0 \leqq x \in R$, then, putting

$$T_0 x = \bigcup_{\lambda \in \Lambda} T_\lambda x \qquad \text{for} \quad 0 \leqq x \in R,$$

$$T_0 x = T_0 x^+ - T_0 x^- \qquad \text{for an arbitrary } x \in R,$$

we see easily that T_0 is a linear operator on R into S. Furthermore, if $T_\lambda \geqq 0$ $(\lambda \in \Lambda)$, then it is evident that $T_0 \geqq 0$, and hence T_0 is bounded. Therefore the totality of bounded linear operators on R into S constitutes a universally continuous linear space, and we have obviously that $T_\lambda \uparrow_{\lambda \in \Lambda} T$ implies

$$Tx = \bigcup_{\lambda \in \Lambda} T_\lambda x \qquad \text{for} \quad 0 \leqq x \in R,$$

We also have proved

Theorem 2.2. *If* $0 \leqq T_\lambda \downarrow_{\lambda \in \Lambda}$ *and* $T_\lambda x (\lambda \in \Lambda)$ *is bounded for every* $0 \leqq x \in R$, *then there exists* T *for which* $T_\lambda \uparrow_{\lambda \in \Lambda} T$.

Now we suppose that both R and S are universally continuous. A linear operator T on R into S is said to be *universally continuous*, if T is bounded and $R \ni x_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies $|T| x_\lambda \downarrow_{\lambda \in \Lambda} 0$. With this definition, it is evident that if $T \geqq 0$ and T is universally continuous, then $x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $Tx = \bigcup_{\lambda \in \Lambda} Tx_\lambda$, and $x_\lambda \downarrow_{\lambda \in \Lambda} x$ implies $Tx = \bigcap_{\lambda \in \Lambda} Tx_\lambda$.

If both T_1 and T_2 are universally continuous, then $\alpha T_1 + \beta T_2$ also is universally continuous for every real numbers α, β. If T_1 is universally continuous and $|T_1| \geqq |T_2|$, then T_2 also is universally continuous. Thus the totality of universally continuous linear operators is obviously a semi-normal manifold of the totality of bounded linear operators.

Theorem 2.3. *If* $0 \leqq T_\lambda \uparrow_{\lambda \in \Lambda} T$ *and all* T_λ $(\lambda \in \Lambda)$ *are universally continuous, then* T *also is universally continuous.*

Proof. We suppose $R \ni a_\gamma \downarrow_{\gamma \in \Gamma} 0$. For each $\gamma_0 \in \Gamma$, we have by Theorem 2.1

$$T_\lambda a_{\gamma_0} \uparrow_{\lambda \in \Lambda} T a_{\gamma_0},$$

and hence

$$(T - T_\lambda) a_{\gamma_0} \downarrow_{\lambda \in \Lambda} 0 .$$

On the other hand, we have for $a_\gamma \leqq a_{\gamma_0}$

$$T a_\gamma = T_\lambda a_\gamma + (T - T_\lambda) a_\gamma \leqq T_\lambda a_\gamma + (T - T_\lambda) a_{\gamma_0} .$$

Since all $T_\lambda (\lambda \in \Lambda)$ are universally continuous by assumption, we obtain hence

$$\bigcap_{\gamma \in \Gamma} T a_\gamma \leqq (T - T_\lambda) a_{\gamma_0} \qquad \text{for every} \quad \lambda \in \Lambda .$$

Consequently we have $\bigcap_{\gamma \in \Gamma} T a_\gamma = 0$. Therefore T is universally continuous by definition.

By virtue of this Theorem 2.3, we obtain

Theorem 2.4. *The totality of universally continuous linear operators is a normal manifold of the totality of bounded linear operators.*

Let A be a complete semi-normal manifold of R. Each operator T on R into S also may be considered as an operator T^A on A into S, that is,

$$T^A x = Tx \qquad \text{for every} \quad x \in A .$$

If T is universally continuous, then T^A also is obviously universally continuous. It is evident that $T\geqq 0$ implies $T^A\geqq 0$. Conversely, if $T^A\geqq 0$ and T is universally continuous, then we have $T\geqq 0$. Because, for every $0\leqq x\in R$, we have $a\uparrow_{x\geqq a\in A} x$, and hence

$$Tx = \bigcup_{x\geqq a\in A} Ta = \bigcup_{x\geqq a\in A} T^A a \geqq 0 .$$

Let F be a positive linear operator on A into S. If we can find a universally continuous linear operator T_0 on R into S such that $T_0^A\geqq F$, then there exists a linear operator T on R into S such that $T^A=F$. Because for every $0\leqq x\in R$, we have

$$T_0 x = \bigcup_{x\geqq a\in A} T_0^A a \geqq \bigcup_{x\geqq a\in A} Fa ,$$

and hence, putting

$$Tx = \bigcup_{x\geqq a\in A} Fa \qquad \text{for} \quad 0\leqq x\in R ,$$

$$Tx = Tx^+ - Tx^- \qquad \text{for an arbitrary } x\in R ,$$

we see easily that T is a linear operator on R into S and $T^A=F$. Thus we can state

Theorem 2.5. *For a complete semi-normal manifold A of R, the totality of universally continuous linear operators on R into S is isomorphic to a semi-normal manifold of the totality of bounded linear operators on A into S by the correspondence $T\rightarrow T^A$.*

Next we suppose that both R and S are semi-regular, and $\bar{R}$, $\bar{S}$ are the conjugate spaces of R and S respectively. If a linear operator T on R into S is universally continuous, then for each $\bar{y}\in\bar{S}$, putting

$$\varphi(x) = \bar{y}(Tx) \qquad \text{for} \quad x\in R ,$$

we obtain a universally continuous linear functional φ on R. In fact, it is obvious that φ is a linear functional on R. As

$$|\varphi(x)| \leqq |\bar{y}|(|T|\,|x|) \qquad \text{for every } x\in R ,$$

$R\ni x_\lambda \downarrow_{\lambda\in\Lambda} 0$ implies $\inf_{\lambda\in\Lambda} |\varphi(x)|=0$, and hence φ is universally continuous, that is, $\varphi\in\bar{R}$. Therefore, putting $\bar{T}\bar{y}=\varphi$, namely

$$\bar{T}\bar{y}(x) = \bar{y}(Tx) \qquad \text{for every } x\in R ,$$

we obtain an operator $\bar{T}$ on $\bar{S}$ into $\bar{R}$. This operator $\bar{T}$ is called the *conjugate operator* of T. The conjugate operator $\bar{T}$ is obviously a linear operator. As

$$|\bar{T}\bar{y}(x)| \leqq |\bar{y}|(|T|\,|x|) = (\overline{|T|})|\bar{y}|(|x|) ,$$

we obtain $|\overline{T}\bar{y}| \leqq (\overline{|T|})|\bar{y}|$, and hence $\overline{T}$ is bounded. Furthermore we conclude by Theorem 2.1.

$$|\overline{T}| \leqq (\overline{|T|}),$$

and $\bar{S} \ni \bar{y}_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies for every $0 \leqq x \in R$

$$(\overline{|T|})\bar{y}_\lambda(x) = \bar{y}_\lambda(|T|x) \downarrow_{\lambda \in \Lambda} 0,$$

and consequently $\overline{|T|}\,\bar{y}_\lambda \downarrow_{\lambda \in \Lambda} 0$. Therefore $\overline{T}$ is universally continuous.

Concerning conjugate operators, we have obviously

$$\overline{\alpha T} = \alpha \overline{T},$$

$$\overline{T_1 + T_2} = \overline{T}_1 + \overline{T}_2,$$

$$\overline{T}_1 \geqq \overline{T}_2 \quad \text{if and only if} \quad T_1 \geqq T_2.$$

Thus we have $\overline{T_1 \smile T_2} \geqq \overline{T}_1, \overline{T}_2$, and hence

$$\overline{T_1 \smile T_2} \geqq \overline{T}_1 \smile \overline{T}_2.$$

From this relation we conclude immediately

$$\overline{T_1 \frown T_2} \leqq \overline{T}_1 \frown \overline{T}.$$

Theorem 2.6. *If R is semi-regular and S is reflexive, then for any universally continuous linear operator K on $\bar{S}$ into $\bar{R}$, we can find uniquely a universally continuous linear operator T on R into S such that $K = \overline{T}$.*

Proof. Since S is reflexive by assumption, corresponding to every $x \in R$, we can find uniquely $y \in S$ such that

$$K\bar{y}(x) = \bar{y}(y) \qquad \text{for every } \bar{y} \in \bar{S},$$

because $K\bar{y}(x)$ $(\bar{y} \in \bar{S})$ may be considered as a universally continuous linear functional on $\underline{S}$. Putting $y = Tx$ for such $x \in R$, $y \in S$, we obtain a universally continuous linear operator T on R into S, and it is obvious that $K = \overline{T}$.

Theorem 2.7. *If R is semi-regular and S is reflexive, then we have*

$$\overline{T_1 \smile T_2} = \overline{T}_1 \smile \overline{T}_2, \qquad \overline{T_1 \frown T_2} = \overline{T}_1 \frown \overline{T}_2.$$

Proof. By virtue of Theorem 2.6, we can find a universally continuous linear operator T_0 on R into S such that $\overline{T}_1 \smile \overline{T}_2 = \overline{T}_0$. Then we have $\overline{T}_1, \overline{T}_2 \leqq \overline{T}_0$, and hence $T_1 \smile T_2 \leqq T_0$. This relation yields

$$\overline{T_1 \smile T_2} \leqq \overline{T}_1 \smile \overline{T}_2.$$

On the other hand, we have $\overline{T_1 \smile T_2} \geqq \overline{T}_1 \smile \overline{T}_2$, as proved already. Thus we obtain the first assertion. We conclude the second assertion immediately from the first.

For $\bar{x}_0 \in \bar{R}$, $y_0 \in S$, putting

$$Tx = \bar{x}_0(x) y_0 \qquad \text{for every } x \in R,$$

we obtain obviously a universally continuous linear operator T on R into S. For this operator T, we have for every $\bar{y} \in \bar{S}$

$$\bar{y}(Tx) = \bar{x}_0(x)\bar{y}(y_0) = (\bar{y}(y_0)\bar{x}_0)(x).$$

Thus we have

$$\bar{T}\bar{y} = \bar{y}(y_0)\bar{x}_0. \qquad \text{for every } \bar{y} \in \bar{S}.$$

§ 3. Universally continuous bilinear functionals

Let φ be a bilinear functional on R and S. For each $a \in R$, putting

$$\varphi^{(a)}(y) = \varphi(a, y) \qquad \text{for every } y \in S,$$

we obtain obviously a linear functional $\varphi^{(a)}$ on S. If φ is bounded, then $\varphi^{(a)}$ also is bounded. Because, for $0 \leqq b \in S$, we have by Theorem 1.1 for $0 \leqq y \leqq b$

$$\begin{aligned} |\varphi^{(a)}(y)| &\leqq |\varphi(a^+, y)| + |\varphi(a^-, y)| \\ &\leqq \{|\varphi(a^+, y)| + |\varphi(a^+, b-y)|\} + \{|\varphi(a^-, y)| + |\varphi(a^-, b-y|\} \\ &\leqq |\varphi|(a^+, b) + |\varphi|(a^-, b). \end{aligned}$$

Therefore $\varphi \in \widetilde{R,S}$ implies $\varphi^{(a)} \in \tilde{S}$ for every $a \in R$, and

$$|\varphi^{(a)}| \leqq |\varphi|^{(a)} \qquad \text{for} \quad 0 \leqq a \in R.$$

Furthermore it is evident by definition that

$$\varphi^{(\alpha a + \beta b)} = \alpha\varphi^{(a)} + \beta\varphi^{(b)}.$$

Thus, putting

$$T_\varphi x = \varphi^{(x)} \qquad \text{for exery } x \in R,$$

we obtain a linear operator T_φ on R into the associated space $\tilde{S}$ of S. Since

$$|T_\varphi x| \leqq |\varphi|^{(x)} \qquad \text{for} \quad 0 \leqq x \in R,$$

T_φ is a bounded linear operator.

Conversely, for a bounded linear operator T on R into $\tilde{S}$, putting

$$\varphi(x, y) = Tx(y) \qquad \text{for} \quad x \in R, y \in S,$$

we obtain obviously a bilinear functional φ on R and S, and φ is bounded, because we have for $0 \leqq x \in R$, $0 \leqq y \in S$.

$$|\varphi(x, y)| \leqq |T| x(y) ,$$

We also see easily that we have $T_{\varphi} \geqq 0$ if and only if $\varphi \geqq 0$. Therefore we have

Theorem 3.1. *The biassociated space* $\widetilde{R,S}$ *is isomorphic to the totality of bounded linear operators on* R *into the associated space* $\widetilde{S}$ *by the correspondence*:

$$\varphi \rightarrow T_{\varphi}, \quad T_{\varphi} x = \varphi^{(x)} \qquad \textit{for every } x \in R .$$

Recalling Theorem 2.1, we obtain by this Theorem 3.1

Theorem 3.2. *For every* $\varphi, \psi \in \widetilde{R,S}$, *we have for* $0 \leqq a \in R$

$$(\varphi \smile \psi)^{(a)} = \bigcup_{0 \leqq x \leqq a} (\varphi^{(x)} + \psi^{(a-x)}) ,$$

$$(\varphi \frown \psi)^{(a)} = \bigcap_{0 \leqq x \leqq a} (\varphi^{(x)} + \psi^{(a-x)}) ,$$

$$|\varphi|^{(a)} = \bigcup_{0 \leqq x \leqq a} |\varphi^{(x)}| .$$

Theorem 3.3. $0 \leqq \varphi, \psi \in \widetilde{R,S}$, $[\varphi] \geqq [\psi]$ *implies*

$$[\varphi^{(x)}] \geqq [\psi^{(x)}] \qquad \textit{for} \quad 0 \leqq x \in R .$$

Proof. If $[\varphi] \geqq [\psi]$, then we have

$$\psi = [\varphi] \psi = \bigcup_{\nu=1}^{\infty} \psi \frown \nu \varphi ,$$

and hence

$$\bigcap_{\nu=1}^{\infty} (\psi - \nu \varphi)^{+} = 0 .$$

Since we have by Theorem 3.2 for $0 \leqq x \in R$

$$(\psi - \nu \varphi)^{+(x)} \geqq (\psi - \nu \varphi)^{(x)+} = (\psi^{(x)} - \nu \varphi^{(x)})^{+} ,$$

we obtain hence

$$\bigcap_{\nu=1}^{\infty} (\psi^{(x)} - \nu \varphi^{(x)})^{+} = 0 ,$$

and consequently $[\varphi^{(x)}] \geqq [\psi^{(x)}]$.

Next we suppose that both R and S are universally continuous. A bounded bilinear functional $\varphi \in \widetilde{R,S}$ is said to be *universally continuous*, if $|\varphi|(x, y)$ is universally continuous as a linear functional on R for fixed $y \in S$ and on S for fixed $x \in R$, that is, $R \ni x_{\lambda} \downarrow_{\lambda \in \Lambda} 0$, $S \ni y_{\gamma} \downarrow_{\gamma \in \Gamma} 0$ implies

$$\inf_{\lambda \in \Lambda} |\varphi| (x_{\lambda}, y) = 0 \qquad \text{for} \quad 0 \leqq y \in S ,$$

$$\inf_{\gamma \in \Gamma} |\varphi|(x, y_\gamma) = 0 \qquad \text{for } 0 \leqq x \in R .$$

Since $|\varphi(x, y)| \leqq |\varphi|(|x|, |y|)$, we see easily by definition that, denoting by $\bar{S}$ the conjugate space of S, we have $\varphi^{(x)} \in \bar{S}$ for every $x \in R$, if φ is universally continuous. Thus, if φ is universally continuous, then T_φ is a universally continuous linear operator on R into $\bar{S}$. Conversely, if T_φ is a universally continuous linear operator on R into $\bar{S}$, then φ is universally continuous. Therefore we obtain by Theorem 2.3 and 3.1

Theorem 3.4. *If both R and S are universally continuous, then the totality of universally continuous bilinear functionals on R and S is a normal manifold of $\widetilde{R,S}$ and isomorphic to the totality of universally continuous linear operators on R into the conjugate space $\bar{S}$ of S.*

Theorem 3.5. *If $0 \leqq \varphi, \psi \in \widetilde{R,S}$, and both φ and ψ are universally continuous, then*

$$\varphi^{(a)} \frown \psi^{(a)} = 0 , \qquad 0 \leqq a \in R$$

implies $\varphi[a] \frown \psi[a] = 0$.

Proof. Since φ and ψ are universally continuous by assumption, we have by definition $\varphi^{(a)}, \psi^{(a)} \in \bar{S}$. Thus, if $\varphi^{(a)} \frown \psi^{(a)} = 0$, then for any $y \in S$ we can find $y_1, y_2 \in S$ such that

$$[y] = [y_1] + [y_2], \qquad \varphi^{(a)}[y_1] = \psi^{(a)}[y_2] = 0 .$$

Then we have by definition

$$(\varphi[y_1])^{(a)} = \varphi^{(a)}[y_1] = 0 ,$$

and hence for $0 \leqq a \in R$ we have

$$\varphi[y_1][a] = 0 .$$

We also obtain likewise

$$\psi[y_2][a] = 0 .$$

Thus we have by Theorem 1.3

$$(\varphi[a] \frown \psi[a])[y] = (\varphi[a][y_1] \frown \psi[a][y_1]) + (\varphi[a][y_2] \frown \psi[a][y_2]) = 0 .$$

Since $y \in S$ may be arbitrary, we obtain therefore $\varphi[a] \frown \psi[a] = 0$.

§ 4. Integral representation

Let both R and S be universally continuous, and $\mathfrak{E}$ and $\mathfrak{F}$ the proper spaces of R and S respectively. We shall denote by $U^{\mathfrak{E}}_{[p]}$ the neighbourhood in $\mathfrak{E}$ corresponding to a projector $[p]$, and by $U^{\mathfrak{F}}_{[q]}$ the neighbourhood in $\mathfrak{T}$ corresponding to a projector $[q]$, and by $U^{\mathfrak{S}}_{[p]}U^{\mathfrak{T}}_{[q]}$ a point set in the product space $\mathfrak{S}\otimes\mathfrak{T}$. The product space $\mathfrak{S}\otimes\mathfrak{T}$ is obviously a locally compact Hauseorff space, and $U^{\mathfrak{S}}_{[p]}U^{\mathfrak{T}}_{[q]}$ $(p\in R, q\in S)$ constitutes a neighbourhood system of $\mathfrak{S}\otimes\mathfrak{T}$.

For $0\leqq\varphi\in\widetilde{R,S}$, $0\leqq a\in R$, $0\leqq b\in S$, putting

$$m(U^{\mathfrak{S}}_{[p]}U^{\mathfrak{T}}_{[q]}) = \varphi([p]a, [q]b) \quad (p\in R,\ q\in S),$$

we obtain a topological measure m on $\mathfrak{S}\otimes\mathfrak{T}$. For a continuous function $\omega(\mathfrak{p},\mathfrak{q})$ on $U^{\mathfrak{S}}_{[p]}U^{\mathfrak{T}}_{[q]}$ we shall denote by

$$\int\limits_{[p]\times[q]} \omega(\mathfrak{p},\mathfrak{q})\,\varphi(d\mathfrak{p}a, d\mathfrak{q}b)$$

the integral of $\omega(\mathfrak{p},\mathfrak{q})$ by m in $U^{\mathfrak{S}}_{[p]}U^{\mathfrak{T}}_{[q]}$.

Theorem 4.1. *If a positive bilinear functional* $\varphi\in\widetilde{R,S}$ is *universally cantinuous, then for* $0\leqq a\in R$, $0\leqq b\in S$, $[p]\leqq[a]$, $[q]\leqq[b]$ *we have*

$$\varphi([p]x, [q]y) = \int\limits_{[p]\times[q]} \left(\frac{x}{a}, \mathfrak{p}\right)\left(\frac{y}{b}, \mathfrak{q}\right)\varphi(d\mathfrak{p}a, d\mathfrak{q}b)$$

for every $x\in R$ *and* $y\in S$.

Proof. Since we have by the spectral theory

$$[p]x = \int\limits_{[p]}\left(\frac{x}{a}, \mathfrak{p}\right)d\mathfrak{p}a,\quad [q]y = \int\limits_{[q]}\left(\frac{y}{b}, \mathfrak{q}\right)d\mathfrak{q}b,$$

we need only prove that $\lim\limits_{\nu\to\infty} x_\nu = x$, $\lim\limits_{\nu\to\infty} y_\nu = y$ implies

$$\lim_{\nu\to\infty}\varphi(x_\nu, y_\nu) = \varphi(x, y).$$

On the other hand we have

$$|\varphi(x_\nu, y_\nu) - \varphi(x, y)| \leqq |\varphi|(|x_\nu|, |y_\nu - y|) + |\varphi|(|x_\nu - x|, |y|).$$

Since φ is universally continuous by assumption, we obtain our assertion.

Theorem 4.2. *For* $0\leqq\varphi$, $\psi\in\widetilde{R,S}$, *if* $[\varphi]\geqq[\psi]$, *then for any* $\varepsilon>0$, $0\leqq a\in R$, $0\in S$, *we can find* $\delta>0$ *such that*

$$\sum_{\nu=1}^{\kappa}\varphi([p_\nu]a, [q_\nu]b)\leqq\delta,$$

$$[p_\nu][p_\mu] = 0 \quad or \quad [q_\nu][q_\mu] = 0 \qquad for \quad \nu\neq\mu$$

implies $\sum\limits_{\nu=1}^{\kappa}\psi([p_\nu]a, [q_\nu]b)\leqq\varepsilon$.

Proof. Since $[\varphi]\geqq[\psi]$ by assumption, we have

$$(\psi-\mu\varphi)^+\downarrow_{\mu=1}^{\infty} 0,$$

and hence for any $\varepsilon>0$, $0\leqq a\in R$, $0\leqq b\in S$ we can find $\rho>0$ by Theorem 1.1 such that

$$(\psi-\rho\varphi)^{+}(a, b)<\frac{1}{2}\varepsilon .$$

For such ρ, putting $\delta=\frac{1}{2\rho}\varepsilon$, we see by Theorem 1.1 that

$$\sum_{\nu=1}^{\kappa}\varphi([p_\nu]a, [q_\nu]b)\leqq\delta ,$$

$$[p_\nu][p_\mu]=0 \quad \text{or} \quad [q_\nu][q_\mu]=0 \qquad \text{for} \quad \nu\neq\mu ,$$

implies

$$\begin{aligned}\sum_{\nu=1}^{\kappa}\psi([p_\nu]a, [q_\nu]b) &\\ \leqq(\psi-\rho\varphi)^{+}(a, b)+\rho\sum_{\nu=1}^{\kappa}\varphi([p_\nu]a, [q_\nu]b) &\\ \leqq\frac{1}{2}\varepsilon+\rho\delta=\varepsilon .&\end{aligned}$$

Theorem 4.3. *For* $\varphi,\psi\in\widetilde{R,S}$, *if* $[\psi]\leqq[\varphi]$, $\varphi\geqq 0$, *and* φ *is universally continuous, then we can find a Borel function* $\Phi(\mathfrak{p}, \mathfrak{q})$ *on the product space* $\mathfrak{E}\otimes\mathfrak{F}$ *such that*

$$\psi(x,y)=\int_{[x]\times[y]}\Phi(\mathfrak{p}, \mathfrak{q})\,\varphi(d\mathfrak{p}x, d\mathfrak{q}y)$$

for every $x\in S$, $y\in S$.

Proof. Recalling the integration theory, if $[\psi]\leqq[\varphi]$, $\psi\geqq 0$, then we conclude by Theorem 4.2 that for every $0\leqq a\in R$, $0\leqq b\in S$ we can find a Borel function $\Phi(\mathfrak{p}, \mathfrak{q})$ on $U^{\mathfrak{E}}_{[a]}U^{\mathfrak{F}}_{[b]}$ such that

$$\psi([p]a, [q]b)=\int_{[p]\times[q]}\Phi(\mathfrak{p}, \mathfrak{q})\,\varphi(d\mathfrak{p}a, d\mathfrak{q}b)$$

for $[p]\leqq[a]$, $[q]\leqq[b]$. Since φ is universally continuous by assumption, ψ also is so by Theorem 3.4, and hence we can conclude further by Theorem 4.1 that

$$\psi([p]x, [q]y)=\int_{[p]\times[q]}\Phi(\mathfrak{p}, \mathfrak{q})\,\varphi(d\mathfrak{p}x, d\mathfrak{q}y)$$

for $[p], [x]\leqq[a]$; $[q], [y]\leqq[b]$; $0\leqq x\in R$, $0\leqq y\in S$. Therefore we obtain a Borel function $\Phi(\mathfrak{p}, \mathfrak{q})$ on $\mathfrak{E}\otimes\mathfrak{F}$ such that

$$\psi(x, y)=\int_{[x]\times[y]}\Phi(\mathfrak{p}, \mathfrak{q})\,\varphi(d\mathfrak{p}x, d\mathfrak{q}y)$$

for every $0\leqq x\in R$, $0\leqq y\in S$. For an arbitrary $\psi\in\widetilde{R,S}$, but $[\psi]\leqq[\varphi]$, we have $[\psi^+]$, $[\psi^-]\leqq[\varphi]$, and hence we obtain our assertion, because $\psi=\psi^+-\psi^-$.

Let $0\leqq\varphi\in\widetilde{R,S}$ be universally continuous, and $\Phi(\mathfrak{p},\mathfrak{q})$ a Borel function on $\mathfrak{S}\otimes\mathfrak{T}$. If $\Phi(\mathfrak{p},\mathfrak{q})$ is integrable by $\varphi(x,y)$ for every $0\leqq x\in R$, $0\leqq y\in S$, then, putting

$$\psi(x,y)=\int_{[x]\times[y]}\Phi(\mathfrak{p},\mathfrak{q})\,\varphi(d\mathfrak{p}x,d\mathfrak{q}y)\qquad\text{for}\quad 0\leqq x\in R,\ 0\leqq y\in S,$$

$$\psi(x,y)=\psi(x^+,y^+)+\psi(x^-,y^-)-\psi(x^+,y^-)-\psi(x^-,y^+),$$

we obtain obviously a bilinear functional ψ on R and S. This bilinear functional ψ will be denoted by

$$\int\Phi(\mathfrak{p},\mathfrak{q})\,\varphi\,d\mathfrak{p}d\mathfrak{q}\,.$$

Theorem 4.4. *For a universally continuous bilinear functional* $0\leqq\varphi\in\widetilde{R,S}$, *if a Borel function* $\Phi(\mathfrak{p},\mathfrak{q})$ *is integrable by* $\varphi(x,y)$ *for every* $0\leqq x\in R$, $0\leqq y\in S$, *then, putting*

$$\psi=\int\Phi(\mathfrak{p},\mathfrak{q})\,\varphi\,d\mathfrak{p}d\mathfrak{q}\,,$$

we obtain a universally continous bilinear functional ψ *on* R *and* S, *and we have*

$$[\psi]\leqq[\varphi]\,,$$

$$\varphi^+=\int\Phi^+(\mathfrak{p},\mathfrak{q})\,\varphi\,d\mathfrak{p}d\mathfrak{q}\,,$$

$$\psi^-=\int\Phi^-(\mathfrak{p},\mathfrak{q})\,\varphi\,d\mathfrak{p}d\mathfrak{q}\,,$$

$$|\psi|=\int|\Phi(\mathfrak{p},\mathfrak{q})|\,\varphi d\mathfrak{p}d\mathfrak{q}$$

for $\Phi^+(\mathfrak{p},\mathfrak{q})=\operatorname{Max}\{\Phi(\mathfrak{p},\mathfrak{q}),0\}$, $\Phi^-(\mathfrak{p},\mathfrak{q})=\{-\Phi(\mathfrak{p},\mathfrak{q})\}^+$.

Proof. We suppose firstly $\Phi(\mathfrak{p},\mathfrak{q})\geqq 0$. Then we have obviously $\psi\geqq 0$. For $0\leqq a\in R$, $0\leqq b\in S$, we have

$$(\psi-\nu\varphi)^+(a,b)\leqq\int_{[a]\times[b]}\operatorname{Max}\{\Phi(\mathfrak{p},\mathfrak{q})-\nu,0\}\,\varphi(d\mathfrak{p}a,d\mathfrak{q}b)$$

and

$$\operatorname{Max}\{\Phi(\mathfrak{p},\mathfrak{q})-\nu,0\}\downarrow_{\nu=1}^{\infty}0$$

up to a measure zero set by the measure $\varphi([p]a,[q]b)$. Thus we obtain

$$(\psi-\nu\varphi)^+(a,b)\downarrow_{\nu=1}^{\infty}0\,.$$

From this relation we conclude $[\psi]\leqq[\varphi]$.

In the general case, we have obviously

$$\psi^{+} \leqq \int \Phi^{+}(\mathfrak{p},\mathfrak{q})\, \varphi\, d\mathfrak{p} d\mathfrak{q},$$

$$\psi^{-} \leqq \int \Phi^{-}(\mathfrak{p},\mathfrak{q})\, \varphi\, d\mathfrak{p} d\mathfrak{q}.$$

Thus we have $[\psi^{+}] \leqq [\varphi]$, $[\psi^{-}] \leqq [\varphi]$, as proved just above. Consequently we obtain

$$[\psi] = [\psi]^{+} \smile [\psi^{-}] \leqq [\varphi].$$

Therefore ψ also is universally continuous by Theorem 3.4.

Furthermore, for any positive bilinear functional $\omega \geqq \psi$, we can find by Theorem 4.3 a Borel function $\Psi(\mathfrak{p},\mathfrak{q})$ on $\mathfrak{E}\otimes\mathfrak{F}$ such that

$$[\varphi]\omega = \int \Psi(\mathfrak{p},\mathfrak{q})\, \varphi\, d\mathfrak{p} d\mathfrak{q}.$$

Then, since $[\varphi]\omega \geqq [\varphi]\psi = \psi$, we see easily that

$$\Psi(\mathfrak{p},\mathfrak{q}) \geqq \Phi^{+}(\mathfrak{p},\mathfrak{q})$$

up to a measure zero set by the measure $\varphi([p]a, [q]b)$ for every $0 \leqq a \in R$, $0 \leqq c \in S$. Thus we have

$$\omega \geqq [\varphi]\omega \geqq \int \Phi^{+}(\mathfrak{p},\mathfrak{q})\, \varphi\, d\mathfrak{p} d\mathfrak{q}.$$

Therefore we obtain

$$\psi^{+} = \int \Phi^{+}(\mathfrak{p},\mathfrak{q})\, \varphi\, d\mathfrak{p} d\mathfrak{q}.$$

From this relation we conclude easily the others.

§ 5. Totally continuous bilinear functionals

Let R and S universally continuous and $\bar{R}$ and $\bar{S}$ the conjugate spaces of R and S respectively. A bilinear functional $\varphi \in \widetilde{R,S}$ is said to be *totally continuous*, if

$$|\varphi| = \bigcup_{\bar{x}\in\bar{R},\, \bar{y}\in\bar{S}} [\bar{x}\bar{y}]|\varphi|$$

With this definition, we see at once that a bilinear functional $\varphi \in \widetilde{R,S}$ is totally continuous if and only if φ is contained in the least normal manifold of $\widetilde{R,S}$ containing all $\bar{x}\bar{y}$ for $\bar{x}\in\bar{R}$, $\bar{y}\in\bar{S}$. Thus the totality of totally continuous bilinear functionals is the least normal manifold of $\widetilde{R,S}$ containing all $\bar{x}\bar{y}$ for $\bar{x}\in\bar{R}$, $\bar{y}\in\bar{S}$. This normal manifold will be denoted by $\overline{R,S}$ in the sequel.

Theorem 5.1. *If $\bar{a}\in\bar{R}$ is complete in $[a]R$ and $\bar{b}\in\bar{S}$ is complete in $[b]S$, then we have for every $\varphi \in \overline{R,S}$*

$$[\bar{a}\bar{b}]\varphi[a][b] = \varphi[a][b].$$

Proof. We need only consider the case: $\varphi \geqq 0$. In this case, we have obviously

$$[\bar{a}\bar{b}]\varphi[a][b] \leqq \varphi[a][b] .$$

On the other hand, for every $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$, we have

$$[\bar{x}\bar{y}]\varphi[a][b] = [(\bar{x}[a])(\bar{y}[b])]\varphi[a][b] .$$

Since $\bar{a}$ is complete in $[a]R$ and $\bar{b}$ is complete in $[b]S$ by assumption, we have

$$[\bar{x}[a]] \leqq [\bar{a}] , \qquad [\bar{y}[b]] \leqq [\bar{b}] ,$$

and hence we obtain

$$[\bar{x}\bar{y}]\varphi[a][b] \leqq [\bar{a}\bar{b}]\varphi[a][b] .$$

Since $\varphi \in \overline{R,S}$ by assumption, we have by definition

$$\varphi = \bigcup_{\bar{x} \in \bar{R},\, \bar{y} \in \bar{S}} [\bar{x}\bar{y}]\varphi .$$

Thus we conclude

$$\varphi[a][b] \leqq [\bar{a}\bar{b}]\varphi[a][b] .$$

Theorem 5.2. *Let a bilinear functional $\varphi \in \overline{R,S}$ be positive and universally continuous. In order that φ be totally continuous, it is necessary and sufficient that* $0 \leqq a_\nu \leqq a (\nu = 1, 2, \cdots)$, $\text{s-}\lim_{\nu \to \infty} a_\nu = 0$ *in R implies*

$$\lim_{\nu \to \infty} \varphi^{(a_\nu)} = 0 \qquad \text{in } \bar{S} .$$

Proof. We need only consider the case where both R and S are semi-regular. We suppose firstly that φ is totally continuous. Let $[a]$ and $[b]$ be regular projectors in R and S respectively, and $0 \leqq \bar{a} \in \bar{R}$, $0 \leqq \bar{b} \in \bar{S}$ complete in $[a]R$, $[b]S$ respectively. Then we have by Theorem 5.1

$$[\bar{a}\bar{b}]\varphi[a][b] = \varphi[a][b] .$$

Thus we can find by Theorem 4.3 a Borel function $\Phi \geqq 0$ on $\mathfrak{S} \otimes \mathfrak{T}$ such that

$$\varphi[a][b] = \int \Phi(\mathfrak{p}, \mathfrak{q})\, \bar{a}\bar{b}\, d\mathfrak{p}\, d\mathfrak{q} .$$

If $0 \leqq a_\nu \leqq a_0 (\nu = 1, 2, \cdots)$, $\text{s-}\lim_{\nu \to \infty} a_\nu = 0$ in R, then we have by the integration theory for every $0 \leqq y \in S$ and $\nu = 1, 2, \cdots$,

$$\varphi([a]a_\nu,[b]y) = \int_{[a]\times[b]} \Phi(\mathfrak{p},\mathfrak{q})\bar{a}(d\mathfrak{p}a_\nu)\bar{b}(d\mathfrak{q}y)$$
$$= \int_{[b]}\{\int_{[a]} \Phi(\mathfrak{p},\mathfrak{q})\,\bar{a}(d\mathfrak{p}a_\nu)\}\,\bar{b}(d\mathfrak{q}y)\,,$$

and further for $\nu=1,2,\cdots$

$$0 \leqq \int_{[a]} \Phi(\mathfrak{p},\mathfrak{q})\bar{a}(d\mathfrak{p}a_\nu) \leqq \int_{[a]} \Phi(\mathfrak{p},\mathfrak{q})\,\bar{a}(d\mathfrak{p}a_0)\,.$$

We denote by R_0 the relative segment of R by a_0, that is, R_0 consists of all those elements x for which we can find $\alpha>0$ such that $|x|\leqq\alpha a_0$. For each $\mathfrak{q}$, if

$$\int_{[a]} \Phi(\mathfrak{p},\mathfrak{q})\,\bar{a}(d\mathfrak{p}a_0) < +\infty\,,$$

then $\int_{[a]} \Phi(\mathfrak{p},\mathfrak{q})\,\bar{a}d\mathfrak{p}$ is a universally continuous linear functional on R_0, and hence

$$\lim_{\nu\to\infty}\int_{[a]} \Phi(\mathfrak{p},\mathfrak{q})\,\bar{a}(d\mathfrak{p}a_\nu) = 0\,.$$

On the other hand, we have

$$\int_{[a]} \Phi(\mathfrak{p},\mathfrak{q})\,\bar{a}(d\mathfrak{p}a_\nu) = \left(\frac{\varphi^{([a]a_\nu)}}{\bar{b}},\ \mathfrak{q}\right)$$

in $U^{\mathfrak{F}}_{[b]}$ up to a nowhere dense closed set, and

$$\varphi^{([a]a_\nu)}([b]y) = \int_{[b]}\{\int_{[a]} \Phi(\mathfrak{p},\mathfrak{q})\,\bar{a}(d\mathfrak{p}a_\nu)\}\,\bar{b}(d\mathfrak{q}y)\,.$$

Thus we obtain $\text{ind-}\lim_{\nu\to\infty}\varphi^{([a]a_\nu)}[b]=0$ in $\bar{S}$. Since a regular projector $[b]$ may be arbitrary, we have hence

$$\text{ind-}\lim_{\nu\to\infty}\varphi^{([a]a_\nu)} = 0 \qquad \text{in } \bar{S}\,.$$

Since $\varphi^{([a]a_\nu)}\leqq\varphi^{([a]a_0)}$ in $\bar{S}$, we conclude further $\lim_{\nu\to\infty}\varphi^{([a]a_\nu)}=0$ in $\bar{S}$. Furthermore, since

$$\varphi^{(a_\nu)} = \varphi^{([a]a_\nu)} + \varphi^{((1-[a])a_\nu)} \leqq \varphi^{([a]a_\nu)} + \varphi^{((1-[a])a_0)}\,,$$

we obtain hence $\overline{\lim}_{\nu\to\infty}\varphi^{(a_\nu)}\leqq\varphi^{((1-[a])a_0)}$ for every regular projector $[a]$ in R. Therefore we conclude $\lim_{\nu\to\infty}\varphi^{(a_\nu)}=0$, because

$$((1-[a])a_0)\downarrow_{[a]}0 \qquad \text{for all regular projector } [a].$$

Conversely, we assume that φ is universally continuous, and $0\leqq$

$a_\nu \leqq a_0$, $\text{s-}\lim_{\nu\to\infty} a_\nu = 0$ implies $\lim_{\nu\to\infty} \varphi^{(a_\nu)} = 0$. Putting

$$\psi = \varphi - \bigcup_{\bar{x}\in\bar{R},\ \bar{y}\in\bar{S}} [\bar{x}\bar{y}]\varphi ,$$

we have obviously $0 \leqq \psi \leqq \varphi$ and

$$\bar{x}\bar{y}_\frown \psi = 0 \qquad \text{for every } 0 \leqq \bar{x} \in \bar{R},\ 0 \leqq \bar{y} \in \bar{S}.$$

Thus we have by Theorem 3.4

$$T_{\bar{x}\bar{y}\frown} T_\psi = 0 \qquad \text{for } 0 \leqq \bar{x} \in \bar{R},\ 0 \leqq \bar{y} \in \bar{S},$$

and hence by Theorem 2.1

$$\bigcap_{0\leqq x\leqq a} (\bar{x}(x)\,\bar{y} + \psi^{(a-x)}) = 0 \qquad \text{in } \bar{S} .$$

If $[b]$ is an arbitrary regular prjector in S and $0 \leqq \bar{x} \in \bar{S}$ is complete in $[b]S$, then we have naturally

$$\bigcap_{0\leqq x\leqq a} (\bar{x}(x)\,\bar{y}[b] + \psi^{(a-x)}[b]) = 0 .$$

Since $[b]S$ is superuniversally continuous, we can find a sequence $0 \leqq x_\nu \leqq a$ ($\nu = 1, 2, \cdots$) such that

$$\bigcap_{\nu=1}^{\infty} (\bar{x}(x_\nu)\,\bar{y}[b] + \psi^{(a-x_\nu)}[b]) = 0 .$$

For such $0 \leqq x_\nu \leqq a$ ($\nu = 1, 2, \cdots$), we have obviously

$$\bar{x}(x_\nu)\,\bar{y}[b] + \psi^{(a-x_\nu)}[b] \geqq \bar{x}(x_\nu)\,\bar{y}[b] ,$$

and hence we conclude naturally

$$\bigcap_{\bar{x}(x_\nu)\leqq\frac{1}{\mu}} (\bar{x}(x_\nu)\,\bar{y}[b] + \psi^{(a-x_\nu)}[b]) = 0$$

for every $\mu = 1, 2, \cdots$. Thus we can find a subsequence ν_μ ($\mu = 1, 2, \cdots$) such that

$$\underline{\lim}_{\mu\to\infty} (\bar{x}(x_{\nu_\mu})\,\bar{y}[b] + \psi^{(a-x_{\nu_\mu})}[b]) = 0 ,$$

$$\lim_{\mu\to\infty} \bar{x}(x_{\nu_\mu}) = 0 .$$

Now let $[a]$ be an arbirary regular projector in R and $\bar{x}$ complete in $[a]R$. Then

$$\lim_{\mu\to\infty} \bar{x}([a]\,x_{\nu_\mu}) = 0$$

implies $\text{s-}\lim_{\mu\to\infty} [a]x_{\nu_\mu} = 0$, and hence

$$\text{s-}\lim_{\mu\to\infty} [a](a - x_{\nu_\mu}) = a ,$$

Thus we have by assumption

$$\lim_{\mu\to\infty}\varphi^{([a](a-x_{\nu\mu}))}[b]=\varphi^{(a)}[b] \qquad \text{in } \bar{S}\,.$$

On the other hand, we have

$$\lim_{\mu\to\infty}(\bar{x}([a]x_{\nu\mu})\,\bar{y}[b]+\varphi^{([a](a-x_{\nu\mu}))}[b])=0\,,$$

as proved just above. Thus we conclude $\varphi^{(a)}[b]=0$ in $\bar{S}$ for every regular projector $[a]$ in R and $[b]$ in S, and hence $\varphi=0$, that is,

$$\varphi=\bigcup_{\bar{x}\in\bar{R},\,\bar{y}\in\bar{S}}[\bar{x}\bar{y}]\,\varphi\,.$$

A bilinear functional $\varphi\in\widetilde{R,S}$ is said to be *absolutely continuous*, if we can find $\bar{a}\in\bar{R}$ and $\bar{b}\in\bar{S}$ such that $|\varphi|\leqq\bar{a}\bar{b}$. With this definition, every absolutely continuous bilinear functional is obviously totally continuous, and the totality of absolutely continuous bilinear functionals constitutes a semi-normal madifold of $\overline{R,S}$.

Theorem 5.3. *If a bilinear functional $\varphi\in\overline{R,S}$ is absolutely continuous, then*

$$\text{w-}\lim_{\nu\to\infty}a_\nu=0 \qquad \textit{in } R$$

implies $\lim_{\nu\to\infty}\varphi^{(a_\nu)}=0$ *in* $\bar{S}$.

Proof. We need only prove the case: $\varphi\geqq0$. If

$$0\leqq\varphi\leqq\bar{a}\bar{b},\ 0\leqq\bar{a}\in\bar{R},\ 0\leqq\bar{b}\in\bar{S},$$

then we can find a Borel function $\Phi(\mathfrak{p},\mathfrak{q})$ on $\mathfrak{E}\otimes\mathfrak{F}$ such that

$$\varphi=\int\Phi(\mathfrak{p},\mathfrak{q})\,\bar{a}\bar{b}\,d\mathfrak{p}\,d\mathfrak{q},$$

$$0\leqq\Phi(\mathfrak{p},\mathfrak{q})\leqq1\,.$$

Then, putting for each $\mathfrak{q}\in\mathfrak{F}$

$$\bar{a}_{\mathfrak{q}}=\int\Phi(\mathfrak{p},\mathfrak{q})\,\bar{a}\,d\mathfrak{p}\,,$$

we have $\bar{a}_{\mathfrak{q}}\in\bar{R}$. Thus, if $\text{w-}\lim_{\nu\to\infty}a_\nu=0$, then we have

$$\lim_{\nu\to\infty}\bar{a}_{\mathfrak{q}}(a_\nu)=0\,.$$

Since $\varphi^{(a_\nu)}=\int\bar{a}_{\mathfrak{q}}(a_\nu)\,\bar{b}\,d\mathfrak{q}$, we obtain hence

$$\text{ind-}\lim_{\nu\to\infty}\varphi^{(a_\nu)}=0 \qquad \text{in } \bar{S}.$$

On the other hand we have for every $0\leqq y\in S$

$$|\varphi(a_\nu, y)| \leqq \varphi(|a_\nu|, y) \leqq \bar{a}(|a_\nu|)\,\bar{b}(y),$$

and hence $|\varphi^{(a_\nu)}| \leqq \bar{a}(|a_\nu|)\bar{b}$ in $\bar{S}$ for every $\nu = 1, 2, \cdots$. Thus, recalling Theorem 46.4 in a book[11], we conclude

$$\lim_{\nu\to\infty} \varphi^{(a_\nu)} = 0 .$$

The totality of absolutely continuous bilinear functionals on R and S will be denoted by $\bar{R}\times\bar{S}$. Then $\bar{R}\times\bar{S}$ is obviously a semi-normal manifold of $\overline{R,S}$ and complete in $\overline{R,S}$.

For a universally continuous semi-ordered linear space W, a system of positive elements $a_\lambda \in W\,(\lambda\in\Lambda)$ is said to be *positively fundamental* in W, if every closed manifold of W containing all linear combinations

$$\sum_{\nu=1}^{\kappa} \alpha_\nu a_{\lambda_\nu}$$

for every finite number of elements $\lambda_\nu \in \Lambda$ and positive numbers α_ν $(\nu=1,2,\cdots,\kappa)$, includes all positive elements of W.

Theorem 5.4. *The totality of bilinear functionals* $\bar{x}\bar{y}$ $(0\leqq\bar{x}\in\bar{R},\ 0\leqq\bar{y}\in\bar{S})$ *on R and S is positively fundamental in* $\bar{R}\times\bar{S}$.

Proof. For each $0\leqq\varphi_0\in\bar{R}\times\bar{S}$, we can find by definition $0\leqq\bar{a}\in\bar{R}$, $0\leqq\bar{b}\in\bar{S}$ for which $\varphi_0\leqq\bar{a}\bar{b}$. Let $0\leqq a\in R$, $0\leqq b\in S$ be arbitrary, and we shall denote by $\bar{A}$ the relative segment of $\bar{R}\times\bar{S}$ by $(\bar{a}[a])(\bar{b}[b])$. Then we have obviously $\varphi_0[a][b]\in\bar{A}$, and for every $\varphi\in\bar{A}$, putting

$$\|\varphi\| = |\varphi|(a,b),$$

we obtain a norm on $\bar{A}$. We see easily that this norm is continuous. Recalling Theorem 4.3, we can find a Borel function $\Phi_0(\mathfrak{p},\mathfrak{q})$ on $\mathfrak{E}\otimes\mathfrak{F}$ such that

$$\varphi_0(a,b) = \int_{[a]\times[b]} \Phi_0(\mathfrak{p},\mathfrak{q})\,\bar{a}(d\mathfrak{p}a)\,\bar{b}(d\mathfrak{q}b),$$

$$0 \leqq \Phi_0(\mathfrak{p},\mathfrak{q}) \leqq 1 .$$

Then, by virtue of the integration theory, for any $\varepsilon>0$ we can find a continuous function $\Psi_0(\mathfrak{p},\mathfrak{q})$ on $U^{\mathfrak{E}}_{[a]}\times U^{\mathfrak{F}}_{[b]}$ such that $0\leqq\Psi_0(\mathfrak{p},\mathfrak{q})\leqq1$,

$$\int_{[a]\times[b]} |\Phi_0(\mathfrak{p},\mathfrak{q})-\Psi_0(\mathfrak{p},\mathfrak{q})|\,\bar{a}(d\mathfrak{p}a)\,\bar{b}(d\mathfrak{q}b) < \varepsilon ,$$

and further a finite number of real numbers $\alpha_{\nu,\mu}$ and of projectors $[p_\nu]$, $[q_\mu]$ such that

11) H. Nakano, Modern spectral theory, (1950), Tokyo, Theorem 46.4.

$$[a] = \sum_{\nu=1}^{\kappa} [p_\nu], \quad [b] = \sum_{\mu=1}^{\sigma} [q_\mu], \quad 0 \leqq \alpha_{\nu,\mu} \leqq 1 ,$$

$$|\Psi_0(\mathfrak{p},\mathfrak{q}) - \alpha_{\nu,\mu}| < \varepsilon \qquad \text{for } \mathfrak{p} \in U^{\mathfrak{R}}_{[p_\nu]}, \ \mathfrak{q} \in U^{\mathfrak{S}}_{[q_\mu]} .$$

Then we have

$$\begin{aligned} &\|\varphi_0[a][b] - \sum_{\nu,\mu} \alpha_{\nu,\mu} (\bar{a}[p_\nu])(\bar{b}[q_\mu])\| \\ &= |\varphi_0[a][b] - \sum_{\nu,\mu} \alpha_{\nu,\mu} (\bar{a}[p_\nu])(\bar{b}[q_\mu])|(a,b) \\ &= \sum_{\nu,\mu} \int_{[p_\nu]\times[q_\mu]} |\Phi_0(\mathfrak{p},\mathfrak{q}) - \alpha_{\nu,\mu}| \, \bar{a}(d\mathfrak{p}a)\, \bar{b}(d\mathfrak{q}b) \\ &\leqq \int_{[a]\times[b]} |\Phi_0(\mathfrak{p},\mathfrak{q}) - \Psi_0(\mathfrak{p},\mathfrak{q})| \, \bar{a}(d\mathfrak{p}a)\, \bar{b}(d\mathfrak{q}b) \\ &\quad + \varepsilon \sum_{\nu,\mu} \bar{a}([p_\nu]a)\, \bar{b}([q_\mu]b) < \varepsilon\, (1 + \bar{a}(a)\bar{b}(b)) . \end{aligned}$$

Therefore we can find a sequence of linear combinations

$$\psi_\nu = \sum_{\mu=1}^{\kappa_\nu} \xi_\mu (\bar{a}[p_{\nu,\mu}])(\bar{b}[q_{\nu,\mu}]) \in \bar{A} \qquad (\nu = 1, 2, \cdots)$$

such that $0 \leqq \xi_\mu \leqq 1, \ \sum_{\mu=1}^{\kappa_\nu} [p_{\nu,\mu}] = [a], \quad \sum_{\mu=1}^{\kappa_\nu} [q_{\nu,\mu}] = [b]$,

$$\lim_{\nu\to\infty} \|\psi_\nu - \varphi_0[a][b]\| = 0 ,$$

and hence

$$\text{s-ind-lim}_{\nu\to\infty} \psi_\nu = \varphi_0[a][b] .$$

On the other hand, we have obviously

$$0 \leqq \psi_\nu \leqq (\bar{a}[a]\, \bar{b}[b]) \qquad \text{for every } \nu = 1, 2, \cdots,$$

and hence we conclude $\text{s-lim}_{\nu\to\infty} \psi_\nu = \varphi_0[a][b]$. Since $0 \leqq a \in R$, $0 \leqq b \in S$ may be arbitrary, we obtain therefore our assertion.

Since $\bar{R} \times \bar{S}$ is a complete semi-normal manifold of $\overline{R,S}$, the totality of positive elements of $\bar{R} \times \bar{S}$ is positively fundamental in $\overline{R,S}$. Thus we conclude immediately by Theorem 5.4.

Theorem 5.5. *The totality of bilinear functionals* $\bar{x}\bar{y}$ $(0 \leqq \bar{x} \in \bar{R}, \ 0 \leqq \bar{y} \in \bar{S})$ *on* R *and* S *is positively fundamental in* $\overline{R,S}$.

Theorem 5.6. *If a bilinear functional* $\bar{\varphi}$ *on* $\bar{R}$ *and* $\bar{S}$ *is totally continuous, then there exists uniquely a univesally continuous linear functional* ψ *on* $\bar{R} \times \bar{S}$ *such that*

$$\bar{\varphi}(\bar{x}, \bar{y}) = \psi(\bar{x}\bar{y}) \qquad \textit{for every } \bar{x} \in \bar{R},\ \bar{y} \in \bar{S}.$$

If $\bar{\varphi}$ is positive, then ψ also is positive.

Proof. We need obviously consider the case: $\bar{\varphi} \geqq 0$. Let $a_\lambda \geqq 0$ $(\lambda \in \Lambda)$ be a complete orthogonal system of R such that $[a_\lambda]R$ is regular for every $\lambda \in \Lambda$, and $b_\gamma \geqq 0$ $(\gamma \in \Gamma)$ a complete orthogonal system of S such that $[b_\gamma]S$ is regular for every $\gamma \in \Gamma$. Putting

$$a_\lambda^{\bar{R}}(\bar{x}) = \bar{x}(a_\lambda) \qquad \text{for every } \bar{x} \in \bar{R},$$

$$b_\gamma^{\bar{S}}(\bar{y}) = \bar{y}(b_\gamma) \qquad \text{for every } \bar{y} \in \bar{S},$$

we have obviously $a_\lambda^{\bar{R}} \in \bar{\bar{R}}$, $b_\gamma^{\bar{S}} \in \bar{\bar{S}}$ and $a_\lambda^{\bar{R}}(\lambda \in \Lambda)$ is a complete orthegonal system of $\bar{\bar{R}}$, and $b_\lambda^{\bar{S}}(\gamma \in \Gamma)$ is a complete orthogonal system of $\bar{\bar{S}}$. Since $\bar{\varphi} \geqq 0$ is totally continuous by assumption, we have by definition

$$\bar{\varphi} = \bigcup_{\gamma \in \Lambda, \gamma \in \Gamma} [a_\lambda^{\bar{R}} b_\gamma^{\bar{S}}]\, \bar{\varphi},$$

$$\bar{\varphi}(\bar{x}, \bar{y}) = \sum_{\lambda \in \Lambda, \gamma \in \Gamma} [a_\lambda^{\bar{R}} b_\gamma^{\bar{S}}]\, \bar{\varphi}(\bar{x}, \bar{y}) \qquad \text{for } 0 \leqq \bar{x} \in \bar{R},\ 0 \leqq \bar{y} \in \bar{S}.$$

Considering the canonical spaces $\mathfrak{E}$ and $\mathfrak{F}$of R and S respectively, we obtain by Theorem 4.3 a Borel function $\Phi_{\lambda,\gamma}(\mathfrak{p}, \mathfrak{q}) \geqq 0$ on the product space $\mathfrak{E} \otimes \mathfrak{F}$ such
that

$$[a_\lambda^{\bar{R}} b_\gamma^{\bar{S}}]\, \bar{\varphi} = \int_{[a_\lambda] \times [b_\gamma]} \Phi_{\lambda,\gamma}(\mathfrak{p}, \mathfrak{q})\, a_\lambda^{\bar{R}} b_\gamma^{\bar{S}}\, d\mathfrak{p}\, d\mathfrak{q}.$$

Then we have for $0 \leqq \bar{x} \in \bar{R}$, $0 \leqq \bar{y} \in \bar{S}$

$$[a_\lambda^{\bar{R}} b_\gamma^{\bar{S}}]\, \bar{\varphi}(\bar{x}, \bar{y}) = \int_{[a_\lambda] \times [b_\gamma]} \Phi_{\lambda,\gamma}(\mathfrak{p}, \mathfrak{q})\, \bar{x}\bar{y}(d\mathfrak{p} a_\lambda, d\mathfrak{q} b_\gamma),$$

and hence

$$\bar{\varphi}(\bar{x}, \bar{y}) = \sum_{\lambda \in \Lambda, \gamma \in \Gamma} \int_{[a_\lambda] \times [b_\gamma]} \Phi_{\lambda,\gamma}(\mathfrak{p}, \mathfrak{q})\, \bar{x}\bar{y}(d\mathfrak{p} a_\lambda, d\mathfrak{q} b_\gamma).$$

Now, putting for $0 \leqq \varphi \in \bar{R} \times \bar{S}$

$$\psi(\varphi) = \sum_{\lambda \in \Lambda, \gamma \in \Gamma} \int_{[a_\lambda] \times [b_\gamma]} \Phi_{\lambda,\gamma}(\mathfrak{p}, \mathfrak{q})\, \varphi(d\mathfrak{p} a_\lambda, d\mathfrak{q} b_\gamma),$$

we have $\psi(\varphi) < +\infty$ for every $0 \leqq \varphi \in \bar{R} \times \bar{S}$, because for each $0 \leqq \varphi \in \bar{R} \times \bar{S}$, we can find $0 \leqq \bar{x} \in \bar{R}$, $0 \leqq \bar{y} \in \bar{S}$ such that $\varphi \leqq \bar{x}\bar{y}$, and then

$$\psi(\varphi) \leqq \bar{\varphi}(\bar{x}, \bar{y}) < +\infty.$$

Furthermore, putting

$$\psi(\varphi) = \psi(\varphi^+) - \psi(\varphi^-) \qquad \text{for every } \varphi \in \bar{R} \times \bar{S},$$

we see easily that ψ is a positive linear functional on $\bar{R}\times\bar{S}$, and

$$\psi(\bar{x}\bar{y}) = \bar{\varphi}(\bar{x},\bar{y}) \qquad \text{for every } \bar{x}\in\bar{R},\ \bar{y}\in\bar{S}.$$

If $\bar{x}_0\bar{y}_0 \geqq \varphi_\rho \downarrow_{\rho\in P} 0$ in $\bar{R}\times\bar{S}$, then for any $\varepsilon>0$ we can find $\delta>0$ such that, putting

$$\Phi_{\lambda,\gamma,\delta}(\mathfrak{p},\mathfrak{q}) = \operatorname{Min}\{\Phi_{\lambda,\gamma}(\mathfrak{p},\mathfrak{q}),\delta\},$$

we have

$$\int_{[a_\lambda]\times[b_\gamma]} \{\Phi_{\lambda,\gamma}(\mathfrak{p},\mathfrak{q})-\Phi_{\lambda,\gamma,\delta}(\mathfrak{p},\mathfrak{q})\}\,\bar{x}_0\bar{y}_0(d\mathfrak{p}a_\lambda, d\mathfrak{q}b_\gamma)<\varepsilon,$$

and hence

$$\int_{[a_\lambda]\times[b_\gamma]} \Phi_{\lambda,\gamma}(\mathfrak{p},\mathfrak{q})\varphi_\rho(d\mathfrak{p}a, d\mathfrak{q}b) \leqq \int_{[a_\lambda]\times[b_\gamma]} \Phi_{\lambda,\gamma,\delta}(\mathfrak{p},\mathfrak{q})\varphi_\rho(d\mathfrak{p}a_\lambda, d\mathfrak{q}b_\gamma)$$
$$+ \int_{[a_\lambda]\times[b_\gamma]} \{\Phi_{\lambda,\gamma}(\mathfrak{p},\mathfrak{q})-\Phi_{\lambda,\gamma,\delta}(\mathfrak{p},\mathfrak{q})\}\,\bar{x}_0\bar{y}_0(d\mathfrak{p}a_\lambda, d\mathfrak{q}b_\gamma) \leqq \delta\varphi_\rho(a,b)+\varepsilon.$$

This relation yields

$$\inf_{\rho\in P}\int_{[a_\lambda]\times[b_\gamma]} \Phi_{\lambda,\gamma}(\mathfrak{p},\mathfrak{q})\,\varphi_\rho(d\mathfrak{p}a_\lambda, d\mathfrak{q}b_\gamma) = 0.$$

Thus we conclude easily that $\inf_{\rho\in P}\psi(\varphi_\rho)=0$, that is, ψ is universally continuous. By virtue of Theorem 5.4, we conclude easily the uniqueness of such a universally continuous linear functional ψ on $\bar{R}\times\bar{S}$.

Theorem 5.7. *$\overline{\bar{R},\bar{S}}$ is isomorphic to the conjugate space $\overline{\bar{R}\times\bar{S}}$ of $\bar{R}\times\bar{S}$ by a correspondence*:

$$\overline{\bar{R},\bar{S}}\ni\bar{\varphi}\to\bar{\varphi}^{\bar{R}\times\bar{S}}\in\overline{\bar{R}\times\bar{S}}, \qquad \bar{\varphi}^{\bar{R}\times\bar{S}}(\bar{x}\bar{y})=\bar{\varphi}(\bar{x},\bar{y}) \text{ for } \bar{x}\in\bar{R},\bar{y}\in\bar{S}.$$

Proof. By virtue of Theorem 5.6, for each $\bar{\varphi}\in\overline{\bar{S},\bar{R}}$ we can find uniquely $\bar{\varphi}^{\bar{R}\times\bar{S}}\in\overline{\bar{R}\times\bar{S}}$ such that $\bar{\varphi}^{\bar{R}\times\bar{S}}(\bar{x}\bar{y})=\bar{\varphi}(\bar{x},\bar{y})$ for $\bar{x}\in\bar{R},\bar{y}\in\bar{S}$, and $\bar{\varphi}\geqq0$ implies $\bar{\varphi}^{\bar{R}\times\bar{S}}\geqq0$. Conversely, if $\bar{\varphi}^{\bar{R}\times\bar{S}}\geqq0$, then we have obviously $\bar{\varphi}\geqq0$. Thus we have $\bar{\varphi}^{\bar{R}\times\bar{S}}\geqq\bar{\psi}^{\bar{R}\times\bar{S}}$ if and only if $\bar{\varphi}\geqq\bar{\psi}$. If $0\leqq\omega\leqq\bar{\varphi}^{\bar{R}\times\bar{S}}$, $\omega\in\overline{\bar{R}\times\bar{S}}$, $\bar{\varphi}\in\overline{\bar{R},\bar{S}}$, then, putting

$$\bar{\omega}(\bar{x},\bar{y}) = \omega(\bar{x}\bar{y}) \qquad \text{for } \bar{x}\in\bar{R},\ \bar{y}\in\bar{S},$$

we obtain obviously a positive bilinear functional $\bar{\omega}$ on $\bar{R}$ and $\bar{S}$. Since

$$0\leqq\bar{\omega}(\bar{x},\bar{y})\leqq\bar{\varphi}(\bar{x},\bar{y}) \qquad \text{for } 0\leqq\bar{x}\in\bar{R},\ 0\leqq\bar{y}\in\bar{S},$$

$\bar{\omega}$ is totally continuous, that is $\bar{\omega}\in\overline{\bar{R},\bar{S}}$. Recalling Theorem 5.4, we see further that $\bar{\omega}^{\bar{R}\times\bar{S}}=\omega$. Therefore $\bar{\varphi}^{\bar{R}\times\bar{S}}(\bar{\varphi}\in\overline{\bar{R},\bar{S}})$ constitutes a semi-normal manifold of $\overline{\bar{R}\times\bar{S}}$. Furthermore, since

$$\bar{\varphi}_\lambda^{\bar{R}\times\bar{S}} \uparrow_{\lambda\in\Lambda} \omega \in \overline{\overline{\bar{R}\times\bar{S}}}, \qquad \bar{\varphi}_\lambda \in \overline{\bar{R}\,\bar{S}}\ (\lambda\in\Lambda)$$

implies

$$\sup_{\lambda\in\Lambda} \bar{\varphi}_\lambda(\bar{x},\bar{y}) = \omega(\bar{x}\bar{y}) < +\infty \quad \text{for} \quad 0\leqq\bar{x}\in\bar{R},\ 0\leqq\bar{y}\in\bar{S},$$

there exists $\bar{\varphi}\in\overline{\bar{R},\bar{S}}$ for which $\bar{\varphi}_\lambda \uparrow_{\lambda\in\Lambda}\bar{\varphi}$. For such $\bar{\varphi}\in\overline{\bar{R},\bar{S}}$ we have

$$\bar{\varphi}(\bar{x},\bar{y}) = \omega(\bar{x}\bar{y}) \qquad \text{for} \quad 0\leqq\bar{x}\in\bar{R},\ 0\leqq\bar{y}\in\bar{S},$$

and hence $\bar{\varphi}^{\bar{R}\times\bar{S}} = \omega$. Thus $\bar{\varphi}^{\bar{R}\times\bar{S}}(\bar{\varphi}\in\overline{\bar{R},\bar{S}})$ constitutes a normal manifold of $\overline{\overline{\bar{R}\times\bar{S}}}$. For any $\omega\in\overline{\overline{\bar{R}\times\bar{S}}}$, if $\bar{\varphi}^{\bar{R}\times\bar{S}}(\omega)=0$ for every $\bar{\varphi}\in\overline{\bar{R},\bar{S}}$, then we have $\omega(x,y)=0$ for every $x\in R$, $y\in S$, because, putting $x^{\bar{R}}(\bar{x})=\bar{x}(x)$ for $\bar{x}\in\bar{R}$ and $y^{\bar{S}}(\bar{y})=\bar{y}(y)$ for $\bar{y}\in\bar{S}$, we have $x^{\bar{R}}y^{\bar{S}}\in\overline{\bar{R},\bar{S}}$, and

$$(x^{\bar{R}}y^{\bar{S}})^{\bar{R}\times\bar{S}}(\omega) = \omega(x,y)$$

by Theorem 5.6. Therefore $\bar{\varphi}^{\bar{R}\times\bar{S}}(\bar{\varphi}\in\overline{\bar{R},\bar{S}})$ coincides with the whole $\overline{\overline{\bar{R}\times\bar{S}}}$.

§ 6. Singular bilinear functionals

Let R and S be universally continuous, and $\bar{R}$ and $\bar{S}$ the conjugate spaces of R and S respectively. A bilinear functional $\varphi\in\widetilde{R,S}$ is said to be *singular*, if $[\bar{x}\bar{y}]\varphi=0$ for every $\bar{x}\in\bar{R}$, $\bar{y}\in\bar{S}$, that is,

$$\bar{x}\bar{y}_\frown|\varphi| = 0 \qquad \text{for every} \quad \bar{x}\in\bar{R},\ \bar{y}\in\bar{S}.$$

With this definition, we see easily that the totality of singular bilinear functionals is a normal manifold of $\widetilde{R,S}$ and the orthogonal complement of the totality of totally continuous bilinear functionals in $\widetilde{R,S}$.

Theorem 6.1. *Let R be semi-regular and $\bar{R}$ the conjugate space of R. If R has no discrete element except for* 0, *then putting*

$$\varphi(\bar{x},x) = \bar{x}(x) \qquad \textit{for} \quad \bar{x}\in\bar{R},\ x\in R,$$

we obtain a singular bilinear functional φ on $\bar{R}$ and R.

Proof. For every $x\in R$, putting

$$x^{\bar{R}}(\bar{x}) = \bar{x}(x) \qquad \text{for evry} \quad \bar{x}\in\bar{R},$$

we obtain a universally continuous linear functional $x^{\bar{R}}$ on $\bar{R}$, and denoting by $\bar{\bar{R}}$ the conjugate space of $\bar{R}$, we see easily that $x^{\bar{R}}(x\in R)$ constitutes a complete semi-normal manifold of $\bar{\bar{R}}$.

The bilinear functional φ: $\varphi(\bar{x},x)=\bar{x}(x)$ for $\bar{x}\in\bar{R}$, $x\in R$, is obviously positive and universally continuous. We shall prove firstly that for every positive $y \in R$, $\bar{y} \in \bar{R}$ we have

$$y^{R}\bar{y} \frown \varphi = 0 .$$

For this purpose, we need by Theorem 3.4 prove

$$T_{y^{R}\bar{y}} \frown T_{\varphi} = 0 ,$$

that is, by Theorem 2.1

$$\bigcap_{0 \leqq \bar{x} \leqq \bar{a}} (\bar{x}(y)\bar{y} + (\bar{a} - \bar{x})) = 0 \qquad \text{for every } 0 \leqq \bar{a} \in \bar{R} ,$$

because $T_{y^{R}\bar{y}}\bar{x} = \bar{x}(y)\bar{y}$, $T_{\varphi}(\bar{a} - \bar{x}) = \bar{a} - \bar{x}$. Putting

$$\bar{b} = \bigcap_{0 \leqq \bar{x} \leqq \bar{a}} (\bar{x}(y)\bar{y} + (\bar{a} - \bar{x})) ,$$

we have obviously $0 \leqq \bar{b} \leqq \bar{a}$. For each point $\bar{\mathfrak{p}}$ of the proper space $\mathfrak{G}$ of $\bar{R}$, we have obviously

$$[\bar{x}]\bar{a} \downarrow_{[\bar{x}] \in \bar{\mathfrak{p}}} 0 ,$$

because R has no discrete element except for 0 by assumption. Thus we have

$$\inf_{[\bar{x}] \in \bar{\mathfrak{p}}} [\bar{x}]\bar{a}(y) = 0 .$$

Since for $[\bar{x}] \in \bar{\mathfrak{p}}$

$$\left(\frac{\bar{b}}{\bar{a}} , \bar{\mathfrak{p}}\right) \leqq \left(\frac{[\bar{x}]\bar{a}(y)\bar{y} + (1 - [\bar{x}])\bar{a}}{\bar{a}} , \bar{\mathfrak{p}}\right) = [\bar{x}]\bar{a}(y)\left(\frac{\bar{y}}{\bar{a}} , \bar{\mathfrak{p}}\right),$$

we obtain hence $\left(\frac{\bar{b}}{\bar{a}} , \bar{\mathfrak{p}}\right) = 0$ for every point $\bar{\mathfrak{p}}$; and consequently $\bar{b} = 0$. Therefore we conclude

$$y^{R}\bar{y} \frown \varphi = 0$$

for every $0 \leqq y \in R$, $0 \leqq \bar{y} \in \bar{R}$. Since $x^{R} (x \in R)$ is complete in $\bar{\bar{R}}$, recalling Theorem 1.8, we see by definition that φ is singular.

Theorem 6.2. *Let both R and S be semi-regular, and $\bar{R}$ and $\bar{S}$ the conjugate spaces of R and S respectively. If R has no discrete element except for* 0, *then for each $y \in S$, putting*

$$\Phi_{y}(\varphi, x) = \varphi(x, y) \qquad (\varphi \in \overline{R,S},\ x \in R) ,$$

we obtain a singular bilinear functional Φ_{y} on $\overline{R,S}$ and R.

Proof. We need only prove that Φ_{y} is a singular bilinear functional on $\overline{R,S}$ and R for $0 \leqq y \in S$. For every $0 \leqq y \in S$, it is obvious that Φ_{y} is a positive universally continuous bilinear functional on $\overline{R,S}$ and R. We shall prove firstly that for every positive universally continuous linear functional $\bar{\varphi}_0$ on $\overline{R,S}$ and $\bar{x}_0$ on $\bar{R}$, we have

$$\bar{\varphi}_0\bar{x}_0{}_\frown\Phi_y = 0 ,$$

that is, by Theorem 3.4

$$T_{\bar{\varphi}_0\bar{x}_0}{}_\frown T_{\Phi_y} = 0 .$$

For arbitrary $0 \leqq \bar{a} \in \bar{R}$, $0 \leqq \bar{b} \in \bar{S}$, putting

$$\bar{c} = (T_{\bar{\varphi}_0\bar{x}_0}{}_\frown T_{\Phi_y})\bar{a}\bar{b} ,$$

we have by Theorem 2.1

$$\bar{c} = \bigcap_{0 \leqq \varphi \leqq \bar{a}\bar{b}} (T_{\Phi_y}\varphi + T_{\bar{\varphi}_0\bar{x}_0}(\bar{a}\bar{b} - \varphi))$$
$$= \bigcap_{0 \leqq \varphi \leqq \bar{a}\bar{b}} (\varphi^{(y)} + \bar{\varphi}_0(\bar{a}\bar{b} - \varphi)\bar{x}_0) .$$

Thus, for every projector $[\bar{p}]$ in $\bar{R}$, putting $\varphi = ((1-[\bar{p}])\bar{a})\bar{b}$, we have

$$0 \leqq \bar{c} \leqq \bar{b}(y)(1-[\bar{p}])\bar{a} + \bar{\varphi}_0(([\bar{p}]\bar{a})\bar{b})\bar{x}_0 .$$

Especially, putting $[\bar{p}] = 0$, we obtain $0 \leqq \bar{c} \leqq \bar{b}(y)\bar{a}$. For every point $\mathfrak{p}$ in the proper space $\mathfrak{E}$ of $\bar{R}$, we have

$$[\bar{p}]\bar{a} \downarrow_{[\bar{p}] \in \mathfrak{p}} 0 ,$$

because R has no discrete element except for 0 by assumption. Consequently we obtain by Theorem 1.8

$$\inf_{[\bar{p}] \in \mathfrak{p}} \bar{\varphi}_0(([\bar{p}]\bar{a})\bar{b}) = 0 .$$

Since $\left(\frac{\bar{c}}{\bar{a}}, \mathfrak{p}\right) \leqq \bar{\varphi}_0(([\bar{p}]\bar{a})\bar{b})\left(\frac{\bar{x}_0}{\bar{a}}, \mathfrak{p}\right)$ for every $[\bar{p}] \in \mathfrak{p}$, we obtain therefore $\left(\frac{\bar{c}}{\bar{a}}, \mathfrak{p}\right) = 0$ for every point $\mathfrak{p}$, and hence $\bar{c} = 0$. Recalling Theorem 5.4 and 2.5, we conclude thus further that $T_{\bar{\varphi}_0\bar{x}_0}{}_\frown T_{\Phi_y} = 0$. Therefore Φ_y is singular by definition.

§ 7. Product spaces

Let R and S be universally continuous. For a universally continuous semi-ordered linear space W, if there exists a correspondence

$$(xy)^W \in W \qquad \text{for every } x \in R \text{ and } y \in S,$$

such that

1) $((\alpha x)y)^W = (x(\alpha y))^W = \alpha(xy)^W$ for every real number α,

2) $((x_1+x_2)y)^W = (x_1y)^W+(x_2y)^W$ for $x_1, x_2 \in R, y \in S$,
$(x(y_1+y_2))^W = (xy_1)^W+(xy_2)^W$ for $x \in R$, $y_1, y_2 \in S$,

3) $R \ni x_\lambda\ (\lambda \in \Lambda), x = \bigcap_{\lambda \in \Lambda} x_\lambda$, $0 \leqq y \in S$ implies $(xy)^W = \bigcap_{\lambda \in \Lambda}(x_\lambda y)^W$,
$0 \leqq x \in R$, $y_\lambda \in S\ (\lambda \in \Lambda)$, $y = \bigcap_{\lambda \in \Lambda} y_\lambda$ implies $(xy)^W = \bigcap_{\lambda \in \Lambda}(xy_\lambda)^W$,

4) $(xy)^W (0 \leqq x \in R,\ 0 \leqq y \in S)$ is positively fundamental in W,

then W is said to be a *product space* of R and S by the correspondence $(xy)^W \in W(x \in R, y \in S)$, and this correspondence is called the *product correspondence*.

Concerning the product correspondence $(xy)^W \in W(x \in R, y \in S)$, we obtain obviously by 1)

$$(0y)^W = (x0)^W = 0 \qquad \text{for every } x \in R, y \in S,$$

and by 3) that

$$x_1 \geqq x_2 \text{ implies } (x_1y)^W \geqq (x_2y)^W \qquad \text{for every } 0 \leqq y \in S,$$
$$y_1 \geqq y_2 \text{ implies } (xy_1)^W \geqq (xy_2)^W \qquad \text{for every } 0 \leqq x \in R.$$

Thus we have obviously that $(xy)^W \geqq 0$ for every $0 \leqq x \in R$, $0 \leqq y \in S$.

Let both R and S be semi-regular and $\overline{R}$ and $\overline{S}$ the conjugate spaces of R and S respectively. A product space W of R and S by a product correspondence $(xy)^W \in W\ (x \in R, y \in S)$ is said to be *normal*, if W is semi-regular and the conjugate space $\overline{W}$ of W is a product space of $\overline{R}$ and $\overline{S}$ by a product correspondence $(\overline{x}\overline{y})^{\overline{W}} \in \overline{W}(\overline{x} \in \overline{R}, \overline{y} \in \overline{S})$ such that

$$(\overline{x}\overline{y})^{\overline{W}}((xy)^W) = \overline{x}(x)\overline{y}(y) \qquad \text{for } x \in R, y \in S, \overline{x} \in \overline{R}, \overline{y} \in \overline{S}.$$

With this definition, if a product space W is normal, then the product correspondence $(\overline{x}\overline{y})^{\overline{W}} \in \overline{W}\ (\overline{x} \in \overline{R}, \overline{y} \in \overline{S})$ is determined uniquely, because $(xy)^W (0 \leqq x \in R, 0 \leqq y \in S)$ is positively fundamental in W.

Let W be a normal product space of R and S by a product correspondence $(xy)^W \in W(x \in R, y \in S)$. For every $\overline{z} \in \overline{W}$, putting

$$\varphi_{\overline{z}}(x, y) = \overline{z}((xy)^W) \qquad \text{for } x \in R,\ y \in S,$$

we obtain obviously a bilinear functional $\varphi_{\overline{z}}$ on R and S. For this bilinear functional $\varphi_{\overline{z}} (\overline{z} \in \overline{W})$, it is obvious by definition that

$$\varphi_{\alpha \overline{z}_1 + \beta \overline{z}_2} = \alpha \varphi_{\overline{z}_1} + \beta \varphi_{\overline{z}_2} \qquad \text{for } \overline{z}_1, \overline{z}_2 \in \overline{W},$$

and $\overline{z} \geqq 0$ implies $\varphi_{\overline{z}} \geqq 0$. Conversely, if $\varphi_{\overline{z}} \geqq 0$, then we have

$$\overline{z}^+((xy)^W) \geqq \overline{z}^-((xy)^W) \qquad \text{for } 0 \leqq x \in R,\ 0 \leqq y \in S.$$

Thus, putting $A = \{z\colon \overline{z}^+(z) \geqq \overline{z}^-(z), z \in W\}$, we have

$$A \ni \sum_{\nu=1}^{\kappa} \alpha_\nu (x_\nu y_\nu)^W$$

for every finite number of elements $0 \leqq x_\nu \in R$, $0 \leqq y_\nu \in S$ and real numbers $\alpha_\nu \geqq 0$ $(\nu = 1, 2, \cdots, \kappa)$. Furthermore we see easily that A is closed in W. Consequently A contains by the condition 4) all positive elements of $\overline{W}$, that is,

$$\bar{z}^+(z) \geqq \bar{z}^-(z) \qquad \text{for every} \quad 0 \leqq z \in W.$$

This relation yields $\bar{z}^- = \bar{z}^+ \frown \bar{z}^- = 0$. Therefore $\varphi_{\bar{z}} \geqq 0$ implies $\bar{z} \geqq 0$. Thus we have $\varphi_{\bar{z}_1} \geqq \varphi_{\bar{z}_2}$ if and only if $\bar{z}_1 \geqq \bar{z}_2$ in $\overline{W}$.

$$\overline{W} \ni \bar{z}_\lambda \downarrow_{\lambda \in \Lambda} \bar{z}_0 \text{ implies } \varphi_{\bar{z}_\lambda} \downarrow_{\lambda \in \Lambda} \varphi_{\bar{z}_0},$$

because, if $\overline{W} \ni \bar{z}_\lambda \downarrow_{\lambda \ni \Lambda} \bar{z}_0$, then we have obviously $\varphi_{\bar{z}_\lambda} \downarrow_{\lambda \in \Lambda}$ and for every $0 \leqq x \in R$, $0 \leqq y \in S$ we have

$$\varphi_{\bar{z}_0}(x, y) = \bar{z}_0((xy)^W) = \inf_{\lambda \in \Lambda} \bar{z}_\lambda((xy)^W) = \inf_{\lambda \in \Lambda} \varphi_{\bar{z}_\lambda}(x, y).$$

We also can prove likewise

$$\overline{W} \ni \bar{z}_\lambda \uparrow_{\lambda \in \Lambda} \bar{z}_0 \text{ implies } \varphi_{\bar{z}_\lambda} \uparrow_{\lambda \in \Lambda} \varphi_{\bar{z}_0}.$$

If we denote by $\overline{W}_1$ the totality of those elements $\bar{z} \in \overline{W}$ for which $\varphi_{\bar{z}}$ is totally continuous, then we have obviously

$$\overline{W}_1 \ni \sum_{\nu=1}^{\kappa} \alpha_\nu (\bar{x}_\nu \bar{y}_\nu)^{\overline{W}}$$

for every finite number of elements $0 \leqq \bar{x}_\nu \in \overline{R}$, $0 \leqq \bar{y}_\nu \in \overline{S}$ and real numbers $\alpha_\nu \geqq 0$ $(\nu = 1, 2, \cdots, \kappa)$. If $\overline{W}_1 \ni \bar{z}_\lambda \downarrow_{\lambda \in \Lambda} \bar{z}_0$, then we have $\varphi_{\bar{z}_\lambda} \downarrow_{\lambda \in \Lambda} \varphi_{\bar{z}_0}$, as proved just above, and hence $\overline{W}_1 \ni \bar{z}_0$. Since $(\bar{x}\bar{y})^{\overline{W}}$ $(0 \leqq \bar{x} \in \overline{R}, 0 \leqq \bar{y} \in \overline{S})$ is positively fundamental in $\overline{W}$, we conclude hence $\overline{W}_1 = \overline{W}$, that is, $\varphi_{\bar{z}}$ is totally continuous for every $\bar{z} \in \overline{W}$.

We denote by $\overline{W}_0$ the totality of those elements $\bar{z} \in \overline{W}$ for which $\varphi_{\bar{z}}$ is absolutely continuous. Then we have obviously $(\bar{x}\bar{y})^{\overline{W}} \in \overline{W}_0$ for every $\bar{x} \in \overline{R}$, $\bar{y} \in \overline{S}$, and hence $\overline{W}_0$ is a complete semi-normal manifold of $\overline{W}$, because $(\bar{x}\bar{y})^{\overline{W}}$ $(0 \leqq \bar{x} \in \overline{R}, 0 \leqq \bar{y} \in \overline{S})$ is positively fundamental in $\overline{W}$ by assumption. Furthermore $\varphi_{\bar{z}}(\bar{z} \in \overline{W}_0)$ is obviously a linear manifold of $\overline{R} \times \overline{S}$.

For every $z \in W$, putting

$$\bar{\varphi}_z(\bar{x}, \bar{y}) = (\bar{x}\bar{y})^{\overline{W}}(z) \qquad \text{for} \quad \bar{x} \in \overline{R}, \ \bar{y} \in \overline{S},$$

we obtain likewise a totally continuous bilinear functional $\bar{\varphi}_z$ on $\overline{R}$ and $\overline{S}$ such that

$$\bar{\varphi}_{\alpha z_1 + \beta z_2} = \alpha \bar{\varphi}_{z_1} + \beta \bar{\varphi}_{z_2} \qquad \text{for} \quad z_1, z_2 \in W,$$

and we have $\bar{\varphi}_{z_1} \geqq \bar{\varphi}_{z_2}$ if and only if $z_1 \geqq z_2$.

For a positive bilinear functional $\bar{\varphi}$ on $\bar{R}$ and $\bar{S}$, if there is a positive element $a \in W$ such that $\bar{\varphi} \leqq \bar{\varphi}_a$, then $\bar{\varphi}$ is obviously totally continuous, and hence there exists by Theorem 5.6 a universally continuous positive linear functional ψ on $\bar{R} \times \bar{S}$ such that

$$\bar{\varphi}(\bar{x}, \bar{y}) = \psi(\bar{x}\bar{y}) \qquad \text{for every } \bar{x} \in \bar{R},\ \bar{y} \in \bar{S}.$$

Since $\varphi_{\bar{z}} (\bar{z} \in \overline{W}_0)$ is a linear manifold of $\bar{R} \times \bar{S}$, putting

$$\bar{\psi}(\bar{z}) = \psi(\varphi_{\bar{z}}) \qquad \text{for every } \bar{z} \in \overline{W}_0,$$

we obtain a positive linear functional $\bar{\psi}$ on $\overline{W}_0$. Furthermore $\bar{\psi}$ is universally continuous, because $\overline{W}_0 \ni \bar{z}_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies $\varphi_{\bar{z}_\lambda} \downarrow_{\lambda \in \Lambda} 0$, as proved just above, and hence

$$\inf_{\lambda \in \Lambda} \bar{\psi}(\bar{z}_\lambda) = \inf_{\lambda \in \Lambda} \psi(\varphi_{\bar{z}_\lambda}) = 0 .$$

For this $\bar{\psi}$, we have for every $0 \leqq \bar{x} \in \bar{R}$, $0 \leqq \bar{y} \in \bar{S}$

$$\bar{\psi}((\bar{x}\bar{y})^{\overline{W}}) = \psi(\bar{x}\bar{y}) = \bar{\varphi}(\bar{x}, \bar{y}) \leqq \bar{\varphi}_a(\bar{x}\bar{y}) = (\bar{x}\bar{y})^{\overline{W}}(a) .$$

Since $(\bar{x}\bar{y})^{\overline{W}}$ $(0 \leqq \bar{x} \in \bar{R},\ 0 \leqq \bar{y} \in \bar{S})$ is positively fundamental in $\overline{W}_0$, we conclude hence

$$0 \leqq \bar{\psi}(\bar{z}) \leqq \bar{z}(a) \qquad \text{for every } 0 \leqq \bar{z} \in \overline{W}_0 .$$

Since $\overline{W}_0$ is a complete semi-normal manifold of $\overline{W}$, we can hence extend $\bar{\psi}$ universally continuous over $\overline{W}$ such that

$$0 \leqq \bar{\psi}(\bar{z}) \leqq \bar{z}(a) \qquad \text{for every } 0 \leqq \bar{z} \in \overline{W} .$$

Thus there exists $b \in W$ such that $0 \leqq b \leqq a$ and

$$\bar{\psi}(\bar{z}) = \bar{z}(b) \qquad \text{for every } \bar{z} \in \overline{W} .$$

For this $b \in W$, we have

$$\bar{\varphi}_b(\bar{x}, \bar{y}) = (\bar{x}\bar{y})^{\overline{W}}(b) = \bar{\psi}((\bar{x}\bar{y})^{\overline{W}}) = \bar{\varphi}(\bar{x}, \bar{y})$$

for every $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$, that is, $\bar{\varphi}_b = \bar{\varphi}$. Therefore $\bar{\varphi}_z (z \in W)$ is a semi-normal manifold of $\overline{\bar{R}, \bar{S}}$.

Since R is isomorphic to a complete semi-normal manifold of $\bar{\bar{R}}$ by the correspondence $R \ni x \rightarrow x^{\bar{R}} \in \bar{\bar{R}}$, $x^{\bar{R}}(\bar{x}) = \bar{x}(x)$ for $\bar{x} \in \bar{R}$, and S is isomorphic to a complete semi-normal manifold of $\bar{\bar{S}}$ by the correspondence $S \ni y \rightarrow y^{S} \in \bar{\bar{S}}$, $y^{S}(\bar{y}) = \bar{y}(y)$ for $\bar{y} \in \bar{S}$, we see easily by Theorems 1.8 and 5.5 that $\bar{\varphi}_{(xy)^W}$ $(0 \leqq x \in R,\ 0 \leqq y \in S)$ is positively fundamental in $\overline{\bar{R}, \bar{S}}$, and hence $\bar{\varphi}_z (z \in W)$ is a complete semi-normal manifold of $\overline{\bar{R}, \bar{S}}$.

Thus we can state that if a product space W of R and S is normal by a product correspondence $(xy)^W \in W\, (x \in R, y \in S)$, then W is isomorphic to a complete semi-normal manifold of $\overline{\bar{R}, \bar{S}}$ by a correspondence $W \ni z \to \bar{\varphi}_z \in \overline{\bar{R}, \bar{S}}$, $\bar{\varphi}_{(xy)^W}(\bar{x}, \bar{y}) = \bar{x}(x)\bar{y}(y)$ for $x \in R$, $y \in S$, $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$.

Conversely, if a product space W of R and S by a product correspondence $(xy)^W \in W\, (x \in R, y \in S)$ is isomorphic to a complete semi-normal manifold of $\overline{\bar{R}, \bar{S}}$ by a correspondence $W \ni z \to \bar{\varphi}_z \in \overline{\bar{R}, \bar{S}}$, $\bar{\varphi}_{(xy)^W}(\bar{x}, \bar{y}) = \bar{x}(x)\bar{y}(y)$ for $x \in R$, $y \in S$, $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$, then, since $\overline{\bar{R}, \bar{S}}$ is by Theorem 5.7 isomorphic to the conjugate space $\overline{\bar{R} \times \bar{S}}$ of $\bar{R} \times \bar{S}$ by a correspondence

$$\overline{\bar{R}, \bar{S}} \ni \bar{\varphi} \to \bar{\varphi}^{\bar{R} \times \bar{S}} \in \overline{\bar{R} \times \bar{S}}, \quad \bar{\varphi}^{\bar{R} \times \bar{S}}(\bar{x}\bar{y}) = \bar{\varphi}(\bar{x}, \bar{y}) \text{ for } \bar{x} \in \bar{R}, \bar{y} \in \bar{S},$$

$\bar{R} \times \bar{S}$ is isomorphic to a complete semi-normal manifold of the conjugate space $\overline{W}$ of W by a correspondence $\bar{R} \times \bar{S} \ni \varphi \to \bar{z}_\varphi \in \overline{W}$, $\bar{\varphi}_z^{\bar{R} \times \bar{S}}(\varphi) = \bar{z}_\varphi(z)$ for $\varphi \in \bar{R} \times \bar{S}$, $z \in W$. Thus putting $(\bar{x}\bar{y})^{\overline{W}} = \bar{z}_{\bar{x}\bar{y}}$ for $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$, we see easily that $\overline{W}$ is a product space of $\bar{R}$ and $\bar{S}$ by this correspondence $(\bar{x}\bar{y})^{\overline{W}} \in \overline{W}$ $(\bar{x} \in \bar{R}, \bar{y} \in \bar{S})$ and we have for every $x \in R, y \in S$

$$(\bar{x}\bar{y})^{\overline{W}}((xy)^W) = \bar{z}_{\bar{x}\bar{y}}((xy)^W) = \bar{\varphi}^{\bar{R} \times \bar{S}}_{(xy)^W}(\bar{x}\bar{y}) = \bar{\varphi}_{(xy)^W}(\bar{x}, \bar{y}) = \bar{x}(x)\bar{y}(y).$$

Therefore the product space W is normal by definition. Now we can state:

Theorem 7.1. *A product space W of R and S by a product correspondence $(xy)^W \in W\, (x \in R, y \in S)$ is normal if and only if W is isomorphic to a complete semi-normal manifold of $\overline{\bar{R}, \bar{S}}$ by a correspondence $W \ni z \to \bar{\varphi}_z \in \overline{\bar{R}, \bar{S}}$,*

$$\bar{\varphi}_{(xy)^W}(\bar{x}, \bar{y}) = \bar{x}(x)\bar{y}(y) \qquad \textit{for} \quad \bar{x} \in \bar{R}, \bar{y} \in \bar{S}.$$

For each $x \in R$, $y \in S$, denoting by xy a bilinear functional on $\bar{R}$ and $\bar{S}$ such that

$$xy(\bar{x}, \bar{y}) = \bar{x}(x)\bar{y}(y) \qquad \text{for} \quad \bar{x} \in \bar{R}, \bar{y} \in \bar{S},$$

we see easily by Theorems 1.8 and 5.5 that $xy\, (0 \leqq x \in R, 0 \leqq y \in S)$ is positively fundamental in $\overline{\bar{R}, \bar{S}}$. Thus $\overline{\bar{R}, \bar{S}}$ is a product space of R and S by a product correspondence $xy\, (x \in R, y \in S)$ and furthermore normal by Theorem 7.1. This product space $\overline{\bar{R}, \bar{S}}$ is called the *upper product space of* R and S, and denoted by RS.

For the upper product space RS, the totality of those elements $z \in RS$ for which we can find $0 \leqq x \in R$ and $0 \leqq y \in S$ such that $z \leqq xy$, constitutes obviously a complete semi-normal manifold of RS which is by Theorem 7.1 a normal product space of R and S. This product space is called the *lower product space* of R and S, and denoted by $R \times S$.

With this definition, we see at once that every normal product space of R and S is isomorphic to a complete semi-normal manifold of the upper product space which includes the lower product space.

Theorem 7.2. *$\overline{R,S}$ is isomorphic to the conjugate space $\overline{R\times S}$ of the lower product space $R\times S$ by a correspondence*

$$\overline{R,S}\ni\varphi\to\varphi^{R\times S}\in\overline{R\times S},\quad \varphi^{R\times S}(xy)=\varphi(x,y)\qquad \textit{for } x\in R,\ y\in S.$$

Proof. For $0\leqq\varphi\in\overline{R,S}$ and $0\leqq y\in S$, we obtain a universally continuous positive linear functional $\varphi(x,y)$ $(x\in R)$ on R, and hence putting

$$\varphi^{\bar R,S}(\bar x,y)=\sup_{x^{\bar R}\leqq\bar x}\varphi(x,y)\qquad\text{for}\quad x\in R,\ \bar x\in\bar R,$$

and for an arbitrary $y\in S$

$$\varphi^{\bar R,S}(\bar x,y)=\varphi^{\bar R,S}(\bar x,y^+)-\varphi^{\bar R,S}(\bar x,y^-),$$

we obtain a universally continuous positive bilinear functional $\varphi^{\bar R,S}$ on $\bar R$ and S. From this $\varphi^{\bar R,S}$, we obtain likewise a universally continuous positive bilinear functional $\varphi^{\bar R,\bar S}$ on $\bar R$ and $\bar S$ such that

$$\varphi^{\bar R,\bar S}(x^{\bar R},y^{\bar S})=\varphi(x,y)\qquad\text{for}\quad x\in R,\ y\in S.$$

Since $x^{\bar R}(0\leqq x\in R)$ is positively fundamental in $\bar R$, and $y^{\bar S}(0\leqq y\in S)$ is positively fundamental in $\bar S$, we see easily that we have $\varphi_1^{\bar R,\bar S}\geqq\varphi_2^{\bar R,\bar S}$ if and only if $\varphi_1\geqq\varphi_2$, and $\varphi_\lambda\downarrow_{\lambda\in\Lambda}0$ implies $\varphi_\lambda^{\bar R,\bar S}\downarrow_{\lambda\in\Lambda}0$. Thus $\varphi_\lambda\downarrow_{\lambda\in\Lambda}\varphi$ implies $\varphi_\lambda^{\bar R,\bar S}\downarrow_{\lambda\in\Lambda}\varphi^{\bar R,\bar S}$, and $\varphi_\lambda\uparrow_{\lambda\in\Lambda}\varphi$ implies $\varphi_\lambda^{\bar R,\bar S}\uparrow_{\lambda\in\Lambda}\varphi^{\bar R,\bar S}$. Denoting by A the totality of those bilinear functionals $0\leqq\varphi\in\overline{R,S}$ for which $\varphi^{\bar R,\bar S}$ is totally continuous, we have hence that $A\ni\varphi_\lambda\downarrow_{\lambda\in\Lambda}\varphi\in\overline{R,S}$ implies $\varphi\in A$, and $A\ni\varphi_\lambda\uparrow_{\lambda\in\Lambda}\varphi\in\overline{R,S}$ implies $\varphi\in A$. Furthermore we have $A\ni\sum_{\nu=1}^{\kappa}\alpha_\nu\bar x_\nu\bar y_\nu$ for every finite number of elements $0\leqq\bar x_\nu\in\bar R$, $0\leqq\bar y_\nu\in\bar S$, and numbers $\alpha_\nu\geqq0$ $(\nu=1,2,\cdots\kappa,)$, because

$$(\bar x\bar y)^{\bar R,\bar S}=\bar x^{\bar R}\bar y^{\bar S}\qquad\text{for}\quad 0\leqq\bar x\in\bar R,\ 0\leqq\bar y\in\bar S.$$

Thus A contains by Theorem 5.5 all $0\leqq\varphi\in\overline{R,S}$, that is, $\varphi^{\bar R,\bar S}$ is totally continuous for every $0\leqq\varphi\in\overline{R,S}$. Therefore there exists by Theorem 5.6 a universally continuous positive linear functional $\varphi^{\bar R\times\bar S}$ on $\bar R\times\bar S$ such that

$$\varphi^{\bar R\times\bar S}(\bar x\bar y)=\varphi^{\bar R,\bar S}(\bar x,\bar y)\qquad\text{for}\quad\bar x\in\bar R,\ \bar y\in\bar S.$$

Since $R\times S\subset\bar R\times\bar S\subset RS$ by definition, for each $\varphi\in\overline{R,S}$ there exists hence $\varphi^{R\times S}\in\overline{R\times S}$ such that

$$\varphi^{R\times S}(xy) = \varphi(x,y) \qquad \text{for} \quad x\in R,\ y\in S.$$

Furthermore we can prove likewise as in Proof of Theorem 5.7 that $\varphi^{R\times S}(\varphi\in\overline{R,S})$ coincides with $\overline{R\times S}$.

For each $\bar{z}\in\overline{R\times S}$ we obtain $\bar{z}^{\bar{R}\times\bar{S}}\in\overline{\bar{S}\times\bar{R}}$, as proved just above. Thus $\overline{R,S}$ is isomorphic to the upper product space $\bar{R}\bar{S}=\overline{\bar{R},\bar{S}}$ of $\bar{R}$ and $\bar{S}$ by a correspondence

$$\overline{R,S}\ni\varphi\rightarrow\varphi^{\bar{R},\bar{S}}\in\overline{\bar{\bar{R}},\bar{\bar{S}}}, \qquad \varphi^{\bar{R},\bar{S}}(x^{\bar{R}},y^{\bar{S}})=\varphi(x,y) \qquad \text{for } x\in R,\ y\in S.$$

Consequently $\bar{R}\times\bar{S}$ is by definition isomorphic to the lower product space of $\bar{R}$ and $\bar{S}$. Thus we can state

Theorem 7.3. *$\overline{R,S}$ is isomorphic to the upper product space $\bar{R}\bar{S}$ of $\bar{R}$ and $\bar{S}$, and $\bar{R}\times\bar{S}$ is isomorphic to the lower product space of $\bar{R}$ and $\bar{S}$.*

Theorem 7.4. *If a product space W of R and S is normal, then the conjugate space $\overline{W}$ of W also is a normal product space of $\bar{R}$ and $\bar{S}$.*

Proof. If a product space W of R and S is normal, then W is by Theorem 7.1 isomorphic to a complete semi-normal manifold of the upper product space RS which includes the lower product space $R\times S$. Thus the conjugate space $\overline{W}$ is isomorphic to a complete semi-normal manifold of tee conjugate space $\overline{R\times S}$ of $R\times S$, and hence $\overline{W}$ is normal by Theoremr 7.1 and 7.2.

§8. Product norms

Let R and S be normed lattice ordered linear spaces. A bilinear functional $\varphi\in\widetilde{R,S}$ on R and S is said to be *norm bounded*, if

$$\sup_{\|x\|\leq 1,\|y\|\leq 1}|\varphi|(x,y) < +\infty \qquad (x\in R,\ y\in S).$$

The totality of norm bounded $\varphi\in\widetilde{R,S}$ is called the *norm biassociated space* of R and S, and denoted by $\widetilde{R,S}^{\|}$. It is obvious that $\widetilde{R,S}^{\|}$ is a semi-normal manifold of $\widetilde{R,S}$. For each $\varphi\in\widetilde{R,S}^{\|}$ we define the norm $\|\varphi\|$ as

$$\|\varphi\| = \sup_{\|x\|\leq 1,\|y\|\leq 1}|\varphi|(x,y) \qquad (x\in R, y\in S).$$

With this definition, we see easily that $\widetilde{R,S}^{\|}$ is a normed semi-ordered linear space.

Theorem 8.1. *The norm of $\widetilde{R,S}^{\|}$ is universally semi-continuous and universally monotone complete.*

Proof. If $0 \leqq \varphi_\lambda \uparrow_{\lambda \in \Lambda}$, $\varphi_\lambda \in \widetilde{R,S}''$ $(\lambda \in \Lambda)$ and

$$\sup_{\lambda \in \Lambda} \|\varphi_\lambda\| < +\infty ,$$

then we have obviously

$$\sup_{\lambda \in \Lambda} \varphi_\lambda (x, y) < +\infty \qquad \text{for } 0 \leqq x \in R,\ 0 \leqq y \in S ,$$

and hence there exists by Theorem 1.2 $\varphi \in \widetilde{R,S}$ for which $\varphi_\lambda \uparrow_{\lambda \in \Lambda} \varphi \in \widetilde{R,S}$. For such $\varphi \in \widetilde{R,S}$, since

$$\varphi (x,y) = \sup_{\lambda \in \Lambda} \varphi_\lambda (x,y) \qquad \text{for } 0 \leqq x \in R,\ 0 \leqq y \in S ,$$

we have $\varphi(x,y) \leqq \sup_{\lambda \in \Lambda} \|\varphi_\lambda\|$ for $\|x\| \leqq 1$, $\|y\| \leqq 1$, and consequently

$$\|\varphi\| \leqq \sup_{\lambda \in \Lambda} \|\varphi_\lambda\| , \quad \varphi \in \widetilde{R,S}'' .$$

Thus the norm of $\widetilde{R,S}''$ is universally monotone complete. Furthermore, it is obvious that $\|\varphi\| \geqq \|\varphi_\lambda\|$ for every $\lambda \in \Lambda$, and hence $\|\varphi\| = \sup_{\lambda \in \Lambda} \|\varphi_\lambda\|$. Thus the norm on $\widetilde{R,S}''$ is universally semi-continuous.

Theorem 8.2. *If both R and S are norm complete together, then* $\widetilde{R,S}''$ *coincides with* $\widetilde{R,S}$.

Proof. If $0 \leqq \varphi \in \widetilde{R,S}$ and $\varphi(x_\nu, y_\nu) \geqq 2^\nu$, $0 \leqq x_\nu \in R$, $0 \leqq y_\nu \in S$, $\|x_\nu\| \leqq 1$, $\|y_\nu\| \leqq 1$ $(\nu = 1,2,\cdots)$, then, since both R and S are norm complete by assumption, we can find $x \in R$, $y \in S$ such that

$$x = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} x_\nu , \quad y = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} y_\nu ,$$

and we have for every $\kappa = 1, 2, \cdots$

$$\varphi(x,y) \geqq \sum_{\nu=1}^{\kappa} \frac{1}{2^{2\nu}} \varphi(x_\nu, y_\nu) \geqq \kappa$$

contradicting $\varphi(x,y) < +\infty$. Therefore every $\varphi \in \widetilde{R,S}$ is norm bounded.

Let two normed semi-ordered linear spaces R and S be semi-regular and the norms be complete and reflexive, that is,

$$\|x\| = \sup_{\|\bar{x}\| \leqq 1} |\bar{x}(x)| \qquad \text{for } x \in R,\ \bar{x} \in \bar{R},$$

$$\|y\| = \sup_{\|\bar{y}\| \leqq 1} |\bar{y}(y)| \qquad \text{for } y \in S.\ \bar{y} \in \bar{S}.$$

Since $\bar{R}$ and $\bar{S}$ are norm complete, every $\bar{\varphi} \in \widetilde{\bar{R},\bar{S}}$ is norm bounded by Theorem 8.2. Thus we can define a norm $\|z\|_u$ on the upper product space RS as

$$\|z\|_u = \sup_{\|\bar{x}\| \leq 1, \|\bar{y}\| \leq 1} |z(\bar{x}, \bar{y})| \qquad \text{for} \quad z \in RS,\ \bar{x} \in \bar{R},\ \bar{y} \in \bar{S}.$$

This norm $\|z\|_u (z \in RS)$ will be called the *upper norm*. We see by Theorem 8.1 that the upper norm is universally semi-continuous and universally monotone complete. Furthermore we have

$$\|xy\|_u = \|x\| \, \|y\| \qquad \text{for} \quad x \in R,\ y \in S,$$

because the norms of R and S are reflexive by assumption. The upper norm also is reflexive. Because, for the conjugate space $\overline{RS}$ of RS and the conjugate norm

$$\|\bar{z}\| = \sup_{\|z\|_u \leq 1} |\bar{z}(z)| \qquad \text{for} \quad z \in RS,\ \bar{z} \in \overline{RS},$$

we have obviously

$$\|z\|_u \geqq \sup_{\|\bar{z}\| \leq 1} |\bar{z}(z)| \qquad \text{for} \quad z \in RS,\ \bar{z} \in \overline{RS}.$$

On the other hand, for every $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$, we have $\bar{x}\bar{y} \in \overline{RS}$ as

$$\bar{x}\bar{y}(z) = z(\bar{x}, \bar{y}) \qquad \text{for} \quad z \in RS,$$

and we have by the definition of the upper norm $\|z\|_u$

$$\|\bar{x}\bar{y}\| = \sup_{\|z\|_u \leq 1} |z(\bar{x}, \bar{y})| \leqq \|\bar{x}\| \, \|\bar{y}\|.$$

Thus we obtain

$$\sup_{\|\bar{z}\| \leq 1} |\bar{z}(z)| \geqq \sup_{\|\bar{x}\| \leq 1, \|\bar{y}\| \leq 1} |z(\bar{x}, \bar{y})| = \|z\|_u.$$

Therefore we have

$$\|z\|_u = \sup_{\|\bar{z}\| \leq 1} |\bar{z}(z)| \qquad \text{for} \quad z \in RS,\ \bar{z} \in \overline{RS},$$

that is, the upper norm is reflexive.

Since R and S are norm complete by assumption, every $\varphi \in \overline{R,S}$ is norm bounded by Theorem 8.2, that is, $\overline{R,S}$ also is a normed semi-ordered linear space. Since $\overline{R,S}$ is isomorphic to the conjugate space $\overline{R \times S}$ of the lower product space $R \times S$, we can define a norm $\|z\|_l$ on the lower product space $R \times S$ as the conjugate norm from $\overline{R,S}$:

$$\|z\|_l = \sup_{\|\varphi\| \leq 1} |\varphi^{R \times S}(z)| \qquad (z \in R \times S,\ \varphi \in \overline{R,S})$$

for the isomorphic correspondence $\overline{R,S} \ni \varphi \rightarrow \varphi^{R \times S} \in \overline{R \times S}$,

$$\varphi^{R\times S}(xy) = \varphi(x,y) \qquad \text{for} \quad x\in R,\ y\in S.$$

This norm $\|z\|_l (z\in R\times S)$ will be called the *lower norm*. The lower norm $\|z\|_l (z\in R\times S)$ is reflexive, because it is defined as the conjugate norm from $\overline{R,S}$, and $R\times S$ is isomorphic to a complete semi-normal manifold of the conjugate space $\overline{\overline{R,S}}$ of $\overline{R,S}$. Furthermore we have

$$\|xy\|_l = \|x\|\,\|y\| \qquad \text{for} \quad x\in R,\ y\in S,$$

because we have obviously by definition

$$\|xy\|_l = \sup_{\|\varphi\|\leqq 1} |\varphi(x,y)| \leqq \|x\|\,\|y\|,$$

$$\sup_{\|\varphi\|\leqq 1} |\varphi(x,y)| \geqq \sup_{\|\bar{x}\bar{y}\|\leqq 1} |\bar{x}(x)\,\bar{y}(y)| \geqq \|x\|\,\|y\|,$$

since $\|\bar{x}\bar{y}\| = \|\bar{x}\|\,\|\bar{y}\|$ by definition.

Now we consider a normal product space W of R and S. Since W is by Theorem 7.1 isomorphic to a complete semi-normal manifold of the upper product space RS including the lower product space $R\times S$. Thus we can assume without loss of generality that W is a semi-normal manifold of RS and

$$R\times S \subset W \subset RS.$$

A norm $\|z\| (z\in W)$ on W is said to be a *product norm*, if

$$\|xy\| = \|x\|\,\|y\| \qquad \text{for} \quad x\in R,\ y\in S.$$

With this definition, it is obvious that both the upper norm and the lower norm are product norms. And we have

$$\|z\|_u \leqq \|z\|_l \qquad \text{for} \quad z\in R\times S,$$

because we have by definition

$$\|z\|_l = \sup_{\|\varphi\|\leqq 1} |\varphi^{R\times S}(z)| \geqq \sup_{\|\bar{x}\bar{y}\|\leqq 1} |z(\bar{x},\bar{y})|$$

$$\geqq \sup_{\|\bar{x}\|\leqq 1,\|\bar{y}\|\leqq 1} |z(\bar{x},\bar{y})| = \|z\|_u.$$

Recalling Theoremr 7.2 and 7.3, we have obviously by definition

Theorem 8.3. The upper norm on RS is the conjugate norm of the lower norm on $\bar{R}\times\bar{S}$, the upper norm on $\bar{R}\bar{S}$ is the conjugate norm of the lower norm on $R\times S$.

For a product norm $\|z\| (z\in W)$, the norm conjugate space $\overline{W}^{\|}$ of W may be considered as a complete semi-normal manifold of the upper product space $\bar{R}\bar{S}$, and we have for the conjugate norm $\|\bar{z}\| (\bar{z}\in\overline{W}^{\|})$

$$\|\bar{z}\| = \sup_{\|z\| \leq 1} |\bar{z}(z)| \geq \sup_{\|xy\| \leq 1} |\bar{z}(xy)| \geq \sup_{\|x\| \leq 1, \|y\| \leq 1} |\bar{z}(xy)| = \|\bar{z}\|_u .$$

Thus, if the conjugate norm $\|\bar{z}\|$ $(\bar{z} \in \overline{W})$ also is a product norm from $\bar{R}$ and $\bar{S}$ then we have by the same reason

$$\|z\| \geq \sup_{\|\bar{z}\| \leq 1} |\bar{z}(z)| \geq \|z\|_u \qquad \text{for} \quad z \in W .$$

Conversely, if $\|z\| \geq \|z\|_u$ for $z \in W$, then we obtain by Theorem 8.3 for every $\bar{z} \in \bar{R} \times \bar{S}$

$$\|\bar{z}\|_l = \sup_{\|z\|_u \leq 1} |\bar{z}(z)| \geq \sup_{\|z\| \leq 1} |\bar{z}(z)| = \|\bar{z}\| ,$$

and $\|\bar{z}\| \geq \|\bar{z}\|_u$ for $\bar{z} \in \overline{W}$, as proved just above. Thus we have

$$\|\bar{x}\| \, \|\bar{y}\| = \|\bar{x}\bar{y}\|_l \geq \|\bar{x}\bar{y}\| \geq \|\bar{x}\bar{y}\|_u = \|\bar{x}\| \, \|\bar{y}\| ,$$

and hence $\|\bar{x}\bar{y}\| = \|\bar{x}\| \, \|\bar{y}\|$, that is, $\|\bar{z}\|$ $(\bar{z} \in \overline{W})$ is a product norm from $\bar{R}$ and $\bar{S}$. Therefore we can state

Theorem 8.4. *The conjugate norm of a product norm* $\|z\|$ $(z \in W)$ *is a product norm from* $\bar{R}$ *and* $\bar{S}$ *if and only if*

$$\|z\| \geq \|z\|_u \qquad \textit{for} \quad z \in W ,$$

and then for the conjngate norm $\|\bar{z}\|$ $(\bar{z} \in \overline{W})$ *we have*

$$\|\bar{z}\|_l \geq \|\bar{z}\| \geq \|\bar{z}\|_u \qquad \textit{for} \quad \bar{z} \in \bar{R} \times \bar{S} .$$

Theorem 8.5. *For a normal product space* W *of* R *and* S, *if there is a complete norm* $\|z\|$ $(z \in W)$ *on* W, *then there exists a product norm* $\|z\|_p$ *on* W *such that* $\|z\|_l \geq \|z\|_p \geq \|z\|_u$ *for every* $z \in R \times S$, $\|z\|_p (z \in W)$ *is reflexive, and every* $\bar{z} \in \overline{W}$ *is norm bounded by this norm* $\|z\|_p$.

Proof. As the norm $\|z\|$ $(z \in W)$ is complete by assumption, we can find $\alpha > 0$ such that

$$\alpha \|z\| \geq \|z\|_u \qquad \text{for} \quad z \in W .$$

For such $\alpha > 0$, putting $\|z\|_\alpha = \alpha \|z\|$, we obtain a norm $\|z\|_\alpha$ on W, and for the conjugate norm $\|\bar{z}\|_\alpha$ $(\bar{z} \in \overline{W})$ of $\|z\|_\alpha$ $(z \in W)$, we have by definition that $\overline{W} \supset \bar{R} \times \bar{S}$ and

$$\|\bar{z}\|_\alpha \leq \|\bar{z}\|_l \qquad \text{for} \quad \bar{z} \in \bar{R} \times \bar{S} .$$

Putting $\|\bar{z}\|_p = \mathrm{Max}\{\|\bar{z}\|_u, \|\bar{z}\|_\alpha\}$ for $\bar{z} \in \overline{W}$, we obtain a norm $\|\bar{z}\|_p$ on $\overline{W}$ such that

$$\|\bar{z}\|_u \leq \|\bar{z}\|_p \leq \|\bar{z}\|_l \qquad \text{for} \quad \bar{z} \in \bar{R} \times \bar{S} .$$

Thus $\|\bar{z}\|_p (\bar{z} \in \overline{W})$ is a product norm from $\bar{R}$ and $\bar{S}$, and equivalent to the norm $\|\bar{z}\|_\alpha$ $(\bar{z}\in\overline{W})$. Because we have obviously

$$\|\bar{z}\|_\alpha \leqq \|\bar{z}\|_p \qquad \text{for} \quad \bar{z}\in\overline{W},$$

and since the conjugate norm $\|\bar{z}\|_\alpha$ $(\bar{z}\in\overline{W})$ is monotone complete, we can find $\beta>0$ such that $\beta\|\bar{z}\|_\alpha \geqq \|\bar{z}\|_p$ for $\bar{z}\in\overline{W}$. Therefore, putting

$$\|z\|_p = \sup_{\|\bar{z}\|_p\leqq 1} |\bar{z}(z)| \qquad \text{for} \quad z\in W,$$

we obtain a product norm $\|z\|_p$ on W such that

$$\|z\|_u \leqq \|z\|_p \leqq \|z\|_l \qquad \text{for} \quad z\in R\times S,$$

and it is obvious that this norm $\|z\|_p$ $(z\in W)$ is reflexive and every $\bar{z}\in\overline{W}$ is norm bounded by $\|z\|_p$ $(z\in W)$.

§ 9. Product modulars

Let R and S be modulared semi-ordered linear spaces, and both R and S are universally continuous. A bilinear functional $\varphi\in\widetilde{R,S}$ is said to be *modular bounded*, if

$$\sup_{m(x)\leqq 1, m(y)\leqq 1} |\varphi|(x,y) < +\infty \qquad \text{for} \quad x\in R,\ y\in S,$$

that is, if φ is norm bounded for the second norm $|||x|||$ $(x\in R)$ and $|||y|||$ $(y\in S)$. With this definition, we can prove easily

Theorem 9.1. *In order that a bilinear functional* $\varphi\in\widetilde{R,S}$ *be modular bounded, it is necessary and sufficient that we can find two positive numbers* α *and* γ *such that*

$$\alpha|\varphi|(x,y) \leqq \gamma + \frac{1}{2}(m(x)+m(y))^2 \qquad \textit{for} \quad x\in R,\ y\in S.$$

The totality of modular bounded bilinear functionals $\varphi\in\widetilde{R,S}$ is called the *modular biassociated space* of R and S, and denoted by $\widetilde{R,S}^m$. For each $\varphi\in\widetilde{R,S}^m$ we define the modular $\overline{m}(\varphi)$ as

$$\overline{m}(\varphi) = \sup_{x\in R, y\in S}\left\{\varphi(x,y) - \frac{1}{2}(m(x)+m(y))^2\right\}.$$

With this definition, we see easily that $\widetilde{R,S}^m$ is a monotone complete modulared semi-ordered linear space.

By virtue of Theorem 8.2, we obtain immediately

Theorem 9.2. *If both* R *and* S *are modular complete, then* $\widetilde{R,S}^m$ *coincides with* $\widetilde{R;S}$.

Let two modulared semi-ordered linear space R and S be semi-regular and modular complete. Since $\overline{R}$ and $\overline{S}$ are modular complete, every $\overline{\varphi} \in \overline{\overline{R},\overline{S}}$ is modular bounded by Theorem 9.2. Thus we can find a modular $m_u(z)$ on the upper product space RS as

$$m_u(z) = \sup_{\overline{x} \in \overline{R}, \overline{y} \in \overline{S}} \left\{ z(\overline{x}, \overline{y}) - \frac{1}{2} (\overline{m}(\overline{x}) + \overline{m}(\overline{y}))^2 \right\}.$$

This modular m_u will be called the *upper modular*. Concerning the upper modular $m_u(z)$ $(z \in RS)$ we have

$$m_u(xy) \leqq \frac{1}{2} \left\{ m(x) + m(y) \right\}^2 \qquad \text{for} \quad x \in R, \ y \in S,$$

because we have by definition

$$\begin{aligned} m_u(xy) &= \sup_{\overline{x} \in \overline{R}, \overline{y} \in \overline{S}} \left\{ \overline{x}(x)\, \overline{y}(y) - \frac{1}{2} (\overline{m}(\overline{x}) + \overline{m}(\overline{y}))^2 \right\} \\ &\leqq \sup_{\overline{x} \in \overline{R}, \overline{y} \in \overline{S}} \left\{ (\overline{m}(\overline{x}) + m(x)) (\overline{m}(\overline{y})) + m(y)) - \frac{1}{2} (\overline{m}(\overline{x}) + \overline{m}(\overline{y}))^2 \right\} \\ &\leqq \frac{1}{2} \left\{ m(x) + m(y) \right\}^2 . \end{aligned}$$

Since R and S are modular complete by assumption, every $\varphi \in \overline{R,S}$ is modular bounded by Theorem 9.2, that is, $\overline{R,S}$ also is a modulared semi-ordered linear space. Since $\overline{R,S}$ is isomorphic to the conjugate space $\overline{R \times S}$ of the lower product space $R \times S$, we can find a modular m_l on the lower product space $R \times S$ as the conjugate modular from $\overline{R,S}$:

$$m_l(z) = \sup_{\varphi \in \overline{R,S}} \{ \varphi^{R \times S}(z) - \overline{m}(\varphi) \} \qquad (z \in R \times S)$$

for the isomorphic correspondence $\overline{R,S} \ni \varphi \to \varphi^{R \times S} \in \overline{R \times S}$,

$$\varphi^{R \times S}(xy) = \varphi(x, y) \qquad \text{for} \quad x \in R, \ y \in S.$$

This modular m_l will be called the *lower modular*. Concerning the lower modular m_l, we also have

$$m_l(xy) \leqq \frac{1}{2} \left\{ m(x) + m(y) \right\}^2 \qquad \text{for} \quad x \in R, \ y \in S,$$

because we have by the definition of $\overline{m}(\varphi)$ $(\varphi \in \overline{R,S})$

$$m_l(xy) = \sup_{\varphi \in \overline{R,S}} \left\{ \varphi(x, y) - \overline{m}(\varphi) \right\} \leqq \frac{1}{2} \left\{ m(x) + m(y) \right\}^2 .$$

Let W be a normal product space of R and S, and $R \times S \subset W \subset RS$. A modular m on W is said to be a *product modular*, if

$$m(xy) \leqq \frac{1}{2} \left\{ m(x) + m(y) \right\}^2 \qquad \text{for} \quad x \in R, \ y \in S.,$$

With this definition, we have obviously that both the upper and the lower modular are product moddulars, the upper modular on RS is the conjugate modular of the lower modular on $\bar{R}\times\bar{S}$, and the upper modular on $\bar{R}\bar{S}$ is the conjugate modular of the lower modular on $R\times S$,

Let $m(z)$ $(z\in W)$ be a product modular on W. Since the modular conjugate space $\overline{W}^m$ of W may be considered as a complete semi-normal manifold of the upper product space $\bar{R}\bar{S}$, we have for the conjugate modular $\bar{m}(\bar{z})$ $(\bar{z}\in\overline{W}^m)$

$$\bar{m}(\bar{z})=\sup_{z\in W}\left\{\bar{z}(z)-m(z)\right\}\geqq\sup_{x\in R,y\in S}\left\{\bar{z}(x,y)-m(xy)\right\}$$
$$\geqq\sup_{x\in R,y\in S}\left\{\bar{z}(x,y)-\frac{1}{2}(m(x)+m(y))^2\right\}=m_u(\bar{z})$$

for every $\bar{z}\in\overline{W}^m$, and hence for every $z\in R\times S$

$$m(z)=\sup_{\overline{W}^m\ni\bar{z}}\left\{\bar{z}(z)-\bar{m}(\bar{z})\right\}\leqq\sup_{\bar{z}\in\bar{R}\bar{S}}\left\{\bar{z}(z)-m_u(\bar{z})\right\}=m_l(z).$$

Conversely, if $m(z)\leqq m_l(z)$ for $z\in R\times S$, then we have obviously for $x\in R$, $y\in S$

$$m(xy)\leqq m_l(xy)\leqq\frac{1}{2}\left\{m(x)+m(y)\right\}^2.$$

Therefore we can state

Theorem 9.3. *A modular $m(z)$ $(z\in W)$ is a product modular, if and only if $m(z)\leqq m_l(z)$ for $z\in R\times S$.*

From this Theorem 9.3 we conclude immediately

Theorem 9.4. *The conjugate modular of $m(z)$ $(z\in W)$ is a product modular from $\bar{R}$ and $\bar{S}$, if and only if $m(z)\geqq m_u(z)$ for $z\in W$.*

§10. Fubini operators

Let R and S be semi-regular, and W a normal product space of R and S. Since W is by Theorem 7.1 isomorphic to a complete semi-normal manifold of the upper product space RS including the lower product space $R\times S$, we can assume $R\times S\subset W\subset RS$. For every $\bar{b}\in\bar{S}$, a linear operator $F_{\bar{b}}$ from W into R is said to be a *Fubini operator* on W into R by $\bar{b}$, if $F_{\bar{b}}$ is universally continuous and

$$F_{\bar{b}}xy=\bar{b}(y)x \qquad \text{for every } x\in R,\ y\in S.$$

Such a linear operator $F_{\bar{b}}$ is determined uniquely, because xy $(0\leqq x\in R$, $0\leqq y\in S)$ is positively fundamental in W, Concerning Fubini operators $F_{\bar{b}}$ ($\bar{b}\in\bar{S}$), we have obviously by definition

$$F_{\beta\bar{b}+\gamma\bar{c}}=\beta F_{\bar{b}}+\gamma F_{\bar{c}} \qquad \text{for} \quad \bar{b}, \bar{c}\in\bar{S},$$

and we have $F_{\bar{b}}\geqq 0$ if and only if $\bar{b}\geqq 0$.

Theorem 10.1. *For a linear operator T on W into R, if T is universally continuous and for any $x\in R$, $y\in S$ there exists a real number $\alpha_{x,y}$ such that $Txy=\alpha_{x,y}x$, then T is a Fubini operator.*

Proof. For two linearly independent elements $x_1, x_2\in R$, since

$$T(x_1+x_1)\,y=Tx_1y+Tx_2y,$$

we have $\alpha_{x_1+x_2,y}(x_1+x_2)=\alpha_{x_1,y}x_1+\alpha_{x_2,y}x_2$, and hence

$$\alpha_{x_1,y}=\alpha_{x_1+x_2,y}=\alpha_{x_2,y}.$$

Thus $\alpha_{x,y}$ is independent from $x\in R$. Therefore we can put $\alpha_y=\alpha_{x,y}$ for every $x\in R$. Since

$$Tx(\beta_1y_1+\beta_2y_2)=\beta_1Txy_1+\beta_2Txy_2,$$

we obtain $\alpha_{\beta_1y_1+\beta_2y_2}=\beta_1\alpha_{y_1}+\beta_2\alpha_{y_2}$. Thus α_y is a linear functional on S. Furthermore, recalling Theorem 1.8, we see that α_y is universally continuous as a linear functional on S, because T is universally continuous by assumption. Therefore T is a Fubini operator.

Theorem 10.2. *If R is reflexive, then for any normal product space of R and S there exists the Fubini operator $F_{\bar{b}}$ on W into R for every $\bar{b}\in\bar{S}$.*

Proof. We need only prove that there exists the Fubini operator $F_{\bar{b}}$ on the upper product space RS into R for every $0\leqq\bar{b}\in\bar{S}$. Every $z\in RS$ may be considered by definition a totally continuous bilinear functional on $\bar{R}$ and $\bar{S}$. Thus, putting

$$\bar{\bar{z}}_{\bar{b}}(\bar{x})=z(\bar{x},\bar{b}) \qquad \text{for} \quad \bar{x}\in\bar{R},$$

we obtain a universally continuous linear functional $\bar{\bar{z}}_{\bar{b}}$ on $\bar{R}$. Since R is reflexive by assumption, there exists then uniquely $x_z\in R$ for which

$$\bar{x}(x_z)=z(\bar{x},\bar{b}) \qquad \text{for every} \quad \bar{x}\in\bar{R},$$

and putting $F_{\bar{b}}z=x_z$ for every $z\in RS$, we obtain a universally continuous linear operator $F_{\bar{b}}$ on RS into R. Furthermore we have for every $\bar{x}\in\bar{R}$

$$\bar{x}(F_{\bar{b}}xy)=\bar{x}(x)\,\bar{b}(y)=\bar{x}(\bar{b}(y)x),$$

and hence $F_{\bar{b}}xy=\bar{b}(y)x$. Therefore $F_{\bar{b}}$ is the Fubini operator on RS into R by $\bar{b}$.

Theorem 10.3. *For every $\bar{b}\in\bar{S}$, there exists the Fubini operator $F_{\bar{b}}$ on the lower product space $R\times S$ into R by $\bar{b}$.*

Proof. By virtue of Theorem 10.2, there exists the Fubini operator $F'_{\bar{b}}$ on the upper product space $\bar{\bar{R}}S$ for every $\bar{b}\in\bar{S}$. Since $\bar{\bar{R}}$ is isomorphic to $\bar{R}$, and R is isomorphic to a complete semi-normal manifold of $\bar{\bar{R}}$, we can consider that RS coincides with $\bar{\bar{R}}S$ and R is a complete semi-normrl manifold of $\bar{\bar{R}}$. Then for each $0\leqq z\in R\times S$, we can find $0\leqq x\in R$, $0\leqq y\in S$, such that $z\leqq xy$, and for every $0\leqq\bar{b}\in\bar{S}$ we have

$$F_{\bar{b}}z\leqq F_{\bar{b}}xy=\bar{b}(y)x\in R\,.$$

Thus we have $F_{\bar{b}}z\in R$ for every $z\in R\times S$. Therefore there exists the Fubini operator $F_{\bar{b}}$ on $R\times S$ for every $\bar{b}\in\bar{S}$.

For a linear operator T on R into S, putting

$$\varphi_T(x,\bar{y})=\bar{y}(Tx)\qquad\text{for}\quad x\in R,\ \bar{y}\in\bar{S},$$

we obain a bilinear functional φ_T on R and $\bar{S}$. If φ_T is singular as a bilinear functional, then T is said to be *singular*.

With this definition we have

Theorem 10.4. *Every Fubini operator $F_{\bar{b}}$ on a normal product space W is singular.*

Proof. By virtue of Theorem 2.5, we need only consider Fubini operators on the upper product space RS. As proved in Proof of Theorem 10.2, we have

$$z(\bar{x},\bar{b})=\bar{x}(F_{\bar{b}}z)\qquad\text{for}\quad\bar{x}\in\bar{R},\ z\in RS.$$

Putting $\varphi(z,\bar{x})=z(\bar{x},\bar{b})$ for $\bar{x}\in\bar{R}$, $z\in RS$, we obtain by Theorem 6.2 a singular bilinear functional φ on RS and $\bar{R}$. Therefore the Fubini operator $F_{\bar{b}}$ is singular by definition.

We also define likewise Fubini operators $F_{\bar{a}}$ on W into S by $\bar{a}\in\bar{R}$, Then we have for every $z\in W$

$$\bar{a}(F_{\bar{b}}z)=z(\bar{a},\bar{b})=\bar{b}(F_{\bar{a}}z)\,,$$

as proved in Proof of Theorem 10.2. Therefore we have

Theorem 10.5. *For Fubini operators $F_{\bar{a}}$ on W into S by $\bar{a}\in\bar{R}$ and $F_{\bar{b}}$ on W into R by $\bar{b}\in\bar{S}$, we have*

$$\bar{a}(F_{\bar{b}}z)=\bar{b}(F_{\bar{a}}z)\qquad\textit{for every}\quad z\in W\,.$$