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Title	A GENERALIZATION OF THE THEOREM OF ORLICZ AND BIRNBAUM
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Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 13(2), 060-064
Issue Date	1956
Doc URL	https://hdl.handle.net/2115/55988
Type	departmental bulletin paper
File Information	JFSHIU_13_N2_060-064.pdf



A GENERALIZATION OF THE THEOREM OF ORLICZ AND BIRNBAUM

By

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ORLICZ and BIRNBAUM have proved in [1] that an ORLICZ-space L_ϕ is finite if and only if the function ϕ satisfies the following condition for some $r > 0$:

$$\phi(2t) \leq r\phi(t) \quad \text{for every } t \geq t_0$$

This fact can be generalized for arbitrary monotone complete modulars¹⁾ on non-discrete spaces, and the modular $m(x)$ satisfying the corresponding condition is said to be *semi-upper bounded*,²⁾ that is, by definition, if for some $\epsilon, r > 0$ we have

$$(1) \quad m(2x) \leq rm(x) \quad \text{for every } x \text{ such that } m(x) \geq \epsilon.$$

(Here always we can take ϵ arbitrarily small varying r).

It is obvious that in this case m is also *uniformly finite*.³⁾

In discrete spaces, a finite modular is not necessarily be semi-upper bounded nor uniformly finite, and we shall give the necessary and sufficient conditions for them.

In the sequel, R is a modularized semi-ordered linear space with the modular m which we suppose to be monotone complete throughout the paper.

1. A set U of R will be said to involve the modular m , if there exists $\epsilon > 0$ for which $m(x) \leq \epsilon$ implies $x \in U$. Such a set U must belong to every filtre which is order-convergent to 0, since m is order-continuous.

1) For the definition of the modular see H. NAKANO [2].

2) The upper bounded modular was defined in [2]. The "conjugate" of "semi-upper bounded" is: for some ϵ, α, r such that $\epsilon > 0$ and $1 < \alpha < r$, we have $m(\alpha x) \geq rm(x)$ for every x such that $m(x) \geq \epsilon$.

3) m is said to be uniformly finite if for every $r > 0$ we have

$$\sup_{\|x\| \leq r} m(x) < +\infty,$$

where $\|x\|$ is the modular norm which is defined as $\|x\| \leq 1$ is equivalent to $m(x) \leq 1$.

Conversely, as the fact playing the fundamental roll in this paper, for every order-closed set U which belongs to every filtre order-convergent to 0, there exist mutually orthogonal normal manifolds $N_i (i=0, 1, \dots, n)$ such that $R = N_0 + N_1 + \dots + N_n$, N_0 is finite-dimensional and the set $N_0 + N_1 \frown U + \dots + N_n \frown U$ involves m .

The last statement is equivalent to say that $N_i \frown U$ involves m in the space N_i for every $i=1, 2, \dots, n$, so what is to prove is the existence of a normal manifold N in which $N \frown U$ involves m and which belongs to an arbitrarily given non-atomic maximal dual ideal of the Boolean algebra of all normal manifolds. Let this given ideal be $\mathfrak{p}$ and suppose there exists no normal manifold in $\mathfrak{p}$ satisfying the condition. Now we construct consecutively an orthogonal sequence of elements $x_\nu \in R$ ($\nu=1, 2, \dots$) such that $m(x_\nu) \leq \frac{1}{2^\nu}$, $x_\nu \notin U$, and $[x_1, x_2, \dots, x_\nu]R \in \mathfrak{p}^{(4)}$ for every $\nu=1, 2, \dots$. If $x_1, x_2, \dots, x_\nu$ had been taken, then we can find x in $(1 - [x_1, x_2, \dots, x_\nu])R$ such that $m(x) \leq \frac{1}{2^{\nu+1}}$ and $x \notin U$, so we can put $x_{\nu+1} = [N]x^{(4)}$ where N is not in $\mathfrak{p}$ and sufficiently large to keep $[N]x$ outside of U . The existence of such a sequence x_ν is, however, a contradiction since the monotone completeness of m implies the convergence of the series $\sum_{\nu=1}^{\infty} x_\nu$ and so x_ν is order-convergent to 0. Thus the proof was completed.

2. If there exists another modular m_1 on R which is finite, then the set $\{x: m_1(x) \leq 1\}$ satisfies the condition of U in 1. Therefore making use of the same notations, there exist $N_i (i=0, 1, \dots, n)$ and since for every x in $N_1 \frown U + N_2 \frown U + \dots + N_n \frown U$ we have obviously $m_1(x) \leq n$ and for some $r > 0$, a set $N_0 \frown \{x: m_1(x) \leq r\}$ involves m in N_0 , we see that a set $\{x: m_1(x) \leq \alpha\}$ involves m for some $\alpha > 0$.

If m is itself finite, a modular $m_1(x) = m(rx)$ is also finite for every $r > 0$, and hence for some ε and $\alpha > 0$, $m(x) \leq \varepsilon$ implies $m(2x) \leq \alpha$.

3. Now we can prove the following

Theorem. *Every monotone complete finite modular on a space without discrete part⁵⁾ is semi-upper bounded and uniformly finite, and in the case there exists no complete element, the modular is upper bounded in some normal manifold different from $\{0\}$.*

4) For a set $A \subset R$, $[A]R$ means the least normal manifold containing A and $[A]$ is its projection.

5) That is to say, there exist no atomic element $\neq 0$ in R .

Let $\varepsilon, r > 0$ be such that $m(x) \leq 2\varepsilon$ implies $m(2x) \leq \varepsilon r$, then every element x such that $m(x) \geq \varepsilon$ can be decomposed orthogonally as $x = x_1 + x_2 + \dots + x_n$ such that $\varepsilon \leq m(x_i) \leq 2\varepsilon$ ($i=1, 2, \dots, n$) and hence we have $m(2x) \leq r m(x)$ summing up the inequalities $m(2x_i) \leq \varepsilon r \leq r m(x_i)$. Therefore m is semi-upper bounded and fortiori uniformly finite.

Now we fix $r > 0$, and for every normal manifold N , let $\varepsilon(N)$ be the least number of $\varepsilon \geq 0$ such that $m(x) \geq \varepsilon$ and $x \in N$ imply $m(2x) \leq r m(x)$. We suppose r is sufficiently large so that $\varepsilon(R)$ exists. For every orthogonal system of normal manifolds N_λ ($\lambda \in A$), we have $\varepsilon(N_\lambda) = 0$ except a countable number of λ , because $\sum_{\lambda \in A} \varepsilon(N_\lambda) > \varepsilon(R)$ is a contradiction as easily be seen. Therefore if m is not upper bounded in every N , then we have $\varepsilon(N_\lambda) > 0$ for every $\lambda \in A$, that is, there exists no orthogonal decomposition of R into more than countable number of factors, and this is equivalent to the existence of complete element in our case. Thus the theorem was proved.

If m is moreover constant and the space has no complete element, then m is upper bounded. This particular case of the theorem was proved in [2] (Theorem. 55.10).

In general m is not upper bounded even for simple finite ORLICZ-spaces.

4. If the modular norm is continuous (in non-discrete case this is equivalent to say that m is finite), then every infinite orthogonal sequence $\{x_\nu\}$ for which $\{\|x_\nu\|\}$ has a non-zero lower bound is not order-bounded, as the same is true for every continuous norm. Here we shall show that the modular norm, in case it is continuous satisfies a stronger condition than above, that is, *for every $\varepsilon > 0$ there exists an integer n such that the norm of the sum of n mutually orthogonal elements having their norm more than ε is always ≥ 1* . To see that the existence of n as above for some $\varepsilon < 1$ is sufficient, because then for ε^m where m is an arbitrary integer we can take n^m in place of n . If the condition is satisfied in both normal manifolds N and M , then we can see easily that the same is also true in $N + M$. Since the sphere $\{x: \|x\| \leq \frac{1}{2}\}$ satisfies the condition of U in 1, and since the finite-dimensional case is trivial, we may restrict ourselves to the case that the sphere involves m , that is, for some $\delta > 0$, $m(x) \leq \delta$ implies $\|x\| \leq \frac{1}{2}$. Then the sum of n mutually orthogonal elements whose norm are all

more than $\frac{1}{2}$ has its modular more than $n\delta$, so in case $n > \frac{1}{\delta}$, more than 1, that is, its norm is ≥ 1 .

5. In the sequel we shall deal with discrete spaces and for this we prefer the concrete treatment as follows: let A be an abstract set and for every $\lambda \in A$, φ_λ be a convex function satisfying the conditions of the modular on real number, R is the totality of real function $x \equiv \{\xi_\lambda\}$ on A such that

$$m(ax) = \sum_{\lambda \in A} \varphi_\lambda(a\xi_\lambda) < +\infty \quad \text{for some } a > 0.$$

The modular m on R is defined by the equation above, and we can see easily that m is monotone complete.

Now we shall prove that m is semi-upper bounded if and only if m is finite and for some $\varepsilon, r > 0$, $\varphi_\lambda(\xi) \geq \varepsilon$ implies $\varphi_\lambda(2\xi) \leq r\varphi_\lambda(\xi)$ for every $\lambda \in A$.

The necessity is obvious. To prove the sufficiency, we proceed as in the proof of the foregoing theorem. For the same $\varepsilon, r > 0$, though we can not, in general, decompose x into x_i such that $\varepsilon \leq m(x_i) \leq 2\varepsilon$, the following decomposition is possible:

$$x = x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_m + z,$$

where $\varepsilon \leq m(x_i) \leq 2\varepsilon$, $y_j (j=1, 2, \dots, m)$ is atomic and $m(y_j) > 2\varepsilon$ and $m(z) \leq \varepsilon$. Then summing up the inequalities $m(2x_i) \leq r m(x_i)$, $m(2y_j) \leq r m(y_j)$ and $m(2z) \leq \varepsilon r \leq r m(x)$ we have $m(2x) \leq 2r m(x)$.

6. Concerning the uniformly finiteness of modulars on discrete spaces, we shall prove that m is uniformly finite if and only if m is finite and

$$\sup_{\lambda \in A} \varphi_\lambda(\xi \alpha_\lambda) < +\infty \quad \text{for every } \xi > 0,$$

where α_λ is the number such that $\varphi_\lambda(\alpha_\lambda) = 1$.

What is to prove is only the sufficiency. If m is finite, then the modular norm is continuous and so the result obtained in 4 is applicable. For every element x such that $\|x\| \leq r$, we can decompose it orthogonally as

$$x = x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_m + z,$$

where $\frac{1}{2} \leq \|x_i\| \leq 1 (i=1, 2, \dots, n)$, $y_j (j=1, 2, \dots, m)$ are atomic and $\|y_j\| > 1$,

and finally $\|z\| < \frac{1}{2}$. In this case the number $n+m$ has an upper bound which depends only to r . If $y \equiv \{\eta_\lambda\}$ is atomic and $\|y\| \leq r$, then $\eta_\lambda = 0$ except for one λ , say λ_0 , and we have $\eta_{\lambda_0} \leq r\alpha_{\lambda_0}$, or $m(y) \leq \varphi_{\lambda_0}(r\alpha_{\lambda_0})$. Therefore we have

$$m(x) \leq n + m \sup_{\lambda \in A} \varphi_\lambda(r\alpha_\lambda) + 1,$$

and this completes the proof.

7. In both case 5 and 6, the conditions for φ_λ are not superfluous and this can be seen by the following fact which displays the essential difference between discrete space and non-discrete one.

For another system $\psi_\lambda (\lambda \in A)$, if we have for every λ

$$\varphi_\lambda(\xi) = \psi_\lambda(\xi) \quad \text{for every } \xi \text{ such that } \varphi_\lambda(\xi) \text{ or } \psi_\lambda(\xi) \leq 1,$$

then we obtain the same space R and in general a different modular $m_1(x) = \sum_{\lambda \in A} \psi_\lambda(\xi_\lambda)$ which coincides with $m(x)$ in case $m(x) \leq 1$, and so their modular norms are the same.

If m is finite and all ψ_λ are finite, then m_1 is also finite. Thus we can vary "larger part of values" of a given finite modular rather arbitrarily without the loss of the finiteness.

Contrarily in non-discrete spaces m is determined by its values on the set $\{x: m(x) \leq \varepsilon\}$ for whatever small $\varepsilon > 0$.

(September, 1954)

References

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