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MODULARS ON SEMI-ORDERED LINEAR SPACES I

By

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In an earlier paper [1], one of the authors defined modulars on linear spaces and discussed their properties: a functional $m(x)$ on a linear space R is said to be a *modular* on R , if

- 1) $m(0) = 0$;
- 2) $m(-a) = m(a)$ for every $a \in R$;
- 3) for any $a \in R$ we can find a positive number α such that

$$m(\alpha a) < +\infty;$$

- 4) $m(\xi a) = 0$ for every positive number ξ implies $a = 0$;
- 5) $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ implies for every $a, b \in R$

$$m(\alpha a + \beta b) \leq \alpha m(a) + \beta m(b);$$

- 6) $m(a) = \sup_{0 \leq \xi < 1} m(\xi a)$ for every $a \in R$.

For universally continuous semi-ordered linear spaces R , modulars were considered with adding conditions: 7) $|a| \leq |b|$ implies $m(a) \leq m(b)$, 8) $|a| \wedge |b| = 0$ implies $m(a+b) = m(a) + m(b)$, and 9) $0 \leq a_\lambda \uparrow_{\lambda \in A} a$ implies $m(a) = \sup_{\lambda \in A} m(a_\lambda)$. (cf. [2])

In this paper we shall discuss modulars on lattice ordered linear spaces only with adding condition 7).

§ 1. Modulars on linear spaces

Firstly we shall give a rough sketch of the properties of modulars on linear spaces which are obtained in [1] and [3], and will be used in this paper. Let $m(x)$ ($x \in R$) be a modular on a linear space R . A linear functional $\tilde{x}(x)$ ($x \in R$) on R is said to be *modular bounded*, if we can find positive numbers α, β such that

$$\alpha \tilde{x}(x) \leq \beta + m(x) \quad \text{for every } x \in R.$$

The totality of modular bounded linear functionals on R also builds a linear space which will be called the *modular associated space* of R and denoted by $\bar{R}$. For each $\tilde{a} \in \bar{R}$, putting

$$\bar{m}(\tilde{a}) = \sup_{x \in R} \{\tilde{a}(x) - m(x)\}$$

we obtain a modular $\bar{m}$ on $\tilde{R}$, which will be called the *conjugate modular* of m . Then we have the reflexive relation:

$$m(a) = \sup_{\tilde{x} \in \tilde{R}} \{\tilde{x}(a) - \bar{m}(\tilde{x})\} \quad (a \in R)$$

Putting

$$(1) \quad \|x\| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \quad (x \in R)$$

we obtain a norm on R , which will be called the *second norm* of m . Concerning the second norm, we have

$$\begin{aligned} m(x) &\leq \|x\| && \text{if } \|x\| \leq 1, \\ m(x) &\geq \|x\| && \text{if } \|x\| \geq 1. \end{aligned}$$

Putting

$$\|a\| = \sup_{\bar{m}(\tilde{x}) \leq 1} |\tilde{x}(a)| \quad (a \in R),$$

we also obtain another norm on R , which will be called the *first norm* of m . Between the first and the second norms there is the relation:

$$\|x\| \leq \|x\| \leq 2 \|x\| \quad (x \in R).$$

The first norm also may be defined as

$$(2) \quad \|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad (x \in R).$$

For the first and the second norm of the conjugate modular $\bar{m}$ we have

$$\begin{aligned} \|x\| &= \sup_{\|\tilde{x}\| \leq 1} |\tilde{x}(x)|, & \|x\| &= \sup_{\|\tilde{x}\| \leq 1} |\tilde{x}(x)| \\ \|\tilde{x}\| &= \sup_{\|x\| \leq 1} |\tilde{x}(x)|, & \|\tilde{x}\| &= \sup_{\|x\| \leq 1} |\tilde{x}(x)| \end{aligned} \quad (x \in R, \tilde{x} \in \tilde{R}).$$

A linear functional $\tilde{x}$ on R is modular bounded if and only if $\tilde{x}$ is norm bounded, that is,

$$\sup_{m(x) \leq 1} |\tilde{x}(x)| < +\infty \quad (x \in R).$$

A sequence $x_\nu \in R$ ($\nu = 1, 2, \dots$) is said to be *modular convergent* to $x \in R$, if

$$\lim_{\nu \rightarrow \infty} m(\xi(x_\nu - x)) = 0 \quad \text{for every } \xi > 0.$$

With this definition we have that a sequence $x_\nu \in R$ ($\nu=1, 2, \dots$) is modular convergent to $x \in R$ if and only if it is *norm convergent*, that is,

$$\lim_{\nu \rightarrow \infty} \|x_\nu - x\| = 0.$$

A modular m on R is said to be *complete*, if

$$\lim_{\nu, \mu \rightarrow \infty} m(\xi(x_\nu - x_\mu)) = 0 \quad \text{for every } \xi > 0$$

implies the modular convergence of the sequence $x_\nu \in R$ ($\nu=1, 2, \dots$). With this definition, a modular m on R is complete if and only if the first or second norm of m is complete. The conjugate modular $\bar{m}$ of any modular m on R is always complete on $\bar{R}$.

From the postulate 5) we conclude easily for $0 < \varepsilon \leq 1$

$$(3) \quad m(x) \leq m(y) + \frac{\varepsilon}{1+\varepsilon} m((1+\varepsilon)y) + \frac{\varepsilon^2}{1+\varepsilon} m\left(\frac{1+\varepsilon}{\varepsilon^2}(x-y)\right).$$

§ 2. Monotone modulars

Let R be a lattice ordered linear space. A modular m on R is said to be *monotone* if $|x| \leq |y|$ implies $m(x) \leq m(y)$. With this definition we have obviously by the formulas (1) and (2) in §1 that if a modular m on R is monotone, then both the first and the second norm of m are monotone too, that is, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ and $\|x\| \leq \|y\|$.

A modular m on R is said to be *upper semi-continuous*, if m is monotone and $0 \leq x_\lambda \uparrow_{\lambda \in A} x$ implies

$$m(x) = \sup_{\lambda \in A} m(x_\lambda).$$

Theorem 2.1. *If a modular m on R is upper semi-continuous, then the second norm of m is semi-continuous, that is, $0 \leq x_\lambda \uparrow_{\lambda \in A} x$ implies $\sup_{\lambda \in A} \|x_\lambda\| = \|x\|$.*

Proof. If $0 \leq x_\lambda \uparrow_{\lambda \in A} x$ and $\sup_{\lambda \in A} \|x_\lambda\| < \|x\|$, then we can find a positive number α such that

$$\sup_{\lambda \in A} \|\alpha x_\lambda\| < 1 < \|\alpha x\|.$$

Thus we have for such α

$$\sup_{\lambda \in A} m(\alpha x_\lambda) \leq 1 < m(\alpha x), \quad 0 \leq \alpha x_\lambda \uparrow_{\lambda \in A} \alpha x.$$

Therefore we obtain our assertion.

A modular m on R is said to be *lower semi-continuous*, if m is monotone and $x_\lambda \downarrow_{\lambda \in A} 0$, $m(x_\lambda) < +\infty$ for every $\lambda \in A$ implies $\inf_{\lambda \in A} m(x_\lambda) = 0$. If a

modular m on R is upper and lower semi-continuous simultaneously, then m is said to be *semi-continuous*.

A modular m on R is said to be *continuous*, if m is monotone and $x_\lambda \downarrow_{\lambda \in A} 0$ implies always $\inf_{\lambda \in A} m(x_\lambda) = 0$.

Theorem 2.2. Every continuous modular is semi-continuous.

Proof. If a modular m on R is continuous, then m is obviously lower semi-continuous by definition. Since m is monotone, we have for $0 \leq x_\lambda \uparrow_{\lambda \in A} x$

$$\sup_{\lambda \in A} m(x_\lambda) \leq m(x).$$

On the other hand we have by the formula (3) for $0 < \varepsilon \leq 1$

$$\begin{aligned} m\left(\frac{1}{1+\varepsilon} x\right) &\leq m\left(\frac{1}{1+\varepsilon} x_\lambda\right) + \frac{\varepsilon}{1+\varepsilon} m(x_\lambda) + \frac{\varepsilon^2}{1+\varepsilon} m\left(\frac{1}{\varepsilon^2}(x-x_\lambda)\right) \\ &\leq \frac{1+2\varepsilon}{1+\varepsilon} \sup_{\lambda \in A} m(x_\lambda) + \frac{\varepsilon^2}{1+\varepsilon} m\left(\frac{1}{\varepsilon^2}(x-x_\lambda)\right). \end{aligned}$$

Since $\frac{1}{\varepsilon^2}(x-x_\lambda) \downarrow_{\lambda \in A} 0$, we obtain by assumption

$$m\left(\frac{1}{1+\varepsilon} x\right) \leq \frac{1+2\varepsilon}{1+\varepsilon} \sup_{\lambda \in A} m(x_\lambda).$$

This relation yields $m(x) \leq \sup_{\lambda \in A} m(x)$, because $\sup_{\varepsilon > 0} m\left(\frac{1}{1+\varepsilon} x\right) = m(x)$ by the postulate 6). Therefore m is upper semi-continuous too.

Theorem 2.3. A monotone modular m on R is continuous, if and only if the first or the second norm of m is continuous: $x_\lambda \downarrow_{\lambda \in A} 0$ implies

$$\inf_{\lambda \in A} \|x\| = 0 \quad \text{or} \quad \inf_{\lambda \in A} \|x\| = 0.$$

Proof. It is obvious that $\inf_{\lambda \in A} \|x_\lambda\| = 0$ is equivalent to $\inf_{\lambda \in A} \|x_\lambda\| = 0$. If m is continuous, then for $x_\lambda \downarrow_{\lambda \in A} 0$ we have $\nu x_\lambda \downarrow_{\lambda \in A} 0$ for every $\nu = 1, 2, \dots$, and hence we can find $\lambda_\nu \in A$ ($\nu = 1, 2, \dots$) such that $m(\nu x_{\lambda_\nu}) \leq 1$ ($\nu = 1, 2, \dots$). Then we have $\|\nu x_{\lambda_\nu}\| \leq 1$, namely $\|x_{\lambda_\nu}\| \leq \frac{1}{\nu}$ for every $\nu = 1, 2, \dots$, and this relation yields $\inf_{\lambda \in A} \|x_\lambda\| = 0$. Thus the second norm of m is continuous.

Conversely, if the second norm of m is continuous, then for $x_\lambda \downarrow_{\lambda \in A} 0$ we can find $\lambda_\nu \in A$ ($\nu = 1, 2, \dots$) such that $\|\nu x_{\lambda_\nu}\| \leq 1$, and hence

$$m(x_{\lambda_\nu}) \leq \frac{1}{\nu} m(\nu x_{\lambda_\nu}) \leq \frac{1}{\nu}$$

for every $\nu = 1, 2, \dots$. This relation yields $\inf_{\lambda \in A} m(x_\lambda) = 0$. Thus m is continuous by definition.

A monotone modular m on R is said to be *monotone complete*, if

$$0 \leq x_\lambda \uparrow_{\lambda \in A}, \quad \sup_{\lambda \in A} m(x_\lambda) < +\infty$$

implies the existence of $\bigcup_{\lambda \in A} x_\lambda$. If m is monotone complete, then R must be universally continuous, because $0 \leq x_\lambda \uparrow_{\lambda \in A}$, $x_\lambda \leq x$ ($\lambda \in A$) implies $\sup_{\lambda \in A} m(\alpha x_\lambda) < +\infty$ for some positive number α such that $m(\alpha x) < +\infty$.

Theorem 2.4. *A monotone modular m on R is monotone complete if and only if the first or the second norm of m is monotone complete.*

Proof. If $\sup_{\lambda \in A} m(x_\lambda) \leq \alpha$ for some $\alpha > 1$, then we have

$$m\left(\frac{1}{\alpha}x_\lambda\right) \leq \frac{1}{\alpha}m(x_\lambda) \leq 1 \quad \text{for every } \lambda \in A$$

and hence $\sup_{\lambda \in A} \left\| \frac{1}{\alpha}x_\lambda \right\| \leq 1$, that is, $\sup_{\lambda \in A} \|x_\lambda\| \leq \alpha$. Conversely if $\sup_{\lambda \in A} \|x_\lambda\| \leq \alpha$ for some $\alpha > 0$, then we have

$$\sup_{\lambda \in A} m\left(\frac{1}{\alpha}x_\lambda\right) \leq 1.$$

Therefore we can conclude our assertion.

Theorem 2.5. *For any monotone modular m on R , its conjugate modular $\bar{m}$ is upper semi-continuous and monotone complete.*

Proof. The modular associated space $\bar{R}$ of R is always universally continuous. (cf. [2]) The conjugate modular $\bar{m}$ is obviously monotone by definition. If $0 \leq \tilde{x}_\lambda \uparrow_{\lambda \in A} \tilde{x}$, then we have

$$\begin{aligned} \bar{m}(\tilde{x}) &= \sup_{x \in \bar{R}} \{\tilde{x}(x) - m(x)\} = \sup_{0 \leq x \in \bar{R}} \left\{ \sup_{\lambda \in A} \tilde{x}_\lambda(x) - m(x) \right\} \\ &= \sup_{\lambda \in A} \left\{ \sup_{0 \leq x \in \bar{R}} \{\tilde{x}_\lambda(x) - m(x)\} \right\} = \sup_{\lambda \in A} \bar{m}(\tilde{x}_\lambda). \end{aligned}$$

Thus $\bar{m}$ is upper semi-continuous. The first norm of $\bar{m}$ is the conjugate norm of the second norm of m , and hence monotone complete. (cf. [2]) Thus $\bar{m}$ is monotone complete by Theorem 2.4.

§ 3. Reflexivity of upper semi-continuous modulars

Now we suppose that R is a universally continuous linear space and m is a monotone modular on R . The totality of universally continuous linear functionals on R , which are modular bounded, is called

the modular conjugate space of R and denoted by $\bar{R}$. $\bar{R}$ is a normal manifold of the modular associated space $\bar{R}$ of R . If m is continuous, then the second norm of m also is continuous by Theorem 2.3, and hence $\bar{R} = \bar{R}$.

Theorem 3.1. If R is semi-regular and m is upper semi-continuous, then m is reflexive, that is, we have for every $x \in R$

$$m(x) = \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(x) - \bar{m}(\bar{x}) \}$$

Proof. For any $0 \neq \bar{a} \in \bar{R}$ and $\nu = 1, 2, \dots$, putting

$$m_\nu(x) = \inf_{|x| = |y| + |z|} \text{Max} \{ m(y), 2^\nu |\bar{a}|(|z|) \} \quad \text{for } x \in [\bar{a}]R,$$

we obtain a monotone modular m_ν on $[\bar{a}]R$. Indeed we see easily that m_ν satisfies the all postulates except for 4). If $m_\nu(x) = 0$ and $x \in [\bar{a}]R$, then we can find $0 \leq y_\mu, z_\mu \in R (\mu = 1, 2, \dots)$ such that

$$|x| = y_\mu + z_\mu, \quad \text{Max} \{ m(y_\mu), 2^\nu |\bar{a}|(|z_\mu|) \} \leq \frac{1}{2^\mu}$$

and putting $u_\mu = \bigcup_{\rho \geq \mu} z_\rho (\mu = 1, 2, \dots)$, we have

$$2^\nu |\bar{a}|(u_\mu) \leq \sum_{\rho \geq \mu} \frac{1}{2^\rho} \quad (\mu = 1, 2, \dots)$$

and hence $2^\nu |\bar{a}| \left(\bigcap_{\mu=1}^{\infty} u_\mu \right) = 0$. This relation yields $\bigcap_{\mu=1}^{\infty} u_\mu = 0$, that is, $u_\mu \downarrow_{\mu=1}^{\infty} 0$. Thus we have $|x| - u_\mu \uparrow_{\mu=1}^{\infty} |x|$ and

$$m(|x| - u_\mu) \leq m(y_\mu) \leq \frac{1}{2^\mu} \quad (\mu = 1, 2, \dots).$$

Therefore we obtain $m(x) = 0$, because m is upper semi-continuous by assumption, and we conclude that $m_\nu(x) = 0$ and $x \in [\bar{a}]R$ implies $m(x) = 0$. Consequently the postulate 4) also is satisfied.

The modular m_ν on $[\bar{a}]R$ is continuous for every $\nu = 1, 2, \dots$, because we have obviously

$$m_\nu(x) \leq 2^\nu |\bar{a}|(|x|) \quad \text{for every } x \in [\bar{a}]R.$$

Thus the modular associated space $\bar{R}_\nu$ of $[\bar{a}]R$ by m_ν coincides with the modular conjugate space of $[\bar{a}]R$ by m_ν and hence $\bar{R}_\nu$ is included in the modular conjugate space $\bar{R}$ of R by m , because we have obviously

$$m_\nu(x) \leq m(x) \quad \text{for every } x \in [\bar{a}]R.$$

Therefore we have for every $x \in [\bar{a}]R$

$$m_\nu(x) = \sup_{\bar{x} \in \bar{R}_\nu} \{ \bar{x}(x) - \bar{m}_\nu(\bar{x}) \} \leq \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(x) - \bar{m}(\bar{x}) \},$$

because we have for $\bar{x} \in \bar{R}_\nu$

$$\bar{m}(\bar{x}) = \sup_{x \in [\bar{x}]R} \{ \bar{x}(x) - m(x) \} \leq \sup_{x \in [\bar{x}]R} \{ \bar{x}(x) - m_\nu(x) \} = \bar{m}_\nu(\bar{x}).$$

On the other hand we have

$$\lim_{\nu \rightarrow \infty} m_\nu(x) = m(x) \quad \text{for every } x \in [\bar{a}]R.$$

Because, for any $x \in [\bar{a}]R$ we can find $0 \leq y_\nu, z_\nu \in R$ ($\nu = 1, 2, \dots$) such that

$$|x| = y_\nu + z_\nu, \quad m(y_\nu) \leq m_\nu(x) + \frac{1}{2^\nu}, \quad 2^\nu |\bar{a}|(z_\nu) \leq m_\nu(x) + \frac{1}{2^\nu}.$$

Then putting $u_\nu = \bigcup_{\rho \geq \nu} z_\rho$ ($\nu = 1, 2, \dots$), we conclude $u_\nu \downarrow_{\nu=1}^\infty 0$ and

$$m(|x| - u_\nu) \leq m(y_\nu) \leq m_\nu(x) + \frac{1}{2^\nu} \leq m(x) + \frac{1}{2^\nu},$$

as obtained above. This relation yields $m(x) = \lim_{\nu \rightarrow \infty} m_\nu(x)$. Therefore we conclude

$$m(x) \leq \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(x) - \bar{m}(\bar{x}) \} \quad \text{for every } x \in [\bar{a}]R.$$

Since R is semi-regular by assumption, we have $[\bar{a}]x \uparrow_{\bar{x} \in \bar{R}} x$, and hence we obtain furthermore

$$m(x) \leq \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(x) - \bar{m}(\bar{x}) \} \quad \text{for every } x \in R.$$

On the other hand it is obvious by definition

$$m(x) \geq \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(x) - \bar{m}(\bar{x}) \} \quad \text{for every } x \in R.$$

Thus we conclude our assertion.

Recalling Theorem 2.4, we obtain immediately

Theorem 3.2. *If R is semi-regular and m is upper semi-continuous and monotone complete, then R is reflexive and the modular conjugate space of R by m coincides with the conjugate space of R .*

§ 4. Semi-additive modulars

A modular m on a lattice ordered linear space R is said to be *upper semi-additive*, if m is monotone and

$$m(a+b) \geq m(a) + m(b) \quad \text{for } 0 \leq a, b \in R.$$

Theorem 4.1. *If an upper semi-additive modular m is upper semi-*

continuous, then m is lower semi-continuous, and hence semi-continuous.

Proof. For $x_\lambda \downarrow_{\lambda \in \Lambda} 0$, $m(x_\lambda) < +\infty$ ($\lambda \in \Lambda$) we have

$$m(x_\lambda) \leq m(x_{\lambda_0}) - m(x_{\lambda_0} - x_\lambda) \quad \text{for } x_\lambda \leq x_{\lambda_0},$$

because m is upper semi-additive by assumption. Since

$$x_{\lambda_0} - x_\lambda \uparrow_{x_\lambda \leq x_{\lambda_0}} x_{\lambda_0}$$

and m is upper semi-continuous by assumption, we have

$$\sup_{x_\lambda \leq x_{\lambda_0}} m(x_{\lambda_0} - x_\lambda) = m(x_{\lambda_0}).$$

Thus we obtain $m(x_\lambda) \downarrow_{\lambda \in \Lambda} 0$.

A modular m on R is said to be *lower semi-additive*, if m is monotone and

$$m(a \smile b) \leq m(a) + m(b) \quad \text{for } 0 \leq a, b \in R.$$

A modular m on R is said to be *additive*, if m is upper and lower semi-additive simultaneously. Additive modulars are discussed in detail already in [2]. When R is universally continuous, if a modular m on R is upper semi-continuous and

$$m(a+b) = m(a) + m(b) \quad \text{for } a \smile b = 0,$$

then m is additive. (cf. [2])

Theorem 4.2. *The conjugate modulars of the upper semi-additive modulars are lower semi-additive, and the conjugate modulars of the lower semi-additive modulars are upper semi-additive.*

Proof. If a modular m on R is upper semi-additive, then for the conjugate modular $\bar{m}$ of m and the modular associated space $\bar{R}$ of R we have by definition for $0 \leq \bar{x}, \bar{y} \in \bar{R}$

$$\begin{aligned} \bar{m}(\bar{x}) + \bar{m}(\bar{y}) &= \sup_{x, y \in R} \{ \bar{x}(x) + \bar{y}(y) - m(x) - m(y) \} \\ &\geq \sup_{0 \leq z \in R} \{ \sup_{\substack{z = x + y, \\ x, y \geq 0}} \{ \bar{x}(x) + \bar{y}(y) \} - m(z) \} \\ &= \sup_{z \in R} \{ \bar{x} \smile \bar{y}(z) - m(z) \} = \bar{m}(\bar{x} \smile \bar{y}), \end{aligned}$$

and hence $\bar{m}$ is lower semi-additive by definition. If m is lower semi-additive, then we have by definition for $0 \leq \bar{x}, \bar{y} \in \bar{R}$

$$\begin{aligned} \bar{m}(\bar{x}) + \bar{m}(\bar{y}) &= \sup_{0 \leq x, y \in R} \{ \bar{x}(x) + \bar{y}(y) - m(x) - m(y) \} \\ &\leq \sup_{0 \leq x, y \in R} \{ \bar{x}(x \smile y) + \bar{y}(x \smile y) - m(x \smile y) \} \end{aligned}$$

$$\leq \sup_{z \in R} \{\tilde{x}(z) + \tilde{y}(z) - m(z)\} = \bar{m}(\tilde{x} + \tilde{y}),$$

and hence $\bar{m}$ is upper semi-additive by definition.

§ 5. Bimodulars

Let R, S be two lattice ordered linear spaces. A functional $M(x, y)$ ($x \in R, y \in S$) is said to be a *bimodular*, if $M(x, y)$ is an additive upper semi-continuous modular on R for every fixed $0 \neq y \in S$,

$$M(x, |y_1| + |y_2|) = M(x, y_1) + M(x, y_2),$$

$$M(x, \beta y) = |\beta| M(x, y),$$

and for any $x \in R$ we can find a positive number α such that

$$M(\alpha x, y) < +\infty \quad \text{for every } y \in S.$$

A bimodular $M(x, y)$ ($x \in R, y \in S$) is said to be *finite*, if

$$M(x, y) < +\infty \quad \text{for every } x \in R, y \in S.$$

If S is a normed space and complete, then putting

$$m(x) = \sup_{\|y\| \leq 1} M(x, y) \quad (x \in R, y \in S),$$

we obtain a modular m on R . This modular m is said to be a *norm-modular* of M by the norm of S .

Theorem 5.1. *Every norm-modular of a bimodular $M(x, y)$ ($x \in R, y \in S$) is lower semi-additive and upper semi-continuous.*

Proof. For $0 \leq x_1, x_2 \in R$ we have by definition

$$\begin{aligned} m(x_1 \vee x_2) &= \sup_{\|y\| \leq 1} M(x_1 \vee x_2, y) \\ &\leq \sup_{\|y\| \leq 1} M(x_1, y) + \sup_{\|y\| \leq 1} M(x_2, y) = m(x_1) + m(x_2), \end{aligned}$$

because $M(x_1 \vee x_2, y) \leq M(x_1, y) + M(x_2, y)$. Thus the norm-modular m is lower semi-additive. For $0 \leq x_\lambda \uparrow_{\lambda \in A} x$ we have by definition

$$\begin{aligned} m(x) &= \sup_{\|y\| \leq 1} M(x, y) = \sup_{\|y\| \leq 1} \left\{ \sup_{\lambda \in A} M(x_\lambda, y) \right\} \\ &= \sup_{\lambda \in A} \left\{ \sup_{\|y\| \leq 1} M(x_\lambda, y) \right\} = \sup_{\lambda \in A} m(x_\lambda). \end{aligned}$$

Thus the norm-modular m is upper semi-continuous by definition.

Theorem 5.2. *If a bimodular $M(x, y)$ ($x \in R, y \in S$) is finite, then the norm-modular of M is finite.*

Proof. For each $x \in R$, since $M(x, y) < +\infty$ by assumption, putting

$$x(y) = M(x, y^+) - M(x, y^-) \quad (y \in S)$$

we obtain a positive linear functional x on S . Since the norm of S is complete by assumption, this linear functional x on S is norm bounded, and hence

$$\sup_{\|y\| \leq 1} M(x, y) < +\infty \quad \text{for every } x \in R.$$

Thus the norm-modular of M is finite by definition.

For an additive complete modular m_s on S , putting

$$m(x) = \sup_{y \in S} \{M(x, y) - m_s(y)\} \quad (x \in R)$$

we obtain a monotone modular m on R . This modular m on R is said to be the *double-modular* of M by m_s .

Theorem 5.3. Every double-modular of a bimodular $M(x, y)$ ($x \in R, y \in S$) is upper semi-additive and semi-continuous.

Proof. For $0 \leq x_1, x_2 \in R$ we have by definition

$$\begin{aligned} m(x_1 + x_2) &= \sup_{y \in S} \{M(x_1 + x_2, y) - m_s(y)\} \\ &\geq \sup_{y \in S} \{M(x_1, y) + M(x_2, y) - m_s(y)\} \\ &\geq \sup_{0 \leq y_1, y_2 \in S} \{M(x_1, y_1) + M(x_2, y_2) - m_s(y_1 \cup y_2)\} \\ &\geq \sup_{0 \leq y_1, y_2 \in S} \{M(x_1, y_1) + M(x_2, y_2) - m_s(y_1) - m_s(y_2)\} = m(x_1) + m(x_2), \end{aligned}$$

because

$$M(x_1 + x_2, y) \geq M(x_1, y) + M(x_2, y),$$

$$m_s(y_1 \cup y_2) \leq m_s(y_1) + m_s(y_2).$$

Thus the double-modular m is upper semi-additive. For $0 \leq x_\lambda \uparrow_{\lambda \in A} x$ we have by definition

$$m(x) = \sup_{y \in S} \left\{ \sup_{\lambda \in A} \{M(x_\lambda, y) - m_s(y)\} \right\} = \sup_{\lambda \in A} m(x_\lambda).$$

Thus m is upper semi-continuous. Recalling Theorem 4.1, we conclude therefore that m is semi-continuous.

Theorem 5.4. Let m_s be a complete, additive modular on S . For a bimodular $M(x, y)$ ($x \in R, y \in S$), denoting by m_a the double-modular of M by m_s and by m_n the norm-modular of M by the first norm of m_s , then we have

$$m_a(x) \leq m_n(x) \quad \text{for } m_n(x) \leq 1,$$

$$m_a(x) \geq m_n(x) \quad \text{for } m_n(x) \geq 1,$$

and the second norm of m_a coincides with that of m_n .

Proof. If $M(x, y) < +\infty$ for every $y \in S$, then putting

$$x(y) = M(x, y^+) - M(x, y^-) \quad (y \in S)$$

we obtain a positive linear functional x on S . Since the modular m_s is complete by assumption, this linear functional x is modular bounded. Thus we have by definition

$$m_a(x) = \bar{m}_s(x), \quad m_n(x) = \|x\|$$

for the conjugate modular $\bar{m}_s$ of m_s and the second norm $\|x\|$ of $\bar{m}_s$. If $M(x, y) = +\infty$ for some $y \in S$, then we have obviously by definition

$$m_a(x) = m_n(x) = +\infty.$$

Therefore we conclude that $m_n(x) \leq 1$ implies $m_a(x) \leq m_n(x)$, and that $m_n(x) \geq 1$ implies $m_a(x) \geq m_n(x)$. Consequently the second norm of m_a coincides with that of m_n .

§ 6. Proper bimodular

Let m be an additive upper semi-continuous modular on a universally continuous semi-ordered linear space R , and $\mathfrak{E}$ the proper space of R . We denote by D_m the totality of such dilatators T in R that for any $x \in R$ we can find a positive number α for which

$$\int_{\mathfrak{E}} \left(\frac{|T|}{1} , p \right) m(dpx) < +\infty.$$

Then, putting

$$M_m(x, T) = \int_{\mathfrak{E}} \left(\frac{|T|}{1} , p \right) m(dpx) \quad (x \in R, T \in D_m)$$

we obtain a bimodular M_m . Here we see easily that D_m is a semi-normal manifold of the dilatator space and $1 \in D_m$, because $M_m(x, 1) = m(x)$. This bimodular M_m is said to be the *proper bimodular* of m .

For a semi-normal manifold D of D_m containing 1, and for a complete norm $\|T\|$ ($T \in D$) on D , putting

$$m_n(x) = \sup_{\|T\| \leq 1, T \in D} M_m(x, T) \quad (x \in R),$$

we obtain a norm-modular m_n of M_m .

Theorem 6.1. *If the modular m on R is monotone complete, then every norm-modular of the proper bimodular M_m of m also is monotone complete.*

Proof. If $0 \leq x_\lambda \uparrow_{\lambda \in A}$, $\sup_{\lambda \in A} m_n(x_\lambda) < +\infty$, then we have by definition

$$\sup_{\lambda \in A} m(x_\lambda) = \sup_{\lambda \in A} M_m(x_\lambda, 1) < +\infty$$

and hence $x_\lambda (\lambda \in A)$ is bounded, because m is monotone complete by assumption. Therefore the norm-modular m_n also is monotone complete.

For a complete, additive modular $m_D(T)$ ($T \in D$) on D , putting

$$m_a(x) = \sup_{T \in D} \{M_m(x, T) - m_D(T)\} \quad (x \in R),$$

we obtain a double-modular m_a of M_m .

Theorem 6.2. Every double-modular of the proper bimodular M_m of m also is additive.

Proof. If $M_m(x, T) < +\infty$ for every $T \in D$, then, putting

$$x(T) = \int_{\mathfrak{G}} \left(\frac{T}{1}, p \right) m(dp x) \quad (T \in D),$$

we obtain a positive linear functional $x(T)$ ($T \in D$) on D . Furthermore if

$$x \sim y = 0, \quad M_m(x, T) < +\infty, \quad M_m(y, T) < +\infty \quad \text{for every } T \in D,$$

then we also have $x \sim y = 0$ considering both x and y linear functionals on D , and hence

$$\bar{m}_D(x+y) = \bar{m}_D(x) + \bar{m}_D(y)$$

for the conjugate modular $\bar{m}_D$ of m_D , because m_D is additive by assumption. On the other hand we have by definition

$$m_a(x) = \begin{cases} \bar{m}_D(x) & \text{if } M_m(x, T) < +\infty \quad \text{for every } T \in D, \\ +\infty & \text{if } M_m(x, T) = +\infty \quad \text{for some } T \in D. \end{cases}$$

Thus we conclude that $x \sim y = 0$ implies $m_a(x+y) = m_a(x) + m_a(y)$. Therefore the double-modular m_a is additive. (cf. [2])

Theorem 6.3. If the modular m on R is monotone complete, then every double-modular of the proper bimodular M_m of m also is monotone complete.

Proof. For an additive complete modular m_D on D , we can find a positive number α such that $m_D(\alpha) < +\infty$ considering α a dilatator in R . If $0 \leq x_\lambda \uparrow_{\lambda \in A}$ and $\sup_{\lambda \in A} m_a(x_\lambda) < +\infty$, then we have

$$\sup_{\lambda \in A} m(x_\lambda) = \frac{1}{\alpha} \sup_{\lambda \in A} M_m(x_\lambda, \alpha) \leq \frac{1}{\alpha} \{ \sup_{\lambda \in A} m_a(x_\lambda) + m_D(\alpha) \} < +\infty,$$

and hence $x_\lambda (\lambda \in A)$ is bounded, because m is monotone complete by

assumption. Therefore the double-modular m_d also is monotone complete by definition.

References

- [1] H. NAKANO: Modulared linear spaces, Jour. Fac. Sci. Univ. Tokyo, Sec. I Vol. 1 (1951) 85-131.
- [2] H. NAKANO: Modulared semi-ordered linear spaces, Tokyo (1950).
- [3] H. NAKANO: Modern spectral theory, Tokyo (1950).