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ON THE MAXIMAL SPECTRALITY

By

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Let R be a Hilbert space and $\mathfrak{B}$ a totally additive set class in an abstract space $\mathcal{Q}$. A system of projection operators $E(\phi) (\phi \in \mathfrak{B})$ in R is called a *spectrality*¹⁾ in R on $\mathfrak{B}$ if (1) $E(\phi) + E(\phi^c) = 1$, and (2) $E\left(\sum_{i=1}^{\infty} \phi_i\right) = \bigcup_{i=1}^{\infty} E(\phi_i)$. We say a spectrality $E(\phi) (\phi \in \mathfrak{B})$ is *maximal* (due to Prof. NAKANO's suggestion) if

1) for any finite measure ν on $\mathfrak{B}$ we can find an element $x \in R$ such that $\nu(\phi) = \|E(\phi)x\|^2 (\phi \in \mathfrak{B})$, and

2) $\mathfrak{R}_E$ is a simple ring, where $\mathfrak{R}_E$ is a closed projection operator ring²⁾ generated by $\{E(\phi); \phi \in \mathfrak{B}\}$.

$\mathfrak{R}_E$ is *simple*³⁾ if and only if for any projection operator P that is commutative to $\mathfrak{R}_E$ we have $P \in \mathfrak{R}_E$.

In this paper we shall show that for any given $\mathcal{Q}$ and $\mathfrak{B}$ we can construct a Hilbert space R and a maximal spectrality $E(\phi) (\phi \in \mathfrak{B})$ in R on $\mathfrak{B}$, and moreover R and $E(\phi) (\phi \in \mathfrak{B})$ are determined uniquely within an unitary isomorphism (Theorem 1). Conversely for any given R we can find $\mathcal{Q}$ and $\mathfrak{B}$ for which there exists a *discrete* maximal spectrality in R on $\mathfrak{B}$. But it is known in WECKEN [1] that if the dimension of R is continuum, there exists in R a maximal spectrality on the Borel sets in the real line. If R is separable, we can prove that there is no maximal spectrality other than a discrete one (Theorem 2).

Theorem 1. *For any given $\mathcal{Q}$ and $\mathfrak{B}$ we can construct a Hilbert space R and a maximal spectrality $E(\phi) (\phi \in \mathfrak{B})$ in R . Furthermore such R and $E(\phi) (\phi \in \mathfrak{B})$ are determined uniquely within unitary isomorphism.*

The method of the proof is essentially same as in [1], so we give an outline only, about details refer [1] or NAKANO [2] Chap. V.

Let $\mathfrak{M}_{\mathfrak{B}}$ be the totality of finite measures on $\mathfrak{B}$. From the property of $\mathfrak{M}_{\mathfrak{B}}$ as a Boolean lattice and Maximal theorem we can find a maximal

1) cf. [2] §28.

2) cf. [2] §14.

3) cf. [2] §20.

system $m_\lambda \in \mathfrak{M}_\mathfrak{B} (\lambda \in \Lambda)$ such that (i) $m_\lambda \perp m_{\lambda'} (\lambda \neq \lambda')$, that is, if $\nu \in \mathfrak{M}_\mathfrak{B}$ is absolutely continuous about m_λ and $m_{\lambda'}$, then $\nu = 0$, (ii) for any $\nu \in \mathfrak{M}_\mathfrak{B}$ we can write $\nu(\Phi) = \sum_{i=1}^{\infty} \mu_{\lambda_i}(\Phi) (\Phi \in \mathfrak{B})$ for suitable countable measures $\mu_{\lambda_i} \in \mathfrak{M}_B$, $\mu_{\lambda_i} \prec m_{\lambda_i}$, $i = 1, 2, \dots$ (μ_{λ_i} is absolutely continuous about m_{λ_i}). Putting $R = \sum_{\lambda \in \Lambda} \oplus L^2(m_\lambda)$ and $E(\Phi) \{f_\lambda\} = \{\chi_\Phi f_\lambda\}$, where $\{f_\lambda\}$ is an element of R , $f_\lambda \in L^2(m_\lambda) (\lambda \in \Lambda)$, and χ_Φ a characteristic function of Φ , we obtain a spectrality $E(\Phi) (\Phi \in \mathfrak{B})$ in R . This spectrality is maximal. Because by the definition of $m_\lambda (\lambda \in \Lambda)$ it is almost evident that the condition 1) of the maximality is satisfied. The condition 2) and the uniqueness of R and $E(\Phi) (\Phi \in \mathfrak{B})$ are obtained by the following facts. Denoting by $[x]_{\mathfrak{R}_E} (x \in R)$ the projection operator of the subspace generated by $\{E(\Phi)x; \Phi \in \mathfrak{B}\}$, then $[x]_{\mathfrak{R}_E} R$ is unitary isomorphic to $L^2(m_x)$, where $m_x \in \mathfrak{M}_\mathfrak{B}$, $m_x(\Phi) = \|E(\Phi)x\|^2 (\Phi \in \mathfrak{B})$. Let $C_{[x]_{\mathfrak{R}_E}}$ be the cover⁴⁾ of $[x]_{\mathfrak{R}_E}$, then $C_{[x]_{\mathfrak{R}_E}} \geq C_{[y]_{\mathfrak{R}_E}}$ is equivalent to $m_x \succ m_y$. And $\mathfrak{R}_E$ is simple if and only if $[x]_{\mathfrak{R}_E} [y]_{\mathfrak{R}_E} = 0$ and $C_{[x]_{\mathfrak{R}_E}} = C_{[y]_{\mathfrak{R}_E}}$ imply $x = y = 0$.

Let R be any Hilbert space and $a_\lambda (\lambda \in \Lambda)$ a complete orthonormal system in R .

Putting $\Omega = \Lambda$ and

$\mathfrak{B} = \{\Phi; \Phi \ni \lambda_0 \text{ and at most countable or } \Phi \ni \lambda_0 \text{ and } \Phi^c \text{ at most countable}\}$, where λ_0 is a fixed index, and for $\Phi \in \mathfrak{B}$ $E(\Phi)$ is the projection operator of the subspace generated by $\{a_\lambda; \lambda \in \Phi\}$, then obviously $\mathfrak{B}$ is a totally additive set class and $E(\Phi) (\Phi \in \mathfrak{B})$ is a spectrality in R on $\mathfrak{B}$. We remark that if R is separable, $\mathfrak{B} = 2^\Lambda$. A projection operator P belongs to $\mathfrak{R}_E$ if and only if P is the projection operator of the subspace generated by $\{a_\lambda; \lambda \in S\}$ for a subset $S \subseteq \Lambda$. For $x \in R$ $[x]_{\mathfrak{R}_E}$ is the projection operator of the subspace generated by $\{a_\lambda; (x, a_\lambda) \neq 0\}$, hence $[x]_{\mathfrak{R}_E} \in \mathfrak{R}_E$. Therefore $\mathfrak{R}_E$ is simple ([2]th. 20. 3). Next for any $\nu \in \mathfrak{M}_\mathfrak{B}$ we put $\Omega_1 = \{\lambda; \nu(\{\lambda\}) \neq 0, \lambda \neq \lambda_0\}$, $\Omega_2 = \Omega - \Omega_1$, then $\Omega_1 \ni \lambda_0$ and at most countable, and we put $x = \sum_{\lambda \in \Omega_1} \sqrt{\nu(\{\lambda\})} a_\lambda + \sqrt{\nu(\Omega_2)} a_{\lambda_0}$, then for $\Phi \in \mathfrak{B}$

$$\begin{aligned} \|E(\Phi)x\|^2 &= \sum_{\lambda \in \Omega_1} \nu(\{\lambda\}) \|E(\Phi)a_\lambda\|^2 + \nu(\Omega_2) \|E(\Phi)a_{\lambda_0}\|^2 \\ &= \sum_{\lambda \in \Phi \cap \Omega_1} \nu(\{\lambda\}) + \nu(\Omega_2) \|E(\Phi)a_{\lambda_0}\|^2 = \nu(\Phi \cap \Omega_1) + \nu(\Phi \cap \Omega_2) = \nu(\Phi), \end{aligned}$$

because $\nu(\Omega_2 \cap \Phi) = \nu(\Omega_2)$ or 0 according to $\Phi \ni \lambda_0$ or $\Phi \not\ni \lambda_0$, hence $m_x = \nu$. Therefore this spectrality is maximal. We call this spectrality *discrete*.

Theorem 2. *In a separable Hilbert space a spectrality is maximal if*

4) cf. [2] § 14.

and only if it is discrete.

Proof. Let $E(\phi) (\phi \in \mathfrak{B})$ be a maximal spectrality in a separable Hilbert space R . As R is separable, by Maximal theorem there are $e_n \in R$ $n=1, 2, \dots$ such that $[e_n]_{\mathfrak{R}_E} [e_m]_{\mathfrak{R}_E} = 0$ ($n \neq m$) and $\bigcup_{n=1}^{\infty} [e_n]_{\mathfrak{R}_E} = I$. If we put $e = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n$, then we have $[e]_{\mathfrak{R}_E} = \bigcup_{n=1}^{\infty} [e_n]_{\mathfrak{R}_E} = I$ ([2]th. 14.5). If $E(\phi)e=0$, then $E(\phi) = E(\phi)[e]_{\mathfrak{R}_E} = [E(\phi)e]_{\mathfrak{R}_E} = 0$, and hence $\phi = \emptyset$ from the condition 1) of a maximal spectrality. From the separability of R we can take out $\phi_n \in \mathfrak{B}$ $n=1, 2, \dots$ such that $\{E(\phi_n)e; n=1, 2, \dots\}$ is dense in $\{E(\phi)e; \phi \in \mathfrak{B}\}$. Putting

$\Psi_{\{\delta_n\}} = \prod_{n=1}^{\infty} \phi_n^{\delta_n}$ ($\delta_n = 1$ or -1), $\phi_n^{\delta_n} = \phi_n$ or ϕ_n^c according to $\delta_n = 1$ or -1 , then evidently $\{\delta_n\} \neq \{\delta'_n\}$ implies $\Psi_{\{\delta_n\}} \Psi_{\{\delta'_n\}} = \emptyset$, and we have $\sum_{\{\delta_n\}} \Psi_{\{\delta_n\}} = \mathcal{Q}$.

Next if $\Psi_{\{\delta_n\}} \neq \emptyset$, then $\Psi_{\{\delta_n\}}$ is an atomic element in $\mathfrak{B}$, because if $\Psi_{\{\delta_n\}} \supsetneq \phi$, $\phi \in \mathfrak{B}$, then

$$\|E(\Psi_{\{\delta_n\}} - \phi)e\|^2 = \|E(\Psi_{\{\delta_n\}})e - E(\phi)e\|^2 = \delta > 0$$

for any positive number $\varepsilon > 0$ ($0 < \varepsilon < \delta$) we can find ϕ_{n_0} such that $\|E(\phi)e - E(\phi_{n_0})e\|^2 < \varepsilon$. For such n_0 we have $\delta_{n_0} = -1$, for if $\delta_{n_0} = 1$, then $\phi_{n_0} \supsetneq \Psi_{\{\delta_n\}}$, therefore

$$\varepsilon > \|E(\phi_{n_0})e - E(\phi)e\|^2 \geq \|E(\Psi_{\{\delta_n\}})e - E(\phi)e\|^2 = \delta > \varepsilon,$$

and it is a contradiction. Therefore $\delta_{n_0} = -1$, and hence $\phi \subseteq \phi_{n_0}^c$, so

$$\|E(\phi_{n_0})e - E(\phi)e\|^2 = \|E(\phi_{n_0})e\|^2 + \|E(\phi)e\|^2 < \varepsilon,$$

hence $\|E(\phi)e\|^2 < \varepsilon$, since ε is arbitrary, $E(\phi)e = 0$, namely $\phi = \emptyset$. As R is separable, we have $\mathfrak{R}_E = \{E(\phi); \phi \in \mathfrak{B}\}$, and hence $E(\Psi_{\{\delta_n\}})$ is atomic in $\mathfrak{R}_E$, and the simplicity of $\mathfrak{R}_E$, we obtain that $E(\Psi_{\{\delta_n\}})R$ has one or zero dimension. From the above mentioned $E(\phi) (\phi \in \mathfrak{B})$ is a discrete spectrality. q. e. d.

Finally I thank Prof. NAKANO for his many suggestions.

Refereces

- [1] F. J. WECKEN: Unitärinvariant selbstadjungierter operatoren, Math. Ann. (1939).
- [2] H. NAKANO: Spectral theory in the Hilbert space, Tokyo Math. Book Series, Vol. IV. (1953).