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Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 13(2), 068-070
Issue Date	1956
Doc URL	https://hdl.handle.net/2115/55993
Type	departmental bulletin paper
File Information	JFSHIU_13_N2_068-070.pdf



ON THE UNIFORMLY BOUNDED COMMUTATIVE GROUP OF LINEAR TRANSFORMATIONS IN THE HILBERT SPACE

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F. RIESZ and B. SZ-NAGY have remarked in [2], p. 393 that an uniformly bounded real parameter group $\{T_t\}$ of linear transformations in Hilbert space is *similar* to a real parameter group $\{U_t\}$ of unitary transformations ([1]), that is, we can find a bicontinuous transformation A such that $T_t = A^{-1}U_t A$ ($-\infty < t < +\infty$). In this paper we prove that this theorem can be generalized for any uniformly bounded commutative group. The proof is based on MARKOFF-KAKUTANI'S *fixed point theorem* ([3], appendix).

Let R be a Hilbert space where an inner product (x, y) is defined. A complex valued functional $\varphi(x, y)$ on $R \times R$ is called a symmetric bilinear functional on R if 1) $\varphi_y(x) = \varphi(x, y)$ is a linear functional on R , 2) $\varphi_x(y) = \overline{\varphi(x, y)}$ is a conjugate linear functional, 3) $\varphi(x, y) = \overline{\varphi(y, x)}$, where $\overline{\varphi(y, x)}$ is a conjugate complex number of $\varphi(y, x)$. Let $\bar{R}$ be the total of symmetric bilinear functionals on R . It is easy to see that $\bar{R}$ is not empty and a real vector space if the addition and the scalar multiplication are defined as usual: $(\alpha\varphi)(x, y) = \alpha\varphi(x, y)$, $(\varphi + \psi)(x, y) = \varphi(x, y) + \psi(x, y)$ ($\varphi, \psi \in \bar{R}$, α is a real number). The following lemma is evident.

Lemma 1. For any positive number $\varepsilon > 0$, and finite points $x_i \in R$, $y_j \in R$, $i=1, 2, \dots, n$, $j=1, 2, \dots, m$, if we put

$$\begin{aligned} V(\varepsilon, x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) \\ = \{\varphi; \varphi \in \bar{R}, |\varphi(x_i, y_j)| < \varepsilon, i=1, 2, \dots, n, j=1, 2, \dots, m\} \end{aligned}$$

then a separated linear topology on $\bar{R}$ is given uniquely such that $\{V(\varepsilon, x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)\}$ is a fundamental system of neighbourhoods at zero.

We can prove the next lemma in the same way as SCHWARZ'S inequality.

Lemma 2. If $\varphi \in \bar{R}$ is positive: $\varphi(x, x) \geq 0$, then we have

$$|\varphi(x, y)|^2 \leq \varphi(x, x) \varphi(y, y) \quad (x, y \in R).$$

Lemma 3. For any two positive numbers c_1, c_2 as $0 < c_1 < c_2$, if we put

$$K_{c_1, c_2} = \{\varphi; c_1 \|x\|^2 \leq \varphi(x, x) \leq c_2 \|x\|^2 (x \in R), \varphi \in \bar{R}\},$$

then K_{c_1, c_2} is convex compact in $\bar{R}$.

Proof. Evidently K_{c_1, c_2} is convex, and if $\varphi \in K_{c_1, c_2}$, then we have from lemma 2 $|\varphi(x, y)| \leq c_2 \|x\| \|y\| (x, y \in R)$. We put

$$S_{(x, y)} = \{\alpha; |\alpha| \leq c_2 \|x\| \|y\|\} (x \neq y), \quad S_{(x, x)} = \{\alpha; c_1 \|x\|^2 \leq \alpha \leq c_2 \|x\|^2\},$$

then by TYCHONOFF's theorem the direct product $S = \prod_{x, y \in R} S_{(x, y)}$ is a compact set. It is easy to see that K_{c_1, c_2} is embedded homeomorphic in S , and K_{c_1, c_2} is closed in S , therefore K_{c_1, c_2} is compact. q. e. d.

Lemma 4. For a linear transformation T on R we have a continuous linear transformation $\tilde{T}$ on $\bar{R}$ such that $(\tilde{T}\varphi)(x, y) = \varphi(Tx, Ty) (\varphi \in \bar{R})$.

Proof. $\tilde{T}$ is obviously linear and from

$$\tilde{T}(V(\varepsilon, Tx_1, \dots, Tx_n, Ty_1, \dots, Ty_m)) \subseteq V(\varepsilon, x_1, \dots, x_n, y_1, \dots, y_m)$$

$\tilde{T}$ is continuous on $\bar{R}$. q. e. d.

If $\mathfrak{G}$ is a group of linear transformations on R , then $\tilde{\mathfrak{G}} = \{\tilde{T}; T \in \mathfrak{G}\}$ is a group of continuous linear transformations on $\bar{R}$. If $\mathfrak{G}$ is commutative, then $\tilde{\mathfrak{G}}$ is commutative, too.

Lemma 5. Let $\mathfrak{G}$ be a group of linear transformations and uniformly bounded: $\|T\| \leq r (T \in \mathfrak{G})$. If we put $K = \{(Tx, Ty); T \in \mathfrak{G}\} \subseteq \bar{R}$ and $\bar{K}$ is the least convex closed set that contains K , then $\bar{K}$ is compact and invariant under $\tilde{\mathfrak{G}}$, that is, if $\varphi \in \bar{K}$, then $\tilde{T}\varphi \in \bar{K} (\tilde{T} \in \tilde{\mathfrak{G}})$.

Proof. As $\mathfrak{G}$ is a group, K is invariant under $\mathfrak{G}$. Let K_1 be the least convex extension of K , then for any $\varphi \in K_1$ we have $\varphi(x, y) = \sum_{i=1}^n \lambda_i (T_i x, T_i y)$ for some $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$, $T_i \in \mathfrak{G} (i=1, 2, \dots, n)$, so K_1 is invariant under $\mathfrak{G}$. From lemma 4 $\tilde{T} (T \in \mathfrak{G})$ is continuous on $\bar{R}$, hence

$$\tilde{T}(\bar{K}) = \tilde{T}(\bar{K}_1) \subseteq \overline{\tilde{T}(K_1)} = \bar{K}_1 = \bar{K} \quad (\tilde{T} \in \tilde{\mathfrak{G}})$$

here a bar shows closure operation, therefore $\bar{K}$ is invariant under $\tilde{\mathfrak{G}}$. On the other hand since $\mathfrak{G}$ is uniformly bounded, we have

$$\frac{1}{r^2} \|x\|^2 \leq (Tx, Ty) \leq r^2 \|x\|^2 \quad (x \in R)$$

hence $K \subseteq K_{(\frac{1}{r^2}), r^2}$. From lemma 3 $K_{(\frac{1}{r^2}), r^2}$ is convex and compact, therefore $\bar{K} \subseteq K_{(\frac{1}{r^2}), r^2}$, and hence $\bar{K}$ is compact. q. e. d.

Especially if $\mathfrak{G}$ is commutative, by MARKOFF-KAKUTANI's fixed point theorem we obtain invariant elements in $\tilde{K}$ under $\tilde{\mathfrak{G}}$. ([3], appendix). Let one of them be $\langle x, y \rangle$, as $\tilde{K} \subseteq K_{(\frac{1}{r})^2, r^2}$ from lemma 5, we have $\frac{1}{r^2} \|x\|^2 \leq \langle x, x \rangle \leq r^2 \|x\|^2$ ($x \in R$), hence $\langle x, y \rangle$ is an inner product on R equivalent to (x, y) . From RIESZ's theorem we can find a self-adjoint operator B such that $\frac{1}{r^2} \leq B \leq r^2$ and $\langle x, y \rangle = (Bx, y)$. If we put $A = \sqrt{B}$, then $\frac{1}{r} \leq A \leq r$ and $\langle x, y \rangle = (Ax, Ay)$. On the other hand as $\langle x, y \rangle$ is invariant under $\tilde{\mathfrak{G}}$, we have $\langle x, y \rangle = \langle Tx, Ty \rangle$. Therefore $(Ax, Ay) = (ATx, ATy)$, hence $T^* A^2 T = A^2$. If we put $U_T = AT A^{-1}$, we have $U_T^* U_T = A^{-1} T^* A^2 T A^{-1} = I$, so U_T is a unitary transformation, and $\mathfrak{U} = \{U_T; T \in \mathfrak{G}\}$ is a group of unitary transformations. Thus we obtain the following.

Theorem. *If $\mathfrak{G}$ is an uniformly bounded commutative group of linear transformations in a Hilbert space R , then $\mathfrak{G}$ is similar to an unitary transformation group $\mathfrak{U}$ in R .*

Bibliography

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