FOURIER SERIES VIII: ABSOLUTE RIESZ SUMMABILITY

By

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1. The series $\sum a_n$ is said to be $|R, \lambda(n), \delta|$ summable, if

$$\sum_{w=2}^{\infty} \frac{\lambda'(w)}{(\lambda(w))^{\delta+1}} \left| \sum_{n=2}^{w-1} \lambda(n)\,(\lambda(w)-\lambda(n))^{\delta-1} a_n \right| < \infty .$$

Many results are known about the $|R, \lambda(n), \delta|$ summability of Fourier series.

If $\lambda(n)=\log n$, then the summability $|R, \log n, \delta|$ has the local property for $\delta>1$, but does not for $0<\delta\leqq 1$ ([1], [2]). The summability $|R, \log n, \delta|$ $(\delta>1)$ is treated by many writers. In [2] and [3], sufficient conditions, of course non-local, for $|R, \log n, 1|$ summability of Fourier series are found. We shall here prove similar theorem for the summability $|R, \log n, \delta|$ $(0<\delta<1)$ (Theorem 1).

We consider more generally the summability $|R, \lambda(n), \delta|$ and prove the corresponding theorem (Theorem 2). Finally we treat the corresponding problem concerning $|C, \delta|$ (Theorem 3).

2. **Theorem 1.** *Let* $0<\delta=1-\alpha<1$ *and* $\varepsilon>0$. *If*

$$(1) \qquad \int_0^{\pi/n} (f(\xi+t)-f(\xi-t))\,dt = O\,(1/n^{1+\alpha}(\log n)^{1+\alpha+\varepsilon})$$

uniformly in ξ *and*

$$(2) \qquad \int_0^{\pi/n} (f(x)-f(x+t))\,dt = O\,(1/n^{1+\alpha}(\log n)^{\alpha+\varepsilon})$$

for a fixed x, *then the Fourier series of* $f(t)$ *is* $|R, \log n, \delta|$ *summable at* x.

Proof. It is sufficient to prove that

$$J = \sum_{w=2}^{\infty} \frac{1}{w(\log w)^{2-\alpha}} \left| \sum_{n=2}^{w-1} \frac{\log n}{(\log w/n)^{\alpha}} \int_0^{\pi} \varphi_x(t) \cos nt\,dt \right| < \infty$$

where

$$\varphi_x(t) = f(x+t)+f(x-t)-2f(x) .$$

The inner sum of J is

$$I(w)=\sum_{n=2}^{w-1}\frac{\log n}{(\log w/n)^{\alpha}}\int_0^{\pi}\varphi_x(t)\cos nt\,dt=\int_0^{\pi}\varphi_x(t)\sum_{n=2}^{w-1}\frac{\log n}{(\log w/n)^{\alpha}}\cos nt\,dt\,.$$

By Abel's lemma we write

$$I(w)=\int_0^{\pi}\varphi_x(t)\Big[\sum_{n=2}^{w-2}\Delta\Big(\frac{\log n}{\log (w/n)^{\alpha}}\Big)\frac{\sin(n+1/2)t}{2\sin t/2}$$
$$+\frac{\log(w-1)}{(\log w/(w-1))^{\alpha}}\frac{\sin(w-1/2)t}{2\sin t/2}\Big]dt = I_1(w)+I_2(w)\ .$$

We shall first estimate $I_2(w)$ and the corresponding sum

$$J_2=\sum_{w=2}^{\infty}\frac{|I_2(w)|}{w(\log w)^{2-\alpha}}\ .$$

Let $w-1/2=v$ and divide the interval $(0,\pi)$ into the following subintervals:

(3) $\quad (0,\ \tau),\ (\tau,\ \tau+\pi/v),\ (\tau+\pi/v,\ \tau+2\pi/v),\cdots,(\tau+(2K-1)\pi/v,\ \pi)$

where $0<\tau<2\pi/v$ and K is an integer such that $\tau+(2K-1)\pi/v=\pi-\pi/v$, and hence

$$\tau=\pi-\Big[\frac{v}{2}\Big]\frac{2\pi}{v}=\pi\Big(1-\Big[\frac{v}{2}\Big]\frac{2}{v}\Big)\,,\quad K=\frac{\pi-\tau}{2\pi}v\ .$$

Then we have

$$I_2(w)=\frac{\log(w-1)}{(-\log(1-1/w))^{\alpha}}\int_0^{\pi}\varphi_x(t)\frac{\sin vt}{2\sin t/2}dt$$
$$=\frac{\log(w-1)}{(-\log(1-1/w))^{\alpha}}\Big[\int_0^{\tau}\varphi_x(t)\frac{\sin vt}{2\sin t/2}dt$$
$$+\sum_{k=1}^{K+1}\int_{\tau}^{\tau+\pi/v}\Big(\frac{\varphi_x(t+(2k-4)\pi/v)}{2\sin(t+(2k-4)\pi/v)/2}-\frac{\varphi_x(t+(2k-3)\pi/v)}{2\sin(t+(2k-3)\pi/v)/2}\Big)\sin vt\,dt\Big]$$
$$=\frac{\log(w-1)}{(-\log(1-1/w))^{\alpha}}\Big[\int_0^{\tau}\varphi_x(t)\frac{\sin vt}{2\sin t/2}dt$$
$$+\sum_{k=1}^{K+1}\int_{\tau}^{\tau+\pi/v}\frac{\varphi_x(t+(2k-4)\pi/v)-\varphi_x(t+(2k-3)\pi/v)}{2\sin(t+(2k-4)\pi/v)/2}\sin vt\,dt$$
$$+\sum_{k=2}^{K+1}\int_{\tau}^{\tau+\pi/v}\varphi_x(t+(2k-3)\pi/v)$$
$$\cdot\Big(\frac{1}{2\sin(t+(2k-4)\pi/v)/2}-\frac{1}{2\sin(t+(2k-3)\pi/v)/2}\Big)\sin vt\,dt\Big]$$
$$=I_{21}(w)+I_{22}(w)+I_{23}(w)\ .$$

We shall begin to estimate $I_{21}(w)$. If $3\pi/2v<\tau<2\pi/v$, then

$$\int_0^\tau \varphi_x(t)\frac{\sin vt}{2\sin t/2}\,dt=\left[\int_0^{3\pi/2v}+\int_{3\pi/2v}^{\tau}\right]\varphi_x(t)\frac{\sin vt}{2\sin t/2}\,dt$$
$$=v\int_0^\xi \varphi_x(t)\,dt-\frac{1}{2\sin 3\pi/v}\int_{3\pi/2v}^{\eta}\varphi_x(t)\,dt$$

by the second mean value theorem, where $0<\xi<3\pi/2v<\eta<\pi$; and then[1]

$$\left|\int_0^\tau \varphi_x(t)\frac{\sin vt}{2\sin t/2}\,dt\right|\leqq Av\max_{0<\zeta<2\pi/v}\left|\int_0^\zeta \varphi_x(t)\,dt\right|.$$

In the case $0<\tau<3\pi/2v$ we get also the same estimation. Thus we get

$$|I_{21}(w)|\leqq Aw^\alpha\log w\cdot v\max_{0<\zeta<2\pi/v}\left|\int_0^\zeta \varphi_x(t)\,dt\right|$$
$$\leqq Aw^{1+\alpha}\log w/w^{1+\alpha}(\log w)^{\alpha+\varepsilon}\leqq A/(\log w)^{\alpha+\varepsilon-1}$$

and then

$$J_{21}=\sum_{w=2}^\infty\frac{|I_{21}(w)|}{w(\log w)^{2-\alpha}}\leqq A\sum_{w=2}^\infty\frac{1}{w(\log w)^{1+\varepsilon}}<\infty .$$

We denote by ξ_k the extremum point of $\sin vt/2\sin(t+(2k-4)\pi/v)/2$ in the interval $(\tau,\tau+\pi/v)$, then we get by the second mean value theorem

$$I_{22}(w)=\frac{\log(w-1)}{(-\log(1-1/w))^\alpha}\sum_{k=2}^{K+1}\int_\tau^{\tau+\pi/v}\frac{\varphi_x(t+(2k-4)\pi/v)-\varphi_x(t+(2k-3)\pi/v)}{2\sin(t+(2k-4)\pi/v)/2}\sin vt\,dt$$
$$=\frac{\log(w-1)}{(-\log(1-1/w))^\alpha}\sum_{k=2}^{K+1}\left[\int_\tau^{\xi_k}+\int_{\xi_k}^{\tau+\pi/v}\right]$$
$$\frac{\varphi_x(t+(2k-4)\pi/v)-\varphi_x(t+(2k-3)\pi/v)}{2\sin(t+(2k-4)\pi/v)/2}\sin vt\,dt$$
$$=\frac{\log(w-1)}{(-\log(1-1/w))^\alpha}\sum_{k=2}^{K+1}\frac{\sin v\xi_k}{2\sin(\xi_k+(2k-4)\pi/v)/2}$$
$$\cdot\left[\int_{\eta_k}^{\xi_k}+\int_{\xi_k}^{\zeta_k}\right]\left[\varphi_x(t+(2k-4)\pi/v)-\varphi_x(t+(2k-3)\pi/v)\right]dt$$

where $\tau\leqq\eta_k<\xi_k\leqq\zeta_k\leqq\tau+\pi/v$, and then

$$|I_{22}(w)|\leqq\frac{A\log(w-1)}{(-\log(1-1/w))^\alpha}\sum_{k=2}^{K+1}\frac{v}{k}\left|\int_{\eta_k}^{\zeta_k}\left[\varphi_x(t+(2k-4)\pi/v)-\varphi_x(t+(2k-3)\pi/v)\right]dt\right|.$$

Thus we have

1) We denote by A an absolute constant, which need not be equal in different occurences.

$$J_{22} = \sum_{w=2}^{\infty} \frac{|I_{22}(w)|}{w(\log w)^{2-\alpha}}$$

$$\leqq A \sum_{w=2}^{\infty} \frac{1}{(w \log w)^{1-\alpha}} \sum_{k=2}^{K+1} \frac{v}{k} \left| \int_{\eta_k}^{\zeta_k} \left[\varphi_x(t+(2k-4)\pi/v) - \varphi_x(t+(2k-3)\pi/v) \right] dt \right|$$

$$\leqq A \sum_{k=2}^{\infty} \frac{1}{k} \sum_{w=2k}^{\infty} \frac{1}{w^{-\alpha}(\log w)^{1-\alpha}} \frac{1}{w^{1+\alpha}(\log w)^{1+\alpha+\varepsilon}}$$

$$= A \sum_{k=2}^{\infty} \frac{1}{k} \sum_{w=2k}^{\infty} \frac{1}{w(\log w)^{2+\varepsilon}} \leqq A \sum_{k=2}^{\infty} \frac{1}{k(\log k)^{1+\varepsilon}} < \infty .$$

By Abel's lemma, we get now

$$I_{23}(w) = \frac{\log(w-1)}{(-\log(1-1/w))^{\alpha}} \sum_{k=2}^{K+1} \int_{\tau}^{\tau+\pi/v} \varphi_x(t+(2k-3)\pi/v) \sin vt$$

$$\cdot \left(\frac{1}{2\sin(t+(2k-4)\pi/v)/2} - \frac{1}{2\sin(t+(2k-3)\pi/v)/2} \right) dt$$

$$= \frac{\log(w-1)}{(-\log(1-1/w))^{\alpha}} \left\{ \int_{\tau}^{\tau+\pi/v} \varphi_x(t+\pi/v) \sin vt \right.$$

$$\cdot \sum_{j=2}^{K+1} \left(\frac{1}{2\sin(t+(2j-4)\pi/v)/2} - \frac{1}{2\sin(t+(2j-3)\pi/v)/2} \right) dt$$

$$+ \sum_{k=3}^{K+1} \int_{\tau}^{\tau+\pi/v} \left[\varphi_x(t+(2k-3)\pi/v) - \varphi_x(t+(2k-5)\pi/v) \right] \sin vt$$

$$\left. \cdot \sum_{j=k}^{K+1} \left(\frac{1}{2\sin(t+(2j-4)\pi/v)/2} - \frac{1}{2\sin(t+(2j-3)\pi/v)/2} \right) dt \right\}$$

$$= I_{231}(w) + I_{232}(w).$$

Then

$$\int_{\tau}^{\tau+\pi/v} \varphi_x(t+\pi/v) \frac{\sin t(+(2j-3)\pi/v)/2 - \sin(t+(2j-4)\pi/v)/2}{2\sin(t+(2j-4)\pi/v)/2 \cdot \sin(t+(2j-3)\pi/v)/2} \sin vt\, dt$$

$$= \sin \pi/4v \int_{\tau}^{\tau+\pi/v} \varphi_x(t+\pi/v) \frac{\sin vt \cdot \cos(2t+(4j-7)\pi/v)/2}{\sin(t+(2j-4)\pi/v)/2 \cdot \sin(t+(2j-3)\pi/v)/2} dt$$

$$= \theta_j \int_{E_j} \varphi_x(t+\pi/v)\, dt$$

by the second mean value theorem, where E_j consists of a finite set of intervals contained in the interval $(\tau, \tau+\pi/v)$

$$|\theta_j| \leqq A \frac{1}{v} \cdot \frac{v^2}{j^2} = A \frac{v}{j^2} .$$

Thus we get

$$|I_{231}(w)| \leqq Aw^{\alpha} \log w \sum_{j=2}^{K+1} \frac{v}{j^2} \left| \int_{E_j} \varphi_x(t+\pi/v)\, dt \right| \leqq A(\log w)^{1-\alpha-\varepsilon}$$

and then

$$J_{231} = \sum_{w=2}^{\infty} \frac{|I_{231}(w)|}{w(\log w)^{2-\alpha}} \leqq A \sum_{w=2}^{\infty} \frac{1}{w(\log w)^{1+\varepsilon}} < \infty .$$

Similarly

$$|I_{232}(w)| \leqq Aw^{\alpha} \log w \sum_{k=3}^{K+1} \sum_{j=k}^{K+1} \frac{w}{j^2} \left| \int_{E_j} \Big[\varphi_x(t+(2k-3)\pi/v) - \varphi_x(t+(2k-5)\pi/v) \Big] dt \right|$$

$$\leqq Aw^{1+\alpha} \log w \sum_{k=3}^{K+1} \sum_{j=k}^{K+1} \frac{1}{j^2} \frac{1}{w^{1+\alpha} (\log w)^{1+\alpha+\varepsilon}}$$

$$\leqq A \frac{1}{(\log w)^{\alpha+\varepsilon}} \sum_{k=3}^{K+1} \frac{1}{k} \leqq A(\log w)^{1-\alpha-\varepsilon} ,$$

and then

$$J_{232} = \sum_{w=2}^{\infty} \frac{|I_{232}(w)|}{w(\log w)^{2-\alpha}} \leqq A \sum_{w=2}^{\infty} \frac{1}{w(\log w)^{1+\varepsilon}} < \infty .$$

Thus we have proved that

$$J_2 \leqq J_{21} + J_{22} + (J_{231} + J_{232}) < \infty .$$

Let us now turn to the estimation of $I_1(w)$ and the corresponding sum $J_1 = \sum |I_1(w)|/w(\log w)^{2-\alpha}$. We have

$$I_1(w) = \int_0^{\pi} \varphi_x(t) \Big(\sum_{n=2}^{w-2} \Delta \Big(\frac{\log n}{(\log w/n)^{\alpha}} \Big) \frac{\sin (n+1/2)t}{2 \sin t/2} \Big) dt$$

$$= \sum_{n=1}^{w-2} \Delta \Big(\frac{\log n}{(\log w/n)^{\alpha}} \Big) \int_0^{\pi} \varphi_x(t) \frac{\sin (n+1/2)t}{2 \sin t/2} dt .$$

We put $m = n+1/2$, and divide the interval $(0, \pi)$ in the following:

$$(0, \pi) = (0, \tau_n) \cup (\tau_n, \tau_n + \pi/m) \cup \cdots \cup (\tau_n + (2K_n - 1)\pi/m, \pi)$$

where $0 < \tau_n < 2\pi/m$ and $\tau_n + (2K_n - 1)\pi/m = \pi - \pi/m$, and then

$$K_n = \frac{\pi - \tau_n}{2\pi} m \leqq \frac{n+1}{2} .$$

We write

$$I_1(w)=\sum_{n=2}^{w-2}\Delta\left(\frac{\log n}{(\log w/n)^{\alpha}}\right)\left\{\int_0^{\tau_n}\varphi_x(t)\,\frac{\sin mt}{2\sin t/2}\,dt\right.$$
$$+\sum_{k=2}^{K_n+1}\int_{\tau_n}^{\tau_n+\pi/m}\frac{\varphi_x(t+(2k-4)\pi/m)-\varphi_x(t+(2k-3)\pi/m)}{2\sin(t+(2k-4)\pi/m)/2}\sin mt\,dt$$
$$+\sum_{k=2}^{K_n+1}\int_{\tau_n}^{\tau_n+\pi/m}\varphi_x(t+(2k-3)\pi/m)\sin mt$$
$$\left.\cdot\left(\frac{1}{2\sin(t+(2k-4)\pi/m)/2}-\frac{1}{2\sin(t+(2k-3)\pi/m)/2}\,dt\right)\right\}$$
$$=I_{11}(w)+I_{12}(w)+I_{13}(w)\ .$$

Then

$$|I_{11}(w)|\leqq A\sum_{n=2}^{w-2}\Delta\left(\frac{\log n}{(\log w/n)^{\alpha}}\right)\left|\int_0^{\tau_n}\varphi_x(t)\,\frac{\sin mt}{2\sin t/2}\,dt\right|$$
$$\leqq A\sum_{n=2}^{w-2}\Delta\left(\frac{\log n}{(\log w/n)^{\alpha}}\right)n\left|\int_0^{\eta}\varphi_x(t)\,dt\right|$$
$$\leqq A\sum_{n=2}^{w-2}\Delta\left(\frac{\log n}{(\log w/n)^{\alpha}}\right)\frac{1}{n^{\alpha}(\log n)^{\alpha+\varepsilon}}$$
$$\leqq A\left[\sum_{n=2}^{w-2}\frac{\log n}{(\log w/n)^{\alpha}}\,\frac{1}{n^{1+\alpha}(\log n)^{\alpha+\varepsilon}}+\frac{\log w}{(\log w/(w-2))^{\alpha}}\,\frac{1}{w^{\alpha}(\log w)^{\alpha+\varepsilon}}\right]$$
$$\leqq A\left[\sum_{n=2}^{w-2}\frac{1}{n^{1+\alpha}(\log n)^{\alpha+\varepsilon-1}(\log w/n)^{\alpha}}+(\log w)^{1-\alpha-\varepsilon}\right].$$

Accordingly

$$|J_{11}|\leqq\sum_{w=2}^{\infty}\frac{|I_{11}(w)|}{w(\log w)^{2-\alpha}}$$
$$\leqq A\left[\sum_{n=2}^{\infty}\frac{1}{n^{1+\alpha}(\log n)^{\alpha+\varepsilon-1}}\sum_{w=n+2}^{\infty}\frac{1}{w(\log w)^{2-\alpha}(\log w/n)^{\alpha}}+\sum_{w=2}^{\infty}\frac{1}{w(\log w)^{1+\varepsilon}}\right]$$

where

$$\sum_{w=n+2}^{\infty}\frac{1}{w(\log w)^{2-\alpha}(\log w/n)^{\alpha}}\leqq\int_{n+2}^{\infty}\frac{dw}{w(\log w)^{2-\alpha}(\log w/n)^{\alpha}}$$
$$=\left[\frac{1}{\alpha\log n}\left(\frac{\log w}{\log w/n}\right)^{\alpha-1}\right]_{w=n+2}^{\infty}\leqq A\left[\frac{1}{n^{1-\alpha}(\log n)^{2-\alpha}}+\frac{1}{\log n}\right]\leqq\frac{A}{\log n}\ ,$$

and then

$$|J_{11}|\leqq A\left[\sum_{n=2}^{\infty}\frac{1}{n^{1+\alpha}(\log n)^{\alpha+\varepsilon}}+\sum_{w=2}^{\infty}\frac{1}{w(\log w)^{1+\varepsilon}}\right]<\infty\ .$$

Now

$$|I_{12}(w)| \leqq A \sum_{n=2}^{w-1} \Delta\left(\frac{\log n}{(\log w/n)^\alpha}\right) \sum_{k=2}^{K_n+1} \frac{m}{k} \frac{1}{n^{1+\alpha}(\log n)^{1+\alpha+\varepsilon}}$$

$$\leqq A \sum_{n=2}^{w-2} \Delta\left(\frac{\log n}{(\log w/n)^\alpha}\right) \frac{1}{n^\alpha (\log n)^{\alpha+\varepsilon}} .$$

Similary as J_{11}, we have

$$|J_{12}| = \sum_{w=2}^{\infty} \frac{|I_{12}(w)|}{w(\log w)^{2-\alpha}} < \infty .$$

Finally

$$I_{13}(w) = \sum_{n=2}^{w-2} \Delta\left(\frac{\log n}{(\log w/n)^\alpha}\right) \int_{\tau_n}^{\tau_n+\pi/m} \varphi_x(t+(2k-3)\pi/m) \sin mt$$

$$\cdot \sum_{j=2}^{K_n+1} \left(\frac{1}{2\sin(t+(2j-4)\pi/m)/2} - \frac{1}{2\sin(t+(2j-3)\pi/m)/2}\right) dt$$

$$+ \sum_{k=3}^{K_n+1} \int_{\tau_n}^{\tau_n+\pi/m} \left[\varphi_x(t+(2k-3)\pi/m) - \varphi_x(t+2k-5)\pi/m)\right] \sin mt$$

$$\cdot \sum_{j=k}^{K_n+1} \left(\frac{1}{2\sin(t+(2j-4)\pi/m)/2} - \frac{1}{2\sin(t+2j-3)\pi/m)/2}\right) dt\Big]$$

$$= I_{131}(w) + I_{132}(w) .$$

Then

$$|I_{131}(w)| \leqq A \sum_{n=2}^{w-2} \Delta\left(\frac{\log n}{(\log w/n)^\alpha}\right) \sum_{j=2}^{K_n+1} \frac{m}{j^2} \left| \int_{\xi_n}^{\eta_n} \varphi_x(t+\pi/m)\, dt \right|$$

$$\leqq A \sum_{n=2}^{w-2} \Delta\left(\frac{\log n}{(\log w/n)^\alpha}\right) \sum_{j=2}^{K_n+1} \frac{m}{j^2} \frac{1}{m^{1+\alpha}(\log m)^{\alpha+\varepsilon}}$$

$$\leqq A \sum_{n=2}^{w-2} \Delta\left(\frac{\log n}{(\log w/n)^\alpha}\right) \frac{1}{m^\alpha (\log m)^{\alpha+\varepsilon}}$$

where $\tau_n \leqq \xi_n < \eta_n \leqq \tau_n + \pi/m$, and further

$$|I_{132}(w)| \leqq A \sum_{n=2}^{w-2} \Delta\left(\frac{\log n}{(\log w/n)^\alpha}\right)$$

$$\cdot \sum_{k=3}^{K_n+1} \sum_{j=k}^{K_n+1} \frac{m}{j^2} \left| \int_{\xi_n'}^{\eta_n'} \left[\varphi_x(t+(2k-3)\pi/m) - \varphi_x(t+(2k-5)\pi/m)\right] \sin mt\, dt \right|$$

$$\leqq A \sum_{n=2}^{w-2} \Delta\left(\frac{\log n}{(\log w/n)^\alpha}\right) \sum_{k=3}^{K_n+1} \sum_{j=k}^{K_n+1} \frac{m}{j^2} \frac{1}{m^{1+\alpha}(\log m)^{1+\alpha+\varepsilon}}$$

$$\leqq A \sum_{n=2}^{w-2} \Delta\left(\frac{\log n}{(\log w/n)^\alpha}\right) \sum_{k=3}^{K_n+1} \frac{1}{k} \frac{1}{m^\alpha (\log m)^{1+\alpha+\varepsilon}}$$

$$\leqq A \sum_{n=2}^{w-2} \Delta\left(\frac{\log n}{(\log w/n^\alpha)}\right) \frac{1}{m^\alpha (\log m)^{\alpha+\varepsilon}}$$

where $\tau_n \leqq \xi'_n < \eta'_n \leqq \tau_n + \pi/m$. Thus we get

$$J_1 \leqq J_{11} + J_{12} + (J_{131} + J_{132}) < \infty .$$

Collecting above estimations, we get the theorem.

3. Let us consider the $|R, \lambda(n), \delta|$ summability. We put

$$\lambda(n) = e^{\mu(n)} .$$

We supose, as usual, that $\lambda(n)$ is monotone increasing and differentiable. Further we suppose that

1° $$A\mu(n) \leqq \mu(2n) < B\mu(n) ,$$

2° $$n(\mu(n))^{\delta} \uparrow \qquad \text{for} \quad 0 < \delta < 1 ,$$

3° $$\mu'(n) \downarrow .$$

These conditions are satisfied in the following cases:

$\lambda(n)$	$e^{(\log\log n)^{\alpha}}$	$e^{(\log n)^{\alpha}}$	$e^{n^{\alpha}}$
$\mu(n)$	$(\log\log n)^{\alpha}$	$(\log n)^{\alpha}$	n^{α}
$\mu'(n)$	$\alpha \dfrac{(\log\log n)^{\alpha-1}}{n\log n}$	$\alpha \dfrac{(\log n)^{\alpha-1}}{n}$	$\alpha n^{\alpha-1}$
α	$\alpha > 0$	$\alpha > 0$	$0 < \alpha < 1$

For such absolute Riesz summability, we prove the following theorem.

Theorem 2. *If,*[2] *for* $0 < \delta < 1$,

(4) $$\int_0^{\pi} \frac{|\varphi_x(t)|}{t^2} (\mu'(1/t))^{\delta}\, dt < \infty$$

and

(5) $$\sum_{n=2}^{\infty} n \log n\, (\mu'(n))^{\delta} M(n) < \infty ,$$

where

$$M(n) = \max_{-\pi \leqq \xi \leqq \pi} \left| \int_0^{\pi/n} (f(\xi+t) - f(\xi-t))\, dt \right| ,$$

2) If we replace the condition (4) by the condition

$$\int_0^{\pi} |\varphi_x(t)|\, dt \int_0^{1/t} u\, (\mu'(u))^{\delta}\, du < \infty ,$$

then the condition 2° is needless.

then the Fourier series of $f(t)$ *is* $|R, \lambda(n), \delta|$ *summable at* $t=x$.

It is sufficient to prove that

$$\sum_{w=2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^{\delta}} \left| \sum_{n=2}^{w-1} \lambda(n)(\lambda(w)-\lambda(n))^{\delta-1} \int_0^{\pi} \varphi_x(t) \cos nt\, dt \right| < \infty .$$

By $I(w)$ we denote the inner sum, then

$$\begin{aligned} I(w) &= \sum_{n=2}^{w-1} \lambda(n)(\lambda(w)-\lambda(n))^{\delta-1} \int_0^{\pi} \varphi_x(t) \cos nt\, dt \\ &= \int_0^{\pi} \varphi_x(t) \left[\sum_{n=2}^{w-1} \lambda(n)(\lambda(w)-\lambda(n))^{\delta-1} \cos nt \right] dt \\ &= \int_0^{\pi} \varphi_x(t) \left[\sum_{n=2}^{w-2} \Delta(\lambda(n)(\lambda(w)-\lambda(n))^{\delta-1}) \frac{\sin(n+1/2)t}{2\sin t/2} \right. \\ &\qquad \left. + \lambda(w-1)(\lambda(w)-\lambda(w-1))^{\delta-1} \frac{\sin(w-1/2)t}{2\sin t/2} \right] dt \\ &= I_1(w) + I_2(w) . \end{aligned}$$

Let us first estimate $I_2(w)$ and the corresponding sum

$$J_2 = \sum_{w=2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^{\delta}} |I_2(w)| .$$

Since

$$\lambda(w)-\lambda(w-1) = \int_{w-1}^{w} \lambda'(w)\, dw = \int_{w-1}^{w} \mu'(w)\lambda(w)\, dw ,$$

$$\mu'(w)\lambda(w-1) \leqq \lambda(w)-\lambda(w-1) \leqq \mu'(w-1)\lambda(w) ,$$

we have, using the notation in §2,

$$\begin{aligned} |I_2(w)| &\leqq (\lambda(w-1))^{\delta} (\mu'(w))^{\delta-1} \left| \int_0^{\pi} \varphi_x(t) \frac{\sin(w-1/2)t}{2\sin t/2} dt \right| \\ &\leqq (\lambda(w-1))^{\delta} (\mu'(w))^{\delta-1} \left[\left| \int_0^{\tau} \varphi_x(t) \frac{\sin vt}{2\sin t/2} dt \right| \right. \\ &\quad + \left| \sum_{k=2}^{K+1} \int_{\tau}^{\tau+\pi/v} \frac{\varphi_x(t+(2k-4)\pi/v)-\varphi_x(t+(2k-3)\pi/v)}{2\sin(t+(2k-4)\pi/v)/2} \sin vt\, dt \right| \\ &\quad + \left| \sum_{k=2}^{K+1} \int_{\tau}^{\tau+\pi/v} \varphi_x(t+(2k-3)\pi/v) \sin vt \right. \\ &\qquad \left. \left. \cdot \left(\frac{1}{2\sin(t+(2k-4)\pi/v)/2} - \frac{1}{2\sin(t+(2k-3)\pi/v)/2} \right) dt \right| \right] \\ &= I_{21}(w) + I_{22}(w) + I_{23}(w) . \end{aligned}$$

We have

$$I_{21}(w) \leqq A(\lambda(w))^{\delta}(\mu'(w))^{\delta-1} w \int_0^{2\pi/w} |\varphi_x(t)|\, dt\,,$$

and hence

$$\begin{aligned}
J_{21} &= \sum_{w=2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^{\delta}} I_{21}(w) \\
&\leqq A \sum_{w=2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^{\delta}} (\lambda(w))^{\delta} (\mu'(w))^{\delta-1} w \int_0^{2\pi/w} |\varphi_x(t)|\, dt \\
&\leqq A \sum_{w=2}^{\infty} (\mu'(w))^{\delta} w \int_0^{2\pi/w} |\varphi_x(t)|\, dt \\
&= A \sum_{w=2}^{\infty} (\mu'(w))^{\delta} w \sum_{k=w/2}^{\infty} \int_{\pi/(k+1)}^{\pi/k} |\varphi_x(t)|\, dt \\
&= A \sum_{k=2}^{\infty} \int_{\pi/(k+1)}^{\pi/k} |\varphi_x(t)|\, dt \sum_{w=2}^{2k} (\mu'(w))^{\delta} w \\
&\leqq A \int_0^{\pi} |\varphi_x(t)|\, dt \sum_{w=2}^{2[\pi/t]+2} w(\mu'(w))^{\delta} \leqq A \int_0^{\pi} \frac{|\varphi_x(t)|}{t^2} (\mu'(1/t))^{\delta}\, dt
\end{aligned}$$

which is finite by the assumption (4), since $w(\mu'(w))^{\delta}$ increases.

Secondly we have

$$\begin{aligned}
I_{22}(w) &\leqq A(\lambda(w))^{\delta}(\mu'(w))^{\delta-1} \sum_{k=2}^{w} \frac{w}{k} \left| \int_{\xi_k}^{\eta_k} (\varphi_x(t+(2k-4)\pi/v) - \varphi_x(t+(2k-3)\pi/v))\, dt \right| \\
&\leqq A(\lambda(w))^{\delta}(\mu'(w))^{\delta-1} w M(w) \sum_{k=2}^{w} \frac{1}{k} \leqq A(\lambda(w))^{\delta}(\mu'(w))^{\delta-1} w \log w\, M(w)\,,
\end{aligned}$$

and then

$$J_{22} = \sum_{w=2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^{\delta}} I_{22}(w) \leqq \sum_{w=2}^{\infty} w \log w\, (\mu'(w))^{\delta} M(w)$$

which is finite by the assumption (5).

Finally, by Abel's lemma,

$$\begin{aligned}
&\sum_{k=2}^{K+1} \int_{\tau}^{\tau+\pi/v} \varphi_x(t+(2k-3)\pi/v) \\
&\qquad \cdot \left(\frac{1}{2\sin(t+(2k-4)\pi/v)/2} - \frac{1}{2\sin(t+(2k-3)\pi/v)/2} \right) \sin vt\, dt \\
&= \int_{\tau}^{\tau+\pi/v} \varphi_x(t+\pi/v) \\
&\qquad \cdot \sum_{j=2}^{K+1} \left(\frac{1}{2\sin(t+(2j-4)\pi/v)/2} - \frac{1}{2\sin(t+(2j-3)\pi/v)/2} \right) \sin vt\, dt
\end{aligned}$$

$$+\sum_{k=3}^{K+1}\int_{\tau}^{\tau+\pi/v}\Big[\varphi_x(t+(2k-3)\pi/v)-\varphi_x(t+(2k-5)\pi/v)\Big]$$

$$\cdot\sum_{j=k}^{K+1}\Big(\frac{1}{2\sin(t+(2j-4)\pi/v)/2}-\frac{1}{2\sin(t+(2j-3)\pi/v)/2}\Big)\sin vt\,dt,$$

accordingly

$$I_{23}(w)\leqq A(\lambda(w))^{\delta}(\mu'(w))^{\delta-1}w\Big[\sum_{j=2}^{w}\frac{1}{j^2}\int_0^{\pi/v}|\varphi_x(t+\tau+\pi/v)|\,dt$$

$$+\sum_{k=3}^{w}\sum_{j=k}^{w}\frac{1}{j^2}\Big|\int_{\xi_k}^{\eta_k}(\varphi_x(t+(2k-3)\pi/v)-\varphi_x(t+(2k-5)\pi/v))\,dt\Big|\Big]$$

$$\leqq A(\lambda(w))^{\delta}(\mu'(w))^{\delta-1}w\Big[\int_0^{\pi/n}|\varphi_x(t)|\,dt+\log w\cdot M(w)\Big]$$

and then, by the condition (4) and (5)

$$J_{23}=\sum_{w=2}^{\infty}\frac{\mu'(w)}{(\lambda(w))^{\delta}}I_{23}(w)$$

$$=A\sum_{w=2}^{\infty}(\mu'(w))^{\delta}w\int_0^{\pi/w}|\varphi_x(t)|\,dt+A\sum_{w=2}^{\infty}(\mu'(w))^{\delta}w\log w\cdot M(w)$$

$$\leqq A(J_{21}+J_{22})<\infty\ .$$

Thus we have proved that

$$J_2\leqq J_{21}+J_{22}+J_{23}\leqq A(J_{21}+J_{22})<\infty\ . \tag{6}$$

Let us now estimate $I_1(w)$ and the corresponding sum

$$J_1=\sum_{w=2}^{\infty}\frac{\mu'(w)}{(\lambda(w))^{\delta}}\,|I_1(w)|\ .$$

If we write

$$\Delta_n=\Delta(\lambda(n)(\lambda(w)-\lambda(n))^{\delta-1})\ ,$$

then

$$I_1(w)=\sum_{n=2}^{w-2}\Delta_n\int_0^{\pi}\varphi_x(t)\,\frac{\sin mt}{2\sin t/2}\,dt\ .$$

Similarly estimating as $I_2(w)$,

$$|I_1(w)|\leqq A\sum_{n=2}^{w-2}\Delta_n\Big[n\int_x^{\pi/n}|\varphi_x(t)|\,dt+n\log n\cdot M(n)\Big]$$

and hence

$$J_1 \leqq A \sum_{w=2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^\delta} \sum_{n=2}^{w-2} \varDelta_n \left[n \int_0^{\pi/n} |\varphi_x(t)|\, dt + n \log n \cdot M(n) \right]$$

$$\leqq A \sum_{n=2}^{\infty} n \left[\int_0^{\pi/n} |\varphi_x(t)|\, dt + \log n \cdot M(n) \right] \sum_{w=n+2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^\delta} \varDelta_n = A(J_{11} + J_{12}).$$

The inner sum is

$$K(n) = \left| \sum_{w=n+2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^\delta} \varDelta(\lambda(n)(\lambda(w)-\lambda(n))^{\delta-1}) \right|$$

$$\leqq A \sum_{w=n+2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^\delta} \left[\lambda'(n)(\lambda(w)-\lambda(n))^{\delta-1} + \lambda'(n)\lambda(n)(\lambda(w)-\lambda(n))^{\delta-2} \right]$$

$$= A\lambda'(n) \sum_{w=n+2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^\delta} (\lambda(w)-\lambda(n))^{\delta-1}$$

$$+ A\lambda'(n)\lambda(n) \sum_{w=n+2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^\delta} (\lambda(w)-\lambda(n))^{\delta-2} = K_1(n) + K_2(n)$$

where

$$|K_1(n)| \leqq A\lambda'(n) \int_{n+2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^\delta} (\lambda(w)-\lambda(n))^{\delta-1}\, dw$$

$$= A\lambda'(n) \left[\frac{1}{\lambda(n)} \left(\frac{\lambda(w)}{\lambda(w)-\lambda(n)} \right)^{-1-\delta} \right]_{w=n+2}^{\infty} \leqq A\mu'(n) ,$$

$$|K_2(n)| \leqq A\lambda'(n)\lambda(n) \int_{n+2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^\delta} (\lambda(w)-\lambda(n))^{\delta-2}\, dw$$

$$\leqq A\lambda'(n) \int_{n+2}^{\infty} \frac{\mu'(w)}{(\lambda(w))^{\delta-1}} (\lambda(w)-\lambda(n))^{\delta-2}\, dw$$

$$= A\lambda'(n) \left[\frac{1}{\lambda(n)} \left(\frac{\lambda(w)}{\lambda(w)-\lambda(n)} \right)^{-\delta} \right]_{w=n+2}^{\infty} \leqq A\mu'(n) .$$

Accordingly $K(n) \leqq A\mu'(n)$, and then

$$J_{11} \leqq A \sum_{n=2}^{\infty} n\mu'(n) \int_0^{\pi/n} |\varphi_x(t)|\, dt = A \sum_{n=2}^{\infty} n\mu'(n) \sum_{k=n}^{\infty} \int_{\pi/(k+1)}^{\pi/k} |\varphi_x(t)|\, dt$$

$$= A \sum_{k=2}^{\infty} \int_{\pi/k(+1)}^{\pi/k} |\varphi_x(t)|\, dt \sum_{n=2}^{k} n\mu'(n) \leqq A \sum_{k=2}^{\infty} \int_{\pi/(k+1)}^{\pi/k} |\varphi_x(t)|\, dt \sum_{n=2}^{[\pi/t]} n(\mu'(n))^\delta$$

$$\leqq A \int_0^{\pi} \frac{|\varphi_x(t)|}{t^2} (\mu'(1/t))^\delta\, dt < \infty ,$$

And hence

$$J_{12} \leqq A \sum_{n=2}^{\infty} n \log n \cdot \mu'(n) \cdot M(n) < \infty .$$

Thus $J_1 < \infty$, which gives with (3) the required.

In the special cases, stated at the begining of this section, we get the following corollaries.

Corollary 1. *If, for* $0<\delta<1$,

$$\int_0^\pi \frac{\varphi_x(t)}{t^{2-\delta}} \frac{(\log\log 1/t)^{(\alpha-1)\delta}}{(\log 1/t)^\delta}\, dt < \infty ,$$

and

$$\sum_{n=2}^\infty (n\log n)^{1-\delta}(\log\log n)^{(\alpha-1)\delta} M(n) < \infty ,$$

then the Fourier series of $f(t)$ *is* $|R, e^{(\log\log n)^\alpha}, \delta|$ *summable at* $t=x$, *for* $\alpha>0$.

Corollary 2. *If, for* $0<\delta<1$,

$$\int_0^\pi \frac{\varphi_x(t)}{t^{2-\alpha}} \left(\log\frac{1}{t}\right)^{(\alpha-1)\delta} dt < \infty ,$$

and

$$\sum_{n=2}^\infty n^{1-\delta} (\log n)^{1+(\alpha-1)\delta}\cdot M(n) < \infty ,$$

then the Fourier series of $f(t)$ *is* $|R, e^{(\log n)^\alpha}, \delta|$ *summable at* $t=x$, *for* $\alpha>0$.

Corollary 3. *If, for* $0<\delta<1$,

$$\int_0^\pi \frac{\varphi_x(t)}{t^{1+\alpha}}\, dt < \infty ,$$

and

$$\sum_{n=2}^\infty \frac{\log n}{n^{(1-\alpha)\delta-1}} M(n) < \infty ,$$

then the Fourier series of $f(t)$ *is* $|R, e^{n^\alpha}, \delta|$ *summable at* $t=x$, *for* $1>\alpha>0$.

4. In the case $\alpha=1$ in Corollary 2, $|R, n, \delta|$ summability is equivalent to $|C, \delta|$ summability. In this case we get better theorem:

Theorem 3. *If*

$$\int_0^\pi \frac{|\varphi_x(t)|}{t}\, dt < \infty , \qquad \sum_{n=1}^\infty n^{1-\delta} M(n) < \infty ,$$

then the Fourier series of $f(t)$ *is summable* $|C, \delta|$ $(0<\delta<1)$ *at* $t=x$.

Proof. Let $\sigma_n^{(\delta)}(t)$ be the n-th (C, δ) mean of the Fourier series of $f(t)$, then it must be proved that

$$\sum_{n=1}^\infty |\sigma_n^{(\delta)}(x)-\sigma_{n-1}^{(\delta)}(x)| < \infty .$$

The left side is

$$\delta \sum_{n=1}^{\infty} |\sigma_n^{(\delta-1)}(x) - \sigma_n^{(\delta)}(x)|/n$$

$$\leqq \sum_{n=1}^{\infty} |\sigma_n^{(\delta-1)}(x) - f(x)|/n + \sum_{n=1}^{\infty} |\sigma_n^{(\delta)}(x) - f(x)|/n = S_1 + S_2 .$$

It is well known [4] that, if we write

$$\sigma_n^{(\delta)}(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t) K_n^{(\delta)}(t)\, dt$$

then

$$|K_n^{(\delta)}(t)| \leqq Cn, \qquad |K_n^{(\delta)}(t)| \leqq C/n^{\delta} t^{1+\delta},$$

and

$$K_n^{(\delta)}(t) = G_n^{(\delta)}(t) + H_n^{(\delta)}(t),$$

where

$$G_n^{(\delta)}(t) = \sin[(n+1/2+\delta/2)t - \delta\pi/2]/A_n^{\delta}(2\sin t/2)^{1+\delta},$$
$$|H_n^{(\delta)}(t)| \leqq C/nt^2$$

for $\delta > -1$. We have

$$\begin{aligned} S_2 &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^{\pi} \varphi_x(t) K_n^{(\delta)}(t)\, dt\right| \\ &\leqq \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\left|\int_0^{\pi/n} \varphi_x(t) K_n^{(\delta)}(t)\, dt\right| + \left|\int_{\pi/n}^{\pi} \varphi_x(t) K_n^{(\delta)}(t)\, dt\right|\right) \\ &\leqq A\sum_{n=1}^{\infty}\left(\int_0^{\pi/n} |\varphi_x(t)|\, dt + \frac{1}{n^{1+\delta}}\int_{\pi/n}^{\pi} \frac{|\varphi_x(t)|}{t^{1+\delta}}\, dt\right) \\ &\leqq A\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\int_{\pi/(k+1)}^{\pi/k} |\varphi_x(t)|\, dt + \frac{1}{n^{1+\delta}}\sum_{k=1}^{n}\int_{\pi/(k+1)}^{\pi/k} \frac{|\varphi_x(t)|}{t^{1+\delta}}\, dt\right) \\ &= A\sum_{k=1}^{\infty} k\int_{\pi/(k+1)}^{\pi/k} |\varphi_x(t)|\, dt + A\sum_{k=1}^{\infty}\frac{1}{k^{\delta}}\int_{\pi/(k+1)}^{\pi/k} \frac{|\varphi_x(t)|}{t^{1+\delta}}\, dt \\ &\leqq A\int_0^{\pi} \frac{|\varphi_x(t)|}{t}\, dt . \end{aligned}$$

Let us estimate S_1.

$$S_1 = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\left|\int_0^{\pi} \varphi_x(t) G_n^{(\delta-1)}(t)\, dt\right| + \left|\int_0^{\pi} \varphi_x(t) H_n^{(\delta-1)}(t)\, dt\right|\right) = S_{11} + S_{12} .$$

Then, as in the estimation for S_2,

$$|S_{12}| \leqq A\int_0^{\pi} \frac{|\varphi_x(t)|}{t}\, dt .$$

Furthermore, putting $m=n+1/2+\delta/2$ and defining τ_n as in §2,

$$S_{11} \leqq \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^{\tau_n} |\varphi_x(t)\, G_n^{(\delta-1)}(t)|\, dt + \left| \int_{\tau n}^{\pi} \varphi_x(t)\, G_n^{(\delta-1)}(t)\, dt \right| \right) = S_{111} + S_{112}.$$

We have easily

$$S_{111} \leqq A \sum_{n=1}^{\infty} \frac{1}{n} \cdot n \int_0^{\pi/n} |\varphi_x(t)|\, dt \leqq A \int_0^{\pi} \frac{|\varphi_x(t)|}{t}\, dt .$$

As in §2, we write $S_{112} = S_{112}' + S_{112}''$ where

$$S_{112}' \leqq A \sum_{n=1}^{\infty} \frac{1}{n^{\delta}} \sum_{k=1}^{m/2} \left| \int_{\tau_n}^{\tau_n + \pi/m} \frac{\varphi_x(t+2k\pi/m) - \varphi_x(t+(2k+1)\pi/m)}{(2\sin(t/2+k\pi/m))^{\delta}} \cdot \sin(mt - \delta\pi/2)\, dt \right|$$

$$\leqq A \sum_{n=1}^{\infty} \frac{1}{n^{\delta}} \sum_{k=1}^{n} \frac{n^{\delta}}{k^{\delta}} \left| \int_{\xi_k}^{\eta_k} (\varphi_x(t+2k\pi/m) - \varphi_x(t+(2k+1)\pi/m))\, dt \right|$$

$$\leqq A \sum_{n=1}^{\infty} n^{1-\delta} M(\delta) < \infty$$

and

$$S_{112}'' \leqq A \sum_{n=1}^{\infty} \frac{1}{n^{\delta}} \int_{\tau_n}^{\tau_n+\pi/m} |\varphi_x(t+\pi/m)|\, dt \cdot \sum_{k=1}^{n} \frac{n^{\delta}}{k^{\delta+1}} \leqq A \int_0^{\pi} \frac{|\varphi_x(t)|}{t}\, dt .$$

Thus the theorem is proved.

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