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ON CERTAIN PROPERTY OF THE NORMS BY MODULARS

By

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Let R be a universally continuous semi-ordered linear space. A functional $m(a) (a \in R)$ is said to be a modular¹⁾ on R if it satisfies the following modular conditions:

- (1) $0 \leq m(a) \leq \infty$ for all $a \in R$;
- (2) if $m(\xi a) = 0$ for all $\xi > 0$, then $a = 0$;
- (3) for any $a \in R$ there exists $\alpha > 0$ such that $m(\alpha a) < \infty$;
- (4) for every $a \in R$, $m(\xi a)$ is a convex function of ξ ;
- (5) $|a| \leq |b|$ implies $m(a) \leq m(b)$;
- (6) $a \wedge b = 0$ implies $m(a+b) = m(a) + m(b)$;
- (7) $0 \leq a_\lambda \uparrow a$ implies $m(a) = \sup_{\lambda \in A} m(a_\lambda)$.

In R , we define functionals $\|a\|$, $\|a\|$ ($a \in R$) as follows

$$\|a\| = \inf_{\xi > 0} \frac{1 + m(\xi a)}{\xi}, \quad \|a\| = \inf_{m(\xi a) < 1} \frac{1}{|\xi|}.$$

Then it is easily seen that both $\|a\|$ and $\|a\|$ are norms on R and $\|a\| \leq \|a\| \leq 2\|a\|$ for all $a \in R$. $\|a\|$ is said to be the first norm by m and $\|a\|$ is said to be the second norm by m . Let $\bar{R}^m$ be the modular conjugate space of R and $\bar{m}$ be the conjugate modular of m ²⁾ then we can introduce the norms by $\bar{m}$ as above. It is known that if R is semi-regular, the first norm by the conjugate modular $\bar{m}$ is the conjugate norm of the second norm by m and the second norm by the conjugate modular $\bar{m}$ is the conjugate norm of the first norm by m . Since $\|a\|$ and $\|a\|$ are semi-continuous by (7), they are reflexive norms (cf. [7]).

If a modular m is of L_p -type, i. e., $m(\xi x) = \xi^p m(x)$ for all $x \in R$, $\xi \geq 0$,

1) We owe the notations and the terminologies using here to the book: H. NAKANO [3].

2) The conjugate modular $\bar{m}$ is defined as $\bar{m}(\bar{a}) = \sup_{x \in R} \{\bar{a}(x) - m(x)\}$ for every $\bar{a} \in \bar{R}^m$, where $\bar{R}^m$ is the space of the modular bounded universally continuous linear functionals on R .

then we have $\frac{\|x\|}{\|x\|} = p^{\frac{1}{p}} q^{\frac{1}{q}}$ for all $0 \neq x \in R$, where $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (In the case of $p=1$, we have $\frac{\|x\|}{\|x\|} = 1$). The converse of this is studied by S. YAMAMURO [5] and I. AMEMIYA [1]. They proved that if the ratios of two norms are constant for all $0 \neq x \in R$, it is of L_p -type essentially. So in the general case, the ratios of two norms are not constant.

A modular m is said to be bounded if there exist real numbers $1 < p_1 \leq p_2 < \infty$, such that

$$\xi^{p_1} m(x) \leq m(\xi x) \leq \xi^{p_2} m(x)$$

for all $\xi \geq 1$ and $x \in R$. In [6], S. YAMAMURO obtained that if a modular m on R is bounded then we have

$$\|x\| \geq r \|x\|$$

for all $x \in R$, where $r > 1$ is a fixed constant.

In this paper we investigate the case when the two norms by a modular m satisfy

$$\inf_{0 \neq x \in R} \frac{\|x\|}{\|x\|} = r > 1 \quad (*)$$

(In this case we say that the norms have property $(*)$ throughout this paper).

As showed above, a bounded modular m has that property $(*)$, but the converse of this is not true in general.

In §1 we prove that if the norms by a modular m satisfy the property $(*)$ then it is uniformly finite and uniformly increasing, provided that R has no atomic element (Theorem 1.1). And we obtain conversely that if a modular m is uniformly finite and uniformly increasing then the norms by m have the property $(*)$ (Theorem 1.4). Thus, we can see that if R has no atomic element, then the property $(*)$ is equivalent to uniform finiteness and uniform increasingness of modular m . Theorem 1.2 shows that uniform simpleness of a modular m implies uniform finiteness, in the case when R has no atomic element. Finally some special cases, where the property $(*)$ is equivalent to boundedness of modular are discussed.

In §2 we define uniform p -properties, that is, uniformly p -finite, p -increasing, p -simple and p -monotone modulars, to determine the degrees of uniform finiteness, increasingness and etc.. Theorems 2.1 and 2.2

show that there exist the conjugate relations between uniformly p -finite modular and uniformly q -increasing modular, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. On the other hand, Theorems 2.3 and 2.4 show the similar relations between uniformly p -simple modular and uniformly q -monotone modular. In the case when R has no atomic element, we have more precisely than in §1, that if a modular m is uniformly p -simple it is uniformly p -finite (Theorem 2.5). There is a modular which is uniformly finite but not uniformly p -finite for any $1 \leq p < \infty$.

In §3 we prove that if the norms by modular m have the property (*) then r (which appears in (*)) determines the degrees of uniform finiteness and uniform increasingness of m . Truly, in the case when R has no atomic element, we obtain that if the norms by a modular m have the property (*), m is uniformly p -increasing and uniformly q -finite, where p, q are positive numbers such that $r = p^{\frac{1}{p}} q^{\frac{1}{q}}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \leq q$ (Theorem 3.1). The converse of this is not true in general. We show an example of this fact at the end of this paper.

§1. Let R be a modular semi-ordered linear space with a modular m .

A modular m is said to be uniformly finite, if

$$\sup_{m(x) \leq 1} m(\xi x) < \infty \quad \text{for all } \xi > 0.$$

A modular m is said to be uniformly increasing, if

$$\liminf_{\xi \rightarrow \infty, m(x) \geq 1} \frac{m(\xi x)}{\xi} = +\infty.$$

In [4; Theorems 5.2, 5.3] it is shown that if a modular m is uniformly finite, then the conjugate modular $\bar{m}$ of m is uniformly increasing and if a modular m is uniformly increasing then the conjugate modular $\bar{m}$ is uniformly finite.

Now we shall prove the following

Theorem 1.1. *Suppose R has no atomic element. If the norms by a modular m have the property (*), then m is uniformly finite and uniformly increasing.*

Proof. 1). Let r be a number, in the sequel, such that $r = \inf_{0 \neq x \in R} \frac{\|x\|}{\|\bar{x}\|}$.

Then we have

$$\inf_{0 \neq \bar{x} \in \bar{R}^m} \frac{\|\bar{x}\|}{\|\bar{\bar{x}}\|} = r \quad (*').$$

In fact we have for every $\bar{x} \in \bar{R}^m$

$$\|\bar{x}\| = \sup_{\|x\| \leq 1} |\bar{x}(x)| \leq \sup_{r\|x\|} |\bar{x}(x)| = \frac{1}{r} \|\bar{x}\|.$$

Since the norms $\|x\|$, $\|\bar{x}\|$ are reflexive, we obtain $(*)'$.

2). If m is not uniformly finite, then there exists a number $\xi_0 \geq 1$ such that

$$\begin{aligned} \sup_{m(x) \leq 1} m(\xi x) &< +\infty & \text{for all } \xi < \xi_0, \\ \sup_{m(x) \leq 1} m(\eta x) &= +\infty & \text{for all } \eta > \xi_0. \end{aligned}$$

Since $r > 1$, we obtain a number α such that $1 > \alpha > 0$ and $\alpha r - 1 > 0$, and we can find also $\varepsilon > 0$ such that $\alpha(\xi_0 + \varepsilon) < \xi_0$.

Then by the definition of ξ_0 , we can find a sequence of elements $\{x_n\}$ ($n=1, 2, \dots$) such that

$$m(x_n) \leq 1, \quad m(\alpha(\xi_0 + \varepsilon)x_n) \leq k, \quad m((\xi_0 + \varepsilon)x_n) \geq n \quad (n=1, 2, \dots),$$

where k is a fixed positive number.

Since R has no atomic element, we can obtain also a sequence of projectors $\{[p_n]\}$ ($n=1, 2, \dots$) such that

$$m(\alpha(\xi_0 + \varepsilon)[p_n]x_n) \leq \frac{k}{n}, \quad m((\xi_0 + \varepsilon)[p_n]x_n) \geq 1.$$

Putting $y_n = (\xi_0 + \varepsilon)[p_n]x_n$, we have

$$m(y_n) \geq 1, \quad m(\alpha y_n) \leq \frac{k}{n} \quad (n=1, 2, \dots).$$

This implies $\lim_{n \rightarrow \infty} \frac{1+m(\alpha y_n)}{\alpha} = \frac{1}{\alpha} < r$ and contradicts $(*)$, because on the other hand, we have $\|y_n\| \geq 1$ and $\|y_n\| \leq \frac{1+m(\alpha y_n)}{\alpha}$ for all $n \geq 1$.

Then by 1) $\bar{m}$ is also uniformly finite, thus m is uniformly increasing³⁾. This completes the proof.

In the proof of the theorem above, we have shown that if a modular m is not uniformly finite, then there exists a sequence of elements y_n such that

3) We note here that $\bar{m}(x) = \sup_{\bar{x} \in \bar{R}^m} \{\bar{x}(x) - m(x)\} \leq m(x)$ for all $x \in R$ by virtue of the definition of conjugate modular. If R is semi-regular, then modular m is reflexive; i.e. $m(x) = \bar{m}(x) = \sup_{\bar{x} \in \bar{R}^m} \{\bar{x}(x) - m(x)\}$ for all $x \in R$ ([3]; §39).

$$m(y_n) \geq 1, \quad \lim_{n \rightarrow \infty} m(\xi y_n) = 0 \quad (n=1,2,\dots)$$

for some $\xi > 0$. Then the sequence $\{y_n\}$ ($n=1,2,\dots$) is conditionally modular convergent to 0, but it is not modular convergent. A modular m is said to be uniformly simple if conditionally modular convergence coincides with modular convergence, i. e., $\lim_{n \rightarrow \infty} m(x_n) = 0$ implies $\lim_{n \rightarrow \infty} m(\xi x_n) = 0$ for every $\xi \geq 0$.

Thus we have

Theorem 1.2. *Suppose that R has no atomic element. If a modular m is uniformly simple, then it is uniformly finite.*

The conjugate property to uniform simpleness of modular is uniform monotonicity.⁴⁾ Therefore we obtain also

Theorem 1.3. *Suppose that R has no atomic element. If a modular m is uniformly monotone, then it is uniformly increasing.*

The converse part of Theorem 1.1 is always true (without the assumption that R has no atomic element). That is, we obtain

Theorem 1.4. *If a modular m is uniformly finite and uniformly increasing, then the norms by m have the property (*).*

Proof. If the property (*) is not satisfied, then we can find $x_n \geq 0$ ($n=1,2,\dots$) such that

$$1 \leq \|x_n\| < 1 + \frac{1}{n}, \quad \|x\| = m(x_n) = 1 \quad (n=1,2,\dots).$$

And we can find also $\xi_n > 0$ such that

$$1 + m(\xi_n x_n) < \left(1 + \frac{1}{n}\right) \xi_n$$

for all $n \geq 1$ by the definition of the first norm.

Considering a subsequence of $\{\xi_n\}$, it is sufficient for us to investigate only the following cases.

1) In this case, $\{\xi_n\}$ satisfies $0 < \xi_n \leq 1$ for all $n \geq 1$. If $\xi_n \leq \xi_0 < 1$ ($n=1,2,\dots$) for some $\xi_0 < 1$, then we obtain

$$\left(1 + \frac{1}{n}\right) > \frac{1 + m(\xi_n x_n)}{\xi_n} \geq \frac{1}{\xi_0} > 1 \quad (n=1,2,\dots).$$

This is a contradiction. Now without a loss of a generality, we may

4) A modular m is said to be uniformly monotone, if $\lim_{\xi \rightarrow 0} \frac{1}{\xi} \sup_{m(x) \leq 1} m(\xi x) = 0$.

assume that

$$\xi_n \uparrow 1, \quad 1 - \xi_n < \frac{1}{n} \quad (n=1, 2, \dots).$$

Since we have

$$m(\xi_n x_n) < \left(1 + \frac{1}{n}\right) \xi_n - 1 \leq \frac{1}{n}$$

and $m(\xi x)$ is a non-decreasing convex function of $\xi \geq 0$, we obtain

$$m((1 + (1 - \xi_n))x_n) \geq 1 + \frac{n-1}{n} \quad (n=1, 2, \dots),$$

and furthermore

$$m((1 + n(1 - \xi_n))x_n) \geq 1 + (n-1) \quad (n=1, 2, \dots).$$

This implies

$$\sup_{m(x) \leq 1} m(2x) \geq \sup_{n=1, 2, \dots} m(2x_n) \geq \sup_{n=1, 2, \dots} (1 + (n-1)) = +\infty,$$

which contradicts that m is uniformly finite.

2). In this case, $\{\xi_n\}$ ($n=1, 2, \dots$) satisfies $1 \leq \xi_n$ for all $n \geq 1$. By definition of $\{\xi_n\}$, we have

$$1 + \frac{1}{n} \geq \frac{1 + m(\xi_n x_n)}{\xi_n} \geq \frac{1}{\xi_n} + 1 \quad \text{for all } n \geq 1.$$

This implies $n \leq \xi_n$ for all $n \geq 1$. Therefore we may assume $\xi_n \uparrow +\infty$ ($n=1, 2, \dots$), so we obtain

$$\liminf_{\xi \rightarrow \infty, m(x) \geq 1} \frac{m(\xi x)}{\xi} \leq \lim_{n \rightarrow \infty} \frac{m(\xi_n x_n)}{\xi_n} \leq \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1,$$

which contradicts that m is uniformly increasing. This completes the proof.

In the case when a modular m on R is of unique spectra ([3]; §54), the property (*) implies boundedness of m . In fact we have

Theorem 1.5. *If a modular m on R is of unique spectra, then boundedness of m is equivalent to the property (*).*

The proof is easily obtained by simple calculations, so it is omitted.

In the case of the constant modular ([3]; §55), the property (*) does not imply simpleness of m , and even in the case of the simple constant modular it does not generally imply the boundedness of m (the examples are easily obtained). Only in the particular case, we have

Theorem 1.6. *If a modular m on R is constant, monotone complete and R has neither complete constant element nor atomic element, then the property $(*)$ is equivalent to boundedness of m .*

Proof. By Theorem 1.1 m is finite, then m is upper bounded by Theorem 55.10 in [3]. Since $\bar{m}$ is constant and has no complete constant element [3; §55], $\bar{m}$ is also upper bounded, that is, m is lower bounded. Thus m is a bounded modular on R .

§ 2. In this section we investigate the degrees of uniform properties of modulars.

Set for $\xi \geq 1$

$$f(\xi) = \sup_{m(x) \leq 1} m(\xi x) \quad \text{and} \quad g(\xi) = \inf_{m(x) \geq 1} m(\xi x),$$

then $f(\xi)$ and $g(\xi)$ are defined in $[1, \infty)$ and non-decreasing functions. In the following, let p be a number such that $1 < p < \infty$.

Definition 2.1. A modular m on R is said to be *uniformly p -finite* if there exist $r > 0$ and $\xi_0 \geq 1$ such that

$$f(\xi) \leq r\xi^p \quad \text{for all } \xi \geq \xi_0.$$

Definition 2.2. A modular m on R is said to be *uniformly p -increasing*, if there exist $r > 0$ and $\xi \geq 1$ such that

$$g(\xi) \geq r\xi^p \quad \text{for all } \xi \geq \xi_0.$$

It is easily seen that if m is uniformly p -finite, it is also uniformly p' -finite for $p \leq p'$, and if m is uniformly p -increasing it is also uniformly p'' -increasing for $1 \leq p'' \leq p$.

In the sequel, we set $q = \frac{p}{p-1}$. Now we have

Theorem 2.1. *If a modular m is uniformly p -finite, then the conjugate modular $\bar{m}$ of m is uniformly q -increasing.*

Proof. We have by the assumption for some $\rho_0 \geq 1$, $r > 0$,

$$f(\xi) \leq r\xi^p \quad (\xi \geq \rho_0 \geq 1).$$

If $\bar{m}(\bar{x}) \geq 1$, $\bar{x} \in \bar{R}^m$ and $0 < \alpha < 1$, we can find x_0 such that $\bar{x}(x_0) > \alpha$, $m(x_0) \leq 1$. For such x_0 , we have by the definition of conjugate modular

$$\bar{m}(\lambda \bar{x}) \geq \lambda \bar{x}(\rho x_0) - m(\rho x_0) \geq \alpha \lambda \rho - r \rho^p$$

for all $\rho \geq \rho_0$. This implies

$$\bar{m}(\lambda \bar{x}) \geq \sup_{\rho \geq \rho_0} \{\alpha \lambda \rho - r \rho^p\}$$

for all $\bar{x} \in \bar{R}^m$ such that $\bar{m}(\bar{x}) \geq 1$.

Then we have for $\lambda \geq \lambda_0 = \frac{rp}{\alpha} \rho_0^{\frac{p}{q}}$,

$$\bar{m}(\lambda \bar{x}) \geq \frac{rp}{q} \left(\frac{\alpha}{p\gamma} \right)^q \lambda^q.$$

Hence the conjugate modular $\bar{m}$ is uniformly q -increasing modular by definition.

Theorem 2.2. *If a modular m is uniformly p -increasing, then the conjugate modular $\bar{m}$ of m is uniformly q -finite.*

Proof. By the assumption we have for some r and ρ_0

$$m(x) \geq 1 \text{ implies } m(\rho x) \geq r\rho^p \quad \text{for } \rho \geq \rho_0.$$

Set $\lambda_0 = \text{Max} \left(\frac{r}{2} \rho_0^{p-1}, 1 \right)$ and for $\lambda \geq \lambda_0$ we define $\rho = \rho(\lambda)$ such that $\rho(\lambda) = \left(\frac{2}{r} \lambda \right)^{\frac{q}{p}}$. Then we have $\rho \geq \rho_0$. Thus we obtain $\frac{m(\rho x)}{\rho} \geq r\rho^{p-1} = 2\lambda$. If $\bar{x} \in \bar{R}^m$, $\bar{m}(\bar{x}) \leq 1$ and $1 \leq m(x) < +\infty$, then there is $\xi > 0$ such that

$$m\left(\frac{1}{\xi} x\right) = 1, \quad 0 < \frac{1}{\xi} < 1$$

and hence by the definition of the conjugate modular $\bar{m}(\bar{x})$ we obtain

$$\bar{x}\left(\frac{1}{\xi} x\right) \leq \bar{m}(\bar{x}) + m\left(\frac{1}{\xi} x\right) \leq 2.$$

For such ξ , if $\xi \geq \rho(\lambda)$, then we have

$$\lambda \bar{x}(x) - m(x) = \xi \left\{ \lambda \bar{x}\left(\frac{1}{\xi} x\right) - \frac{1}{\xi} m\left(\xi \frac{1}{\xi} x\right) \right\} \leq 0,$$

and if $0 < \xi \leq \rho(\lambda)$, then we have

$$\lambda \bar{x}(x) - m(x) \leq \xi \lambda \bar{x}\left(\frac{1}{\xi} x\right) \leq 2\rho\lambda = 2\lambda \left(\frac{2}{r} \lambda \right)^{\frac{q}{p}}.$$

If $\bar{m}(\bar{x}) \leq 1$, $m(x) \leq 1$, we have also

$$\lambda \bar{x}(x) - m(x) \leq \lambda (\bar{m}(\bar{x}) + m(x)) - m(x) \leq 2\lambda.$$

Therefore we obtain consequently

$$\bar{m}(\lambda \bar{x}) \leq 2\lambda\rho = r_0 \lambda^q \quad \text{for all } \lambda \geq \lambda_0$$

where $r_0 = 2^q \left(\frac{1}{r} \right)^{\frac{q}{p}}$. Hence the conjugate modular $\bar{m}$ is uniformly q -

finite modular.

As similarly as uniformly p -finite modulars, we can define uniformly p -simple and uniformly p -monotone modular. In order to define them, we set for $0 \leq \xi \leq 1$

$$\varphi(\xi) = \sup_{m(x) \leq 1} m(\xi x), \quad \psi(\xi) = \inf_{m(x) \geq 1} m(\xi x).$$

Then $\varphi(\xi)$, $\psi(\xi)$ are defined in $[0, 1]$ and finite non-decreasing functions.

Definition 2.3. A modular m on R is said to be *uniformly p -simple* if there exist $r > 0$, and $0 < \xi_0 \leq 1$, such that

$$\psi(\xi) \geq r\xi^p \quad \text{for all } 0 \leq \xi \leq \xi_0.$$

Definition 2.4. A modular m on R is said to be *uniformly p -monotone*, if there exist $r > 0$ and $0 < \xi_0 \leq 1$, such that

$$\varphi(\xi) \leq r\xi^p \quad \text{for all } 0 \leq \xi \leq \xi_0.$$

It is easily seen that if m is uniformly p -simple, it is also uniformly p' -simple for $p \leq p'$, and if m is uniformly p -monotone, it is also uniformly p'' -monotone for $1 \leq p'' \leq p$.

Concerning uniformly p -simple and uniformly q -monotone modulars there exist the conjugate relations, in fact we have

Theorem 2.3. *If a modular m on R is uniformly p -monotone, then the conjugate modular $\bar{m}$ of m is uniformly q -simple.*

Theorem 2.4. *If a modular m on R is uniformly p -simple, then the conjugate modular $\bar{m}$ of m is uniformly q -monotone.*

The proofs of these theorems are analogous to those of Theorems 4.9, 4.10 in [4] and of Theorems 2.1, 2.2, so it is omitted.

Concerning uniform simpleness and uniform finiteness we proved in Theorem 2.2 that uniform simpleness implies uniform finiteness, provided that R has no atomic element. On uniformly p -simple modular we obtain more precisely

Theorem 2.5. *Let R has no atomic element. If a modular m on R is uniformly p -simple, then it is uniformly p -finite.*

Proof. It is known already that m is uniformly finite. If it is not uniformly p -finite, then there exists a sequence of real numbers $\xi_n \geq 0$ ($n=1, 2, \dots$) such that

$$+\infty > f(\xi_n) > n\xi_n^p, \quad \xi_n \uparrow +\infty \quad (n=1, 2, \dots).$$

And by definition of $f(\xi)$, we can choose a sequence of elements $\{x_n\}$ ($n=1, 2, \dots$) such that

$$m(\xi_n x_n) > n \xi_n^p, \quad m(x_n) = 1 \quad (n=1, 2, \dots).$$

Here, we can assume without a loss of generality that

$$m(\xi_n x_n) = N_n$$

where N_n is a natural number, for every $n \geq 1$. Because, if there are $r > 0$ and $\xi_0 \geq 1$ satisfying $m(\xi x) \leq r \xi^p$ for every $\xi \geq \xi_0$ such that $m(\xi x)$ is a natural number, then we have $m(\xi x) \leq (r+1) \xi^p$ for all $\xi \geq \xi_0$. This shows that m is uniformly p -finite.

Then we can find a sequence of projectors $\{[p_n]\}$ ($n=1, 2, \dots$) by orthogonal decompositions of x_n ($n=1, 2, \dots$) such that

$$m([p_n] \xi_n x_n) = 1, \quad m([p_n] x_n) < \frac{1}{n \xi_n^p} \quad (n=1, 2, \dots),$$

since $m(\xi_n x_n)$ is natural number for all $n \geq 1$. Set $y_n = [p_n] \xi_n x_n$ and $\eta_n = \frac{1}{\xi_n}$ for every $n \geq 1$, then we have $m(y_n) = 1$ and $m(\eta_n y_n) < \frac{\eta_n^p}{n}$. Since $\lim_{n \rightarrow \infty} \eta_n = 0$, we show that m is not uniformly p -simple. Thus the proof is completed.

Corresponding to Theorem 2.5 we have

Theorem 2.6. *Let R have no atomic element. If a modular m on R is uniformly p -monotone, then it is uniformly p -increasing.*

It will be conjectured that if a modular m is uniformly finite, then it is uniformly p -finite for some $1 < p < +\infty$. But the following example shows that it is not true.

Example. Set
$$\phi(u) = \begin{cases} \frac{1}{2} u & u \leq 2 \\ e^{u-2} & u > 2 \end{cases}$$

and consider ORLICZ sequence space l_ϕ . Then l_ϕ is uniformly finite as easily seen, but not uniformly p -finite for any $1 < p < \infty$. This example shows at the same time that there exists a modular m which is uniformly increasing but not uniformly p -increasing for any $1 < p < \infty$.

I. AMEMIYA proved in [2] that if a modular m on R is monotone complete and finite, then m is semi-upper bounded, i.e., $m(2x) \leq r m(x)$ for every x such that $m(x) \geq \varepsilon$ for some fixed $r, \varepsilon > 0$, provided that R has no atomic element. Applying this result, it is seen that the above conjecture is affirmative, in the case when m is monotone complete and R has no atomic element. In fact we have

Theorem 2.7. Suppose that R has no atomic element and m is monotone complete. If m is uniformly finite (finite) then it is uniformly p -finite for some $p > 1$.

§3. To any r such that $1 < r \leq 2$, there exist a unique pair of positive numbers (p, q) satisfying the following

- 1) $r = p^{\frac{1}{p}} q^{\frac{1}{q}}$
- 2) $\frac{1}{p} + \frac{1}{q} = 1$
- 3) $1 \leq p \leq 2 \leq q$.

This correspondence is unique and it is easily seen that if r_n is convergent increasingly to 2, then the corresponding $p_n(q_n)$ is also convergent increasingly (decreasingly) to 2.

If the norms of modular m have the property (*) we can find a pair of numbers such that $r = p^{\frac{1}{p}} q^{\frac{1}{q}}$. It is already seen that m is uniformly finite and uniformly increasing, provided that R has no atomic element. Now we shall show that (p, q) gives the degrees of uniform finiteness and increasingness. In fact we can state

Theorem 3.1. Suppose that R has no atomic element. If the norms by a modular m have the property (*), then m is uniformly p -increasing and uniformly q -finite.

Proof. Set $\alpha = \left(\frac{p}{q}\right)^{\frac{1}{q}}$, then $r\alpha - 1 = \alpha^q$.

Thus we obtain by assumption,

$$m(x) = 1 \quad \text{implies} \quad m(\alpha x) \geq \alpha^q.$$

If $m(x) = 1 + \frac{m}{n}$ (for natural numbers $m < n$), we can decompose orthogonally $x = x_1 + x_2 + \dots + x_{n+m}$ such that

$$m(x_i) = m(x_j) = \frac{1}{n} \quad (i, j = 1, 2, \dots, n+m).$$

The numbers of i such that $m(\alpha x_i) < \alpha^q m(x_i)$ are less than n , because if there exists $(i_1, i_2, \dots, i_n)$ such that $m(\alpha x_{i_\nu}) > \alpha^q m(x_{i_\nu})$ ($\nu = 1, 2, \dots, n$), then we have $m\left(\alpha \sum_{\nu=1}^n x_{i_\nu}\right) < \alpha^q m\left(\sum_{\nu=1}^n x_{i_\nu}\right)$ and $m\left(\sum_{\nu=1}^n x_{i_\nu}\right) = 1$. This is a contradiction.

Thus there exists $\{i_k\}$ ($k = 1, 2, \dots, m$) such that $m(\alpha x_{i_k}) \geq \alpha^q m(x_{i_k})$ ($k =$

$1, 2, \dots, m$). Putting $y = \sum_{k=1}^m x_{i_k}$ we have $m(x-y)=1$ and

$$m(\alpha y) \geq \alpha^q m(y)$$

$$m(\alpha(x-y)) \geq \alpha^q m(x-y).$$

Hence we obtain $m(\alpha x) \geq \alpha^q m(x)$. Generally, if $1 \leq m(x) < 2$, since $m(\xi x)$ is continuous function of ξ , we have also

$$m(\alpha x) \geq \alpha^q m(x).$$

Since $m(x)$ is finite for all $x \in R$ and R has no atomic element, we have for x such that $m(x)=1$

$$m(\alpha \xi x) \geq \alpha^q m(\xi x) \quad \text{for all } \xi \geq 1.$$

Here, putting $\beta = \frac{1}{\alpha} > 1$, we obtain

$$m(\beta^n x) \leq \beta^{q \cdot n} m(x) \quad (n=1, 2, \dots)$$

for all x such that $m(x)=1$. From this we have

$$m(\xi x) \leq \beta^q \xi^q \quad \xi \geq \beta,$$

which shows that m is uniformly q -finite. By Theorem 2.1 and (*) we can see m is uniformly p -increasing.

Remark 1. The converse of the theorem is not true. For example, set

$$\phi(u) = \begin{cases} u^{\frac{3}{2}} & u \leq 2 \\ \frac{1}{\sqrt{2}} u^2 & u > 2. \end{cases}$$

Then the ORLICZ space $L[0, 1]$ is uniformly 2-finite and uniformly 2-increasing, but it is easily seen that there is an element such that $\frac{\|x\|}{\|x\|} < 2$. And for any $1 < \alpha < 2$, we can get the example of modular space such that m is uniformly 2-finite and uniformly 2-increasing but the norms by m do not satisfy $\inf_{x \neq 0} \frac{\|x\|}{\|x\|} \geq \alpha$.

Remark 2. If R is a discrete modular semi-ordered linear space, the property (*) does not imply finiteness of m , and even if in the case where m is finite, the property (*) does not imply uniform finiteness of m . The examples are obtained easily. In this case the equivalent

condition to the property (*) is unknown.

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