POSITIVE LINEAR OPERATORS IN SEMI-ORDERED LINEAR SPACES

By

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Since in 1907 O. PERRON [9] discovered a remarkable spectral property of positive matrices and shortly later G. FROBENIUS [1], [2] and R. JENTZSCH [4] investigated and generalized it further, many authors have considered special properties of positive linear operators. Especially M. KREIN and M. A. RUTMAN [5]*) considered with success a generalization to Banach spaces with a cone. They obtained particularly important results, when the space is lattice ordered or the cone has an interior point. In this paper, we consider spectral properties of positive compact (=completely continuous) linear operators on a universally continuous Banach space (= conditionally complete Banach lattice). Our main aim is to generalize the results of G. FROBENIUS [2] to infinite dimensional spaces.

In §1 preliminary definitions are summarized. In §2 the fundamental theorem on the maximum positive spectrum is proved (Theorem 2.1). In §3 we define *completely positive* linear operators. The operators of this class play a similar rôle as strongly positive operators in [5]. In §4 we obtain under some additional conditions a necessary and sufficient condition for that a positive compact linear operator is quasi-nilpotent (Theorem 4.7). In §5 the proper values with maximum modulus of a positive compact linear operator are determined (Theorem 5.2).

§ 1. **Preliminaries.** We recall briefly definitions from the theory of semi-ordered linear spaces and linear operators. A lattice ordered linear space (with real scalar) R is said to be *universally continuous,* if for any $a_\lambda \geqq 0$ $(\lambda \in \Lambda)$ there exists $\bigcap_{\lambda \in \Lambda} a_\lambda$. A linear manifold N of R is said to be *normal* if there exists a positive linear projection $[N]$ of R onto N such that $|x-[N]x| \frown |y| = 0$ for $x \in R$ and $y \in N$. The projection

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onto the normal manifold generated by a is denoted by $[a]$. Normal manifolds and order-projections correspond to each other in one-to-one way.

When we consider spectral problems, it is convenient to define a *complex extension* $\hat{R}$ of R, whose elements consist of all pair of elements of R, $(a, b) \equiv a+ib$, the *absolute value* of $a+ib$ is defined by

$$(1.1) \qquad |a+ib| = \bigcup_{0 \leqq \theta \leqq 2\pi} |a\cos\theta + b\sin\theta| .$$

When R is normed, the norm, in this paper, satisfies the following additional condition:

$$(1.2) \qquad |a| \leqq |b| \quad \text{implies} \quad \|a\| \leqq \|b\| ,$$

The norm on $\hat{R}$ is defined, when R is normed, by

$$(1.3) \qquad \|a+ib\| = \| \, |a+ib| \, \| .$$

A norm on R is said to be *continuous* if $a_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies $\|a_\lambda\| \downarrow_{\lambda \in \Lambda} 0$. A bounded linear functional $\tilde{a}$ is said to be *universally continuous*, if $a_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies $\inf_{\lambda \in \Lambda} |\tilde{a}(a_\lambda)| = 0$. $\tilde{R}$ and $\bar{R}$ denote the space of all bounded linear functionals on R and that of all universally continuous linear functionals respectively. For the other notations and definitions, we refer to [6].

For a bounded linear operator A on a complex Banach space R into itself, $\sigma(A)$ denotes the set of all spectra of A, and $\rho(A)$ *the resolvent set.* If $(\lambda I - A)x = 0$ has a non-trivial solution in R, λ is said to be a *proper value* and its solutions are *proper elements.* We put

$$(1.4) \qquad r(A) = \sup_{\xi \in \sigma(A)} |\xi|$$

$$(1.5) \qquad R(\lambda) = (\lambda I - A)^{-1} \qquad \text{for } \lambda \in \rho(A).$$

It is well known (cf. [3]) that

$$(1.6) \qquad r(A) = \lim_{\nu \to \infty} \|A^\nu\|^{\frac{1}{\nu}}$$

A is said to be *quasi-nilpotent,* if $r(A) = 0$.

A bounded linear operator A is said to be *compact* if the unit sphere is mapped by A into a compact set. We assume in this paper the results of F. Riesz ([10], chap. IV and V) concerning the spectral properties of compact linear operators. If A is a compact linear operator, every non-zero spectrum is a proper value and the corresponding proper manifold is each finite dimensional. $\sigma(A)$ constitutes a totally disconnected set with the only possible limiting point 0. For a non-zero complex number λ we can define a bounded linear projection operator $E(\lambda)$ relative to A by

$$E(\lambda)=\frac{1}{2\pi i}\int R(\zeta)d\zeta \tag{1.6}$$

where the integration is formed along a Jordan curve surrounding λ, whose boundary and interior intersect $\sigma(A)$ in λ alone. The *index* $\mu(\lambda)$ of λ is the samallest integer n satisfying $(\lambda I-A)^n E(\lambda)=0$ For other definitions in operator theories, we refer to [3].

Examples of universally continuous Banach spaces are: L_p, l_p $(1\leqslant p\leqslant\infty)$, and more generally *modulared spaces* studied in [6].

As we develop in the following a theory of positive compact linear operators in a universally continuous Banach space, it has immediate applications to the theory of integral equations or linear equations with infinite unknowns in the spaces mentioned above.

§ 2. **The maximum positive spectrum.** *Throughout the paper R denotes a universally continuous semi-ordered Banach space and A a linear operator on R into itself, if the contrary is not mentioned.*

Though some of results are known under weaker conditions (see [5]), we prove them for the sake of completenss, since under our conditions proofs are sometimes simple.

A is said to be *positive*, if $a\geqq 0$ implies $Aa\geqq 0$.

Lemma 2.1. *A positive linear operator is necessarily bounded.*

Proof. For $\tilde{a}\in\tilde{R}$, putting $\tilde{b}(x)=\tilde{a}(Ax)$, $\tilde{b}$ is an (o)-bounded linear functional on R, so by Theorem 31.3 in [6], is norm-bounded. This means that the image of the unit sphere by A is weakly bounded. The assertion follows from the known theorem on weakly bounded sets.

Theorm 2.1. *If A is positive, $r(A)$ is in $\sigma(A)$.*

Proof. Suppose $r=r(A)$ is not in $\sigma(A)$. Then $R(\lambda)=\sum_{\nu=0}^{\infty}\frac{A^\nu}{\lambda^{\nu+1}}$ for $\lambda>r$ and $\sup_{r<\lambda}\|R(\lambda)\|<\infty$. Since, considering $\hat{R}$, by (1.3)

$$\|R(\lambda e^{i\theta})x\|\leqslant\|R(\lambda)x\| \qquad \text{for } \lambda>r,\ 0\leqslant\theta\leqslant 2\pi \text{ and } x>0,$$

hence $\sup_{\lambda>r}\|R(\lambda e^{i\theta})\|<\infty$, this implies $re^{i\theta}\notin\sigma(A)$, $0\leqslant\theta\leqslant 2\pi$, contradicting the assumption (cf. [3]).

Corollary 2.1.1. *If A is positive, $R(\lambda)\geqq 0$* if and only if *$\lambda>r$.*

Proof. If $R(\lambda)\geqq 0$, λ is apparently real and by the resolvent equation

([3], p. 99) $R(\lambda)-R(\mu)=\dfrac{R(\lambda)\,R(\mu)}{\mu-\lambda}\geqq 0$ for $\mu>$ Max(λ, r). Hence as in Theorem 2.1 $\lambda>r(A)$. The converse part is obvious.

Corollary 2.1.2. *Let A be positive. If for a* $\lambda>0$, *there exists* $x\gneqq 0$, *such that* $\lambda x\geqq Ax$, *then* $\lambda\leqslant r(A)$.

Proof. If $\lambda\in\sigma(A)$, $\lambda\leqslant r(A)$ by Theorem 2.1. If $\lambda\in\rho(A)$, from the hypothesis $R(\lambda)$ is not positive, the assertion follows from Corollary 2.1.1.

Concerning the indice on the circle of radius $r=r(A)$, we obtain

Theorem 2.2. *If A is positive compact with* $r=r(A)>0$,

$$\mu(\lambda)\leqslant\mu(r) \qquad \textit{for } |\lambda|=r \tag{2.1}$$

Proof. By the Laurent resolution ([3], p. 109) $\sup\limits_{0<\varepsilon<\delta}\varepsilon^{\mu(r)}\|R(r+\varepsilon)\|<\infty$ for a small δ. Since from (1.3) $\varepsilon^{\mu(r)}\|R(re^{i\theta}+\varepsilon e^{i\theta})\|\leqslant\varepsilon^{\mu(r)}\|R(r+\varepsilon)\|$, we obtain $\mu(\lambda)\leqslant\mu(r)$.

Theorem 2.3. *If A is positive compact* $r=r(A)>0$, *A has a positive proper element corresponding to the proper value r.*

Proof. We know that $\lim\limits_{\varepsilon\to 0}\varepsilon^{\mu(r)}R(r+\varepsilon)=(A-rI)^{\mu(r)-1}E(r)$ ([3], p. 109). Since $R(r+\varepsilon)\geqq 0$, $(A-rI)^{\mu(r)-1}E(r)\geqq 0$, there exists $x>0$ with $y=(A-rI)^{\mu(r)-1}E(r)x>0$. We obtain that $Ay=ry$ and $y>0$.

§ 3. **Completely positive linear operators.** A linear operator A is said to be *universally continuous*, if $a_\lambda\downarrow_{\lambda\in\Lambda}0$ implies $\bigcap\limits_{\lambda\in\Lambda}|Aa_\lambda|=0$.

Lemma 3.1. *If A is positive, compact and universally continuous, the range of the conjugate operator* A^* *is contained in* $\bar{R}$.

Proof. Since A is positive and compact, for any $a_\lambda\downarrow_{\lambda\in\Lambda}0$ $\{Aa_\lambda\}_{\lambda\in\Lambda}$ has a limiting point and it must be equal to 0. So $\lim\limits_{\lambda}Aa_\lambda=0$. This implies that $A^*\tilde{a}$ is universally continuous for every $\tilde{a}\in\tilde{R}$.

An element a of R is said to be *complete*, if $[a]=I$. A bonuded linear functional $\tilde{a}$ is said to be *complete* if $|\tilde{a}|(|a|)=0$ implies $a=0$. We remark that if a is complete, $\bar{a}\in\bar{R}$ $|\bar{a}|(|a|)=0$ implies $\bar{a}=0$.

Theorem 3.1. *Let A be universally continuous, positive and compact with* $r=r(A)>0$. *If A has a positive complete proper element, then* $\mu(r)=1$ *and the proper element corresponds to r.*

Proof. Let $Aa=\lambda a>0$ and $[a]=I$. If $r\neq\lambda$, $\tilde{a}(a)=0$ for the positive proper element of A^* corresponding to r, which exists by Theorem 2.3 and is universally continuous by Lemma 3.1. This implies $\tilde{a}=0$. So

λ must be equal to r. Suppose that $\mu(r)\geqq 2$. There exists a positive linear functional $\tilde{b}=(A^*-rI)^{\mu(r)-1}E^*(r)\tilde{x}$ as in Theorem 2.3. But by Lemma 3.1 $\tilde{b}(a)=0$ and $\tilde{b}\in\bar{R}$, this implies $\tilde{b}=0$, contradicting the assumption.

The special property of the proper manifold corresponding to is contained in:

Theorem 3.2. *If A is positive compact and A^* has a positive complete proper element $\tilde{a}$, the proper manifold corresponding to r of A is a linear lattice manifold.*

Proof. $\tilde{a}$ corresponds to r by Theorem 3.1. If $Aa=ra$, $Aa^+\geqq ra^+$, so we obtain $\tilde{a}(Aa^+-ra^+)=0$. Since $\tilde{a}$ is complete, this implies that $Aa=ra$. Hence the proper manifold corresponding to r is a linear lattice manifold.

Corollary 3.2. *Under the same assumption as Theroem* 3.2, *if $a\in\hat{R}$ is a proper element corresponding to ξ with $|\xi|=r$, $A|a|=r|a|$.*

Proof. By the definition (1.1), $Aa=\xi a$ we have $A\cdot|a|\geqq r|a|$. The assertion follows as above.

As a special class of positive linear operators, we define: a positive universally continuous linear operator A is said to be *completely positive* if

$$\bigcup_{\nu=1}^{\infty}[A^\nu x]=I \qquad \textit{for every } x>0 \tag{3.1}$$

This means that for a positive $x>0$, $a\frown A^\nu x=0$ ($\nu=1,2,\cdots$) implies $a=0$.

Lemma 3.2. *If A is positive and universally continuous, for $a>0$, putting $\bigcup_{\nu=1}^{\infty}[A^\nu a]=[N]$, we obtain an invariant normal manifold, that is,*

$$A[N]=[N]A[N] \tag{3.2}$$

Proof. Since for $x>0$, $[N]x=\bigcup_{\nu=1}^{\infty}(x\frown\nu(Aa+\cdots+A^\nu a))$ and A is universally continuous, $A[N]x=\bigcup_{\nu=1}^{\infty}A(x\frown\nu(Aa+\cdots+A^\nu a))$ so we obtaine $[A[N]x]\leqslant\bigcup_{\nu=1}^{\infty}[Aa+\cdots+A^\nu a]\leqslant[N]$.

Complete positiveness corresponds to "*Unzerlegbarkeit*" in [2], as is seen in the following:

Theorem 3.3. *A positive universally continuous linear operator is completely positive if and only if it has no non-trivial invariant normal manifold.*

This is an immediate consequence of Lemma 3.2.

Lemma 3. 3. *If A is compact and completely positive, every positive proper element of A (and A^*) is complete.*

Proof. If a is a proper element of A, $[a]=[A^\nu a]$ $(\nu=1,2,\cdots)$ and so $[a]=I$. If $\tilde{a}$ is a proper element of A^*, $\tilde{a}(a)=0$ implies $\tilde{a}(A^\nu a)=0$ $(\nu=1,2,\cdots)$. Since $\tilde{a}$ is in $\bar{R}$ by Lemma 3.1, $a=0$.

Theorem 3. 4. *If A is compact and completely positive with $r=r(A)>0$, the multiplicity of the proper value r is equal to* 1 (cf. §5 later).

Proof. By Theorem 3.2 and Lemma 3.3, the proper space corresponding to r is a linear lattice manifold. Since A is completely positive, all positive proper elements are complete, so the multiplicity must be equal to 1.

Next we consider the distribution of proper values corresponding to positive proper elements.

For a bounded linear operator A on a complex Banach space R into itself, the *spectral radius* of x, $r(x, A)$ or $r(x)$ (if there is no confusion), is defined by

$$r(x, A) \equiv r(x) = \overline{\lim_{\nu\to\infty}} \| A^\nu x \|^{\frac{1}{\nu}} \tag{3.2}$$

This functional satisfies the following properties:

1) $0 \leqslant r(x) \leqslant r(A)$

2) $\sup_{x\in R} r(x) = r(A)$

3) $r(\alpha x) = r(x)$ for $\alpha \neq 0$

4) $r(x+y) \leqslant \mathrm{Max}\,\{r(x),\ r(y)\}$

5) $r(Ax) = r(x)$

We remark that $r(x)$ *is nothing but the maximum modulus of singularities of analytic continuation of $R(\lambda)x$ $(\lambda\in\rho(A))$.*

Since the set of all spectra of a compact operator is a totally disconnected set with the only possible limiting point 0, the functional $r(x)$ is rather convenient.

Lemma 3. 4. *If A is compact, the functional $r(x)$ satisfies the following*:

a) $r(x) = \mathrm{Max}\,\{|\lambda| : E(\lambda)x \neq 0\}$ for $x \neq 0$

b) *$r(x)$ is lower semi-continuous,*

c) *the range of $r(x)$ coincides with the set $\{|\lambda|;\ \lambda\in\sigma(A)\}$.*

Proof. a) is an immediate consequence of remarks stated above. b) follows from a). c) is evident, since every non-zero spectrum is a proper value.

For a positive compact linear operator, we obtain:

Treorem 3.5. *If A is positive, universally continuous and compact, the set $\{r(x);\ r(x)>0,\ x>0\}$ coincides with the set of all non-zero proper values corresponding to positive proper elements.*

Proof. For $\lambda>0$, the set $S_\lambda=\{x;\ r(|x|)\leqslant\lambda\}$ is a closed linear manifold by Lemma 3.4. If $0\leqslant x_\rho\uparrow_\rho x\ (x_\rho\in S_\lambda)$, $x\in S_\lambda$, because $E(\xi)$ is universally continuous, hence S_λ is normal. If there exists $x\in S_\lambda$ such that $r(|x|)=\lambda$, by formula (5) $A[S_\lambda]=[S_\lambda]A[S_\lambda]$ and $r(A[S_\lambda])=\lambda$. Hence by Theorem 2.1 there exists $0\leqslant a_\lambda\in S_\lambda$ such that $Aa_\lambda=\lambda a_\lambda$. Conversely if $Aa=\rho a>0$, $r(a)=\rho$.

Theorem 3.6. *Let A be positive, universally continuous and compact, with $r=r(A)>0$, A is completely positive if and only if*

a) *A has a unique positive proper element (up to scalar) which is complete,*

b) $$r(x)>0 \qquad \textit{for every}\quad x>0.$$

Proof. If A is completely positive, then by Theorems 3.2 and 3.4 the positive proper element is unique and complete. Since A^* has a positive complete proper element $\tilde{a}$ corrresponding to r, for any positive $a>0$, $r(a)\geqq\overline{\lim\limits_{\nu\to\infty}}|\tilde{a}(A^\nu a)|^{\frac{1}{\nu}}=r$. Conversely suppose that A satisfies a) and b). If a normal manifold N is invariant relative to A, by b) $r(A[N])>0$ and by Theorem 2.3 there exists a positive proper element in N which is not complete.

We may replace the condition b) by the condition

b′) *A^* has a complete positive proper element.*

Lemma 3.5. *For a compact linear operator A on a Banach space R*

$$\sup_{|\lambda|=r}\mu(\lambda)\leqslant 1 \ \textit{ if and only if }\ \|A^\nu\|\leqslant M\cdot r^\nu \qquad (\nu=1,2,\cdots)$$

for some M, where $r=r(A)$. And in this case

$$E(r)=\lim_{\nu\to\infty}\frac{1}{\nu}\sum_{k=1}^{\nu}\left(\frac{A}{r}\right)^k \tag{3.3}$$

The proof is well known (cf. [10] and [5]).

If the maximum spectrum is not simple, the following decomposition holds:

Theorem 3.7. *Let A be positive, universally continuous and compact. If A and A^* both have positive complete proper elements, there exists a decomposition of identity such that*

$$(3.4)\qquad I=\sum_{\nu=1}^{n}[a_\nu],\quad [a_\nu][a_\mu]=0 \qquad (\nu\neq\mu)$$
$$A[a_\nu]=[a_\nu]A \qquad (\nu=1,2,\cdots,n)$$

and A is completely positive on $[a_\nu]R$ $(\nu=1,2,\cdots,n)$.

Proof. By Lemma 3.5 and Theorem 3.2 the positive projection $E(r)$ is written in a form $E(r)x=\sum_{\nu=1}^{n}\tilde{a}_\nu(x)a_\nu$ such that

$$Aa_\nu=ra_\nu>0,\ A^*\tilde{a}_\nu=r\tilde{a}_\nu>0,\ \tilde{a}_\nu(a_\mu)=\delta_{\nu\mu} \qquad (\nu,\mu=1,2,\cdots,n).$$

Since $\bigcup_{\nu=1}^{n}[a_\nu]=\bigcup_{\nu=1}^{n}[\tilde{a}_\nu]^R=I$ and $A[a_\nu]=[a_\nu]A[a_\nu]$, $A^*[\tilde{a}_\nu]=[\tilde{a}_\nu]A^*[\tilde{a}_\nu]$ $(\nu=1,2,\cdots,n)$, A is completely positive on $[a_\nu]R$ $(\nu=1,2,\cdots,n)$.

We define a somewhat weaker condition than complete positiveness: a positive linear operator A is said to be *natually decomposable,* if there exists a decomposition of identity $[N_\rho]$ $(\rho\in\Lambda)$ and $[M]$ such that

$$(3.5)\qquad I=\sum_{\rho\in\Lambda}[N_\rho]+[M],\ [N_{\rho_1}][N_{\rho_2}]=0\ (\rho_1\neq\rho_2),\ [N_\rho][M]=0$$
$$A[N_\rho]=[N_\rho]A \quad\text{and}\quad A[M]=[M]A,$$
$$r(A[N_\rho])>0\ (\rho\in\Lambda) \quad\text{and}\quad r(A[M])=0,$$

and A acts as a completely positive operator on $[N_\rho]R$ $(\rho\in\Lambda)$.

Theorem 3.8 *Let A be positive, universally continuous and compact. A is naturally decomposable if and only if*

$$\cup[a]=\cup[\tilde{a}]^R$$

where a) (and $\tilde{a}$) varies in all positive proper elements of A (and of A^* respectively).

Proof. If A is naturally decomposable with the decomposition (3.5), then by Lemma 3.3. $\cup[a]=\cup[\tilde{a}]^R=\bigcup_{\rho\in\Lambda}[N_\rho]$. Conversely, let $\cup[a]=\cup[\tilde{a}]^R$. We arrange the proper values of A corresponding to positive proper elements in the decending order $\lambda_1>\lambda_2>\cdots$. Since each corresponding proper manifold is finite dimensional, there exist maximum projectors $[p_\nu]$ corresponding to λ_ν $(\nu=1,2,\cdots)$. Analogously we obtain $\tilde{q}_\nu$ relative to A^* corresponding to $\mu_1>\mu_2>\cdots$. By hypothesis $\bigcup_{\nu=1}^{\infty}[p_\nu]=\bigcup_{\nu=1}^{\infty}[\tilde{q}_\nu]^R$ and $\tilde{q}_k(p_\nu)=0$ if $\lambda_\nu\neq\mu_k$. We obtain $[p_\nu]=[\tilde{q}_\nu]^R$ $(\nu=1,2,\cdots)$ and $[p_\nu]A=A[p_\nu]$.

Since $\left(1-\bigcup_{\nu=1}^{\infty}[p_\nu]\right)A$ is quassi-nilpotent by Theorem 2.3, we obtain the assertion by Theorem 3.7.

Next we consider a characterization of natural decomposability analogous to Theorem 3.6. For a projection $[N]$, we put

$$r([N]) = \sup_{[N]x=x} r(x) \quad \text{and} \quad r^*([N]) = \sup_{\bar{x}[N]=\bar{x}} r^*(\bar{x})$$

where
$$r^*(\bar{x}) = \overline{\lim_{\nu\to\infty}} \| A^{*\nu}\bar{x}\|^{\frac{1}{\nu}}$$

Theorem 3.9. *Let A be positive, universally continuous and compact, and R semi-regular. A is naturally decomposable, if and only if it satisfies*

a) $\qquad r([N]) = r^*([N]) \qquad$ *for every projection* $[N]$,

b) $\qquad \sup_{\nu=1,2,\cdots} \dfrac{\|A^\nu x\|}{r(x)^\nu} < \infty \qquad$ *for $x>0$ with $r(x)>0$.*

Proof. If A is naturally decomposable with the decomposition (3.5), it is easy to see that for a projection $[N]$ and $x>0$

$$r([N]) = \sup_{[N][N_\rho]\neq 0} r(A[N_\rho]) = r^*([N])$$
$$r(x) = \sup_{[N_\rho]x\neq 0} r(A[N_\rho])$$

b) follows from Lemma 3.5. Converesely, suppose that a) and b) are satisfied. Put $S_\lambda=\{x;\, r(|x|)\leqslant\lambda\}$ and $\bar{S}_\lambda=\{\bar{a};\bar{a}\in\bar{R},\, r^*(|\bar{a}|)\leqslant\lambda\}$. By Lemma 3.4 and a) we have $[S_\lambda]=[\bar{S}_\lambda]^R$, hence $A[S_\lambda]=[S_\lambda]A$. If a and b are positive proper elements corresponding to different positive proper values λ_1 and λ_2 respectively. If $[p]=[a][b]\neq 0$, as in the proof of Theorem 3.6,

$$r\;([p])\leqslant \mathrm{Min}\left\{r([a]), r([b])\right\} = \mathrm{Min}\left\{\lambda_1,\, \lambda_2\right\}$$

and

$$r^*([p])\geqq \mathrm{Max}\left\{r([a]), r([b])\right\} = \mathrm{Max}\left\{\lambda_1,\, \lambda_2\right\}$$

contradicting a). So $a\frown b=0$. Let $\lambda_1>\lambda_2>\cdots$ be proper values of A corresponding to positive proper elements in the descending order. Putting $[S_\nu]=[S_{\lambda_\nu}]-[S_{\lambda_{\nu+1}}]$, we have $A[S_\nu]=[S_\nu]A$ $(\nu=1,2,3,\cdots)$. By Lemma 3.5 and b) we can prove that $E(\lambda_\nu)x>0$ and $r(x)=\lambda_\nu$ for $0<x\in[S_\nu]R$. Hence there exists $0<\bar{a}\in\bar{R}$ such that

$$A^*\bar{a} = \lambda_\nu\bar{a} \quad \text{and} \quad [\bar{a}]^R = [S_\nu]$$

Again using a), we obtain that there exists $0<a\in R$ such that $Aa=\lambda_\nu a$

and $[a]=[S_\nu]$. Now Theorem 3.7 is applicable.

§ **4. Quasi-nilpotent operators.** In this § we consider relations between A and its restriction to normal manifolds.

Lemma 4.1. *Let A be a compact linear operator on a Banach space R. If F is a bounded linear operator such that $F^2=F$ and $AF=FAF$, then*

$$\sigma(A)=\sigma(AF)\smile\sigma((1-F)A(I-F))\,, \tag{4.1}$$

$$r(A)=\mathrm{Max}\,\{r(AF),\ r((I-F)A(I-F))\} \tag{4.2}$$

Proof. Let $0\neq\lambda\in\rho(AF)\frown\rho((I-F)A(I-F))$. If $(\lambda I-A)x=0$, $(I-F)(\lambda I-A)x=(I-F)(\lambda-(I-F)A(I-F))x=0$ hence $(I-F)x=0$ and similarly $Fx=0$, so, $x=0$, hence $\lambda\in\rho(A)$. Conversely, if $\lambda\in\rho(A)$, $(\lambda I-A)F'$ is one-to-one on FR, hence by RIESZ's theorem $\lambda\in\rho(AF)$. Similarly $\lambda\in\rho(A^*(I-F^*))=\rho((I-F)A(I-F))$.

For positive linear operators A and B such that $A\geqq B$, it is evident that $r(A)\geqq r(B)$. In particular, for any projection $[N]$, $r([N]A[N])\leqslant r(A)$.

Theorem 4.1. *Let A be positive, universally continuous and compct with $r=r(A)>0$. A is completely positive, if and only if for a projection $[N]$ $r([N]A[N])=r(A)$ implies $[N]=I$.*

Proof. Suppose first that A is completely positive. If $r([N]A[N])=r$, by Theorem 2.3 there exists $a>0$ such that $[N]A[N]a=ra$. Since $Aa\geqq ra$, as in the proof of Theorem 3.2, we obtain $Aa=ra$. Complete positiveness implies $[N]=[a]=I$. Next suppose that A is not completely positive. There exists by Theorem 3.3 a non-trivial normal manifold $[N]$ such that $A[N]=[N]A[N]$. Lemma 4.1 shows that $\mathrm{Max}\,\{r(A[N]), r((I-[N])A(I-[N]))\}=r$.

Between a positive proper value distinct from $r=r(A)$ and $r([N]A[N])$ the following relation holds:

Theorem 4.2. *Let A be positive and compact, and λ a positive proper value distinct from $r=r(A)$. For any non-zero projection $[N]$ there exists a non-zero projection $[M]$ such that $[N]\geqq[M]$ and $r((I-[M])A(1-[M]))\geqq\lambda$.*

Proof. There exists $a\neq0$ such that $Aa=\lambda a$, so $Aa^+\geqq\lambda a^+$ and $Aa^-\geqq\lambda a^-$. By Corollay 2.1.2, $r([a^+]A[a^+])\geqq\lambda$ (and $r([a^-]A[a^-])\geqq\lambda$) if $[a^+]\neq0$ (and $[a^-]\neq0$). If $[N][a]=0$, we put $[M]=[N]$. If $[N][a]\neq0$ and $a^-=0$, we have $Aa=\lambda a>0$ and $r([a]A[a])=\lambda$. Since by Lemma 4.1 Max $\{r(A[a]), r((I-[a])A(1-[a]))\}=r$, we put $[M]=[N][a]$. If $[N][a^+]\neq0$ and $[a^-]\neq0$, we put $[M]=[N][a^+]$. The other case is treated similarly.

Corollary 4.2. *Under the same conditiosns as in Theorem 4.2, if p is* *a discrete element of* R, $r((1-[p])A(1-[p]))\geqq\lambda$.

Proof. Since p is a discrete element, putting $[N]=[p]$, $[M]$ must coincides with $[N]$.

Lemma 4.2. *If* A_λ $(\lambda\in\Lambda)$, (Λ *being a directed set*), *are compact linear operators defined on a Banach space, such that* $\lim_\lambda\|A_\lambda-A\|=0$, *then*

$$\lim_\lambda \sigma(A_\lambda)=\sigma(A) \quad \textit{in the sense of metric,} \tag{4.3}$$

$$\lim_\lambda r(A_\lambda)=r(A). \tag{4.4}$$

The proof is found in [8].

Lemma 4.3. *Let* F_λ $(\lambda\in\Lambda)$, (Λ *being a directed set*), *be bounded linear operators on a Banach space* R *such that* $F_\lambda^2=F_\lambda$ $(\lambda\in\Lambda)$, $F^2=F$ $\sup_{\lambda\in\Lambda}\|F_\lambda\|<\infty$ *and* $\lim_{\lambda\in\Lambda}F_\lambda x=Fx$ $(x\in R)$. *If* A *is a compact linear operator on* R, *then* $\lim_{\lambda\in\Lambda} r(F_\lambda AF_\lambda)=r(FAF)$.

Proof. Since the image of the unit sphere by a compact linear operator is relatively compact, it is easy to see that $\lim_\lambda\|F_\lambda AF_\lambda A-FAFA\|=0$. By Lemma 4.2, $\lim_\lambda r(F_\lambda AF_\lambda A)=r(FAFA)$. But since $r(FAFA)=\lim_{\nu\to\infty}\|(FAFA)^\nu\|^{\frac{1}{\nu}}=\lim_{\nu\to\infty}\|(FAF)^{2\nu-1}A\|^{\frac{1}{\nu}}\leqslant r(FAF)^2$ and $r(FAFA)=\lim_{\nu\to\infty}\|(FAFA)^\nu\|^{\frac{1}{\nu}}\geqslant\lim_{\nu\to\infty}\left\{\frac{1}{\|F\|}\|(FAF)^{2\nu}\|\right\}^{\frac{1}{\nu}}=r(FAF)^2$, we obtain $\lim_\lambda r(F_\lambda AF_\lambda A)=r(FAF)^2$. Hence $\lim_\lambda r(F_\lambda AF_\lambda)=r(FAF)$.

Theorem 4.3. *Let* R *have no discrete element and be of continuous norm. If* A *is positive compact, there exist* $p_\lambda\in R$ $(0\leqslant\lambda\leqslant r(A))$, *such that*

$$[p_\lambda]\leqslant[p_\mu] \qquad (0\leqslant\lambda\leqslant\mu) \tag{4.5}$$

and

$$r([p_\lambda]A[p_\lambda])=\lambda \qquad (0\leqslant\lambda\leqslant r(A))$$

Proof. We choose, Zorn's lemma, a maximal linearly ordered family of projections $[q_\rho]$. The assumptions on R and Lemma 4.3 imply that $\sup_{\xi<\rho} r([q_\xi]A[q_\xi])=\inf_{\xi>\rho} r([q_\xi]A[q_\xi])$. We can choose $[p_\lambda]$ from $[q_\rho]$ with $r([q_\rho]A[q_\rho])=\lambda$, the remaining part is easily proved.

In the operator theory, it is important to study conditions assuring non-quasi-nilpotentness. Naturally a problem arises whether complete positiveness implies non-quasi-nilpotentness. We have been able to solve this problem only under some additional conditions.

A bounded linear operator A is said to be *totally continuous*, if for any positive $a \in R$ and positive $\bar{a} \in \bar{R}$

$$(4.6) \quad [\bar{p}]\bar{x}(A[p]x) = \int_{[p]\times[\bar{p}]^R} \Phi(\mathfrak{q},\mathfrak{p})\,\bar{a}(d\mathfrak{p}x)\,\bar{x}(d\mathfrak{q}a) \quad \text{for } [p]\leqslant[a] \text{ and } [\bar{p}]\leqslant[\bar{a}],$$

for a fixed Borel function $\Phi(\mathfrak{q},\mathfrak{p})$ on the product space $\mathfrak{E}\times\mathfrak{E}$ of the proper space of R (see [7] §5). H. NAKANO [7] proved that A *is totally continuous if and only if* $|a_\nu|\leqslant a$ $(\nu=1,2,\cdots)$ $s-\lim_{\nu\to\infty} a_\nu=0$ (*star-convergence*) *implies* $(o)-\lim_{\nu\to\infty} Aa_\nu=0$. (It is easy to prove that here star-convergence may be replaced by weak-convergence).

Lemma 4.4 *For a bounded linear operator A on a Banach space, put* $B=\sum_{\nu=1}^{\infty}\frac{A^\nu}{\lambda^{\nu+1}}$ *for some* $\lambda>r(A)$. *If B has a non-zero spectrum, A has one also. If A is positive and compact, B is so.*

Proof. Since $R(\lambda)=\frac{1}{\lambda}I+B$, by the spectral mapping theorem (cf. [3] p. 122) $\sigma(A)=\left\{\lambda-\frac{1}{\frac{1}{\lambda}+\xi}\ ;\ \xi\in\sigma(B)\right\}$, if $\sigma(B)$ contains non-zero number, $\sigma(A)$ does also.

Lemma 4.5. *Let R be reflexive as a Banach space. If* A_ν, A $(\nu=1,2,\cdots)$ *are positive compact such that*

$$A_1\leqslant A_2\leqslant A_3\leqslant\cdots \quad \textit{and} \quad \lim_{\nu\to\infty} A_\nu a = Aa \qquad (a\in R),$$

then

$$\lim_{\nu\to\infty}\|A_\nu - A\| = 0$$

Proof. Considering A as a continuous function $\tilde{a}(Aa)$ on $S\times\tilde{S}$, where S and $\tilde{S}$ are the positive unit spheres of R and $\bar{R}$ respectively, topologized by weak topologies. Since $S\times\tilde{S}$ is compact, the assertion follows from the well-known theorem of Dini and the definition of norms of operators.

Theorem 4.5. *Let R be reflexive as a Banach space. If A is compact, totally continuous and completely positive, then A is not quasi-nilpotent.*

Proof. By Lemma 4.4 considering the compact positive operator B, we may assume that $[Ax]=I$ for every $x>0$. Further we may assume that for some $a>0$ and $\bar{a}>0$ $[a]=[\bar{a}]^R=I$ and $\bar{a}(a)=1$. Suppose that A is represented in a form (4.6). A^2 satisfies the same conditions as A and the corresponding function may be given by

$$\Psi(\mathfrak{q},\mathfrak{p}) = \int \Phi(\mathfrak{q},\mathfrak{s})\,\Phi(\mathfrak{s},\mathfrak{p})\,\bar{a}(d\mathfrak{s}a)$$

If $\Psi(\mathfrak{q}, \mathfrak{p})$ vanishes on a set of positive measure, from the measure theory, there exists a measurable subset $\mathfrak{A} \subset \mathfrak{G}$ such that

a) the measure of $\mathfrak{A}$ is positive, (the measure being defined by $\bar{a}([p]a))$.

b) $$\text{meas}\,(\mathfrak{B}_{\mathfrak{p}}) > 0 \qquad \text{for } \mathfrak{p} \in \mathfrak{A}, \quad \text{where}$$
$$\mathfrak{B}_{\mathfrak{p}} = \left\{\mathfrak{q}\,;\, \int \Phi(\mathfrak{q}, \mathfrak{s})\, \Phi(\mathfrak{s}, \mathfrak{p})\, \bar{a}(d\mathfrak{s}a) = 0\right\}$$

c) $\Phi(\mathfrak{q}, \mathfrak{p})$ is measurable with respect to $\mathfrak{q}$ for $\mathfrak{p} \in \mathfrak{A}$.

If $\Phi(\mathfrak{s}, \mathfrak{p}) > 0$ on a set of positive measure $\mathfrak{C}$ for some $\mathfrak{p} \in \mathfrak{A}$ $\Phi(\mathfrak{q}, \mathfrak{s}) = 0$ almost everywhere on $\mathfrak{B}_{\mathfrak{p}} \times \mathfrak{C}$, contradicting the assumption that $[Ax] = I$ for every $x > 0$. Thus $\Phi(\mathfrak{p}, \mathfrak{q}) = 0$ almost everywhere on $\mathfrak{G} \times \mathfrak{A}$, also contrading the assumption. Hence $\Psi(\mathfrak{q}, \mathfrak{p}) > 0$ almost everywhere. It is known [7] that $\bar{a} \otimes a = \bigcap_{\nu=1}^{\infty} (\nu A^2 \frown \bar{a} \otimes a)$ and $r(\bar{a} \otimes a) = 1$. Lemmas 4.5 and 4.2 imply $r(A^2) = r(A)^2 > 0$.

Next theorem is proved in [11], but we give a somewhat different proof.

Theorem 4.6. *Let F_λ $(0 \leqslant \lambda \leqslant 1)$ be bounded linear operators defined on a Banach space R such that $F_0 = 0$, $F_1 = I$, $F_\lambda F_\mu = F_{\min(\lambda,\mu)}$ and $\lim_{\rho \uparrow \lambda} F_\rho x = \lim_{\rho \downarrow \lambda} F_\rho x = F_\lambda x$ $(x \in R)$. If A is a compact linear operator on R such that $AF_\lambda = F_\lambda A F_\lambda$ $(0 \leqslant \lambda \leqslant 1)$, then A is quasi-nilpotent.*

Proof. Since by Lemma 4.1 Max $\{r(F_\lambda A F_\lambda),\ r((I - F_\lambda) A (I - F_\lambda))\} = r(A)$, there exists, by induction, sequences $\lambda_\nu \uparrow_{\nu=1}^{\infty} \alpha$ and $\mu_\nu \downarrow_{\nu=1}^{\infty} \beta$ such that $r((F_{\mu_\nu} - F_{\lambda_\nu}) A (F_{\mu_\nu} - F_{\lambda_\nu})) = r(A)$, $(\nu = 1, 2, \cdots)$. But by Lemma 4.3 $r(A) = r((F_\beta - F_\alpha) A (F_\beta - F_\alpha))$. Continuing this method, we obtain $r(A) = 0$.

Combining Theorems 4.5 and 4.6, we obtain a generalization of criteria of Voltera-type ([10] p. 147).

Theorem 4.7. *Let R be reflexive as a Banach space and have no discrete element, and A be positive, compact and totally continuous. Then A is quasi-nilpotent if and only if there exist projections $[N_\lambda]$ $(0 \leqslant \lambda \leqslant 1)$, such that*

$$(4.7) \qquad [N_0] = 0,\ [N_1] = \bigcup_{x \in R} [Ax],\ \bigcap_{\rho > \lambda} [N_\rho] = \bigcup_{\rho < \lambda} [N_\rho] = [N_\lambda]$$
$$A[N_\lambda] = [N_\lambda] A [N_\lambda] \qquad (0 \leqslant \lambda \leqslant 1).$$

Proof. Since by Theorem 4.5 any non-trivial invariant normal manifold contains the other non-trivial one, the proof proceeds as in Theorem 4.3.

§ 5. **Proper values with maximum modulus.** Here the distribution of proper values with maximum modulus of a positive compact linear operator is considered.

Lemma 5.1. *If A is compact and completely positive with $r=r(A)>0$, the proper values with maximum modulus are all simple and are the solutions of the equation*

$$\xi^k - r^k = 0 \qquad \text{for some } k. \tag{5.1}$$

Proof. Since by Theorem 3.1 $\mu(r)=1$, the assertion follows from Theorem 8.1 of [5].

Lemma 5.2. *For a compact linear operator A on a Banach space R, the following conditions are equivalent to each other*

1) $w-\lim_{\nu\to\infty} A^\nu x = E(1)x \qquad (x\in R)$

2) $\lim_{\nu\to\infty} A^\nu x = E(1)x \qquad (x\in R)$

3) $\lim_{\nu\to\infty} \| A^\nu - E(1)\| = 0$

4) $r(A)\leqslant 1$ *and* 1 *is the only possible proper value with modulus* 1.

The proof is similar to that of Lemma 3.5.

Theorem 5.1. *Let A be compact and completly positive with $r(A)=1$. The following conditions are equivalent:*

1) 1 *is the unique proper value with maximum modulus,*

2) $\lim_{\nu\to\infty} A^\nu x$ *exists and is complete for every* $x>0$,

3) A^ν $(\nu=1, 2, \cdots)$ *are all completely positive.*

Proof. By Theorem 2.1 and Lemma 5.2, 1) and 2) are equivalent. Let 2) be satisfied. Since $\lim_{n\to\infty} A^{\nu n}x=E(1)x$, 3) follows from 2) by the definition. Finally if A^ν $(\nu=1, 2, \cdots)$ are all completely positive and λ is a proper value with $|\lambda|=1$ distinct from 1, then by Lemma 5.1 there exists a positive integer k such that $\lambda^k=1$, so 1 is a proper value of A^k of multiplicity greater than 1, contradicting the assumption by Theorem 3.4.

Lemma 5.3. *Let R be reflexive as a Banach space and A be compact. If $[N_\lambda]\downarrow_{\lambda\in\Lambda}[N]$ and $\sigma([N_\lambda]A[N_\lambda])\subseteq\sigma(A)$, then $\sigma([N]A[N])\subseteq\sigma(A)$.*

Proof. The reflexivity implies that that the norms of R and $\widetilde{R}$ are both continuous. Since $\lim_{\lambda\in\Lambda} \|[N_\lambda]A[N_\lambda]-[N]A[N]\|=0$, by Lemma 4.2 $\sigma([N]A[N])=\lim_{\lambda\in\Lambda}\sigma([N_\lambda]A[N_\lambda])\subseteq\sigma(A)$.

Theorem 5.2. *Let R be reflexive as a Banach space. If A is positive* *compact, the proper values with maximum modulus coincides with the solutions of the equation*

$$\prod_{i=1}^{n} (\xi^{k_i} - r^{k_i}) = 0 \tag{5.2}$$

where k_i $(i=1,2,\cdots,n)$ *are some positive integers.*

Proof. Reflexivity of R implies universal continuity of A. Let λ be a proper value with maximum modulus. Considering, by ZORN's lemma, a maximal linearly ordered family $[N_\rho]_{\rho\in\Lambda}$ such that $\lambda\in\sigma([N_\rho]A[N_\rho])\subseteq\sigma(A)$. Putting $[N]=\bigcap_\rho[N_\lambda]$, by Lemma 5.3, we obtain $\lambda\in\sigma([N]A[N])\subseteq\sigma(A)$. Since $r([N]A[N])\leqslant r(A)$, λ is a proper value of $[N]A[N]$ with maximum modulus. By Lemma 4.1, the maximal hypothesis implies that there is no $0<[M]<[N]$ with $[N]A[M]=[M]A[M]$, that is, $[N]A[N]$ is completely positive. Hence by Lemma 5.1 λ is a solution of an equation $\xi^k=r^k$ for some k and all its solutions are contained in $\sigma([N]A[N])\subseteq\sigma(A)$. Since A has only a finite number of proper values on the circle with radius r, this completes the proof.

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