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ON SEMI-LOWER BOUNDED MODULARS

By

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W. Orlicz and Z. Birnbaum proved in [7] that an Orlicz space $L_\phi(G)$ is finite if and only if the function ϕ satisfies the following condition: for some $\gamma > 0$ and $t_0 > 0$, $\phi(2t) \leq \gamma \phi(t)$ for every $t \geq t_0$. (In case of $\text{mes}(G) = +\infty$, $\phi(2t) \leq \gamma \phi(t)$ for all $t \geq 0$.)

This fact was generalized for arbitrary monotone complete modulars on non-atomic space by I. Amemiya in [1], that is, suppose that R is a universally continuous semi-ordered linear space and has no atomic element, then every monotone complete finite modular on R is semi-upper bounded.

T. Shimogaki showed in [8] a new simple proof of this Amemiya's Theorem. In this paper we investigate the properties of the conjugate modular of a semi-upper bounded modular, i.e. the semi-lower bounded modular. Throughout this paper we use the terminologies and notations used in [5].

In §1 we give corollaries of Amemiya's Theorem and a theorem relate to Amemiya's Theorem. In §2 we investigate the relations between a modular or the modular norms and semi-lower bounded modular. In §3 we express the properties of a semi-upper and semi-lower bounded modular.

§1. Let R be a universally continuous semi-ordered linear space and m be a modular on R^1 . A modular m is said to be "*finite*", if $m(x) < +\infty$ for every $x \in R$. A modular m is said to be "*monotone complete*", if for $0 \leq a_\lambda \uparrow_{\lambda \in A}$, $\sup_{\lambda \in A} m(a_\lambda) < +\infty$ there exists $a \in R$ for which $a_\lambda \uparrow_{\lambda \in A} a$. And a modular m is said to be "*semi-upper bounded*", if for every $\varepsilon > 0$ there exists $\gamma = \gamma(\varepsilon) > 0$ such that $m(x) \geq \varepsilon$ implies $m(2x) \leq \gamma m(x)$.

In [1] I. Amemiya proved:

Theorem 1.1. *Suppose that R has no atomic element, then every monotone complete, finite modular on R is semi-upper bounded.*

We say a modular m on R to be "*domestic*", if for any $a \in \{a : m(a) < +\infty, a \in R\}$ there exists $\xi = \xi(a) > 1$ such that $m(\xi a) < +\infty$. On R , we define the two functionals $\|a\|$, $\|a\|$ ($a \in R$) as follows:

1) For the definition of the modular see H. Nakano [5].

$$\|a\| = \inf_{\xi > 0} \frac{1+m(\xi a)}{\xi}, \quad |||a||| = \inf_{m(\xi a) \leq 1} \frac{1}{|\xi|}.$$

Then it is easily seen that both $\|a\|$ and $|||a|||$ are norms on R and satisfy always $|||a||| \leq \|a\| \leq 2 |||a|||$ for all $a \in R$ (cf. [6]). The norms $\|a\|$ and $|||a|||$ are called the *first norm* and the *second (or modular) norm* by m respectively.

Remark 1.1. (i) If a modular m on R is finite, then m is domestic; (ii) if m is domestic, then $\inf_{0 \neq x \in R} m\left(\frac{x}{|||x|||}\right) = 1$; (iii) $\inf_{0 \neq x \in R} m\left(\frac{x}{|||x|||}\right) > 0$ implies $|||\cdot|||$ is continuous; (iv) if $|||\cdot|||$ is continuous, then m is finite, when R has no atomic element.

Because, (i) is trivial. (iii) and (iv) is well known²⁾. Therefore we have only to prove (ii). If $m\left(\frac{x}{|||x|||}\right) < 1$ for some $x \in R$, there exists $\varepsilon > 0$ by domesticness such that

$$1 < m\left((1+\varepsilon)\frac{x}{|||x|||}\right) < +\infty.$$

Thus there exists $\gamma < 1$, for which $m\left(\gamma(1+\varepsilon)\frac{x}{|||x|||}\right) = 1$. Therefore we obtain $\gamma(1+\varepsilon) = |||\gamma(1+\varepsilon)\frac{x}{|||x|||}||| = 1$, and hence $m\left(\frac{x}{|||x|||}\right) = 1$, contradicting $m\left(\frac{a}{|||x|||}\right) < 1$.

A modular norm $|||x|||$ ($x \in R$) is said to be "*finitely monotone*" (cf. [9]), if for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that $x = \bigoplus_{i=1}^n x_i$, $|||x||| \leq 1$, $|||x_i||| \geq \varepsilon$ ($i=1, 2, \dots, n$) implies $n \leq n_0$. A modular m is said to be "*uniformly finite*", if

$$\sup_{m(x) \leq 1} m(\xi x) < +\infty \quad \text{for all } \xi \geq 0.$$

In [9, Theorems 1.1, 2.1 and 2.2], it is shown that if a norm on R is uniformly monotone³⁾, then it is finitely monotone; if a modular m is uniformly finite, then the modular norm by m is finitely monotone; if the modular norm by m is finitely monotone, then m is uniformly finite when R has no atomic element; if a norm is finitely monotone, then the every norms which is equivalent to it is also finitely monotone.

2) T. Andô obtained (iii). For (iv) see [1].

3) A norm on R is said to be uniformly monotone, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $a \wedge b = 0$, $\|a\| = 1$, $\|b\| \geq \varepsilon$ implies $\|a+b\| \geq 1+\delta$ (cf. [4]).

$\bar{R}^m$ denotes the totality of all universally continuous linear functionals⁴⁾ on R which are bounded under the modular norm $|||\cdot|||$ by m . On $\bar{R}^m$ the conjugate modular of $m(x)$ is defined as follows

$$\bar{m}(\bar{a}) = \sup_{x \in R} \{\bar{a}(x) - m(x)\} \quad \text{for every } \bar{a} \in \bar{R}^m.$$

$\bar{m}(\bar{a})$ satisfies the modular conditions and is monotone complete (cf. [5, §38]).

It has been known that if R is semi-regular⁵⁾, the first norm by the conjugate modular $\bar{m}$ is the conjugate norm of the second norm by m and the second norm by the conjugate modular $\bar{m}$ is the conjugate norm of the first norm by m .

Lemma 1 ([5, Theorem 39.4]). *If R is semi-regular, then R is isometric⁶⁾ to a complete semi-normal manifold of the conjugate space $\bar{\bar{R}}^m$ of $\bar{R}^m$ by the correspondence*

$$R \ni a \rightarrow a^{\bar{R}^m} \in \bar{\bar{R}}^m, \quad a^{\bar{R}^m}(\bar{x}) = \bar{x}(a) \quad \text{for } \bar{x} \in \bar{R}^m.$$

Corollary 1 of Theorem 1.1. *Suppose that R has no atomic element. If the modular norm $|||\cdot|||$ by m is finitely monotone, then m is semi-upper bounded.*

Proof. Since m is uniformly finite by assumption, $\bar{m}$ is uniformly finite on $\bar{R}^m$ ([5, Theorems 48.4, 48.5]). Since $\bar{m}$ is monotone complete and $\bar{R}^m$ has no atomic element, we obtain by Theorem 1.1 $\bar{m}$ is semi-upper bounded on $\bar{R}^m$. Therefore m is semi-upper bounded by Lemma 1. Q.E.D.

Remark 1.2. *If a modular m is semi-upper bounded and semi-simple, then m is uniformly finite.*

Because, if for some $\gamma > 1$ we have $m(2x) \leq \gamma m(x)$ for every x such that $m(x) \geq 1$, then we have obviously $m(2^\nu x) \leq \gamma^\nu m(x)$ ($\nu = 1, 2, \dots$) for every x such that $m(x) \geq 1$. Since m is finite by assumption, we obtain

$$\begin{aligned} \sup_{m(x) \leq 1} m(2^\nu x) &\leq \sup_{1 \leq m(x) \leq 2} m(2^\nu x) \\ &\leq \sup_{1 \leq m(x) \leq 2} \gamma^\nu m(x) \leq 2\gamma^\nu < +\infty \quad (\nu = 1, 2, \dots). \end{aligned}$$

4) A linear functional L on R is said to be universally continuous, if for any $a_\lambda \downarrow_{\lambda \in \Lambda} 0$ we have $\inf_{\lambda \in \Lambda} |L(a_\lambda)| = 0$.

5) R is said to be semi-regular, if $\bar{a}[p] = 0$ for all $\bar{a} \in \bar{R}^m$ implies $p = 0$. For $p \in R$, $[p]$ denotes the projection operator defined by $[p]x = \bigcup_{\nu=1}^{\infty} (x \wedge \nu | p |)$ for all $x \geq 0$.

6) A modular space R with a modular m is said to be isometric to a modular space $\hat{R}$ with a modular $\hat{m}$ by a correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$, if R is isomorphic to $\hat{R}$ by this correspondence and $m(a) = \hat{m}(a^{\hat{R}})$ for all $a \in R$.

Thus, m is uniformly finite.

A norm on R is said to be "monotone", if $0 \leq a < b$ implies $\|a\| < \|b\|$. A norm on R is said to be "universally monotone complete", if for $0 \leq a_\lambda \uparrow_{\lambda \in A}$, $\sup_{\lambda \in A} \|a_\lambda\| < +\infty$ there exists $a \in R$ such that $a_\lambda \uparrow_{\lambda \in A} a$; if $A = \{1, 2, \dots\}$ we say to be "monotone complete".

Corollary 2 of Theorem 1.1. *If the modular norm $\|\cdot\|$ by m is monotone and monotone complete, then m is uniformly simple⁷⁾, and m is semi-upper bounded when R has no atomic element.*

Proof. (i) If the modular norm $\|\cdot\|$ by m is monotone, then $\|\cdot\|$ is continuous.

Because, if $\inf_{0 \neq x \in R} m\left(\frac{x}{\|x\|}\right) < 1$, there exists $a \in R$ such that $\|a\| = 1$ and $m(a) < 1$, therefore we can suppose $[a] < 1$ without difficulty, and hence there exists $0 < b \in R$ such that $a \wedge b = 0$, $m(a+b) \leq 1$. Thus we obtain obviously $\|a+b\| = \|a\| = 1$, which is contradicting $\|\cdot\|$ is monotone. Consequently we obtain $\inf_{0 \neq x \in R} m\left(\frac{x}{\|x\|}\right) = 1$, and hence $\|\cdot\|$ is continuous by Remark 1.1.

(ii) If the modular norm $\|\cdot\|$ by m is monotone, then m is simple⁸⁾.

Because, if m is not simple there exists $a \in R$ such that $0 < a$ and $m(a) = 0$, then $m(a+b) = m(b) \leq 1$ for any $0 < b$, $a \wedge b = 0$ and $\|b\| = 1$. Thus we have $\|a+b\| = \|b\| = 1$, contradicting assumption that $\|\cdot\|$ is monotone. Thus m is simple.

If the modular norm $\|\cdot\|$ by m is continuous and monotone complete, then m is monotone complete (cf. [5, Theorems 30.20, 40.7]). Thus we obtain m is monotone complete, simple and $\|\cdot\|$ is continuous by (i) and (ii). Therefore m is uniformly simple (cf. [11, Theorem 2.1]).

If R has no atomic element, then uniformly simple modular m is uniformly finite ([10, Theorem 1.2]), and hence we obtain m is semi-upper bounded by Corollary 1 of Theorem 1.1. Q.E.D.

Theorem 1.2. *Suppose that R has no atomic element. Each of the following conditions implies that m is semi-upper bounded*

$$(1): \quad \inf_{0 \neq x \in R} m\left(\frac{\alpha}{\|x\|}x\right) > 0 \quad \text{for some } 0 < \alpha < 1,$$

7) A modular m is said to be uniformly simple, if $\inf_{m(x) \geq 1} m(\xi x) > 0$ for all $\xi > 0$, that is, $\lim_{\nu \rightarrow \infty} m(a_\nu) = 0$ implies $\lim_{\nu \rightarrow \infty} \|a_\nu\| = 0$.

8) A modular m on R is said to be simple, if $m(a) = 0$ implies $a = 0$.

$$(2): \quad \sup_{0 \neq x \in R} m\left(\frac{\alpha}{|||x|||} x\right) > 0 \quad \text{for some } \alpha \geq 1.$$

Proof. (1): We prove first that the condition:

$$\inf_{0 \neq x \in R} m\left(\frac{1-\varepsilon}{|||x|||} x\right) = \xi > 0 \quad \text{for some } 1 > \varepsilon > 0$$

implies the condition:

$$\inf_{0 \neq \bar{x} \in \bar{R}^m} \bar{m}\left(\frac{1-\varepsilon'}{|||\bar{x}|||} \bar{x}\right) > 0 \quad \text{for some } \varepsilon > \varepsilon' > 0.$$

For $\bar{x} \in \bar{R}^m$ with $|||\bar{x}||| = 1$ there exists $x_\lambda \in R$ ($\lambda \in A$) such that $x_\lambda \uparrow_{\lambda \in A} \bar{x}$ (cf. [5, Theorem 5.34]), because R is a complete semi-normal manifold of $\bar{R}^m$ by Lemma 1. Since the modular norm is semi-continuous and reflexive (cf. [3]), we obtain $|||x_\lambda||| \uparrow_{\lambda \in A} |||\bar{x}|||$, and hence we have

$$\left(1 - \frac{\varepsilon}{2}\right) |||x_\lambda||| \uparrow_{\lambda \in A} \left(1 - \frac{\varepsilon}{2}\right).$$

Consequently there exists λ_0 such that $\left(1 - \frac{\varepsilon}{2}\right) |||x_\lambda||| \geq 1 - \varepsilon$ for $\lambda \geq \lambda_0$.

If $\inf_{0 \neq x \in R} m\left(\frac{1-\varepsilon}{|||x|||} x\right) = \xi > 0$, we obtain easily $m(x) \geq \xi$ for every x such that $|||x||| \geq 1 - \varepsilon$, thus we have obviously $m\left(\left(1 - \frac{\varepsilon}{2}\right)x_\lambda\right) \geq \xi$ for $\lambda \geq \lambda_0$.

Therefore we have
$$\inf_{0 \neq \bar{x} \in \bar{R}^m} \bar{m}\left(\frac{1-\frac{\varepsilon}{2}}{|||\bar{x}|||} \bar{x}\right) > 0.$$

Therefore, we obtain $|||\bar{a}|||$ ($\bar{a} \in \bar{R}^m$) is continuous by Remark 1.1, and, since $\bar{R}^m$ is non-atomic, $\bar{m}$ is finite on $\bar{R}^m$ by Remark 1.1. As $\bar{m}$ is monotone complete, we obtain $\bar{m}$ is semi-upper bounded by Theorem 1.1, and hence we obtain finally that m is semi-upper bounded by Lemma 1.

The proof for the condition (2) is similar.

Q.E.D.

§2. Let R be a modular semi-ordered linear space with a modular m and be semi-regular. In this section, our aim is to consider the relations between properties of a modular or the modular norms and its semi-lower boundedness.

A modular m on R is said to be “semi-lower bounded” if for every $\varepsilon > 0$, there exist $1 < \alpha = \alpha(\varepsilon) < \gamma(\varepsilon) = \gamma$ such that $m(x) \geq \varepsilon$ implies $m(\alpha x) \geq \gamma m(x)$.

Theorem 2.1. *If a modular m is semi-upper bounded and semi-simple, then the conjugate modular $\bar{m}$ of m is semi-lower bounded.*

Proof. Since the case $\bar{m}(\bar{a}) = +\infty$ is trivial, we can assume that $\bar{m}(\bar{a}) < +\infty$. For every $\varepsilon > 0$ there exists $\gamma = \gamma(\varepsilon) > 0$ such that $m(x) \geq \frac{\varepsilon}{3}$ implies $m(2x) \leq \gamma m(x)$, by assumption. Then we have definition

$$\begin{aligned} \bar{m}\left(\frac{\gamma}{2}\bar{a}\right) &= \sup_{x \in R} \left\{ \frac{\gamma}{2}\bar{a}(2x) - m(2x) \right\} \geq \sup_{m(x) \geq \frac{\varepsilon}{3}} \left\{ \frac{\gamma}{2}\bar{a}(2x) - m(2x) \right\} \\ &\geq \gamma \sup_{m(x) \geq \frac{\varepsilon}{3}} \{ \bar{a}(x) - m(x) \} \quad (\bar{a} \in \bar{R}^m). \end{aligned}$$

For every $0 \leq \bar{a} \in \bar{R}^m$ such that $\varepsilon \leq \bar{m}(\bar{a}) < +\infty$, we have to consider the case

$$\bar{m}(\bar{a}) = \sup_{m(x) < \frac{\varepsilon}{3}} \{ \bar{a}(x) - m(x) \}.$$

For any $\delta > 0$ there exists $x \in R$ such that $m(x) < \frac{\varepsilon}{3}$ and $\bar{a}(x) - m(x) \geq \bar{m}(\bar{a}) - \delta$.

Since m is uniformly finite by Remark 1.2 there exists $\beta = \beta(a) > 1$ such that

$$m(\beta x) = \frac{\varepsilon}{3}.$$

Therefore we obtain $\bar{a}(\beta x) - m(\beta x) \geq \bar{a}(x) - m(x) - m(\beta x) \geq \bar{m}(\bar{a}) - \delta - \frac{\varepsilon}{3}$.

Thus we have

$$\gamma \sup_{m(x) \geq \frac{\varepsilon}{3}} \{ \bar{a}(x) - m(x) \} \geq \gamma \left(\bar{m}(\bar{a}) - \frac{\varepsilon}{3} \right) \geq \gamma \left(\bar{m}(\bar{a}) - \frac{\bar{m}(\bar{a})}{3} \right) = \frac{2}{3} \gamma \bar{m}(\bar{a}),$$

and hence $\bar{m}\left(\frac{\gamma}{2}\bar{a}\right) \geq \frac{2}{3} \gamma \bar{m}(\bar{a})$ for every $\bar{a}$ such that $\bar{m}(\bar{a}) \geq \varepsilon$. Q.E.D.

Theorem 2.2. *If a modular m is semi-lower bounded, then $\bar{m}$ is semi-upper bounded.*

Proof. If for every $\varepsilon > 0$ there exist $\gamma > \alpha > 1$ such that $m(x) \geq \varepsilon$ implies $m(\alpha x) \geq \gamma m(x)$, then we have by definition

$$\begin{aligned} \bar{m}\left(\frac{\gamma}{\alpha}\bar{a}\right) &= \sup_{x \in R} \left\{ \frac{\gamma}{\alpha}\bar{a}(\alpha x) - m(\alpha x) \right\} \leq \gamma \sup_{m(x) \geq \varepsilon} \{ \bar{a}(x) - m(x) \} + \sup_{m(x) < \varepsilon} \{ \gamma \bar{a}(x) - m(\alpha x) \} \\ &\leq \gamma \bar{m}(\bar{a}) + \gamma \sup_{m(x) < \varepsilon} \{ \bar{a}(x) \} \leq \gamma \bar{m}(\bar{a}) + \gamma \sup_{m(x) < \varepsilon} \{ \bar{m}(\bar{a}) + m(x) \} \\ &\leq \gamma \bar{m}(\bar{a}) + \gamma (\bar{m}(\bar{a}) + \varepsilon) = \gamma \{ 2\bar{m}(\bar{a}) + \varepsilon \}, \end{aligned}$$

since by definition $|\bar{a}(x)| \leq \bar{m}(\bar{a}) + m(x)$ for $\bar{a} \in \bar{R}^m$, $x \in R$.

Thus we have $\bar{m}\left(\frac{\gamma}{\alpha}\bar{a}\right) \leq 3\gamma \bar{m}(\bar{a})$ for every $\bar{a}$ such that $\bar{m}(\bar{a}) \geq \varepsilon$. Q.E.D.

The “conjugate” of “uniformly finite” is “uniformly increasing”,
 i.e.
$$\lim_{\xi \rightarrow \infty} \inf_{m(x) \geq 1} \frac{m(\xi x)}{\xi} = +\infty \quad (\text{cf. [5, §48]}).$$

Theorem 2.3. *If a modular m is semi-lower bounded, then m is uniformly increasing.*

Proof. By assumption there exist $1 < \alpha < \gamma$ such that $m(x) \geq 1$ implies $m(\alpha^\nu x) \geq \gamma^\nu m(x)$ ($\nu = 1, 2, \dots$).

Therefore we obtain $\frac{1}{\alpha^\nu} m(\alpha^\nu x) \geq \left(\frac{\gamma}{\alpha}\right)^\nu m(x)$ ($\nu = 1, 2, \dots$) for every x such that $m(x) \geq 1$, and consequently m is uniformly increasing. Q.E.D.

Since the “conjugate” of “finitely monotone” is “finitely flat”, i.e. for every $\gamma > 0$ there exists $\varepsilon = \varepsilon(\gamma)$ such that

$$x = \bigoplus_{i=1}^n x_i, \quad \|x\| \geq 1, \quad \|x_i\| \leq \varepsilon \quad (i=1, 2, \dots, n)$$

implies $n \geq \frac{\gamma}{\varepsilon} \|x\|$ (cf. [9, §1]), we have immediately by Corollary 1 of Theorem 1.1, Theorem 2.1 and Lemma 1 the following

Theorem 2.4. *Suppose that R has no atomic element. If the modular norm $|||\cdot|||$ by m is finitely flat, then m is semi-lower bounded.*

Remark 2.1. *If a modular m is uniformly increasing, then the modular norm is finitely flat. The converse of this is valid, if we suppose that R has no atomic element (cf. [9]).*

A norm $\|\cdot\|$ on R is said to be “flat”, if for any $a \neq 0$, $a \wedge b = 0$ we have

$$\lim_{\xi \rightarrow 0} \frac{\|a + \xi b\| - \|a\|}{\xi} = 0.$$

The “conjugate” of “uniformly simple” is “uniformly monotone”,
 i.e.
$$\lim_{\xi \rightarrow 0} \frac{1}{\xi} \sup_{m(x) \leq 1} m(\xi x) = 0 \quad (\text{cf. [5, §48]}).$$

Theorem 2.5. *If the first norm $\|\cdot\|$ by m is flat and the first norm $|||\cdot|||$ by conjugate modular $\bar{m}$ of m is continuous, then m is uniformly monotone, and m is semi-lower bounded when R has no atomic element.*

Proof. Using Banach’s theorem (cf. [6, §44]) and reflexivity of the norm $\|\cdot\|$, we can prove that flatness of $\|\cdot\|$ implies monotony of $|||\cdot|||$. Thus we have $\bar{m}$ is simple by (ii) in proof of Corollary 2 of Theorem 1.1. Since $||\bar{a}||$ is continuous by assumption and $\bar{m}$ is monotone complete, we obtain $\bar{m}$ is uniformly simple ([11, Theorem 2.1]). Thus m is uniformly monotone.

On the other hand, if m is uniformly monotone then m is uniformly increasing when R has no atomic element ([10, Theorem 1.3]). By Theorem 2.4 and Remark 2.1 the proof is completed. Q.E.D.

A manifold K of R is said to be “*equi-continuous*”, if for any $\bar{a}_\nu \downarrow_{\nu=1}^\infty 0$, $\bar{a}_\nu \in \bar{R}^m$ and $\varepsilon > 0$ there exists ν_0 for which we have $\bar{a}_{\nu_0}(x) \leq \varepsilon$ for all $x \in K$.

Theorem 2.6. *If a modular m is semi-lower bounded, then a manifold $K = \{x : m(x) \leq 1, x \in R\}$ is equi-continuous. The converse of this is true, if we suppose that R has no atomic element.*

Proof. If m is semi-lower bounded, m is uniformly increasing by Theorem 2.3. Then we have $\bar{m}$ is uniformly finite, and hence the conjugate norm of the modular norm by m is continuous by Remark 1.1. Therefore we obtain for any $\varepsilon > 0$ and $\bar{R}^m \ni \bar{a}_\nu \downarrow_{\nu=1}^\infty 0$ there exists ν_0 such that $\bar{a}_{\nu_0}(x) \leq \varepsilon$ for all $x \in K$ ([5, Theorem 31.12]). That is, K is equi-continuous.

Conversely we suppose that R has no atomic element and the manifold $K = \{x : m(x) \leq 1\}$ is equi-continuous. Since we have obviously by definition $\{x : |||x||| \leq 1\} = \{x : m(x) \leq 1\}$, the first norm by $\bar{m}$ is continuous ([5, Theorem 31.12]). Thus we obtain $\bar{m}$ is monotone complete and finite, because $\bar{R}^m$ is non-atomic by assumption. Thus we have $\bar{m}$ is semi-upper bounded by Theorem 1.1, therefore we obtain by Theorem 2.1 and Lemma 1 m is semi-lower bounded. Q.E.D.

A manifold K of R is said to be “*weakly bounded*”, if

$$\sup_{x \in K} |\bar{a}(x)| < +\infty \quad \text{for all } \bar{a} \in \bar{R}^m.$$

Theorem 2.7. *If a modular m is semi-lower bounded, then every weakly bounded manifold is equi-continuous. The converse of this is truth, if we suppose that R has no atomic element.*

Proof. If m is semi-lower bounded, the conjugate norm of a norm by m is continuous. Consequently every manifold K for which $\sup_{x \in K} ||x|| < +\infty$ is equi-continuous ([5, Theorem 33.10]). Therefore we have $\sup_{x \in K} |\bar{a}(x)| \leq \sup_{x \in K} |||\bar{a}||| \cdot ||x||$ for all $\bar{a} \in \bar{R}^m$, and hence K is weakly bounded by definition.

Conversely we suppose that R has no atomic element. Since the norm $|||\cdot|||$ is reflexive (cf. [3]), if a manifold K is weakly bounded, then K is norm bounded, i.e. $\sup_{x \in K} |||x||| < +\infty$ ([5, Theorem 32.6]), and equi-continuous by assumption. Then the first norm by the conjugate modular $\bar{m}$ of m is continuous ([5, Theorem 33.10]). Thus we have obviously our conclusion by the method applied to Theorem 2.6. Q.E.D.

Theorem 2.7. Suppose that R has no atomic element. Each of the following conditions implies m is semi-lower bounded

$$(1) \quad \inf_{0 \neq x \in R} \frac{1}{r} m\left(\frac{r}{|||x|||} x\right) \geq 1 + \delta \quad \text{for some } r, \delta > 0,$$

$$(2) \quad \sup_{0 \neq x \in R} m\left(\frac{x}{||x||}\right) < 1.$$

Proof. (1) For every $\bar{a} \in \bar{R}^m$ with $||\bar{a}|| = 1$, we have

$$(1 + \delta)\bar{a}(\xi a) - m(\xi a) \leq \xi(1 + \delta) - \xi(1 + \delta) = 0$$

for every $a \in R$, $|||a||| = 1$ and $\xi \geq r$.

Thus we have $\bar{m}((1 + \delta)\bar{a}) = \sup_{|||x||| \leq r} \{(1 + \delta)\bar{a}(x) - m(x)\} \leq r(1 + \delta)$.

Suppose that $\bar{R}^m \ni \bar{a}_\nu \downarrow_{\nu=1}^\infty 0$ and $\inf_{\nu \geq 1} ||\bar{a}_\nu|| = \alpha > 0$,

then there exist $\varepsilon_0 > 0$, ν_0 such that

$$\left\| \frac{\bar{a}_\nu}{\alpha - \varepsilon_0} \right\| \leq 1 + \delta \quad \text{for every } \nu \geq \nu_0.$$

Since we have $1 + \bar{m}\left(\frac{\bar{a}_\nu}{\alpha - \varepsilon_0}\right) \geq \left\| \frac{\bar{a}_\nu}{\alpha - \varepsilon_0} \right\| \geq \frac{\alpha}{\alpha - \varepsilon_0}$ for every $\nu \geq \nu_0$.

we obtain $1 + \lim_{\nu \rightarrow \infty} \bar{m}\left(\frac{\bar{a}_\nu}{\alpha - \varepsilon_0}\right) \geq \frac{\alpha}{\alpha - \varepsilon_0} > 1$.

Since $\lim_{\nu \rightarrow \infty} \bar{m}\left(\frac{\bar{a}_\nu}{\alpha - \varepsilon_0}\right) = 0$, this is a contradiction.

Therefore $||\bar{a}||$ is continuous. Thus we have our conclusion by the method applied to Theorem 2.6.

The proof for the condition (2) is similar.

Q.E.D.

§3. Let R be a modular semi-ordered linear space with a semi-simple modular m . In this section, we express the properties of a semi-upper and semi-lower bounded modulars.

If a modular m is semi-upper and semi-lower bounded, then m is said to be “semi-bounded”.

Lemma 3.1. Suppose that R has no atomic element. If the norms by a modular m have the property:

$\inf_{0 \neq x \in R} \frac{||x||}{|||x|||} = r$, where $r > 1$ is a fixed constant, then m is semi-bounded.

Proof. We have m is uniformly finite and uniformly increasing by the assumption (cf. [10, Theorem 1.1]). Therefore we obtain our conclu-

sion by Corollary 1 of Theorem 1.1 and Theorem 2.4.

Q.E.D.

Lemma 3.2. *If a modular m is semi-bounded, then the norms by m have the property:*

$$\inf_{0 \neq x \in R} \frac{\|x\|}{\|x\|} = \gamma \quad \text{for some } \gamma > 1.$$

Proof. Since m is uniformly finite and uniformly increasing by Remark 1.2 and Theorem 2.3, we have our conclusion (cf. [10, Theorem 1.4]).

Q.E.D.

From these Lemmata, we obtain the following theorem

Theorem 3.1. *Suppose that R has no atomic element. A modular is semi-bounded, if and only if the norms by the modular have the property:*

$$\inf_{0 \neq x \in R} \frac{\|x\|}{\|x\|} = \gamma \quad \text{for some } \gamma > 1.$$

In the case when a modular m on R is of unique spectra (cf. [5, §54]), semi-boundedness of m implies boundedness⁹⁾ of m . In fact we have

Theorem 3.2. *If a modular m on R is of unique spectra¹⁰⁾, then semi-boundedness of m is equivalent to boundedness of m .*

Proof. If m is semi-bounded, then m is uniformly finite and uniformly increasing by Remark 1.2 and Theorem 2.3. Therefore m has the upper exponent¹⁰⁾ ρ_u and the lower exponent¹⁰⁾ ρ_l such that $1 \leq \rho_l \leq \rho_u < +\infty$ (cf. [5, Theorems 54.8, 54.10]). Thus m is bounded ([5, Theorems 54.4, 54.5]).

Q.E.D.

A modular m of unique spectra is uniformly convex¹⁰⁾ (or uniformly even¹⁰⁾) if and only if $1 < \rho_l \leq \rho_u < +\infty$ for the upper exponent ρ_u and the lower exponent ρ_l (cf. [5, §50, §54]). Therefore we obtain also:

Theorem 3.3. *A modular m of unique spectra is uniformly convex (or uniformly even), if and only if m is semi-bounded.*

Theorem 3.4. *Suppose that R has no atomic element. If a modular m is uniformly convex (or uniformly even), then m is semi-bounded.*

Proof. Let m be uniformly convex. Then m is uniformly simple ([5, Theorem 50.1]). Since R is non-atomic by assumption, m and $\bar{m}$ are uniformly finite ([10, Theorem 1.2]), and hence m and $\bar{m}$ are semi-upper

9) A modular m on R is said to be upper bounded, if there exist $\omega, \gamma > 1$, for which we have $m(\omega x) \leq \gamma m(x)$ for all $x \in R$; and m is said to be lower bounded, if there exist $\gamma > \omega > 1$ such that $m(\omega x) \geq \gamma m(x)$ for all $x \in R$; if a modular m is upper and lower bounded, then m is said to be bounded.

10) For the definitions see [5].

bounded by Corollary 1 of Theorem 1.1. Thus m is semi-bounded.

Let m be uniformly even. Then m is uniformly finite and uniformly monotone ([5, Theorems 51.1, 51.2]), and hence m is semi-bounded by Corollary 1 of Theorem 1.1 and Theorem 2.4. Q.E.D.

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