HOKKAIDO UNIVERSITY

Title	ON GROUPS OF ROTATIONS IN MINKOWSKI SPACE II
Author(s)	Nagai, Tamao
Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 15(1-2), 046-061
Issue Date	1960
Doc URL	https://hdl.handle.net/2115/56006
Type	departmental bulletin paper
File Information	JFSHIU_15_N1-2_046-061.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

ON GROUPS OF ROTATIONS IN MINKOWSKI SPACE II

By

Tamao NAGAI

§1. **Introduction.** The present paper is a continuation of the one with the same title "On groups of rotations in Minkowski space I." [7][1] Let M_{n+1} be an $n+1$-dimensional Minkowski space whose indicatrix I_n is defined by $F(X^1, X^2, \cdots, X^{n+1})=1$. As I_n is an n-dimensional hypersurface in M_{n+1}, we may represent I_n by $n+1$ equations involving n parameters as

$$X^i = X^i(u^\alpha). \quad (i=1,2,\cdots,n+1;\ \alpha=1,2,\cdots,n)^{2)}$$

Putting $A_{ijk}(X)=F(X)C_{ijk}$, where $C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial X^k}$ and $g_{ij}=\frac{1}{2}\partial^2 F/\partial X^i \partial X^j$, A_{ijk} is a symmetric covariant tensor in M_{n+1}. If we denote by $g_{\alpha\beta}$ and $A_{\alpha\beta\gamma}$ the induced components of g_{ij} and A_{ijk} respectively, i.e.

$$g_{\alpha\beta}=g_{ij}X^i_\alpha X^j_\beta, \quad A_{\alpha\beta\gamma}=A_{ijk}X^i_\alpha X^j_\beta X^k_\gamma \quad (X^i_\alpha=\partial X^i/\partial u^\alpha),$$

I_n may be considered as an n-dimensional compact Riemannian space with the fundamental metric tensor $g_{\alpha\beta}$, and the Riemannian space is characterized by existence of a symmetric covariant tensor $A_{\alpha\beta\gamma}$. Moreover, it is remarkable that in I_n we have

$$R_{\alpha\beta\gamma\delta}=S_{\alpha\beta\gamma\delta}+(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}), \tag{1.1}$$

where $R_{\alpha\beta\gamma\delta}=g_{\alpha\varepsilon}R^{\varepsilon}_{\cdot\beta\gamma\delta}$ and

$$R^{\alpha}_{\cdot\beta\gamma\delta}=\frac{\partial\left\{{\alpha \atop \beta\delta}\right\}}{\partial u^\gamma}-\frac{\partial\left\{{\alpha \atop \beta\gamma}\right\}}{\partial u^\delta}+\left\{{\varepsilon \atop \beta\delta}\right\}\left\{{\alpha \atop \varepsilon\gamma}\right\}-\left\{{\varepsilon \atop \beta\gamma}\right\}\left\{{\alpha \atop \varepsilon\delta}\right\},$$

$$S_{\alpha\beta\gamma\delta}=A_{\alpha\delta}^{\cdot\cdot\sigma}A_{\sigma\gamma\beta}-A_{\alpha\gamma}^{\cdot\cdot\sigma}A_{\sigma\delta\beta}.$$

Also, $A_{\alpha\beta\gamma}$ satisfies the relation $A_{\alpha\beta\gamma;\delta}=A_{\alpha\beta\delta;\gamma}$ [6].[3]

When a centro-affine transformation in M_{n+1}, with its centre at origin, preserves I_n, the transformation is called a *rotation* in M_{n+1}.

1) Numbers in brackets refer to the references at the end of the paper.

2) Throughout the present paper, the Greek indices $\alpha, \beta, \gamma, \cdots$ are supposed to run over the range $1, 2, \cdots, n$.

3) Semi-colon is used to represent the covariant differentiation with respect to the Christoffel symbols made by $g_{\alpha\beta}$.

By this transformation every point on I_n is transformed into a point on the same. When an infinitesimal point transformation on I_n coincides with the transformation of such a class, we call it an *infinitesimal rotation on* I_n [7]. In the previous paper, the present author derived the fundamental equation of an infinitesimal rotation on I_n, i.e.

$$\mathfrak{L} g_{\alpha\beta}=\eta_{\alpha;\beta}+\eta_{\beta;\alpha}=2\eta^{\gamma}A_{\alpha\beta\gamma}\,, \tag{1.2}$$

and discussed about the integrability conditions of the transformation, where $\mathfrak{L}$ denotes Lie differential with respect to an infinitesimal transformation $\bar{u}^{\alpha}=u^{\alpha}+\eta^{\alpha}(u)\delta t$.

The purpose of the present paper is to study the properties of infinitesimal rotations on I_n and develop the structures of r-parameter Lie groups of the transformations, mainly in connection with the properties of the symmetric tensor $A_{\alpha\beta\gamma}$.

In §2 we shall state a few remarks on the connections between properties of I_n and those of the tensor $A_{\alpha\beta\gamma}$. §3 contains theorems concerning with properties of r-parameter groups of infinitesimal rotations on I_n. In §4 we shall discuss about the special cases in which the infinitesimal rotation on I_n coincides with a motion or transformation of other classes. §5 devoted to give the order of the groups when the infinitesimal rotations on I_n preserve the tensor $A_{\alpha\beta\gamma}$.

The present author wishes to express his sincere thanks to Dr. A. Kawaguchi and Dr. Y. Katsurada for their constant guidances and criticisms, and also thanks to Mr. T. Sumitomo who gave the author many valuable suggestions.

§2. Preliminary remarks on the structures of I_n. Let us denote by $\|A_{\alpha\beta\gamma}\|$ the matrix with elements $A_{\alpha\beta\gamma}$, where γ denotes the columns, and α and β the rows. If we consider about the rank of the matrix $\|A_{\alpha\beta\gamma}\|$, the followings are exceptional cases in our discussions.

Case I. Rank $\|A_{\alpha\beta\gamma}\|=0$. In this case, at every point on I_n, $A_{\alpha\beta\gamma}=0$ for every α, β and γ. Therefore, as we have $A_{ijk}=A_{\alpha\beta\gamma}X_i^{\alpha}X_j^{\beta}X_k^{\gamma}$ [6], it follows that $A_{ijk}=0$ and this means that the considered Minkowski space is essentially a Euclidean space.

Case II. Rank $\|A_{\alpha\beta\gamma}\|=1$. In this case, we have

$$\text{det.}\begin{vmatrix} A_{\alpha\beta\tau} & A_{\alpha\beta\omega} \\ A_{\gamma\delta\tau} & A_{\gamma\delta\omega}\end{vmatrix}=0\,. \quad (\alpha, \beta, \gamma, \delta, \tau, \omega=1, 2,\cdots, n) \tag{2.1}$$

On the other hand, by means of the symmetric properties of $A_{\alpha\beta\gamma}$, it is readily seen that

$$S_{\alpha\beta\gamma\delta}=A_{\alpha\delta}^{\cdot\cdot\sigma}A_{\sigma\gamma\beta}-A_{\alpha\gamma}^{\cdot\cdot\sigma}A_{\sigma\delta\beta}=g^{\sigma\tau}(A_{\alpha\tau\delta}A_{\beta\sigma\gamma}-A_{\alpha\tau\gamma}A_{\beta\sigma\delta})\,. \tag{2.2}$$

Comparing (2.1) and (2.2), we find that $S_{\alpha\beta\gamma\delta}=0$. This also implies that the considered Minkowski space is essentially a Euclidean space [6].

According to the definition, as a rotation in M_{n+1} leaves invariant the indicatrix I_n, the transformation preserves the metric of M_{n+1}. However, as we see from the fundamental equation (1.2), in general an infinitesimal rotation on I_n does not preserve the metric of I_n, which is induced from the one of M_{n+1}. If an infinitesimal rotation $\bar{u}^\alpha=u^\alpha+\eta^\alpha(u)\delta t$ coincides with a motion in I_n, the generating vector η^α should satisfy the equation $\eta^\gamma A_{\alpha\beta\gamma}=0$. Therefore if an infinitesimal rotation coincides with a motion in I_n, the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than the maximum value n, and otherwise the transformation must be an identity transformation.

Theorem 2.1. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is equal to the maximum value n, an infinitesimal rotation on I_n does not coincide with a motion in I_n, except an identity transformation.*

§ 3. Groups of infinitesimal rotations on I_n.

Let us denote by G_r the r-parameter group of infinitesimal rotations on I_n and by $X_a f\equiv\eta^\alpha_{(a)}\dfrac{\partial f}{\partial u^\alpha}$ $(a=1, 2,\cdots, r)$[4] its r linearly independent symbols. Then by means of the fundamental equations (1.2), we get

$$\mathfrak{L}_a g_{\alpha\beta}=\eta_{(a)\alpha;\beta}+\eta_{(a)\beta;\alpha}=2\eta^\gamma_{(a)}A_{\alpha\beta\gamma}\,, \tag{3.1}$$

where $\mathfrak{L}_a$ denotes Lie differentiation with respect to an infinitesimal transformation generated by the vector $\eta^\alpha_{(a)}$.

On the other hand, from the theory of Lie derivatives, we have

$$\mathfrak{L}_a\eta^\alpha_{(b)} = c_{ab}^{\,c}\eta^\alpha_{(c)}\,, \tag{3.2}$$

$$(\mathfrak{L}_a, \mathfrak{L}_b)T^{\cdot\lambda}_{\cdot\mu\nu}\equiv(\mathfrak{L}_a\mathfrak{L}_b-\mathfrak{L}_b\mathfrak{L}_a)T^{\cdot\lambda}_{\cdot\mu\nu}=c_{ab}^{\,c}\mathfrak{L}_c T^{\lambda}_{\mu\nu}\,, \tag{3.3}$$

where $T^{\cdot\lambda}_{\cdot\mu\nu}$ is an arbitrary tensor and $c_{ab}^{\,c}$ is a structural constant of G_r and satisfies

$$c_{ab}^{\,e}=-c_{ba}^{\,e}\,,\qquad c_{ab}^{\,e}c_{ec}^{\,f}+c_{bc}^{\,e}c_{ea}^{\,f}+c_{ca}^{\,e}c_{eb}^{\,f}=0\,.$$

Applying the formula (3.3) to the fundamental tensor $g_{\alpha\beta}$, we get

$$(\mathfrak{L}_b, \mathfrak{L}_c)g_{\alpha\beta} = c_{bc}^{\,a}\mathfrak{L}_a g_{\alpha\beta}\,. \tag{3.4}$$

---

4) Throughout the present paper the Latin indices $a, b, c,\cdots$ take the values $1, 2,\cdots, r$.

Substituting (3.1) and (3.2) into the left hand side of (3.4), it follows that

$$(\mathfrak{L}_b, \mathfrak{L}_c)g_{\alpha\beta}=2\mathfrak{L}_b(\eta^\gamma_{(c)}A_{\alpha\beta\gamma})-2\mathfrak{L}_c(\eta^\gamma_{(b)}A_{\alpha\beta\gamma})$$
$$=4c^a_{bc}\eta^\gamma_{(a)}A_{\alpha\beta\gamma}+2(\eta^\gamma_{(c)}\mathfrak{L}_bA_{\alpha\beta\gamma}-\eta^\gamma_{(b)}\mathfrak{L}_cA_{\alpha\beta\gamma}).$$

Since we have $c^a_{bc}\mathfrak{L}_a g_{\alpha\beta}=2c^a_{bc}\eta^\gamma_{(a)}A_{\alpha\beta\gamma}$, in consequence of the above equation, (3.4) gives rise to the relation

$$2c^a_{bc}\eta^\gamma_{(a)}A_{\alpha\beta\gamma}+2(\eta^\gamma_{(c)}\mathfrak{L}_bA_{\alpha\beta\gamma}-\eta^\gamma_{(b)}\mathfrak{L}_cA_{\alpha\beta\gamma})=0. \tag{3.5}$$

If X_af $(a=1, 2,\cdots, r)$ are r generators of G_r, it is satisfied that

$$(X_b, X_c)f\equiv X_bX_cf-X_cX_bf=c^a_{bc}X_af.$$

Now, consider the symbols $X_{bc}f$ $(b, c=1, 2,\cdots, r)$ defined by $X_{bc}f=(X_b, X_c)f$, then each of them determines a subgroup G_1 of G_r and its generating vector is $c^a_{bc}\eta^\alpha_{(a)}$. Since c^a_{bc} is constant, it is evident that

$$\mathfrak{L}_{bc}g_{\alpha\beta}=c^a_{bc}\eta_{(a)\alpha;\beta}+c^a_{bc}\eta_{(a)\beta;\alpha}=2c^a_{bc}\eta^\gamma_{(a)}A_{\alpha\beta\gamma}, \tag{3.6}$$

where $\mathfrak{L}_{bc}$ denotes Lie differentiation with respect to an infinitesimal transformation determined by a vector $c^a_{bc}\eta^\alpha_{(a)}$. By means of (3.5) and (3.6), if each transformation of G_r leaves invariant the tensor $A_{\alpha\beta\gamma}$, namely $\mathfrak{L}_aA_{\alpha\beta\gamma}=0$ $(a=1, 2,\cdots, r)$, we must have $\mathfrak{L}_{bc}g_{\alpha\beta}=0$. Hence we have the following

Theorem 3.1. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than n and each one-parameter group G_1, determined by r generators X_af, leaves invariant the covariant tensor $A_{\alpha\beta\gamma}$, then $X_{bc}f$ are symbols of one-parameter groups of motions on I_n.*

It is well known that, as it follows that

$$(X_{ab}, X_{cd})f=(c^e_{ab}X_e, c^f_{cd}X_f)f=c^e_{ab}c^h_{cd}X_{eh}f,$$

symbols $X_{ab}f$ determine a group which is either G_r itself, or an invariant subgroup of G_r called the derived group of G_r. The order of the derived group is determined by the rank of the matrix $\|c^e_{ab}\|$ where e denotes the columns, and a and b the rows. Specially if the rank of the matrix $\|c^e_{ab}\|$ is equal to r, the derived group coincides with the given group G_r. Then, in consequence of Theorem 3.1 we have

Theorem 3.2. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than n and an r-parameter group G_r of infinitesimal rotations on I_n leaves invariant the covariant tensor $A_{\alpha\beta\gamma}$, the derived group is a subgroup of motions on I_n, and G_r itself is a group of motions if the rank of the matrix $\|c^e_{ab}\|$ is equal to r.*

Since Lie derivatives of $g_{\alpha\beta}$ do not vanish for an infinitesimal rotation on I_n, relations $\mathfrak{L}A_{\alpha\beta\gamma}=0$, $\mathfrak{L}A^{\alpha}_{\cdot\beta\gamma}=0$, $\mathfrak{L}A^{\alpha\beta}_{\cdot\cdot\gamma}=0$ and $\mathfrak{L}A^{\alpha\beta\gamma}=0$ are not equivalent each other. However, in consequence of direct calculations, we can obtain the following analogous results concerning the mixed type of tensors $A^{\alpha}_{\cdot\beta\gamma}$, $A^{\alpha\beta}_{\cdot\cdot\gamma}$ and the contravariant tensor $A^{\alpha\beta\gamma}$.

Theorem 3.3. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than n and an r-parameter group G_r of infinitesimal rotations on I_n leaves invariant the tensor $A^{\alpha}_{\cdot\beta\gamma}$, the derived group is a subgroup of motions on I_n, and itself is a group of motions if rank the of the matrix $\|c^{e}_{ab}\|$ is equal to r.*

Proof. As r vectors $\eta^{\alpha}_{(a)}$ $(a=1, 2,\cdots, r)$ generate infinitesimal rotations on I_n, we have the following equation which is equivalent to the fundamental equation (3.1):

$$\mathfrak{L}_a g_{\alpha\beta} = 2\eta_{(a)\gamma}A^{\gamma}_{\cdot\alpha\beta}\,. \tag{3.7}$$

Therefore, we may put the right hand side of (3.4) as follows:

$$c^{a}_{bc}\mathfrak{L}_a g_{\alpha\beta} = 2c^{a}_{bc}\eta_{(a)\gamma}A^{\gamma}_{\cdot\alpha\beta}\,. \tag{3.8}$$

On the other hand, if we assume that $\mathfrak{L}_a A^{\alpha}_{\cdot\beta\gamma}=0$ $(a=1, 2,\cdots, r)$, by means of the symmetric property of $A_{\alpha\beta\gamma}$ and (3.1), the left member of (3.4) is reduced as follows:

$$\begin{aligned}(\mathfrak{L}_b, \mathfrak{L}_c)g_{\alpha\beta}&=\mathfrak{L}_b(2\eta_{(c)\gamma}A^{\gamma}_{\cdot\alpha\beta})-\mathfrak{L}_c(2\eta_{(b)\gamma}A^{\gamma}_{\cdot\alpha\beta})\\ &=2(\eta_{(c)\gamma;\varepsilon}\eta^{\varepsilon}_{(b)}-\eta_{(c)\varepsilon;\gamma}\eta^{\varepsilon}_{(b)}+\eta^{\varepsilon}_{(c)}\eta_{(b)\varepsilon;\gamma}-\eta^{\varepsilon}_{(c)}\eta_{(b)\gamma;\varepsilon})A^{\gamma}_{\cdot\alpha\beta}\\ &=4(\eta_{(c)\gamma;\varepsilon}\eta^{\varepsilon}_{(b)}-\eta^{\varepsilon}_{(c)}\eta_{(b)\gamma;\varepsilon})A^{\gamma}_{\cdot\alpha\beta}\,.\end{aligned} \tag{3.9}$$

However, as r vectors $\eta^{\alpha}_{(a)}$ $(a=1, 2,\cdots, r)$ satisfy the Maurer-Cartan equation

$$\eta^{\varepsilon}_{(b)}\frac{\partial\eta^{\gamma}_{(c)}}{\partial u^{\varepsilon}}-\eta^{\varepsilon}_{(c)}\frac{\partial\eta^{\gamma}_{(b)}}{\partial u^{\varepsilon}}=c^{a}_{bc}\eta^{\gamma}_{(a)}\,,$$

we obtain from (3.9)

$$(\mathfrak{L}_b, \mathfrak{L}_c)g_{\alpha\beta}=4c^{a}_{bc}\eta_{(a)\gamma}A^{\gamma}_{\cdot\alpha\beta}\,. \tag{3.10}$$

Making use of (3.4), (3.8) and (3.10), it follows that

$$\mathfrak{L}_{bc}g_{\alpha\beta}=2c^{a}_{bc}\eta^{\gamma}_{(a)}A_{\alpha\beta\gamma}=0\,.$$

The last equation implies the result of Theorem 3.3.

Theorem 3.4. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than n and an r-parameter group G_r of infinitesimal rotations on I_n leaves invariant the tensor $A^{\alpha\beta}_{\cdot\cdot\gamma}$, the derived group is a subgroup of motions on I_n, and itself is a group of motions if the rank of the matrix $\|c^{e}_{ab}\|$ is equal to r.*

Proof. Multiplying the fundamental equation (3.1) by $g^{\alpha\delta}$ and summing for α, we have

$$(3.11) \qquad g^{\alpha\delta}(\mathfrak{L}_a g_{\alpha\beta}) = 2\eta_{(a)\gamma} A^{\gamma\delta}_{\cdot\cdot\beta}.$$

Because of our assumption $\mathfrak{L}_a A^{\gamma\delta}_{\cdot\cdot\beta}=0$ $(a=1, 2, \cdots, r)$, from (3.11) it follows that

$$\mathfrak{L}_b g^{\alpha\delta}\mathfrak{L}_a g_{\alpha\beta} - \mathfrak{L}_a g^{\alpha\delta}\mathfrak{L}_b g_{\alpha\beta} + g^{\alpha\delta}(\mathfrak{L}_b, \mathfrak{L}_a) g_{\alpha\beta} = 2A^{\gamma\delta}_{\cdot\cdot\beta}(\mathfrak{L}_b \eta_{(a)\gamma} - \mathfrak{L}_a \eta_{(b)\gamma}).$$

On the other hand it is evident that we have from (3.1)

$$(3.12) \qquad \mathfrak{L}_a g^{\alpha\beta} = -2\eta^{\gamma}_{(a)} A^{\alpha\beta}_{\cdot\cdot\gamma}.$$

Making use of (3.4), (3.12) and the Maurer-Cartan equation, the above equation gives us

$$(3.13) \qquad 4\eta^{\gamma}_{(a)}\eta^{\delta}_{(b)} S_{\alpha\beta\gamma\delta} = 2c_{ba}^{\cdot\cdot c}\eta^{\gamma}_{(c)} A_{\alpha\beta\gamma}.$$

However, from the definition of $S_{\alpha\beta\gamma\delta}$, we must have $S_{\alpha\beta\gamma\delta} = -S_{\beta\alpha\gamma\delta}$. Since $A_{\alpha\beta\gamma}$ is a symmetric covariant tensor, comparing both sides of (3.13), we should have

$$\eta^{\gamma}_{(a)}\eta^{\delta}_{(b)} S_{\alpha\beta\gamma\delta} = 0 \quad \text{and} \quad c_{ba}^{\cdot\cdot c}\eta^{\gamma}_{(c)} A_{\alpha\beta\gamma} = 0.$$

The above equations enable us to obtain the result of Theorem 3.4.

Finally, with regard to the contravariant tensor $A^{\alpha\beta\gamma}$, we have the following

Theorem 3.5. *If the rank of the matrix* $\|A_{\alpha\beta\gamma}\|$ *is less than* n *and an* r*-parameter group* G_r *of infinitesimal rotations on* I_n *leaves invariant the tensor* $A^{\alpha\beta\gamma}$, *the derived group is a subgroup of motions on* I_n, *and itself is a group of motions if the rank of the matrix* $\|c_{ab}^{\cdot\cdot e}\|$ *is equal to* r.

Proof. Multiplying the fundamental equation (3.1) by $g^{\alpha\delta}$ and $g^{\beta\varepsilon}$, and summing with respect to α and β, it follows that

$$(3.14) \qquad g^{\alpha\delta}g^{\beta\varepsilon}\mathfrak{L}_a g_{\alpha\beta} = 2\eta_{(a)\gamma} A^{\gamma\delta\varepsilon}.$$

Because of our assumption $\mathfrak{L}_a A^{\gamma\delta\varepsilon}=0$, we have from (3.14)

$$(\mathfrak{L}_b g^{\alpha\delta})g^{\beta\varepsilon}(\mathfrak{L}_a g_{\alpha\beta}) - (\mathfrak{L}_a g^{\alpha\delta})g^{\beta\varepsilon}(\mathfrak{L}_b g_{\alpha\beta}) + g^{\alpha\delta}(\mathfrak{L}_b g^{\beta\varepsilon})(\mathfrak{L}_a g_{\alpha\beta}) - g^{\alpha\delta}(\mathfrak{L}_a g^{\beta\varepsilon})(\mathfrak{L}_b g_{\alpha\beta})$$
$$= 2A^{\gamma\delta\varepsilon}(\mathfrak{L}_b \eta_{(a)\gamma} - \mathfrak{L}_a \eta_{(b)\gamma}) - g^{\alpha\delta}g^{\beta\varepsilon}(\mathfrak{L}_b, \mathfrak{L}_a) g_{\alpha\beta}.$$

Substituting (3.12) into the above equation, and making use of (3.4) and the Maurer-Cartan equation, we find that

$$(3.15) \qquad 4\eta^{\gamma}_{(a)}\eta^{\delta}_{(b)}(S^{\alpha\beta}_{\cdot\cdot\gamma\delta} + S^{\beta\alpha}_{\cdot\cdot\gamma\delta}) = 2c_{ba}^{\cdot\cdot c}\eta^{\gamma}_{(c)} A^{\alpha\beta}_{\cdot\cdot\gamma}.$$

Since $S^{\alpha\beta}_{\cdot\cdot\gamma\delta} = -S^{\beta\alpha}_{\cdot\cdot\gamma\delta}$ holds good, (3.15) gives rise to the relation $c_{ba}^{\cdot\cdot c}\eta^{\gamma}_{(c)} A_{\alpha\beta\gamma} = 0$. This implies the result of Theorem 3.5.

§ 4. Special calsses of infinitesimal rotations on I_n. Since A_α is a gradient vector [6], $A_{\alpha;\beta}=A_{\beta;\alpha}$ holds good, where $A_\alpha=g^{\beta\gamma}A_{\alpha\beta\gamma}$. Then it follows that

$$\mathfrak{L}A_\alpha=A_{\alpha;\beta}\eta^\beta+A_\beta\eta^\beta_{;\alpha}=(A_\beta\eta^\beta)_{;\alpha}\,. \tag{4.1}$$

On the other hand, if η^α is the generating vector of an infinitesimal rotation on I_n, by means of the fundamental relation (1.2), we get $\eta^\delta_{;\delta}=A_\delta\eta^\delta$. As I_n is a compact Riemannian space, if it is an orientable, in consequence of the theorem of Green, we find from (4.1) and the last relation that

$$\int_{I_n}\eta^\delta_{;\delta}d\sigma=\int_{I_n}A_\delta\eta^\delta d\sigma=0\,. \tag{4.2}$$

If the transformation preserves the vector A_α, that is to say, $\mathfrak{L}A_\alpha=0$, by means of (4.1) it should be satisfied that $(A_\beta\eta^\beta)_{;\alpha}=0$, i.e. $A_\beta\eta^\beta=\text{const.}$ Hence, from (4.2) we get $A_\delta\eta^\delta=0$.

Conversely, from (4.1) it is evident that if $A_\delta\eta^\delta=0$ holds good, we have $\mathfrak{L}A_\delta=0$. Consequently we have the following theorem:

Theorem 4.1. *In order that an infinitesimal rotation on I_n preserves the covariant vector A_α, it is necessary and sufficient that the generating vector of the transformation be orthogonal to the vector A^α.*

Let us suppose that an infinitesimal rotation generated by a vector η^α is a motion on I_n. Then, we should have

$$\mathfrak{L}g_{\alpha\beta}=2\eta^\gamma A_{\alpha\beta\gamma}=0\,. \tag{4.3}$$

In this case the vector η^α satisfies the relation $\eta^\gamma A_\gamma=0$. In consequence of Theorem 4.1, this implies that $\mathfrak{L}A_\delta=0$ holds good. Therefore we have

Theorem 4.2. *If an infinitesimal rotation on I_n is a motion on I_n, the generating vector is orthogonal to the vector A^α and the transformation preserves the vector A^α.*

According to the definition of Lie derivatives, we have

$$\mathfrak{L}A_{\alpha\beta\gamma}=A_{\alpha\beta\gamma;\delta}\eta^\delta+A_{\delta\beta\gamma}\eta^\delta_{;\alpha}+A_{\alpha\delta\gamma}\eta^\delta_{;\beta}+A_{\alpha\beta\delta}\eta^\delta_{;\gamma}\,.$$

However, by means of the fundamental equation (1.2), we can obtain

$$A_{\delta\beta\gamma}\eta^\delta_{;\alpha}=\frac{1}{2}(\mathfrak{L}g_{\beta\gamma})_{;\alpha}-\eta^\delta A_{\delta\beta\gamma;\alpha}\,. \tag{4.4}$$

Making use of the relation $A_{\alpha\beta\gamma;\delta}=A_{\alpha\beta\delta;\gamma}$ (cf. § 1) and (4.4) we find that

$$\mathfrak{L}A_{\alpha\beta\gamma}=-2A_{\alpha\beta\gamma;\delta}\eta^\delta+\frac{1}{2}[(\mathfrak{L}g_{\beta\gamma})_{;\alpha}+(\mathfrak{L}g_{\alpha\gamma})_{;\beta}+(\mathfrak{L}g_{\alpha\beta})_{;\gamma}]\,.$$

Therefore, if $A_{\alpha\beta\gamma;\delta}=0$ holds good in I_n, $\mathfrak{L}g_{\alpha\beta}=0$ gives rise to the relation $\mathfrak{L}A_{\alpha\beta\gamma}=0$.

On the other hand, considering I_n as an affine hypersurface in an $n+1$-dimensional affine space and applying the theory of affine differential geometry, as A. Kawaguchi has shown in [6], it follows that

$$\mathfrak{A}_{\alpha\beta\gamma}=-\left\{A_{\alpha\beta\gamma}-\frac{1}{n+2}(g_{\alpha\beta}A_\gamma+g_{\beta\gamma}A_\alpha+g_{\gamma\alpha}A_\beta)\right\}, \tag{4.5}$$

$$\mathfrak{A}_{\alpha\beta\gamma}^{\ \ \ \gamma}=\mathfrak{A}_{\alpha\beta\gamma;}^{\ \ \ \ \gamma}-\frac{1}{2}(2-n)\mathfrak{A}_{\alpha\beta\gamma}\sigma^\gamma, \tag{4.6}$$

where $\mathfrak{A}_{\alpha\beta\gamma}^{\ \ \ \gamma}\equiv\mathfrak{A}_{\alpha\beta\gamma\delta}g^{\gamma\delta}$, and $\mathfrak{A}_{\alpha\beta\gamma\delta}$ denotes covariant derivative of $\mathfrak{A}_{\alpha\beta\gamma}$ with respect to $\mathfrak{G}_{\alpha\beta}$.[5] According to the result obtained by A. Kawaguchi, if $\mathfrak{A}_{\alpha\beta\gamma}^{\ \ \ \gamma}=0$ holds good at every point on I_n, then I_n ($n\geqq 2$) is a hyperquadric. Therefore, in consequence of (4.5) and (4.6), if $A_{\alpha\beta\gamma;\delta}=0$ holds good in I_2, then the considered Minkowski space M_3 is equivalent with a Euclidean space. Thus we have

Theorem 4.3. *When $A_{\alpha\beta\gamma;\delta}=0$ is satisfied in I_n ($n\geqq 3$), if an infinitesimal rotation is a motion in I_n, the transformation leaves invarian the tensor $A_{\alpha\beta\gamma}$.*

In the previous paper [7], the present author has shown that the system of linear partial differential equations determining the group of infinitesimal rotations on I_n is completely integrable and then the maximum order of the group of infinitesimal rotations on I_n is equal to $\frac{1}{2}n(n+1)$.

If an infinitesimal rotation on I_n coincides with a motion on I_n, the generating vector η^α should satisfy the condition $\eta^\gamma A_{\alpha\beta\gamma}=0$. Now, let us suppose that the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is equal to p, then there exist p independent conditions $\eta^\gamma A_{\alpha\beta\gamma}=0$. Therefore, the order of a subgroup of motions does not exceed $\frac{1}{2}n(n+1)-p$. However, as was stated in §2, since $p\geqq 2$ we can see that the order of the subgroup of motions is less than $\frac{1}{2}n(n+1)-1$.

On the other hand, according to the theorem proved by K. Yano [10], in an n-dimensional Riemannian space for $n\neq 4$, there exists no group of motions of order r such that

$$\frac{1}{2}n(n-1)+1<r<\frac{1}{2}n(n+1).$$

5) $\mathfrak{G}_{\alpha\beta}$ denotes the affine fundamental quantity of the second order, and $\mathfrak{A}_{\alpha\beta\gamma}$ the fundamental tensor of the third order defined on I_n. Cf. W. Blaschke [1].

and if the Riemannian space admits a group of motions of order $\frac{1}{2}n(n-1)+1$, then the group is transitive.

If we call our attention to the fact that a group of infinitesimal rotations on I_n is intransitive [6], we have the following

Theorem 4.4. *In I_n $(n \neq 4)$, the order of the subgroup of motions, contained in the group of infinitesimal rotations on I_n, does not exceed $\frac{1}{2}n(n-1)$.*

Remembering the fact that the indicatrix I_n is a compact Riemannian space, in consequence of the theorems concerning affine motion or homothetic transformation in a compact Riemannian space [2], [8], we get

Lemma 1. *If an infinitesimal rotation on I_n is an affine motion, it is necessarily a motion on I_n.*

Lemma 2. *If an infinitesimal rotation on I_n is a homothetic transformation, it is necessarily a motion on I_n.*

Now, we shall consider the case when an infinitesimal rotation on I_n is a conformal transformation on I_n. Then according to the theory of Lie derivatives, it should be satisfied that [9]

$$\mathfrak{L} g_{\alpha\beta} = 2\eta^\gamma A_{\alpha\beta\gamma} = 2\phi g_{\alpha\beta}\,, \tag{4.7}$$

where ϕ is a scalar function. Contracting $g^{\alpha\beta}$ to the second and third terms of (4.7), we have $\eta^\gamma A_\gamma = n\phi$. Then as $A_{\gamma;\delta} = A_{\delta;\gamma}$ it follows that

$$\mathfrak{L} A_\delta = A_{\delta;\gamma}\eta^\gamma + A_\gamma \eta^\gamma_{;\delta} = n\phi_{;\delta}\,.$$

The last relation shows us that if the transformation leaves invariant the covariant vector A_δ, the scalar function ϕ becomes constant, and from (4.7) the transformation is a homothetic one. Then, by means of Lemma 2, we have

Theorem 4.5. *When an infinitesimal rotation on I_n is a conformal transformation, if the transformation preserves the covariant vector A_δ, it is necessarily a motion.*

Next, we shall consider the case when an infinitesimal rotation on I_n is a projective transformation on I_n. In this case it must be satisfied that [9]

$$\mathfrak{L}\left\{{}^{\alpha}_{\beta\gamma}\right\} = \delta^\alpha_\beta \varphi_\gamma + \delta^\alpha_\gamma \varphi_\beta\,, \tag{4.8}$$

where φ_α be an arbitrary covariant vector.

On the other hand, if the vector η^α is the generating vector of an infinitesimal rotation on I_n, by means of (1.2), we have from the symmetric properties of $A_{\alpha\beta\gamma}$ and the relation $A_{\alpha\beta\gamma;\delta} = A_{\alpha\beta\delta;\gamma}$,

$$(4.9)\qquad \begin{aligned}\mathfrak{L}\{{}^{\alpha}_{\beta\gamma}\} &= \frac{1}{2} g^{\alpha\delta}[(\mathfrak{L}g_{\delta\beta})_{;\gamma}+(\mathfrak{L}g_{\delta\gamma})_{;\beta}-(\mathfrak{L}g_{\beta\gamma})_{;\delta}] \\ &= A^{\alpha}_{\cdot\beta\gamma;\delta}\eta^{\delta}+\eta^{\delta}_{;\lambda}(\delta^{\lambda}_{\gamma}A^{\alpha}_{\cdot\beta\delta}+\delta^{\lambda}_{\beta}A^{\alpha}_{\cdot\gamma\delta}-g^{\alpha\lambda}A_{\beta\gamma\delta})\,.\end{aligned}$$

Comparing (4.8) with (4.9) we find that

$$(4.10)\qquad \delta^{\alpha}_{\beta}\varphi_{\gamma}+\delta^{\alpha}_{\gamma}\varphi_{\beta}=A^{\alpha}_{\cdot\beta\gamma;\delta}\eta^{\delta}+\eta^{\delta}_{;\lambda}(\delta^{\lambda}_{\gamma}A^{\alpha}_{\cdot\beta\delta}+\delta^{\lambda}_{\beta}A^{\alpha}_{\cdot\gamma\delta}-g^{\alpha\lambda}A_{\beta\gamma\delta})\,.$$

Therefore, if we contract α and β in (4.10), it follows that $(n+1)\varphi_{\gamma}=\mathfrak{L}A_{\gamma}$. The last relation gives us that if the transformation leaves invariant A_{γ}, it becomes that $\varphi_{\gamma}=0$, and from (4.8) we have $\mathfrak{L}\{{}^{\alpha}_{\beta\gamma}\}=0$. Therefore the transformation becomes an affine motion. Then, by means of Lemma 1, we have

Theorem 4.6. *When an infinitesimal rotation on I_n is a projective transformation, if the transformation preserves the covariant vector A_{δ}, it is necessarily a motion.*

If an n-dimensional Riemannian space I_n is a space of constant curvature, I_n is equivalent with a hypersphere in an $n+1$-dimensional Euclidean space and the considered Minkowski space M_{n+1} is essentially a Euclidean space [6]. Although a space of constant curvature is an Einstein space, there exists an Einstein space which is not a space of constant curvature. Then, we shall consider the case when I_n is an Einstein space.

If an infinitesimal rotation $\overline{u}^{\alpha}=u^{\alpha}+\eta^{\alpha}(u)\delta t$ is a motion or a conformal transformation, as η^{α} satisfies (4.3) or (4.7) respectively, it follows that $\eta^{\alpha}S_{\alpha\beta\gamma\delta}=0$. Then, we have from (1.1)

$$(4.11)\qquad \eta^{\alpha}R_{\alpha\beta\gamma\delta}=\eta^{\alpha}(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})\,.$$

Multiplying (4.11) by $g^{\beta\gamma}$ and summing for β and γ, we get $\eta^{\alpha}\{R_{\alpha\gamma}-(1-n)g_{\alpha\gamma}\}=0$. If I_n is an Einstein space, substituting the relation $R_{\alpha\gamma}=\dfrac{R}{n}g_{\alpha\gamma}$ into the last equation, we get $\eta_{\gamma}\{R-n(1-n)\}=0$. Hence, at any point, if $R\neq n(1-n)$, then η_{γ} must be zero at the same point. Thus we have

Theorem 4.7. *If I_n is an Einstein space and $R\neq n(1-n)$ is satisfied at every point on I_n, an infinitesimal rotation does not coincide with a motion or a conformal transformation with respect to the metric on I_n.*

K. Yano and T. Nagano [12] has shown the following theorem concerning the transformation in a compact Einstein space:

Theorem I. *In a compact Einstein space with negative scalar curvature, there does not exist an infinitesimal projective transformation.*

Also, T. Sumitomo [8] has proved the following theorem concerning the conformal transformation in a compact Riemannian space:

Theorem II. *In a compact Riemannian manifold M with non-pasitive constant scalar curvature: $R=const.\leqq 0$, $C_0(M)$ coincides with $I_0(M)$.*[6]

In consequence of Theorem I and Theorem II, we have

Theorem 4.8. *If I_n is an Einstein space with $R=n(1-n)$, an infinitesimal rotation on I_n does not coincide with a conformal and a projective transformation in I_n.*

§ 5. Order of groups of infinitesimal rotations on I_n preserving $A_{\alpha\beta\gamma}$.

In the following, we shall consider the groups of infinitesimal rotations on I_n such that each element of the groups preserves one of the tensors $A_{\alpha\beta\gamma}$, $A^{\alpha}_{\cdot\beta\gamma}$, $A^{\alpha\beta}_{\cdot\cdot\gamma}$ and $A^{\alpha\beta\gamma}$.[7]

Case I. $\mathfrak{L}A^{\alpha}_{\cdot\beta\gamma}=0$. In this case we have

$$A_{\beta\gamma;\delta}^{\cdot\cdot\alpha}\eta^{\delta}+A_{\delta\gamma}^{\cdot\cdot\alpha}\eta^{\delta}_{;\beta}+A_{\beta\delta}^{\cdot\cdot\alpha}\eta^{\delta}_{;\gamma}-A_{\beta\gamma}^{\cdot\cdot\delta}\eta^{\alpha}_{;\delta}=0\,.$$

Putting

$$T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\gamma}=\delta^{\varepsilon}_{\beta}A_{\delta\gamma}^{\cdot\cdot\alpha}+\delta^{\varepsilon}_{\gamma}A_{\beta\delta}^{\cdot\cdot\alpha}-\delta^{\alpha}_{\delta}A_{\beta\gamma}^{\cdot\cdot\varepsilon}\,, \tag{5.1}$$

and we denote by $\left\|T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\gamma}\right\|$ the matrix of elements $T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\gamma}$, where $\binom{\varepsilon}{\delta}$ denotes the columns and $\binom{\alpha}{\beta\gamma}$ the rows.

Lemma 1. *If the rank of the matrix $\left\|T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\gamma}\right\|$ is less than n, it follows that $A_{\alpha_3\alpha_2}^{\cdot\cdot\alpha_1}=0$.*

Proof. From (5.1) we can easily see that $T\binom{\alpha_1}{\alpha_j}\binom{\alpha_i}{\alpha_3\alpha_2}=-\delta^{\alpha_i}_{\alpha_j}A_{\alpha_3\alpha_2}^{\cdot\cdot\alpha_1}$,[8] where $\alpha_i\neq\alpha_j$ for $i\neq j$. Then we can select from $\left\|T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\delta}\right\|$ the following n-rowed square matrix:

$$\begin{array}{c|cccc}
 & \binom{\alpha_1}{\alpha_3\;\alpha_2} & \binom{\alpha_2}{\alpha_3\;\alpha_2} & \cdots\cdots & \binom{\alpha_n}{\alpha_3\;\alpha_2} \\
\hline
\binom{\alpha_1}{\alpha_1} & -A_{\alpha_3\alpha_2}^{\cdot\cdot\alpha_1} & 0 & \cdots\cdots & 0 \\
\binom{\alpha_1}{\alpha_2} & 0 & -A_{\alpha_3\alpha_2}^{\cdot\cdot\alpha_1} & & \vdots \\
\vdots & \vdots & & * \quad * \quad * & \vdots \\
\binom{\alpha_1}{\alpha_n} & 0 & 0 & \cdots\cdots & -A_{\alpha_3\alpha_2}^{\cdot\cdot\alpha_1}
\end{array}\,.$$

6) $I_0(M)$ denotes the connected component of the identical transformations in $I(M)$, where $I(M)$ is the totality of isometric transformations of M. $C_0(M)$ denotes the connected component of the conformal transformations in $C(M)$, where $C(M)$ is the totality of conformal transformations of M.

7) In the theory of groups in an asymmetric affine connection space, I. P. Egorov [3] proved that "If the rank of matrix $\left\|T\binom{\rho}{\sigma}\binom{\kappa}{\mu\lambda}\right\|\equiv\left\|\delta^{\sigma}_{\mu}S_{\mu\sigma}^{\cdot\cdot\kappa}+\delta^{\rho}_{\lambda}S_{\mu\sigma}^{\cdot\cdot\kappa}-\delta^{\kappa}_{\sigma}S_{\mu\lambda}^{\cdot\cdot\rho}\right\|$ is less than n, then $S_{\mu\lambda}^{\cdot\cdot\kappa}=0$.", where $S_{\mu\lambda}^{\cdot\cdot\kappa}=\frac{1}{2}(\Gamma^{\kappa}_{\mu\lambda}-\Gamma^{\kappa}_{\lambda\mu})$ and $\Gamma^{\kappa}_{\mu\lambda}$ are parameters of an affine connection. The methods of our proof are analogous as those of I. P. Egorov.

8) Indices α_i, α_j take the values $1, 2, \cdots, n$.

Therefore, if the rank of the matrix $\|T\binom{\xi}{\delta}\binom{\alpha}{\beta\gamma}\|$ is less than n, we must have $(-A_{\alpha_3\alpha_2}^{\cdot\,\cdot\,\alpha_1})^n=0$ and this implies $A_{\alpha_3\alpha_2}^{\cdot\,\cdot\,\alpha_1}=0$. Q.E.D.

Lemma 2. *If the rank of the matrix* $\|T\binom{\xi}{\delta}\binom{\alpha}{\beta\gamma}\|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2}=0$.[9]

Proof. Noting that $\alpha_1\neq\alpha_2$, from the symmetric properties of $A_{\beta\gamma}^{\cdot\,\cdot\,\alpha}$, we have from (5.1) that $T\binom{\alpha_i}{\alpha_1}\binom{\alpha_2}{\alpha_j\alpha_2}=\delta_{\alpha_j}^{\alpha_i}A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2}+\delta_{\alpha_2}^{\alpha_i}A_{\alpha_1\alpha_j}^{\cdot\,\cdot\,\alpha_2}$. Then, if we suppose that $\alpha_1<\alpha_2$, we can select from $\|T\binom{\xi}{\delta}\binom{\alpha}{\beta\gamma}\|$ the following n-rowed square matrix:

$$
\begin{array}{c|cccccccc}
 & \binom{\alpha_2}{1\ \alpha_2} & \binom{\alpha_2}{2\ \alpha_2} & \cdots & \binom{\alpha_2}{\alpha_1\ \alpha_2} & \cdots & \binom{\alpha_2}{\alpha_2\ \alpha_2} & \cdots & \binom{\alpha_2}{n\ \alpha_2} \\
\hline
\binom{1}{\alpha_1} & A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\binom{2}{\alpha_1} & 0 & A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2} & & & & & & \vdots \\
\vdots & \vdots & & * & & & & & \vdots \\
\binom{\alpha_1}{\alpha_1} & \vdots & & & A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2} & & & & \vdots \\
\vdots & 0 & & & & * & & & \vdots \\
\binom{\alpha_2}{\alpha_1} & A_{1\alpha_1}^{\cdot\,\cdot\,\alpha_2} & A_{2\alpha_1}^{\cdot\,\cdot\,\alpha_2} & * & A_{\alpha_1\alpha_1}^{\cdot\,\cdot\,\alpha_2} & * & 2A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2} & * & A_{n\alpha_1}^{\cdot\,\cdot\,\alpha_2} \\
\vdots & \vdots & & & & & & * & 0 \\
\binom{n}{\alpha_1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2}
\end{array}
$$

Therefore, by virtue of Lemma 1, if the rank of the matrix $\|T\binom{\xi}{\delta}\binom{\alpha}{\beta\gamma}\|$ is less than n, we must have $2(A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2})^n=0$ and this implies $A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2}=0$. By the same way we can prove $A_{\alpha_1\alpha_2}^{\cdot\,\cdot\,\alpha_2}=0$ for the case $\alpha_1>\alpha_2$. Q.E.D.

Lemma 3. *If the rank of the matrix* $\|T\binom{\xi}{\delta}\binom{\alpha}{\beta\gamma}\|$ *is less than* n, *it follows that* $A_{\alpha_2\alpha_2}^{\cdot\,\cdot\,\alpha_1}=0$.

Proof. Noting that $\alpha_1\neq\alpha_2$, from the symmetric properties of $A_{\beta\gamma}^{\cdot\,\cdot\,\alpha}$, we have from (5.1) that $T\binom{\alpha_1}{\alpha_j}\binom{\alpha_i}{\alpha_2\alpha_2}=-\delta_{\alpha_j}^{\alpha_i}A_{\alpha_2\alpha_2}^{\cdot\,\cdot\,\alpha_1}$. Then, if we suppose that $\alpha_1<\alpha_2$, we can select from $\|T\binom{\xi}{\delta}\binom{\alpha}{\beta\gamma}\|$ the following n-rowed square matrix:

$$
\begin{array}{c|cccccccc}
 & \binom{1}{\alpha_2\ \alpha_2} & \binom{2}{\alpha_2\ \alpha_2} & \cdots & \binom{\alpha_1}{\alpha_2\ \alpha_2} & \cdots & \binom{\alpha_2}{\alpha_2\ \alpha_2} & \cdots & \binom{n}{\alpha_2\ \alpha_2} \\
\hline
\binom{\alpha_1}{1} & -A_{\alpha_2\alpha_2}^{\cdot\,\cdot\,\alpha_1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\binom{\alpha_1}{2} & 0 & -A_{\alpha_2\alpha_2}^{\cdot\,\cdot\,\alpha_1} & & & & & & \vdots \\
\vdots & \vdots & & * & & & & & \vdots \\
\binom{\alpha_1}{\alpha_1} & \vdots & & & -A_{\alpha_2\alpha_2}^{\cdot\,\cdot\,\alpha_1} & & & & \vdots \\
\vdots & \vdots & & & & * & & & \vdots \\
\binom{\alpha_1}{\alpha_2} & \vdots & & & & & -A_{\alpha_2\alpha_2}^{\cdot\,\cdot\,\alpha_1} & & \vdots \\
\vdots & \vdots & & & & & & * & 0 \\
\binom{\alpha_1}{n} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -A_{\alpha_2\alpha_2}^{\cdot\,\cdot\,\alpha_1}
\end{array}
$$

9) In the following, we do not sum up with respect to the indices α_i $(i=1,2,\cdots,n)$.

Therefore, if the rank of the matrix $\| T({}^{\varepsilon}_{\delta})({}^{\alpha}_{\beta\gamma}) \|$ is less than n, we must have $(-A_{\alpha_2\alpha_2}^{\cdot\ \cdot\ \alpha_1})^n = 0$ and this implies $A_{\alpha_2\alpha_2}^{\cdot\ \cdot\ \alpha_1} = 0$. By the same way, we can prove $A_{\alpha_2\alpha_2}^{\cdot\ \cdot\ \alpha_1} = 0$ for the case $\alpha_1 > \alpha_2$. Q.E.D.

Lemma 4. *If the rank of the matrix* $\| T({}^{\varepsilon}_{\delta})({}^{\alpha}_{\beta\gamma}) \|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_1}^{\cdot\ \cdot\ \alpha_1} = 0$.

Proof. From (5.1), it follows that $T({}^{\alpha_1}_{\alpha_i})({}^{\ \ \alpha_j}_{\alpha_1\alpha_1}) = 2A_{\alpha_1\alpha_i}^{\cdot\ \cdot\ \alpha_j} - \delta^{\alpha_j}_{\alpha_i} A_{\alpha_1\alpha_1}^{\cdot\ \cdot\ \alpha_1}$. Then we can select from $\| T({}^{\varepsilon}_{\delta})({}^{\alpha}_{\beta\gamma}) \|$ the following n-rowed square matrix:

$$
\begin{array}{c|cccccc}
 & ({}^{\ \ 1}_{\alpha_1\alpha_1}) & ({}^{\ \ 2}_{\alpha_1\alpha_1}) & \cdots\cdots & ({}^{\ \ \alpha_1}_{\alpha_1\alpha_1}) & \cdots\cdots & ({}^{\ \ n}_{\alpha_1\alpha_1}) \\
\hline
({}^{\alpha_1}_{1}) & -A_{\alpha_1\alpha_1}^{\cdot\ \cdot\ \alpha_1} & 0 & \cdots & \cdots & \cdots & 0 \\
({}^{\alpha_1}_{2}) & 0 & A_{\alpha_1\alpha_1}^{\cdot\ \cdot\ \alpha_1} & & & & \vdots \\
\vdots & \vdots & & * & & & \vdots \\
({}^{\alpha_1}_{\alpha_1}) & \vdots & & & A_{\alpha_1\alpha_1}^{\cdot\ \cdot\ \alpha_1} & & \vdots \\
\vdots & \vdots & & & & * & 0 \\
({}^{\alpha_1}_{n}) & 0 & \cdots & \cdots & \cdots & 0 & -A_{\alpha_1\alpha_1}^{\cdot\ \cdot\ \alpha_1}
\end{array}
$$

Therefore, if the rank of the matrix $\| T({}^{\varepsilon}_{\delta})({}^{\alpha}_{\beta\gamma}) \|$ is less than n, we must have $-(-A_{\alpha_1\alpha_1}^{\cdot\ \cdot\ \alpha_1})^n = 0$ and this implies $A_{\alpha_1\alpha_1}^{\cdot\ \cdot\ \alpha_1} = 0$. Q.E.D.

Summarizing the results of above Lemmas, we see that if the rank of the matrix $\| T({}^{\varepsilon}_{\delta})({}^{\alpha}_{\beta\gamma}) \|$ is less than n, it follows that $A_{\beta\gamma}^{\cdot\cdot\alpha} = 0$ for every α, β and γ. Consequently, as the maximum order of groups of the infinitesimal rotations on I_n is equal to $\frac{1}{2}n(n+1)$ [7], it follows that

Theorem 5.1. *The order of the group of infinitesimal rotations on* I_n, *preserving the tensor* $A_{\beta\gamma}^{\cdot\cdot\alpha}$, *does not exceed* $\frac{1}{2}n(n-1)$.

Case II. $\mathfrak{L}A^{\alpha\beta}_{\cdot\cdot\gamma} = 0$. In this case we have

$$A^{\alpha\beta}_{\cdot\cdot\gamma;\delta}\eta^\delta - A^{\varepsilon\beta}_{\cdot\cdot\gamma}\eta^\alpha_{;\varepsilon} - A^{\alpha\varepsilon}_{\cdot\cdot\gamma}\eta^\beta_{;\varepsilon} + A^{\alpha\beta}_{\cdot\cdot\delta}\eta^\delta_{;\gamma} = 0 .$$

We put $T({}^{\varepsilon}_{\delta})({}^{\alpha\beta}_{\ \gamma}) = \delta^\alpha_\delta A^{\varepsilon\beta}_{\cdot\cdot\gamma} + \delta^\beta_\delta A^{\alpha\varepsilon}_{\cdot\cdot\gamma} - \delta^\varepsilon_\beta A^{\alpha\beta}_{\cdot\cdot\delta}$, and consider the rank of the matrix $\| T({}^{\varepsilon}_{\delta})({}^{\alpha\beta}_{\ \gamma}) \|$. Making use of the analogous way as those in Case I, we obtain

Lemma 5. *If the rank of the matrix* $\| T({}^{\varepsilon}_{\delta})({}^{\alpha\beta}_{\ \gamma}) \|$ *is less than* n, *it follows that* $A^{\alpha\beta}_{\cdot\cdot\gamma} = 0$.

Consequently, we obtain the following

Theorem 5.2. *The order of the group of infinitesimal rotations on* I_n, *preserving the tensor* $A^{\alpha\beta}_{\cdot\cdot\gamma}$, *does not exceed* $\frac{1}{2}n(n-1)$.

Case III. $\mathfrak{L}A_{\alpha\beta\gamma} = 0$. In this case we have

$$A_{\alpha\beta\gamma;\delta}\eta^\delta + A_{\delta\beta\gamma}\eta^\delta_{;\alpha} + A_{\alpha\delta\gamma}\eta^\delta_{;\beta} + A_{\alpha\beta\delta}\eta^\delta_{;\gamma} = 0 .$$

We put

(5.2) $$T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\equiv\delta^{\lambda}_{\alpha}A_{\delta\beta\gamma}+\delta^{\lambda}_{\beta}A_{\alpha\delta\gamma}+\delta^{\lambda}_{\gamma}A_{\alpha\beta\delta}$$

and denote by $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ the matrix of elements $T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})$, where $\binom{\lambda}{\delta}$ denotes the columns and $(_{\alpha\beta\gamma})$ the rows.

Lemma 6. *If the rank of the matrix* $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_1\alpha_1}=0$.

Proof. Making use of the symmetric properties of $A_{\alpha\beta\gamma}$, we have from (5.2) that $T\binom{\alpha_i}{\alpha_1}(_{\alpha_1\alpha_1\alpha_j})=2\delta^{\alpha_i}_{\alpha_1}A_{\alpha_1\alpha_1\alpha_j}+\delta^{\alpha_i}_{\alpha_j}A_{\alpha_1\alpha_1\alpha_1}$. Then, we can select from $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ the following n-rowed square matrix:

$$\begin{array}{c|cccccc}
 & (_{\alpha_1\alpha_1 1}) & (_{\alpha_1\alpha_1 2}) & \cdots\cdots & (_{\alpha_1\alpha_1\alpha_1}) & \cdots\cdots & (_{\alpha_1\alpha_1 n}) \\
\hline
\binom{1}{\alpha_1} & A_{\alpha_1\alpha_1\alpha_1} & 0 & \cdots & \cdots & \cdots & 0 \\
\binom{2}{\alpha_1} & 0 & A_{\alpha_1\alpha_1\alpha_1} & & & & \vdots \\
\vdots & \vdots & & * & & & \vdots \\
\binom{\alpha_1}{\alpha_1} & * & * & * & 3A_{\alpha_1\alpha_1\alpha_1} & * & \vdots \\
\vdots & \vdots & & & & * & \vdots \\
\binom{n}{\alpha_1} & 0 & \cdots & \cdots & \cdots & 0 & A_{\alpha_1\alpha_1\alpha_1}
\end{array}$$

Therefore, if the rank of the matrix $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ is less than n, we must have $3(A_{\alpha_1\alpha_1\alpha_1})^n=0$, i.e. $A_{\alpha_1\alpha_1\alpha_1}=0$. Q.E.D.

Lemma 7. *If the rank of the matrix* $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_1\alpha_2}=0$.

Proof. From (5.2), by means of the symmetric properties of $A_{\alpha\beta\gamma}$, we have $T\binom{\alpha_i}{\alpha_1}(_{\alpha_j\alpha_2\alpha_2})=\delta^{\alpha_i}_{\alpha_j}A_{\alpha_1\alpha_2\alpha_2}+2\delta^{\alpha_i}_{\alpha_2}A_{\alpha_1\alpha_2\alpha_j}$. Then, if we suppose that $\alpha_2<\alpha_2$, we can select from $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ the following n-rowed square matrix:

$$\begin{array}{c|cccccccc}
 & (_{1\alpha_2\alpha_2}) & (_{2\alpha_2\alpha_2}) & \cdots\cdots & (_{\alpha_1\alpha_2\alpha_2}) & \cdots\cdots & (_{\alpha_2\alpha_2\alpha_2}) & \cdots\cdots & (_{n\alpha_2\alpha_2}) \\
\hline
\binom{1}{\alpha_1} & A_{\alpha_1\alpha_2\alpha_2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\binom{2}{\alpha_1} & 0 & A_{\alpha_1\alpha_2\alpha_2} & & & & & & \vdots \\
\vdots & \vdots & & * & & & & & \vdots \\
\binom{\alpha_1}{\alpha_1} & \vdots & & & A_{\alpha_1\alpha_2\alpha_2} & & & & \vdots \\
\vdots & \vdots & & & & * & & & \vdots \\
\binom{\alpha_2}{\alpha_1} & * & * & * & * & * & 3A_{\alpha_1\alpha_2\alpha_2} & * & * \\
\vdots & \vdots & & & & & & * & 0 \\
\binom{n}{\alpha_1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & A_{\alpha_1\alpha_2\alpha_2}
\end{array}$$

Therefore, if the rank of the matrix $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ is less than n, we must have $3(A_{\alpha_1\alpha_2\alpha_2})^n=0$, i.e. $A_{\alpha_1\alpha_2\alpha_2}=0$. By the same way, we can prove $A_{\alpha_1\alpha_2\alpha_2}=0$ for the case $\alpha_1<\alpha_2$. Q.E.D.

Lemma 8. *If the rank of the matrix* $\| T(^{\lambda}_{\delta})(_{\alpha\beta\gamma}) \|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_2\alpha_3}=0$.

Proof. In consequence of the symmetric property of $A_{\alpha\beta\gamma}$, we may assume that $\alpha_1<\alpha_2<\alpha_3$ without loss of generality. Thus, we have from (5.2) that $T(^{\alpha_i}_{\alpha_1})(_{\alpha_j\alpha_2\alpha_3})=\delta^{\alpha_i}_{\alpha_j}A_{\alpha_1\alpha_2\alpha_3}+\delta^{\alpha_i}_{\alpha_2}A_{\alpha_j\alpha_1\alpha_3}+\delta^{i}_{3}A_{\alpha_j\alpha_2\alpha_1}$. Then we can select from $\| T(^{\lambda}_{\delta})(_{\alpha\beta\gamma}) \|$ the following n-rowed square matrix:

$$
\begin{array}{c|cccccccccc}
 & (_{1\alpha_1\alpha_2}) & (_{2\alpha_2\alpha_3}) & \cdots & (_{\alpha_1\alpha_2\alpha_3}) & \cdots & (_{\alpha_2\alpha_2\alpha_3}) & \cdots & (_{\alpha_3\alpha_2\alpha_3}) & \cdots & (_{n\alpha_2\alpha_3}) \\
\hline
(^{1}_{\alpha_1}) & A_{\alpha_1\alpha_2\alpha_3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
(^{2}_{\alpha_1}) & * & A_{\alpha_1\alpha_2\alpha_3} & & & & & & & & \vdots \\
\vdots & * & & * & & & & & & & \vdots \\
(^{\alpha_1}_{\alpha_1}) & * & & & A_{\alpha_1\alpha_2\alpha_3} & & & & & & \vdots \\
\vdots & * & & & & * & & & & & \vdots \\
(^{\alpha_2}_{\alpha_3}) & * & & & & & 2A_{\alpha_1\alpha_2\alpha_3} & & & & \vdots \\
\vdots & * & & & & & & * & & & \vdots \\
(^{\alpha_3}_{\alpha_1}) & * & & & & & & & 2A_{\alpha_1\alpha_2\alpha_3} & & \vdots \\
\vdots & * & & & & & & & & * & 0 \\
(^{n}_{\alpha_1}) & * & * & * & * & * & * & * & * & * & A_{\alpha_1\alpha_2\alpha_3}
\end{array}
$$

Therefore, if the rank of the matrix $\| T(^{\lambda}_{\delta})(_{\alpha\beta\gamma}) \|$ is less than n, we must have $4(A_{\alpha_1\alpha_2\alpha_3})^n=0$, i.e. $A_{\alpha_1\alpha_2\alpha_3}=0$. Q.E.D.

Summarizing the results of the above Lemmas, we see that if the rank of the matrix $\| T(^{\lambda}_{\delta})(_{\alpha\beta\gamma}) \|$ is less than n, then it follows that $A_{\alpha\beta\gamma}=0$ for every α, β and γ. Consequently we have

Theorem 5.3. *The order of the group of infinitesimal rotations on* I_n, *preserving the tensor* $A_{\alpha\beta\gamma}$, *does not exceed* $\frac{1}{2}n(n-1)$.

Case IV. $\mathfrak{L}A^{\alpha\beta\gamma}=0$. In this case we can easily obtain the following theorem by means of the analogous way as in Case III:

Theorem 5.4. *The order of the group of infinitesimal rotations on* I_n, *preserving the tensor* $A^{\alpha\beta\gamma}$, *does not exceed* $\frac{1}{2}n(n-1)$.

Department of Mathematics, Faculty of Science,
Hokkaido University, Japan.

References

[1] W. BLASCHKE: Vorlesungen über Differentialgeometrie, Bd. II, Springer., Berlin, (1923).

[2] S. BOCHNER and K. YANO: Curvature and Bettinumber, Princeton Univ. Press, 32 (1953).

[3] I. P. EGOROV: On groups of motion of spaces with asymmetric affine connection, Doklady Akad. Nauk SSSR (N. S.), 53 (1947).

[4] L. P. EISENHART: Riemannian Geometry, Princeton (1949).

[5] L. P. EISENHART: Continuous groups of transformations, Princeton (1933).

[6] A. KAWAGUCHI: On the theory of non-linear connections II. Theory of Minkowski spaces and of non-linear connections in a Finsler space, Tensor, New Series, 6 (1956), 165–199.

[7] T. NAGAI: On groups of rotations in Minkowski spaces I, Jour. Fac. Sci. Hokkaido Univ., Ser. I, 15 (1960), 29–45.

[8] T. SUMITOMO: Projective and conformal transformations in compact Riemannian manifolds, Tensor, New Series, 9 (1959), 113–135.

[9] K. YANO: Groups of transformations in generalized spaces, Tokyo (1949).

[10] K. YANO: On n-dimensional Riemannian spaces admitting a group of motions of order $n(n-1)/2-1$, Trans. Amer. Soc., 74 (1953), 260–279.

[11] K. YANO: Lie derivatives and its applications, Amsterdam (1957).

[12] K. YANO and T. NAGANO: Some theorems on projective and conformal transformations, Indag. Math., 19 (1957), 451–458.