



HOKKAIDO UNIVERSITY

Title	ON THE RATIOS OF THE NORMS DEFINED BY MODULARS
Author(s)	Shimogaki, Tetsuya
Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 15(1-2), 001-012
Issue Date	1960
Doc URL	https://hdl.handle.net/2115/56009
Type	departmental bulletin paper
File Information	JFSHIU_15_N1-2_001-012.pdf



ON THE RATIOS OF THE NORMS DEFINED BY MODULARS

By

Tetsuya SHIMOGAKI

§ 1. Let R be a modularized semi-ordered linear space and $m(x)$ ($x \in R$) be a modular¹⁾ on R . Since $0 \leq m(\xi x)$ is a non-trivial convex function of real number $\xi \geq 0$ for every $0 \neq x \in R$, we can define two kinds of norms by the modular m as follows:

$$(1.1) \quad \|x\| = \inf_{\xi > 0} \frac{1+m(\xi x)}{\xi}, \quad |||x||| = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|} \quad (x \in R).$$

The former of them is said to be the *first norm* by m and the latter to be the *second (or modular) norm* by m .

Let $\bar{R}^m$ be the modular conjugate space of R and $\bar{m}$ be the conjugate modular²⁾ of m . Then we can also define the norms on $\bar{R}^m$ by $\bar{m}$ as above. It is well-known [4; § 40] that if R is semi-regular³⁾ the first norm by the conjugate modular $\bar{m}$ is the conjugate one of the second norm by m and the second norm by $\bar{m}$ is the conjugate one of the first norm by m . Since $\|\cdot\|$ and $|||\cdot|||$ are semi-continuous, they are reflexive

[3]. We have always $|||x||| \leq \|x\| \leq 2|||x|||$ for all $x \in R$, that is, $1 \leq \frac{\|x\|}{|||x|||} \leq 2$ for all $0 \neq x \in R$.

When the ratios of these two norms are equal to a constant number, i.e. $\frac{\|x\|}{|||x|||} = \gamma$ holds for each $0 \neq x \in R$, S. Yamamuro [8] and I. Amemiya [1] succeeded in showing that the modular m is of L^p -type essentially, i.e. $m(\xi x) = \xi^p m(x)$ for all $x \in R$ and $\xi \geq 0$, where $1 \leq p$.

In the earlier paper [7] the author investigated the case that the

1) For the definition of a modular see [4]. The notations and terminologies used here are the same as in [4 or 7].

2) $\bar{R}^m$ is the totality of all linear functionals $\bar{a}$ on R such that $\inf_{\lambda \in A} |\bar{a}(x_\lambda)| = 0$ for every $x_\lambda \downarrow_{\lambda \in A} 0$ and $\sup_{m(x) \leq 1} |\bar{a}(x)| < +\infty$. The conjugate modular $\bar{m}$ of m on $\bar{R}^m$ is defined as

$$\bar{m}(\bar{a}) = \sup_{x \in R} \{\bar{a}(x) - m(x)\} \quad (\bar{a} \in \bar{R}^m).$$

3) R is said to be semi-regular, if $\bar{a}(x) = 0$ for all $\bar{a} \in \bar{R}^m$ implies $x = 0$.

ratios satisfy the condition:

$$(1.2) \quad \inf_{0 \neq x \in R} \frac{\|x\|}{\|x\|} > 1,$$

and proved that it is equivalent to uniform finiteness of both m and $\bar{m}$, provided that R is non-atomic.

In an Orlicz space $L^*_\phi(G)$ ⁴⁾, which is one of the concrete examples of modularized semi-ordered linear spaces, the similar results concerning the ratios were found independently by D. V. Salekhov in [6] under more restricted circumstances.

In this paper we shall consider the following conditions on the ratios of the norms by a modular m :

$$(1.3) \quad \frac{\|x\|}{\|x\|} < 2 \quad \text{for all } 0 \neq x \in R;$$

or

$$(1.4) \quad \sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|} < 2,$$

and study their relations to the properties of the modular m . We shall show in § 2 that if the condition (1.3) is satisfied, then either $m(\xi x) < \xi^2 m(x)$ (for all $\xi > 1$ and $x \in R$ with $m(x) \geq 1$), or $m(\xi x) > \xi^2 m(x)$ (for all $\xi > 1$ and $x \in R$ with $+\infty > m(x) \geq 1$) holds, provided that R is non-atomic. And as for (1.4) we shall show in § 3 that (1.4) implies that either $m(\xi x) \leq \xi^p m(x)$ for all $\xi \geq 1$ and $x \in R$ with $m(x) \geq 1$ or $m(\xi x) \geq \xi^{p'} m(x)$ for all $\xi \geq 1$ and $x \in R$ with $m(x) \geq 1$ holds, where p, p' are real numbers with $1 \leq p < 2 < p' \leq +\infty$, provided that R is non-atomic.

The difference between the conditions (1.2) and (1.4) exists in the point of their topological properties, that is, the former of them remains valid for any modular m' equivalent⁵⁾ to the original one except a finite dimensional space, but the later does not hold in general. Thus we can not obtain the explicit conditions equivalent to (1.4) with respect to the modular m in general case. For a modular of unique spectra, however, we shall estimate $\sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|}$ and $\inf_{0 \neq x \in R} \frac{\|x\|}{\|x\|}$ exactly in § 4 by applying the results obtained in §§ 2 and 3.

Throughout this paper we denote by R a modularized semi-ordered linear space and by m a modular on R . For any $p \in R$ we denote by $[p]$

4) For the definition of Orlicz space $L^*_\phi(G)$ see [2] or [9].

5) Two modulars m and m' on R are called *equivalent*, if their norms are equivalent to each other.

a projection operator defined by $p: [p]x = \bigcup_{n=1}^{\infty} (n|p| \wedge x)$ for all $0 \leq x \in R$. R is called to be *non-atomic*, if any $0 \neq a \in R$ is decomposed into $a = b + c$ such that $|b| \wedge |c| = 0$, $b \neq 0$ and $c \neq 0$. Since $m(x+y) = m(x) + m(y)$ for any $x, y \in R$ with $|x| \wedge |y| = 0$, $a \in R$ with $m(a) < +\infty$ can be decomposed into $a = [p]a + (1 - [p])a$ for some $p \in R$ such that $m([p]a) = m((1 - [p])a)$, if R is non-atomic. Here we note that $m(\xi x)$ is a *continuous function* of $\xi \in [0, \eta]$ for each $x \in R$, if $m(\eta x) < +\infty$, because $m(\xi x)$ is a positive convex function of $\xi \geq 0$ for each $x \in R$.

§ 2. We put for every $x \in R$ with $m(x) < +\infty$

$$(2.1) \quad \pi_+(x) = \inf_{\varepsilon > 0} \frac{1}{\varepsilon} \{m((1+\varepsilon)x) - m(x)\}$$

and

$$(2.2) \quad \pi_-(x) = \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \{m(x) - m((1-\varepsilon)x)\},$$

and for $x \in R$ with $m(x) = +\infty$ we put

$$(2.3) \quad \pi_+(x) = \pi_-(x) = +\infty.$$

Then it follows from the definitions that $0 \leq \pi_-(x) \leq \pi_+(x)$ for all $x \in R$ and both $\pi_+(\xi x)$ and $\pi_-(\xi x)$ are *non-decreasing functions* of $\xi \geq 0$ for every $x \in R$ and are *orthogonally additive*, that is, $\pi_{\pm}(x+y) = \pi_{\pm}(x) + \pi_{\pm}(y)$ if $x \perp y$ ⁶⁾, $x, y \in R$. Furthermore $\pi_+(\xi x)$ is a *right-hand continuous function* of $\xi \geq 0$ for every $x \in R$, since $m(\xi x)$ is a convex function of $\xi \geq 0$. In fact, we have for each $\xi_0 \geq 0$ $\lim_{\xi \downarrow \xi_0} \pi_+(\xi x) = \inf_{\xi > \xi_0} \pi_+(\xi x) = \inf_{\xi > \xi_0} \left[\inf_{\varepsilon > 0} \frac{1}{\varepsilon} \{m((1+\varepsilon)\xi x) - m(\xi x)\} \right] = \inf_{\varepsilon > 0} \left[\inf_{\xi > \xi_0} \frac{1}{\varepsilon} \{m((1+\varepsilon)\xi x) - m(\xi x)\} \right] = \inf_{\varepsilon > 0} \frac{1}{\varepsilon} \{m((1+\varepsilon)\xi_0 x) - m(\xi_0 x)\} = \pi_+(\xi_0 x)$, if $m(\alpha \xi_0 x) < +\infty$ for some $\alpha > 1$. If $m(\alpha \xi_0 x) = +\infty$ for all $\alpha > 1$ or $m(\xi_0 x) = +\infty$, we have $\pi_+(\xi_0 x) = +\infty$ and $\lim_{\xi \downarrow \xi_0} \pi_+(\xi x) = \pi_+(\xi_0 x)$. Similarly $\pi_-(\xi x)$ is a *left-hand continuous function* of ξ for every $x \in R$, that is, $\lim_{\xi \uparrow \xi_0} \pi_-(\xi x) = \pi_-(\xi_0 x)$ for each $\xi_0 \geq 0$.

We put also

$$S = \{x : x \in R, |||x||| = 1\},$$

$$S_m = \{x : x \in R, m(x) = 1\} \quad \text{and} \quad S_c = \{x : x \in S, m(x) < 1\}.$$

From the definition of the second norm it is clear that $S = S_m \cup S_c$, $S_m \cap S_c = \emptyset$ and $m(\xi x) = +\infty$ for all $x \in S_c$ and $\xi > 1$.

6) Two elements x, y are called *mutually orthogonal*, if $|x| \cap |y| = 0$ and then we write $x \perp y$. For a subset A of R , $A^\perp$ denotes the set of all $x \in R$ with $x \perp y$ for all $y \in A$.

Lemma 1. We have $\|x\|=2$ for $x \in S$ if and only if $x \in S_m$ and $\pi_-(x) \leq 2 \leq \pi_+(x)$.

Proof. If $\|x\|=2$ for $x \in S$, then we have by the formula (1.1) $2\xi \leq 1+m(\xi x)$ for every $\xi > 0$. This implies $m(x)=1$ and $\frac{1-m(\xi x)}{1-\xi} \leq 2 \leq \frac{m(\eta x)-1}{\eta-1}$ for every $0 < \xi < 1 < \eta$. It follows therefore $\pi_-(x) \leq 2 \leq \pi_+(x)$. Conversely $\pi_-(x) \leq 2 \leq \pi_+(x)$ implies $\frac{m(x)-m(\xi x)}{1-\xi} \leq 2 \leq \frac{m(\eta x)-m(x)}{\eta-1}$ for every $0 < \xi < 1 < \eta$, which yields $2\xi \leq 1+m(\xi x)$ for all $\xi > 0$ in virtue of $m(x)=1$. Hence we have $\|x\|=2$. Q.E.D.

Lemma 2. We have $\pi_-(x) > 2$ for $x \in S_m$, if and only if $\|x\| < 2$ and $\|x\| = \inf_{\frac{1}{2} \leq \xi < 1} \frac{1+m(\xi x)}{\xi}$.

Proof. If $\pi_-(x) > 2$ for $x \in S_m$, then we have for some $0 < \xi < 1$ $2 < \frac{1-m(\xi x)}{1-\xi}$, which implies $2 > \frac{1+m(\xi x)}{\xi}$ and $\|x\| < 2$. Since $\pi_+(x) \geq \pi_-(x) > 2$, we obtain $\frac{m(\eta x)-1}{\eta-1} \geq \pi_-(x) > 2$ for all $\eta > 1$. It follows from above that $\frac{1+m(\eta x)}{\eta} > 2$ for $\eta > 1$ and $\|x\| = \inf_{\frac{1}{2} < \xi < 1} \frac{1+m(\xi x)}{\xi}$. Conversely let $\|x\| < 2$, $\|x\| = \inf_{\frac{1}{2} < \xi < 1} \frac{1+m(\xi x)}{\xi}$ and $x \in S_m$, then there exists ξ_0 ($0 < \xi_0 < 1$) such that $2 > \frac{1+m(\xi_0 x)}{\xi_0}$. This implies $2 < \frac{1-m(\xi_0 x)}{1-\xi_0}$ and $\pi_-(x) > 2$. Q.E.D.

Lemma 3. We have $\pi_+(x) < 2$ for $x \in S$ if and only if $\|x\| < 2$ and $\|x\| = \inf_{1 < \xi} \frac{1+m(\xi x)}{\xi}$.

Proof. If $x \in S$ and $\pi_+(x) < 2$, then we have $m(\xi_0 x) < +\infty$ for some $1 < \xi_0$ by the definition of $\pi_+(x)$. Thus we have $x \in S_m$. The remainder of the proof can be obtained by the similar way as above. Q.E.D.

Now we put

$$S_* = \left\{ x : x \in S, \|x\| = \inf_{\frac{1}{2} \leq \xi < 1} \frac{1+m(\xi x)}{\xi} \right\}$$

and

$$S^* = \left\{ x : x \in S, \|x\| = \inf_{1 < \xi} \frac{1+m(\xi x)}{\xi} \right\}.$$

It is clear by Lemmata 1-3 that $S_* \cup S^* = S$, $S_c \subset S_*$ and that $x \in S_* \cap S^*$ implies $\|x\|=2$.

The following lemma plays essential rôle in this paper.

Lemma 4. *If there exist mutually orthogonal elements $x, y \in R$ with $x \in S_*$ and $y \in S^*$, then there exists an element $z \in S$ such that $\|z\|=2$.*

Proof. If $\|x\|=2$ (or $\|y\|=2$) holds, then the above assertion is clearly true. Hence we suppose $\|x\|<2$ and $\|y\|<2$. We put for every positive number α with $0 \leq \alpha \leq 1$

$$(2.4) \quad \varphi(\alpha) = \sup_{\| \alpha x + \beta y \| = 1} \beta.$$

Since $y \in S^*$ implies $m(y)=1$, it is easily seen that $\varphi(\alpha)$ is a continuous function of α ($0 \leq \alpha \leq 1$) and if α runs decreasingly from 1 to 0, $\varphi(\alpha)$ does increasingly from ξ_0 to 1, where $\xi_0 = \sup_{m(\xi y)=1-m(x)} \xi$.

Now we put $z_\alpha = \alpha x + \varphi(\alpha)y$. It is clear that $z_\alpha \in S_m$ and

$$(2.5) \quad \pi_\pm(z_\alpha) = \pi_\pm(\alpha x) + \pi_\pm(\varphi(\alpha)y) \quad \text{for all } 0 \leq \alpha \leq 1.$$

By Lemma 3, $\|y\|<2$ and $y \in S^*$ imply

$$(2.6) \quad \pi_-(z_0) \leq \pi_+(z_0) = \pi_+(y) < 2.$$

And if $x \in S_m$, then we have by Lemma 2

$$(2.7) \quad \pi_-(z_1) = \pi_-(x) + \pi_-(\xi_0 y) \geq \pi_-(x) > 2.$$

On the other hand, $x \in S_c$ implies $\pi_+(x) = \pi_+(z_1) = +\infty$, because $m(\xi x) = +\infty$ for all $\xi > 1$. Thus we have $\|z_1\|=2$ by Lemma 1, if $\pi_-(z_1) \leq 2$. Therefore, we may also suppose that both (2.6) and (2.7) hold good.

In virtue of (2.6) we can put $\alpha_0 = \sup_{\pi_+(z_\alpha) < 2, 0 \leq \alpha \leq 1} \alpha$. For such $\alpha_0 \geq 0$ there exists a sequence of positive numbers such that $0 \leq \alpha_n \uparrow_{n=1}^\infty \alpha_0$ and $\pi_-(z_{\alpha_n}) \leq \pi_+(z_{\alpha_n}) < 2$. We have by (2.5) and by the fact that $1 \geq \alpha' \geq \alpha \geq 0$ implies $0 \leq \varphi(\alpha') \leq \varphi(\alpha) \leq 1$

$$\pi_-(z_{\alpha_n}) = \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_n)y) \geq \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_0)y).$$

Since $\pi_-(\xi x)$ is left-hand continuous, we obtain

$$2 \geq \overline{\lim}_{n \rightarrow \infty} \pi_-(z_{\alpha_n}) \geq \lim_{n \rightarrow \infty} \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_0)y) = \pi_-(\alpha_0 x) + \pi_-(\varphi(\alpha_0)y),$$

which implies $\pi_-(z_{\alpha_0}) \leq 2$. On the other hand, since $\pi_-(z_{\alpha_0}) \leq 2$ implies $\alpha_0 < 1$, we can find a sequence $\{\alpha'_n\}_{n=1}^\infty$ such that $1 \geq \alpha'_n \downarrow_{n=1}^\infty \alpha_0$ and $\pi_+(z_{\alpha'_n}) \geq 2$. We have also by (2.5)

$$2 \leq \pi_+(z_{\alpha'_n}) = \pi_+(\alpha'_n x) + \pi_+(\varphi(\alpha'_n)y) \leq \pi_+(\alpha'_n x) + \pi_+(\varphi(\alpha_0)y).$$

Since $\pi_+(\xi x)$ is right-hand continuous, we obtain

$$2 \leq \underline{\lim}_{n \rightarrow \infty} \pi_+(z_{\alpha'_n}) \leq \lim_{n \rightarrow \infty} \pi_+(\alpha'_n x) + \pi_+(\varphi(\alpha_0)y) = \pi_+(\alpha_0 x) + \pi_+(\varphi(\alpha_0)y),$$

which yields $\pi_+(z_{\alpha_0}) \geq 2$. Therefore we have $\pi_-(z_{\alpha_0}) \leq 2 \leq \pi_+(z_{\alpha_0})$ and a

fortiori $\|z_{\alpha_0}\|=2$ by Lemma 1.

Q.E.D.

We denote by $S_{x,y}$ the totality of all elements $z \in S$ such that $z = \alpha x + \beta y$ for some $0 \leq \alpha, \beta$.

Lemma 5. *If there exist mutually orthogonal elements $x, y \in S$ and $z \in S_{x,y}$ such that $x, y \in S^*$ and $z \in S_*$, then there exists $z_0 \in S_{x,y}$ such that $\|z_0\|=2$.*

Proof. Here we may also assume without loss of generality that $\pi_+(x) < 2$, $\pi_+(y) < 2$ and $\pi_-(z) > 2$. We also define $\varphi(\alpha)$ by the formula (2.4) and put $z_\alpha = \alpha x + \varphi(\alpha)y$ for $0 \leq \alpha \leq 1$. It follows from $x, y \in S^*$ that z_α belongs to S_m for each $0 \leq \alpha \leq 1$. Since there exists some $1 > \alpha' > 0$ such that $\pi_-(z) = \pi_-(z_{\alpha'}) > 2$ we can put $\alpha_0 = \inf_{\pi_-(z_\alpha) \geq 2} \alpha$. Then we can find a sequence of positive numbers $\{\alpha_n\}_{n=1}^\infty$ such that $1 \geq \alpha_n \downarrow_{n=1}^\infty \alpha_0$ and $\pi_-(z_{\alpha_n}) \geq 2$ ($n=1, 2, \dots$). Since

$$2 \leq \pi_+(z_{\alpha_n}) = \pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_n)y) \leq \pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_0)y)$$

implies

$$2 \leq \lim_{n \rightarrow \infty} \{\pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_0)y)\} = \pi_+(\alpha_0 x) + \pi_+(\varphi(\alpha_0)y) = \pi_+(z_{\alpha_0}),$$

we have $2 \leq \pi_+(z_{\alpha_0})$. On the other hand, $\pi_+(z_{\alpha_0}) \geq 2$ implies $\alpha_0 > 0$ and hence we can find also a sequence of positive numbers $\{\alpha'_n\}_{n=1}^\infty$ such that $0 \leq \alpha'_n \uparrow_{n=1}^\infty \alpha_0$ and $\pi_-(z_{\alpha'_n}) < 2$ ($n=1, 2, \dots$). Since

$$2 > \pi_-(z_{\alpha'_n}) = \pi_-(\alpha'_n x) + \pi_-(\varphi(\alpha'_n)y) \geq \pi_-(\alpha'_n x) + \pi_-(\varphi(\alpha_0)y)$$

implies

$$2 \geq \lim_{n \rightarrow \infty} \{\pi_-(\alpha'_n x) + \pi_-(\varphi(\alpha_0)y)\} = \pi_-(\alpha_0 x) + \pi_-(\varphi(\alpha_0)y),$$

we have $2 \geq \pi_-(z_{\alpha_0})$. Therefore we obtain $\pi_-(z_{\alpha_0}) \leq 2 \leq \pi_+(z_{\alpha_0})$, which implies $\|z_{\alpha_0}\|=2$.

Q.E.D.

Here we note that if there exist mutually orthogonal elements $x, y \in S$ and $z \in S_{x,y}$ such that $z \in S^*$ then $m(x+y) > 1$. Hence applying the similar method as in the proof of Lemma 5, we have

Lemma 6. *If there exist mutually orthogonal elements $x, y \in S$ and $z \in S_{x,y}$ such that $x, y \in S_*$ and $z \in S^*$, then there exists $z_0 \in S_{x,y}$ with $\|z_0\|=2$.*

Collecting the results of the above Lemmata, we have

Theorem 2.1. *In order that the condition (1.3) is satisfied, that is,*

$$\frac{\|x\|}{\|x\|} < 2 \text{ for all } 0 \neq x \in R, \text{ it is necessary and sufficient that either}$$

$$(2.8) \quad \pi_+(x) < 2 \quad \text{for all } x \in S$$

or

$$(2.9') \quad \pi_-(x) > 2 \quad \text{for all } x \in S_m$$

holds.

Proof. Necessity. When R is one-dimensional, the assertion comes directly from Lemmata 2 and 3. Thus we may assume that the dimension of R is greater than two. Now let $R = N_1 \oplus N_2$, where N_i ($i=1, 2$) are normal manifolds and $N_1^\perp = N_2$. For an element $x_0 \in N_1 \cap S$ the condition (1.3) implies either $x_0 \in S_*$ or $x_0 \in S^*$.

First let $x_0 \in S_*$. Then Lemma 4 and the condition (1.3) imply $N_2 \cap S \subseteq S_*$, which implies also $N_1 \cap S \subseteq S_*$ by Lemma 4. Therefore we obtain $S \subseteq S_*$ by Lemma 6. Thus we can see that (2.9') holds good in virtue of Lemma 2.

Secondly let $x_0 \in S^*$, then we have by the same manner $N_2 \cap S = S^*$ and $N_1 \cap S = S^*$. This implies that (2.8) holds good in virtue of Lemmata 3 and 5. Q.E.D.

Sufficiency. Since $x \in S_c$ implies $1 + m(x) < 2$, we have $\|x\| < 2$ for all $x \in S_c$. Thus we can see that Lemmata 2 and 3 assure that (2.8) (or (2.9')) implies the condition (1.3). Q.E.D.

A modular m on R is said to be *finite* if $m(x) < +\infty$ for all $x \in R$. Since we have $S = S_m$, in case R is finite, we have immediately from Theorem 2.1

Corollary 1. *Let a modular m be finite. In order that the condition (1.3) holds, it is necessary and sufficient that either the condition (2.8) or*

$$(2.9) \quad \pi_-(x) > 2 \quad \text{for all } x \in S$$

holds.

In order that we discuss the condition (1.3) more precisely in case of non-atomic R , we need to prove

Lemma 7. *If R non-atomic and $S = S^*$, then the modular m is finite.*

Proof. If there exists $0 \leq x \in R$ with $m(x) = +\infty$, we put $a_0 = \inf_{m(\xi x) = +\infty} \xi$ and $x_0 = a_0 x$. It follows $m(\eta x_0) = +\infty$ for all $\eta > 1$ and $m(\xi x_0) < +\infty$ for all $0 \leq \xi < 1$. When $m(x_0) < +\infty$ holds, we can find an element $p \in R$ such that $m(\eta[p]x_0) = +\infty$ for all $\eta > 1$ and $m([p]x_0) \leq 1$, since R has no atomic elements. For such $[p]x_0$ we have $[p]x_0 \in S_c$, which is inconsistent with $S^* = S$. Now let $m(x_0) = +\infty$ hold. Since R is non-atomic, we can find $p \in R$ with $m\left(\frac{2}{3}[p]x_0\right) \leq \frac{1}{4}$ and $\lim_{\xi \uparrow 1} m(\xi[p]x_0) = +\infty$. Now we put

$\alpha = \frac{1}{\left\| \frac{2}{3}[p]x_0 \right\|}$ and $y = \alpha \frac{2}{3}[p]x_0$. Then we have $1 < \alpha < \frac{3}{2}$ and $y \in S_m$,

hence

$$\pi_-(y) = \pi_-\left(\alpha \frac{2}{3}[p]\pi_0\right) \geq \frac{m\left(\alpha \frac{2}{3}[p]x_0\right) - m\left(\frac{2}{3}[p]x_0\right)}{\frac{\alpha-1}{\alpha}} \geq \frac{1-\frac{1}{4}}{\frac{1}{3}} > 2.$$

This contradicts the assumption: $S=S^*$ by Lemmata 1 and 2. Therefore we have proved that $m(x) < +\infty$ for all $x \in R$. Q.E.D.

Theorem 2.2. *Let R be non-atomic and the condition (1.3) be satisfied, then the modular m satisfies one of the following conditions:*

$$(2.10) \quad m(\xi x) < \xi^2 m(x) \quad \text{for all } \xi > 1 \text{ and } x \in R \text{ with } m(x) \geq 1;$$

$$(2.11) \quad m(\xi x) > \xi^2 m(x) \quad \text{for all } \xi > 1 \text{ and with } +\infty > m(x) \geq 1.$$

Proof. In virtue of the foregoing theorem we know that one of the conditions (2.8) or (2.9') is true. First we suppose that (2.8) holds. Then Lemma 7 together with Lemma 3 implies that m is finite. If $m(x) = N + \frac{m}{n}$ (where N, m and n are natural numbers with $m \leq n$), we can decompose orthogonally x into $x = \sum_{i=1}^{N-1} x_i + \sum_{j=1}^{n+m} y_j$ such that $m(x_i) = 1$ ($i=1, 2, \dots, N-1$) and $m(y_j) = \frac{1}{n}$ ($j=1, 2, \dots, n+m$). The number of j satisfying $\pi_+(y_j) \geq 2m(y_j)$ is less than n , because if there exist $j_1, j_2, \dots, j_n$ with $\pi_+(y_{j_k}) \geq 2m(y_{j_k})$ ($k=1, 2, \dots, n$), we have $\sum_{k=1}^n y_{j_k} \in S_m$ and $\pi_+\left(\sum_{k=1}^n y_{j_k}\right) = \sum_{k=1}^n \pi_+(y_{j_k}) \geq \sum_{k=1}^n 2m(y_{j_k}) = 2m\left(\sum_{k=1}^n y_{j_k}\right) = 2$, which is inconsistent with (2.8). Hence we can find $\{j_p\}$ ($1 \leq p \leq m$) such that $\pi_+(y_{j_p}) < 2m(y_{j_p})$ ($p=1, 2, \dots, m$). Putting $y_0 = \sum_{p=1}^m y_{j_p}$, we obtain $m\left(\sum_{j=1}^{m+n} y_j - y_0\right) = 1$ and $\pi_+\left(\sum_{j=1}^{m+n} y_j - y_0\right) < 2$. Therefore we have

$$\begin{aligned} \pi_+(x) &= \sum_{i=1}^{N-1} \pi_+(x_i) + \pi_+\left(\sum_{j=1}^{m+n} y_j - y_0\right) + \pi_+(y_0) \\ &< 2(N-1) + 2 + 2m(y_0) = 2\left(N + \frac{m}{n}\right) = 2m(x), \end{aligned}$$

hence $\pi_+(x) < 2m(x)$. In general, if $1 < m(x)$, we can find $\{\alpha_n\}_{n=1}^\infty$ with $\alpha_n \downarrow_{n=1}^\infty 1$ and $m(\alpha_n x)$ is a rational number for each $n \geq 1$. It follows

$$\pi_+(x) = \lim_{n \rightarrow \infty} \pi_+(\alpha_n x) \leq \lim_{n \rightarrow \infty} 2m(\alpha_n x) = 2m(x),$$

hence

$$(2.12) \quad \pi_+(x) \leq 2m(x) \quad \text{for all } x \in R \text{ with } 1 \leq m(x).$$

Now we put

$$(2.13) \quad m'_x(\xi) = \lim_{\varepsilon \rightarrow 0} \frac{m((\xi + \varepsilon)x) - m(\xi x)}{\varepsilon}$$

for each $x \in R$ and $\xi \geq 0$. It is clear that $\xi \cdot m'_x(\xi) = \pi_+(\xi x)$ for all $\xi > 0$ and $x \in R$. (2.12) implies

$$(2.14) \quad \frac{m'_x(\xi)}{m(\xi x)} \leq \frac{2}{\xi} \quad (\xi > 1)$$

for every $x \in R$ with $m(x) \geq 1$. Integrating both sides of (2.14) from 1 to $\eta > 1$ with respect to ξ , we have

$$(2.15) \quad \log \frac{m(\eta x)}{m(x)} \leq 2 \log \eta \quad (\eta > 1).$$

In formula (2.15), however, the equal sign does not hold in any case. Indeed, since as is shown above, the set of all ξ satisfying $\xi m'_x(\xi) = \pi_+(\xi x) < 2m(\xi x)$ is dense in $[1, +\infty)$ and $m(\xi x)$ is a continuous function of ξ , there exists an interval $(\xi_0, \eta_0) \subseteq (1, \eta)$ such that

$$\xi m'_x(\xi) = \pi_+(\xi x) < 2m(\xi x)$$

holds for all $\xi \in (\xi_0, \eta_0)$. Therefore we have

$$m(\eta x) < \eta^2 m(x)$$

for all $\eta > 1$ and $x \in R$ with $m(x) \geq 1$.

By the quite same manner we can prove that the condition (2.11) is satisfied, if we assume that the condition (2.9') is true. Q.E.D.

§ 3. Here we consider the case that the norms defined by a modular m satisfy (1.4), that is, $\sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|} < 2$. From the results proved in § 2 we have

Theorem 3.1. *If the condition (1.4) is satisfied, then either*

$$(3.1) \quad \sup_{x \in S} \pi_+(x) < 2$$

or

$$(3.2) \quad \inf_{x \in S_m} \pi_-(x) > 2$$

holds.

Proof. In virtue of Theorem 2.1, we can see that either (2.8) or (2.9') holds. First let (2.8) be true and set $\gamma = \sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|}$. Then for

each $x \in S$ and $\varepsilon > 0$, there exists $\xi > 1$ such that $\frac{1+m(\xi x)}{\xi} < \gamma + \varepsilon$ by Lemma 3. From this

$$\pi_+(x) \leq \frac{m(\xi x) - 1}{\xi - 1} < \gamma + \varepsilon$$

follows, if $\gamma + \varepsilon < 2$. Hence we have $\gamma \geq \pi_+(x)$ for all $x \in S$.

On the other hand, we can prove by the same way⁷⁾ that (2.9') together with (1.4) implies (3.2). Q.E.D.

Remark 1. The converse of Theorem 3.1 does not remain true in general. It is easily verified that there exists a modular which does not fulfil (1.4) but satisfies (3.1) (or (3.2)).

Remark 2. As is seen in the proof of Theorem 3.1, it is clear that $\sup_{x \in S} \pi_+(x) \leq \gamma$ or $\inf_{x \in S_m} \pi_-(x) \geq \frac{\gamma}{\gamma - 1}$ holds respectively, where $\gamma = \sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|}$ < 2 .

As for non-atomic R , corresponding to Theorem 2.2, we have

Theorem 3.2. *Let R be non-atomic and the condition (1.4) be satisfied, then either*

$$(3.3) \quad m(\xi x) \leq \xi^p m(x) \quad \text{for all } \xi \geq 1 \text{ and } x \in R \text{ with } m(x) \geq 1;$$

$$(3.4) \quad m(\xi x) \geq \xi^{p'} m(x) \quad \text{for all } \xi \geq 1 \text{ and } x \in R \text{ with } m(x) \geq 1;$$

where p and p' are real numbers with $1 \leq p < 2 < p' \leq +\infty$ ⁸⁾.

Proof. In virtue of the preceding theorem we need only to verify implications: (3.1) $\rightarrow$ (3.3) and (3.2) $\rightarrow$ (3.4). And these implications can be ascertained by the same manner as in the proof of Theorem 2.2.

Here we may choose p, p' as $p = \gamma$ and $p' = \frac{\gamma}{\gamma - 1}$ respectively, where

$$\gamma = \sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|}.$$

Q.E.D.

Remark 3. As is easily verified by calculating $\|x\|$ of $x \in S$, the condition (3.3) is the sufficient one for (1.4) at the same time. On the other hand, (3.4) is not such a one in general.

§ 4. At last we deal with a modular of unique spectra [4; § 54] and estimate exactly $\sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|}$ and $\inf_{0 \neq x \in R} \frac{\|x\|}{\|x\|}$ as applications of the

7) We note that $2 - r\xi \geq \frac{\gamma}{\gamma - 1}(1 - \xi)$ holds if $1 < r < 2$ and $1 < r\xi$.

8) When $p' = +\infty$, we put $\xi^\infty = \infty$ if $\xi > 1$.

preceding theorems and those of [7]. An element $0 \leq s \in R$ is said to be *simple*, if $m(s) < +\infty$ and $m([p]s) = 0$ implies $[p]s = 0$. And a modular m is said to be of *unique spectra* if $m(\xi s) = \int_{[s]}^{\xi^{\rho(p)}} m(dps)^{9)}$ for all $\xi \geq 0$ and simple elements $s \in R$. Function spaces $L^{p(t)}_{[s]}$ [5] (where $p(t)$ is a measurable function with $p(t) \geq 1$ ($0 \leq t \leq 1$)): the totality of all measurable functions $\varphi(t)$ such that

$$(4.1) \quad \int_0^1 |\alpha \varphi(t)|^{p(t)} dt < +\infty \quad \text{for some } \alpha > 0,$$

and sequence spaces l^{p_ν} (where $p_\nu \geq 1$ ($\nu \geq 1$)): the totality of all sequences $x = (\xi_\nu)_{(\nu \geq 1)}$ such that

$$(4.2) \quad \sum_{\nu=1}^{\infty} |\alpha \xi_\nu|^{p_\nu} < +\infty \quad \text{for some } \alpha > 0.$$

are the examples of modular spaces whose modulars are of unique spectra, where the modulars are defined as $m(\varphi) = \int_0^1 |\varphi(t)|^{p(t)} dt$ and $m(x) = \sum_{\nu=1}^{\infty} |\xi_\nu|^{p_\nu}$ respectively. When m is of unique spectra, we denote by ρ_u, ρ_l the *upper exponent* of m : $\rho_u = \sup_{p \in \epsilon} \rho(p)$ and the *lower exponent* of m : $\rho_l = \inf_{p \in \epsilon} \rho(p)$ respectively. There exist normal manifolds N_1, N_2 such that $R = N_1 \oplus N_2$ and $\rho(p)$ is finite for all $p \in U_{[N_1]}$ and $\rho(p) = +\infty$ for all $p \in U_{[N_2]}$, that is, m is singular in N_2 . For any $0 \neq x \in N_2$ we have $\|x\| = |||x|||$ and $S = S_*$. Therefore we obtain

Theorem 4.1. *If a modular m is of unique spectra, then we have (with the conventions $\frac{1}{\infty} = 0$ and $\infty^0 = 1$)*

$$\sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||} \begin{cases} = 2, & \text{if } \rho_l \leq 2 \leq \rho_u, \\ = \rho_u^{\frac{1}{p_u}} q_u^{\frac{1}{q_u}}, & \text{if } \rho_u < 2, \\ = \rho_l^{\frac{1}{p_l}} q_l^{\frac{1}{q_l}}, & \text{if } \rho_l > 2, \end{cases}$$

where q_u and q_l are real numbers with $\frac{1}{\rho_u} + \frac{1}{q_u} = 1$ and $\frac{1}{\rho_l} + \frac{1}{q_l} = 1$.

Proof. As $\frac{\|x\|}{|||x|||} = 1$ for all $0 \neq x \in N_2$, we may consider only the

9) p is a point of representation space ϵ of R , i.e. the maximal ideal of normal manifolds of R . For N , we denote by $U_{[N]}$ the totality of all $p \in \epsilon$ with $N \in p$. $\rho(p)$ is a continuous function on ϵ with $\rho(p) \geq 1$.

ratios of the norms in N_1 . When $\gamma \leq \rho(p) \leq \gamma'$ for all $p \in U_{[p]} \subseteq U_{[N_1]}$, then we have for all $x \in R$ $\xi' m([p]x) \leq m(\xi[p]x) \leq \xi' m([p]x)$ ($\xi \geq 1$) and $\eta' m([p]x) \geq m(\eta[p]x) \geq \eta' m([p]x)$ ($0 \leq \eta \leq 1$). From this and Lemma 4 we have $\sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|} = 2$, if $\rho_l \leq 2 \leq \rho_u$. Since $\sup_{x \in S} \pi_+(x) < 2$ if and only if $\rho_u < 2$ and $\inf_{x \in S} \pi_-(x) > 2$ if and only if $\rho_l > 2$ for m , we have by Lemmata 2 and 3 that $\|x\| \leq \rho_u^{\frac{1}{p_u}} q_u^{\frac{1}{q_u}} (x \in S)$ and $\|x\| \leq \rho_l^{\frac{1}{p_l}} q_l^{\frac{1}{q_l}}$ respectively according to (3.1) and (3.2). Therefore we complete the proof. Q.E.D.

Similarly we can conclude by Theorem 3.1 in [7]

Theorem 4.2. *If a modular m is of unique spectra, then we have*

$$\inf_{0 \neq x \in R} \frac{\|x\|}{\|x\|} = \text{Min} \left\{ \rho_l^{\frac{1}{p_l}} q_l^{\frac{1}{q_l}}, \rho_u^{\frac{1}{p_u}} q_u^{\frac{1}{q_u}} \right\}.$$

Department of Mathematics,
Hokkaido University, Sapporo..

References

- [1] I. AMEMIYA, *A characterization of the modulars of L_p -type*, Jour. Fac. Sci. Hokkaido Univ., 13 (1954).
- [2] W. A. J. LUXEMBURG, *Banach function spaces*, Thesis, Delft (Netherlands) (1955).
- [3] T. MORI, I. AMEMIYA and H. NAKANO, *On the reflexivity of semi-continuous norms*, Proc. Japan Acad., 31 (1955).
- [4] H. NAKANO, *Modulared semi-ordered linear spaces*, Tokyo (1950).
- [5] H. NAKANO, *Topologies and linear topological spaces*, Tokyo (1951).
- [6] D. V. SALEKHOV, *On the norm of the linear functional in the Orlicz space and on a certain internal characteristic of the L_p -spaces*, Dok. Acad. Nauk, URSS, 111, No. 5 (1956).
- [7] T. SHIMOGAKI, *On certain property of the norms by modulars*, Jour. Fac. Sci. Hokkaido Univ., 13 (1957).
- [8] S. YAMAMURO, *On linear modulars*, Proc. Japan Acad. 27 (1951).
- [9] A. C. ZAAANEN, *Linear analysis*, Amsterdam. Groningen, New York (1953).