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ON THE FINITENESS OF MODULARED SPACES

By

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Let R be *a modulared space*[1] with the modular $m(x)$[2] ($x \in R$). R or m is said to be *finite* if $m(x) < \infty$ for every $x \in R$.

Since the finite modulars are rather convenient to be treated, they were studied from earlier steps of investigation on these spaces by some authors.

W. Orlicz and Z. Birnbaum [6] found a necessary and sufficient condition (so-called Δ_2-condition) in order that Orlicz spaces are finite. After that, this fact was generalized for an arbitrary monotone complete[3] modular by means of finding a formula which characterizes the finite modular, on non-atomic and atomic spaces by I. Amemiya [1] and by T. Shimogaki [7] respectively.

On the other hand, H. Nakano [3] defined *a modulared fuction space*, a kind of the generalizations of the Orlicz spaces, and showed that an arbitrary modulared space could be represented by a modulared function space. Hence, the modulared function space may be considered as the most general space among the concrete examples of the modulared space.

In this paper, we shall first characterize the finite modular by its another formula in the both cases of non-atomic (§ 1) and atomic (§ 2) spaces. Next, relating to the finiteness of the modular, we consider the continuity of the modular norms (§ 3). However, our main purpose lies in the application of the formula in question to the modulared function space to get a generalization of Orlicz-Birnbaum's Δ_2-condition (§ 4). Moreover, we discuss the conjugate property of the finite modular in connection with those characterizations (§ 5).

Unless otherwise stated, m is always monotone complete throughout this paper. And that is not too restrictive, because our problems are ultimately concerned with the modulared function space.

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1) The modulared space is defined by H. Nakano [3] and have been studied mainly by him and his school.

2) Terminologies and notations in this paper are also due to [3].

3) m is said to be monotone complete if $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda}$ with $\sup_{\lambda \in \Lambda} m(x_\lambda) < \infty$ implies the existence of $\bigcup_{\lambda \in \Lambda} x_\lambda$.

§ 1. Finite Modulars on Non-atomic Spaces.

$0<x\in R$ is said to be *an atomic element* if $[x]$[4]$\geqq[y]\geqq 0$ implies $[x]=[y]$ or $[y]=0$. If for any $x\neq 0$ there exists an atomic element y with $[y]\leqq[x]$, R is said to be *atomic*. If R does not contain any atomic element, it is said to be *non-atomic*.

W. Orlicz and Z. Birnbaum proved in [6] that a non-atomic Orlicz space $L_{M(\xi)}(\Omega)$[5] is finite if and only if the function M satisfies, for some $\gamma>0$ and $\xi_0\geqq 0$, the condition:

(Δ_2) $$M(2\xi)\leqq\gamma M(\xi) \quad \text{for all } \xi\geqq\xi_0$$

where ξ_0 must be zero if $\mu(\Omega)=\infty$.

In a non-atomic modulared space R, the result of I. Amemiya [1] enables us to obtain the following main theorem.

Theorem 1. *Let R be non-atomic. m is finite if and only if there exist $K>0$ and $c\in R$ with $m(c)<\infty$ such that*

(F) $$m(2x)\leqq Km(x)+m([x]c) \quad \textit{for all } x\in R.$$

It is evident that (F) implies the finiteness of m. In order to prove the converse, we show the next two lemmas.

Lemma 1. *If m is finite, then there exist $\varepsilon,\delta>0$ such that*

$$m(2x)\leqq\delta \quad \textit{for all } x \textit{ with } m(x)\geqq\varepsilon.$$

Proof. If otherwise, there exists a sequence of elements $x_\nu\geqq 0$ such that

$$m(x_\nu)\leqq\frac{1}{2^\nu} \quad \text{and} \quad m(2x_\nu)\geqq\nu \ (\nu=1,2,\cdots)$$

and consequently, that implies the existence of $x_0=\bigcup_{\nu=1}^{\infty} x_\nu$ with $m(x_0)\leqq 1$ by the monotone completeness of m. On the contrary, we have

$$m(2x_0)\geqq m(2\bigcup_{\nu=1}^{n} x_\nu)\geqq m(2x_n)\geqq n \quad (n\geqq 1)$$

which contradicts the finiteness of m.

Lemma 2. (I. Amemiya). *If R is non-atomic and is finite, then there exist $\varepsilon,\gamma>0$ such that*

$$m(2x)\leqq\gamma m(x) \quad \textit{for all } x \textit{ with } m(x)\geqq\varepsilon.$$

Proof. Let ε be the same as in Lemma 1. If $m(x)\geqq\varepsilon$ there exists an integer n such that

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4) $[x]$ denotes the projector by x : $[x]y=\bigcup_{\nu=1}^{\infty}(\nu\,|\,x\,|\frown y)$ for all $0\leqq y\in R$.

5) See § 4.

$$\varepsilon n \leqq m(x) < \varepsilon(n+1)\,.$$

Since R is non-atomic, we can decompose x orthogonally into $\{x_\nu\}$ such that

$$x=\sum_{\nu=1}^{n+1}\oplus x_\nu \quad \text{with} \quad m(x_\nu)\leqq\varepsilon \quad (\nu=1, 2, \cdots, n+1).$$

Therefore

$$m(2x)=\sum_{\nu=1}^{n+1} m(2x_\nu)\leqq(n+1)\delta\leqq\frac{2\delta}{\varepsilon}m(x)$$

by Lemma 1.

Remark 1. The proofs for Lemma 1 and 2 are simpler than those given for Lemma 2 by both [1] and [7].

Proof of Theorem 1.

Let ε and γ be the same as in Lemma 2. Putting

$$(\sharp) \qquad \boldsymbol{H}\equiv\{x;\ x\geqq 0,\ m(2[p]x)\geqq(\gamma+1)m([p]x) \text{ for all } p\in R\}\,,$$

$\boldsymbol{H}$ is a directed system. Because, for any two elements $x, y\in\boldsymbol{H}$, since by putting $[p]=[(x-y)^+]$ it follows $x\smile y=[p]x+(\mathbf{1}^{6)}-[p])y$, we have, for all $q\in R$

$$\begin{aligned} &m(2[q](x\smile y))\\ &\quad=m(2[q][p]x)+m(2[q](\mathbf{1}-[p])y)\\ &\quad\geqq(\gamma+1)\{m([q][p]x)+m([q](\mathbf{1}-[p])y)\}\\ &\quad=(\gamma+1)m([q](x\smile y)) \end{aligned}$$

by (♯). That is, $x\uparrow_{x\in\mathfrak{H}}$ and $m(x)<\varepsilon$ $(x\in\boldsymbol{H})$ by Lemma 2. Therefore there exists $x_0=\bigcup_{x\in\boldsymbol{H}} x$ with $m(x_0)\leqq\varepsilon$ by the monotone completeness of m.

Now, for any $x\geqq 0$, by Hahn's decomposition[7], there exists $[p_x]\leqq[x]$ such that

$$m(2[p]x)\geqq(\gamma+1)m([p]x) \quad \text{for all } [p]\leqq[p_x]$$

with $m(2(\mathbf{1}-[p_x])x)\leqq(\gamma+1)m((\mathbf{1}-[p_x])x)$ i.e. $[p_x]x\in\boldsymbol{H}$ which implies $[p_x]x\leqq x_0$. Thus, we have

$$\begin{aligned} m(2x)&\leqq m(2[p_x]x)+m(2(\mathbf{1}-[p_x])x)\\ &\leqq m(2[x]x_0)+(\gamma+1)m((\mathbf{1}-[p_x])x)\\ &\leqq m(2[x]x_0)+(\gamma+1)m(x)\,. \end{aligned}$$

Hence the proof completes by taking $2x_0=c$ and $\gamma+1=K$ respectively.

Q.E.D.

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6) 1 denotes the identity operator on R.

7) For Hahn's decomposition theorem, see, for instance, [4; Th. 16.1].

Remark 2. An immediate consequence of the condition (F) is that m is upper bounded[8] on $(1-[c])R$. So, if m is finite and R has no complete element[9] then there exists a normal manifold[10] $N(\neq 0)$ of R such that m is upper bounded on N.

In virtue of Th. 1, we can give another proof for Th. 55.10 by H. Nakano [3].

Corollary (H. Nakano). *Let R be non-atomic and having no constant[11] complete element. If m is finite and constant[12] then m is upper bounded.*

Proof. For any constant element $c\in R$, we have

$$m(2c)\leqq Km(c)\,.$$

If otherwise, i.e. there exists a constant c with $m(2c)>Km(c)$, then there exists a complete orthogonal system of constant elements c_λ $(\lambda\in\Lambda)$ such that

$$\frac{m(2c_\lambda)}{m(c_\lambda)}=\frac{m(2c)}{m(c)}>K \quad (\lambda\in\Lambda)$$

by Th. 55.5 in [3], and so, we have

$$m(2[p]c_\lambda)\geqq Km([p]c_\lambda) \quad \text{for all } [p].$$

Therefore, for $\boldsymbol{H}$ in the proof of Th. 1, we have $\boldsymbol{H}\ni c_\lambda$ $(\lambda\in\Lambda)$, which implies the existence of $c_0=\sum_{\lambda\in\Lambda}\oplus c_\lambda$ because $\boldsymbol{H}$ is order bounded. That contradicts an assumption because c_0 is a complete constant element.

Next, if we put $\xi_c=\sup_{m(\xi c)=0}\xi$ for any constant $c\in R$, we have

$$m(2\xi c)\leqq Km(\xi c) \quad \text{for all } \xi\geqq\xi_c,$$

because if $0<m(\xi c)<\infty$ then ξc is also constant. Here, we have $\xi_c=0$ because if $\xi_c>0$ we have for some $\xi_1>\xi_c$

$$0<Km(\xi_1 c)<m(2\xi_c c)<m(2\xi_1 c)\,,$$

which is a contradiction.

Therefore, we have for any constant c

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8) m is said to be upper bounded if $m(2x)\leqq\gamma m(x)$ $(x\in R)$ for some $\gamma>0$.

9) An element $x\in R$ is said to be complete if any $y\in R$ with $x\perp y$ implies $y=0$.

10) A manifold $N\subset R$ is said to be normal if for any $a\in R$, there exist $x, y\in R$ with $a=x+y$ $(x\in N,\ y\in N^\perp)$.

11) A simple element (see 15)) $c\in R$ is said to be constant if

$$\frac{m(\xi[p]c)}{m([p]c)}=\frac{m(\xi c)}{m(c)} \quad \text{for all } \xi\geqq 0 \text{ and } [p].$$

12) m is said to be constant if for any $x, y\neq 0$ there exists a constant element $c\in R$ with $[c]x,\ [c]y\neq 0$.

$$m(2\xi c)\leqq Km(\xi c) \quad \text{for all } \xi\geqq 0$$

which implies that m is upper bounded on $[c]R\neq 0$ where K does not depend on constant c. Therefore m is upper bounded on R.

Q.E.D.

§ 2. Finite Modulars on Atomic Spaces.

As for an Orlicz sequence space $l_{M(\xi)}$, W. Orlicz and Z. Birnbaum proved in [6] that $l_{M(\xi)}$ defined by a function M with $0<M(\xi)<\infty$ for all $\xi>0$ is finite if and only if there exist $\gamma, \xi_0>0$ such that

$(\Delta_2 s)$ $$M(2\xi)\leqq\gamma M(\xi) \quad \text{for all } 0\leqq\xi\leqq\xi_0.$$

In the atomic modulared space, we shall, for the characterization of the finite modular, obtain the condition (f) which has a closer form than that by T. Shimogaki [7] in generalizing $(\Delta_2 s)$ as is shown in the next theorem.

Theorem 2. *R is atomic, then almost finite*[13] *modular m is finite if and only if there exist $\varepsilon, \gamma>0$ and $c\in R$ with $m(c)<\infty$ such that*

(f) $$m(2x)\leqq\gamma m(x)+m([x]c)$$

for all x with $m(x)\leqq\varepsilon$.

Proof. In order to prove this theorem, we first show that the finiteness of m implies (f).

Since m is finite, for $\varepsilon, \delta>0$ in Lemma 1, we have easily

(*) $$\frac{\varepsilon}{3}\leqq m(x)\leqq\varepsilon \text{ implies } m(x)\leqq\frac{3\delta}{\varepsilon}m(x).$$

If we put

(♯) $$\boldsymbol{K}=\Big\{x;\ x\geqq 0,\ m(x)\leqq\frac{\varepsilon}{3}$$
$$\text{and } m(2[p]x)\geqq\frac{4\delta}{\varepsilon}m([p]x) \text{ for all } [p]\leqq[x]\Big\}$$

then $\boldsymbol{K}$ is a directed system. In fact, for any two $x_1, x_2\in\boldsymbol{K}$ we have

(**) $$m(x_1\smile x_2)\leqq m(x_1)+m(x_2)\leqq 2\cdot\frac{\varepsilon}{3}\leqq\varepsilon.$$

On the other hand, putting $[p]=[(x_1-x_2)^+]$, we have

$$m(2(x_1\smile x_2))=m(2[p]x_1)+m(2(1-[p])x_1)$$
$$\geqq\frac{4\delta}{\varepsilon}m([p]x_1)+\frac{4\delta}{\varepsilon}m((1-[p])x_2)=\frac{4\delta}{\varepsilon}m(x_1\smile x_2)$$

13) See § 3.

by (♯). Therefore, by (*) and (**), $m(x_1 \smile x_2) < \frac{\varepsilon}{3}$. Similarly, we have

$$m(2[p](x_1 \smile x_2)) \geqq \frac{4\delta}{\varepsilon} m([p](x_1 \smile x_2)) \quad \text{for all } [p].$$

That implies $x_1 \smile x_2 \in \boldsymbol{K}$. Thus, since $x \uparrow_{x \in K}$ and $m(x) \leqq \frac{\varepsilon}{3} (x \in \boldsymbol{K})$, there exists $x = \bigcup_{x \in K} x$ with $m(x_0) \leqq \varepsilon$ by the monotone completeness of m.

For any $x \geqq 0$ with $m(x) \leqq \varepsilon$, by Hahn's decomposition, there exists $[p_x] \leqq [x]$ such that

$$m(2[p])x \geqq \frac{4\delta}{\varepsilon} m([p]x) \quad \text{for all } [p] \leqq [p_x]$$

with $m(2(\mathbf{1}-[p_x])x) \leqq \frac{4\delta}{\varepsilon} m((\mathbf{1}-[p_x])x)$.

By (*), $m([p_x]x) < \frac{\varepsilon}{3}$, and so, $[p_x]x \in \boldsymbol{K}$.

Therefore

$$\begin{aligned} m(2x) &= (2(\mathbf{1}-[p_x])x) + m(2[p_x]x) \\ &\leqq \frac{4\delta}{\varepsilon} m((\mathbf{1}-[p_x])x) + m(2[x]x_0) \\ &\leqq \frac{4\delta}{\varepsilon} m(x) + m(2[x]x_0). \end{aligned}$$

Thus, the proof completes by putting $2x_0 = c$ and $\frac{4\delta}{\varepsilon} = \gamma$.

Next, we prove that (f) implies the finiteness of m. For any x with $m(x) < \infty$, there exists $[p_x] \leqq [x]$ such that $m([p_x]x) \leqq \varepsilon$ and $([x]-[p_x])R$ is finite dimensional. Since

$$m(2([x]-[p_x])x) < \infty$$

and

$$m(2[p_x]x) \leqq \gamma\varepsilon + m([x]c) < \infty$$

by the almost finiteness of m and by (f) respectively, we have $m(2x) < \infty$. Q.E.D.

Remark 3. In Th. 2, the assumption that R is atomic is needed only to prove that (f) implies the finiteness of m.

§ 3. Continuity of Modular Norms Relating to Finiteness of Modulars.

In the modulared space R, we can define the norm:

$$\|x\|^{14)} = \inf_{m(\xi x) \leqq 1} \frac{1}{|\xi|} \quad (x \in R).$$

14) We can define two kinds of mutually equivalent norms on R. That stated here is called the second norm and denoted by $|||\ |||$ usually.

$||\ ||$ is said to be *continuous* if $\inf_{\nu\geqq 1}||x_\nu||=0$ for all $0\leqq x_\nu\downarrow_{\nu=1}^{\infty}0$. m is said to be *almost finite* if for any $0<x\in R$ there exists $0<y\leqq x$ such that $m(\xi y)<\infty$ for all $\xi\geqq 0$.

H. Nakano [3; Th. 44.9] proved that if a modular m is almost finite then the continuity of $||\ ||$ is equivalent to the finiteness of m.

In this section, we shall show that the continuity of $||\ ||$ is characterized by the same condition as (f) except for a finite dimensional normal manifold of R.

Theorem 3. *$||\ ||$ is continuous if and only if there exist $\varepsilon, \gamma>0$ and a normal manifold $N\subset R$ such that*

$$(C)\qquad \begin{cases} \text{i)} & \textit{the dimension of } N^{\perp} \textit{ is finite,} \\ \text{ii)} & x\in N,\ m(x)\leqq\varepsilon \textit{ imply } m(2x)\leqq\gamma. \end{cases}$$

Proof.

1. The continuity of $||\ ||$ implies (C).

Let N_λ $(\lambda\in\Lambda)$ be the totality of the normal manifolds in each of which ii) holds for some $\varepsilon, \gamma>0$ (these two numbers may be taken depending on every N_λ). Then for two $\lambda_1, \lambda_2\in\Lambda$, there exist $\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2>0$ such that

$$x\in N_{\lambda_1},\ m(x)\leqq\varepsilon_1 \quad \text{imply } m(2x)\leqq\gamma_1$$
$$y\in N_{\lambda_2},\ m(y)\leqq\varepsilon_2 \quad \text{imply } m(2y)\leqq\gamma_2$$

and so, for $z\in([N_{\lambda_1}]\cup[N_{\lambda_2}])R$

$$m(z)\leqq \operatorname{Min}\{\varepsilon_1, \varepsilon_2\} \quad \text{imply } m(2z)\leqq\gamma_1+\gamma_2.$$

Therefore the system is directed: $N_\lambda\uparrow_{\lambda\in\Lambda}$ and we set

$$[N]^{15)}=\bigcup_{\lambda\in\Lambda}[N_\lambda].$$

Proof of i). Suppose the contrary. We can construct a mutually disjoint sequence of normal manifolds M_ν $(\nu=1, 2, \cdots)$ such that

$$[N^{\perp}]\geqq[M_\nu]>0 \quad \text{and} \quad [M_i][M_j]=0 \ (i\neq j)$$

and such that ii) is not valid on any $[M_\nu]R$. Then there exists a sequence of elements $\{x_\nu\}$ such that

$$[M_\nu]R\ni x_\nu\geqq 0,\quad m(x_\nu)\leqq\frac{1}{2^\nu} \quad \text{and} \quad m(2x_\nu)\geqq\nu \ (\nu=1, 2, \cdots).$$

By the monotone completeness of m, there exists $x_0=\sum_{\nu=1}^{\infty}\oplus x_\nu$ because $m\left(\sum_{\nu=1}^{\infty}\oplus x_\nu\right)=\sum_{\nu=1}^{\infty}m(x_\nu)\leqq 1$.

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15) $[N]$ denotes the projection operator onto N.

This implies

$$y_n=\sum_{\nu=n}^{\infty} x_\nu \downarrow_{n=1}^{\infty} 0 \quad \text{and} \quad m(2y_n)\geqq\sum_{\nu=n}^{\infty} m(2x_\nu)=\infty$$

which contradicts the continuity of $||\ ||$.

Proof of ii). Whithout loss of generality, we can assume $[N]=\mathbf{1}$. If ii) does not hold on R, there exists $0<y_1\in R$ such that

$$m(y_1)\leqq\frac{1}{2} \quad \text{and} \quad m(2y_1)>1.$$

Since $[N_\lambda]\uparrow_{\lambda\in\Lambda}\mathbf{1}$, there exists $[N_{\lambda_1}]$ such that

$$m([N_{\lambda_1}]y_1)\leqq\frac{1}{2} \quad \text{and} \quad m(2[N_{\lambda_1}]y_1)>1.$$

We set $x_1=[N_{\lambda_1}]y_1$. Next, since ii) does not hold too in $[N_{\lambda_1}^{\perp}]R$ (if otherwise, it contradicts the assumption that ii) is not valid on the whole R), there exists also $0\leqq y_2\in[N_{\lambda_1}^{\perp}]R$ such that

$$m(y_2)\leqq\frac{1}{2^2} \quad \text{and} \quad m(2y_2)>2$$

and so, there exists $[N_{\lambda_2}]$ $(\geqq[N_{\lambda_1}])$ such that

$$m([N_{\lambda_2}])\leqq\frac{1}{2} \quad \text{and} \quad m(2[N_{\lambda_2}]y_2)>2.$$

We set $x_2=[N_{\lambda_2}]y_2$ again. Then, we have clearly $x_1, x_2\in[N_{\lambda_2}]R$ and $x_1\perp x_2$.

Thus, we can construct consecutively by induction an orthogonal sequence of elements $\{x_\nu\}$ such that

$$m(x_\nu)\leqq\frac{1}{2^\nu} \quad \text{and} \quad m(2x_\nu)\geqq\nu \ (\nu=1, 2, \cdots).$$

This contradicts the continuity of $||\ ||$ by the same reason as is used in the proof for i).

2. (C) implies the continuity of $||\ ||$.

By the definition of the modular, for any $0\leqq x\in R$ there exists an integer $\nu_0=\nu_0(x)$ such that $m\left(\frac{1}{2^{\nu_0}}x\right)<\infty$. For any $[p_\nu]\downarrow_{\nu=1}^{\infty}0$, there exists $\nu_1>0$ such that

$$[p_{\nu_1}][N^{\perp}]=0 \text{ i.e. } [p_\nu]x\in N \quad \text{for all } \nu\geqq\nu_1,$$

because $N^{\perp}$ is finite dimensional. Therefore there exists $n_1\geqq\nu_1$ such that $m\left(\frac{1}{2^{\nu_0}}[p_{n_1}]x\right)\leqq\varepsilon$ and this implies

$$m\left(\frac{2}{2^{\nu_0}}[p_{n_1}]x\right)\leqq\gamma$$

by (C). Further, we can find $n_2 \geqq n_1$ such that

$$[p_{n_2}] \leqq [p_{n_1}] \quad \text{with} \quad m\left(\frac{2}{2^{\nu_0}}[p_{n_2}]x\right) \leqq \varepsilon$$

i.e.

$$m\left(\frac{2}{2^{\nu_0}}[p_{n_2}]x\right) \leqq \gamma$$

also by (C). Proceeding consecutively, there exists n_{ν_0} such that

$$m([p_{n_{\nu_0}}]x) = m\left(\frac{2^{\nu_0}}{2^{\nu_0}}[p_{n_{\nu_0}}]x\right) \leqq \gamma < \infty.$$

From this, we get $\inf_{\nu \geqq 1} m([p_\nu]x) = 0$ which implies the continuity of $\| \|$ as is easily seen. Q.E.D.

Now next two corollaries are immediate consequences.

Corollary 1. *$\| \|$ is continuous if and only if there exist $\varepsilon, \gamma > 0$ and $c \in R$ with $m(c) < \infty$ and a normal manifold $N \subset R$ such that*

i) *the dimension of N is finite,*

ii) *$x \in N$, $m(x) \leqq \varepsilon$ imply $m(2x) \leqq \gamma m(x) + m([x]c)$.*

Corollary 2. *Let R be non-atomic. The continuity of $\| \|$ is equivalent to the condition* (F) *on R.*

Remark 4. T. Andô [2; Th. 5] found a necessary and sufficient condition in order that $\| \|$ is continuous. That condition is nothing but to say, that there exist $\varepsilon, \xi > 0$ and a normal manifold $N \subset R$ such that the dimension of N is finite and $m(\xi x) \leqq 1$ for all x with $m(x) \leqq \varepsilon$.

Remark 5. If m is simple[16] and satisfies the condition (C) then m is uniformly simple[17] by Th. 2.1 by S. Yamamuro [8].

§ 4. Applications to the Modulared Function Spaces.

Let Ω be an abstract space and μ be a totally additive measure defined on a Borel field $\mathfrak{B}$ of subsets of Ω with $\Omega = \bigcup_{\mu(E)<\infty} E$.

Let $\Phi(\xi, \omega)$ $(\xi \geqq 0,\ \omega \in \Omega)$ be a function satisfying the following conditions: 1) Φ is a measurable function Ω (for all $\xi \geqq 0$) and a non-decreasing convex function of $\xi \geqq 0$ (for all $\omega \in \Omega$); 2) Φ is continuous from the left as a function of $\xi \geqq 0$ with $\Phi(0, \omega) = 0$ (for all $\omega \in \Omega$); 3) $\inf_{\xi > 0} \Phi(\xi, \omega) = 0$ and $\sup_{\xi > 0} \Phi(\xi, \omega) = \infty$ (for all $\omega \in \Omega$).

16) $s \in R$ is said to be simple if $m(s) < \infty$ and $m([p]s) = 0$ imply $[p]s = 0$ for all $[p]$.

17) m is said to be uniformly simple if $\inf_{\|x\| \geqq 1} m(\xi x) > 0$ for all $\xi > 0$.

For any measurable function $x(\omega)$ on Ω, $\Phi(|x(\omega)|, \omega)$ is also measurable. In stead of writing $\Phi(|\xi|, \omega)$ for $\xi<0$, we denote merely $\Phi(\xi, \omega)$ too. We denote by L_Φ (or by $L_\Phi(\Omega)$ if necessary) the totality of all measurable functions $x(\omega)$ on Ω such that for some $\alpha=\alpha(x)<\infty$

$$\int_\Omega \Phi(\alpha x(\omega), \omega)d\mu<\infty .$$

With the order $x\leqq y$ $(x, y\in L_\Phi)$ if $x(\omega)\leqq y(\omega)$ a.e.[18] on Ω, L_Φ is a universally continuous semi-ordered linear space. We get, moreover, a monotone complete modular m_Φ defined by

$$m_\Phi(x)=\int_\Omega \Phi(x(\omega), \omega)d\mu$$

on L_Φ. So, L_Φ with m_Φ is said to be *a modulared function space.* Every semi-normal manifold[19] of L_Φ is also considered as a modulared space, however, in general, it is not monotone complete.

If $\Phi(\xi, \omega)=\xi^{p(\omega)}$ $(\xi\geqq 0,\ \omega\in\Omega)$ for a measurable function $p(\omega)$ with $1\leqq p(\omega)\leqq\infty$ on Ω, then L_Φ is denoted by $L_{p(\omega)}$[20] and if furthermore $p(\omega)=p=$*constant* $(\omega\in\Omega)$ then L_p is the usual L_p-space.

If $\Phi(\xi, \omega)=M(\xi)$ $(\xi\geqq 0,\ \omega\in\Omega)$ i.e. the value of Φ depends only on $\xi\geqq 0$, then L_Φ is said to be *an Orlicz space* and in this paper we denoted it by $L_{M(\xi)}$.

$L_{p(\omega)}$ and L_M are the two special types of L_Φ.

Let us start by stating the next lemma which makes brief the discussions in this section.

Lemma. *m_Φ is almost finite if and only if*

(A) $$\Phi(\xi, \omega)<\infty \quad \textit{for all } \xi\geqq 0 \textit{ and a.e. on } \Omega.$$

Proof. It is clear that (A) is valid if m_Φ is almost finite. For the proof of the converse, we put, for any fixed $E\in\mathfrak{B}$ with $\mu(E)<\infty$,

$$E_{n,\nu}=\{\omega\,;\ \Phi(n, \omega)>\nu\}\frown E \quad (\nu, n=1, 2, \cdots).$$

Since $\mu\left(\bigcap_{\nu=1}^{\infty} E_{n,\nu}\right)=0$ $(n\geqq 1)$, we can find a subsequence $\{\nu_n\}$ such that

$$\mu(E_{n,\nu_n})\leqq\frac{\mu(E)}{2^{n+1}} \quad (n\geqq 1).$$

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18) "a.e. (almost everywhere)" means always, in this paper, that "except on some $A\in\mathfrak{B}$ which $\mu(E\frown A)=0$ for all $\mu(E)<\infty$".

19) A linear manifold $M\subset R$ is said to be semi-normal if $x\in M$, $|x|\geqq|y|$ imply $y\in M$.

20) $L_{p(\omega)}$ is defined and discussed mainly by H. Nakano [3 and 5].

Putting $E_1=\bigcup_{n=1}^{\infty} E_{n,\nu_n}$ and $E_2=E-E_1$ respectively, we have

$$\mu(E_1)\leqq\sum_{n=1}^{\infty}\frac{\mu(E)}{2^{n+1}}\leqq\frac{\mu(E)}{2} \quad \text{i.e. } \mu(E_2)\geqq\frac{\mu(E)}{2}.$$

And so, $E_2\subseteqq E-\bigcup_{n=1}^{\infty}E_{n,\nu_n}\subseteqq E-E_{n,\nu_n}$ $(n\geqq 1)$ implies

$$m_\Phi(n\chi_{E_2})^{21)}=\int_{E_2}\Phi(n,\omega)d\mu\leqq\int_{E-E_{n,\nu_n}}\Phi(n,\omega)d\mu\leqq\nu_n\cdot\mu(E)<\infty$$

i.e, χ_{E_2} is a finite element. Thus, for any $0<x\in L_\Phi$, there exists a finite element $0<y_x$ $(\leqq x)$ and this is nothing but the definition of the almost finiteness of m_Φ. Q.E.D.

Now, on the modulared function space, we shall present the formula corresponding to (Δ_2). First, on the case that R is non-atomic, we have as a consequence of (F) in Th. 1

Theorem 1′. *Let μ be non-atomic. m_Φ is finite if and only if there exist $K>0$ and $h(\omega)\in L_1(\Omega)$ such that*

$$(\Delta_2') \qquad \Phi(2\xi,\omega)\leqq K\Phi(\xi,\omega)+h(\omega)$$

for all $\xi\geqq 0$ and a.e. on Ω.

Proof. That the finiteness of m_Φ follows from (Δ_2) is evident. Conversely, it is easy to see that if m_Φ is finite then (Δ_2) holds by (F) in Th. 1 and by the property of integration. Q.E.D.

Remark 6. If we apply (Δ_2') only to the Orlicz space $L_{M(\xi)}$, we have that L_M is finite if and only if there exist $\gamma>0$, $\alpha\geqq 0$ such that

$$(\Delta_2') \qquad M(2\xi)\leqq\gamma M(\xi)+\alpha \quad \text{for all } \xi\geqq 0$$

where α must be zero if $\mu(\Omega)=\infty$ because $\alpha\in L_1(\Omega)$. And it is clear that (Δ_2') is equivalent to (Δ_2) restricted in a Orlicz space.

Next, we shall consider the case μ is atomic. If μ is atomic, without loss of generality, we can assume $\mu(\omega)=1$ for all $\omega\in\Omega$ and so,

$$m_\Phi(x)=\sum_{\omega\in\Omega}\Phi(x(\omega),\omega) \quad \text{for all } x\in L_\Phi.$$

We find for this case the formula corresponding to $(\Delta_2 s)$ to the Orlicz sequence space as in the next theorem.

Theorem 2′. *Let μ be atomic. m_Φ defined by Φ with $\Phi(\xi,\omega)<\infty$ for all $\xi\geqq 0$ and $\omega\in\Omega$ is finite if and only if there exist $\gamma>0$ and $\xi_\omega,\eta_\omega\geqq 0$ $(\omega\in\Omega)$ such that*

21) χ_E denotes the characteristic function of $E\in\mathfrak{B}$.

$$(\Delta_2'\text{s}) \quad \begin{cases} \text{a)} & \sum_{\omega\in\Omega} \eta_\omega < \infty\,, \\ \text{b)} & \inf_{\omega\in\Omega} \Phi(\xi_\omega, \omega) > 0\,, \\ \text{c)} & \Phi(2\xi, \omega) \leqq \gamma\Phi(\xi, \omega)+\eta_\omega \text{ for all } 0\leqq\xi\leqq\xi_\omega \text{ and } \omega\in\Omega\,. \end{cases}$$

Proof. For the proof, by Lemma it suffices to show that $(\Delta_2'\text{s})$ in Th. 2′ is equivalent to (f) in Th. 2.

1. (f) implies a), b), and c) of $(\Delta_2'\text{s})$.

We take $\varepsilon, \gamma>0$ and $c\in L_\Phi$ the same ones as in (f). Putting

$$\eta_\omega=\Phi(c(\omega), \omega) \quad \text{and} \quad (\sharp) \quad \xi_\omega = \sup_{\Phi(\xi,\omega)\leqq\varepsilon} \xi$$

for all $\omega\in\Omega$, then a) and c) are evident. Since $\Phi(\xi_\omega, \omega)\leqq\varepsilon$, we have

$$\Phi(2\xi_\omega, \omega) \leqq \gamma\varepsilon+\eta_\omega < \infty$$

by c). And by $\varepsilon<\Phi(2\xi_\omega, \omega)$, there exists $0<\xi<2\xi_\omega$ with $\Phi(\xi, \omega)=\varepsilon$ for all $\omega\in\Omega$, because $\Phi(\xi_\omega, \omega)>0$ by $(\sharp)$. Thus $\Phi(\xi_\omega, \omega)=\varepsilon$ for all $\omega\in\Omega$ which implies b).

2. $(\Delta_2'\text{s})$ implies (f).

Let ε be any fixed $0<\varepsilon<\inf_{\omega\in\Omega} \Phi(\xi_\omega, \omega)$ then $m_\Phi(x)\leqq\varepsilon$ implies $|x(\omega)|\leqq\xi_\omega$ $(\omega\in\Omega)$. And so

$$(*) \qquad \Phi(2x(\omega), \omega) \leqq \gamma\Phi(x(\omega), \omega)+\eta_\omega \qquad (\omega\in\Omega)\,.$$

Putting $c(\omega) = \sup_{\Phi(\xi,\omega)\leqq\eta_\omega} \xi$, we have $\eta_\omega\geqq\Phi(c(\omega), \omega)$. Then (*) implies

$$\Phi(2x(\omega, \omega)) \leqq \gamma\Phi(x(\omega), \omega)+\Phi(c(\omega), \omega)\,.$$

Therefore $$m_\Phi(2x)\leqq\gamma m_\Phi(x)+m_\Phi([x]c)$$

where $\{c(\omega)\}=c\in L_\Phi$. Q.E.D.

Remark 7. By applying the condition $(\Delta_2'\text{s})$ to an Orlicz sequence space $l_{M(\xi)}$, we obtain that l_M defined by M with $0<M(\xi)<\infty$ for all $\xi>0$ is finite if and only if there exist $\gamma, \delta>0$ such that

$$(\Delta_2\text{s}) \qquad M(2\xi) \leqq \gamma M(\xi) \quad \text{for all } 0\leqq\xi\leqq\delta\,,$$

because $\inf_{\omega\in\Omega} \xi_\omega=0$ and $\inf_{\omega\in\Omega} \eta_\omega>0$ by a) and b) of $(\Delta_2'\text{s})$ respectively (where, of couse, l_M is assumed to be infinite dimentional).

It is known by Th. 1 that if R is non-atomic and if m is monotone complete and finite then m is *uniformly finite*[22] (I. Amemiya [1]). Therefore, (F) in Th. 1 gives already a necessary and sufficient condition in order that m is uniformly finite.

22) m is said to be uniformly finite if $\sup_{\|x\|\leqq1} m(\xi x)<\infty$ for all $\xi>0$.

On the contrary, if R is atomic, the finiteness of R does not always imply the uniform finiteness of m. I. Amemiya showed a necessary and sufficient condition in order that a modular m is uniformly finite in the atomic space R. Relating to the result, we shall show the next corollary.

Corollary. *Let μ be atomic. m_Φ is uniformly finite if and only if*

i) *there exist $\alpha_\omega>0$ $(\omega\in\Omega)$ with $\Phi(\alpha_\omega,\omega)=1$ such that $\sup_{\omega\in\Omega}\Phi(\xi\alpha_\omega,\omega)<\infty$ for all $\xi\geqq 0$,*

ii) the same as $(\Delta_2'\text{s})$.

Proof. It is clear that the uniform finiteness of m_Φ implies i) and ii). For the converse, in order to prove the uniform finiteness of m_Φ, it suffice to show that for any $\alpha>1$ there exists $\beta>0$ such that

$$x\in L_\Phi,\quad m_\Phi(x)\leqq\alpha \quad \text{imply } m_\Phi(2x)\leqq\beta.$$

Let $\varepsilon=\inf_{\omega\in\Omega}\Phi(\xi_\omega,\omega)>0$ by b) of $(\Delta_2'\text{s})$.

Then, if $m_\Phi(x)\leqq\alpha$, the number of the elements in the set $\{\omega;|x(\omega)|>\xi_\omega\}$ is less than $\frac{\alpha}{\varepsilon}$ as well as it follows $|x(\omega)|\leqq\alpha\cdot\alpha_\omega$ $(\omega\in\Omega)$ by i) too. Therefore, we have

$$\begin{aligned} m_\Phi(2x) &= \sum_{|x(\omega)|>\xi_\omega}\Phi(2x(\omega),\omega)+\sum_{|x(\omega)|\leqq\xi_\omega}\Phi(2x(\omega),\omega) \\ &\leqq\frac{\alpha}{\varepsilon}\sup_{\omega\in\Omega}\Phi(2\alpha\cdot\alpha_\omega,\omega)+\sum_{\omega\in\Omega}\gamma\Phi(x(\omega),\omega)+\sum_{\omega\in\Omega}c_\omega \\ &=\frac{\alpha}{\varepsilon}\sup_{\omega\in\Omega}\Phi(2\alpha\cdot\alpha_\omega,\omega)+\gamma\cdot\alpha+\sum_{\omega\in\Omega}c_\omega=\beta \end{aligned}$$

by ii). Q.E.D.

We denote by $\|\ \|_\Phi$ the norm by m_Φ, and we consider on the continuity of $\|\ \|_\Phi$. In the case μ is non-atomic, the continuity of $\|\ \|_\Phi$ is equivalent to the condition (Δ_2') by Cor. 2 in § 3. For the case μ is atomic, we get the next (C').

Theorem 3'. *Let μ be atomic. $\|\ \|_\Phi$ is continuous if and only if there exist $\gamma>0$ and $\xi_\omega,\eta_\omega\geqq 0$ $(\omega\in\Omega)$ such that*

(C') a) $\sum_{\omega\in\Omega}\eta_\omega<\infty$,

b) $\inf_{\omega\in\Omega_0}\Phi(\xi_\omega,\omega)>0$ *for some $\Omega_0\subset\Omega$ such that $\Omega-\Omega_0$ is finite set.*

c) $\Phi(2\xi,\omega)\leqq\gamma\Phi(\xi,\omega)+\eta_\omega$ *for all $0\leqq\xi\leqq\xi_\omega$ and $\omega\in\Omega$.*

Proof. By the same way as in the proof for Th. 2', we see easily

that (C′) is equivalent to the fact that (Δ_2's) holds on

$$N\equiv\{x;\ x\in L_\varphi,\ x(\omega)=0 \quad \text{for } \omega\in\Omega-\Omega_0\}.$$

Therefore, in view of Cor. 1 in § 3, we can conclude that (C′) is equivalent to the continuity of $\|\ \|_\varphi$.

Remark 8. By (C′), we see that the modular norm $\|\ \|_M$ of the Orlicz sequence space l_M defined by M with $0<M(\xi)<\infty$ for all $\xi>0$ is continuous if and only if there exist $\gamma, \xi_0>0$ such that

$$M(2\xi)\leqq\gamma M(\xi) \quad \text{for all } 0\leqq\xi\leqq\xi_0.$$

§ 5. The Conjugate Property of Uniformly Finite Modulars.

Let m be finite throughout this section. The conjugate space $\overline{R}$ of R, the totality of universally continuous[23] linear functionals on R, coincides with the totality of modular bounded[24] linear functionals, because m is monotone complete. $\overline{R}$ is moreover a modulared space with the conjugate modular m:

$$\overline{m}(\overline{x})=\sup_{x\in R}\{\overline{x}(x)-m(x)\} \quad (\overline{x}\in\overline{R}).$$

It was proved by H. Nakano [5; § 86] that the conjugate modular $\overline{m}$ of a modular m is *uniformly increasing*[25] if and only if m is uniformly finite.

Our aim in this section is to characterize the uniformly increasing modular by the formula which is the conjugate of (F).

Theorem 4. *Let R be non-atomic. m is finite if and only if for the conjugate modular $\overline{m}$ of m, there exist $\gamma>2$ and $\overline{c}\in\overline{R}$ with $\overline{m}(\overline{c})<\infty$ such that*

(UI) $$\overline{m}\left(\frac{\gamma}{2}\overline{x}\right)+2\overline{m}([\overline{x}]\overline{c})\geqq\gamma\overline{m}(\overline{x}) \quad \textit{for all } \overline{x}\in\overline{R}.$$

Proof. We state only that (F) on m implies (UI) on its conjugate modular $\overline{m}$, because, in virtue of the reflexibility[26] of the modular m,

23) A linear functional $\overline{x}$ on R is said to be universally continuous if, for any $x_\lambda\downarrow_{\lambda\in\Lambda}0$ $(x_\lambda\in R)$, we have $\inf_{\lambda\in\Lambda}|\overline{x}(x_\lambda)|=0$.

24) A linear functional $\overline{x}$ on R is said to be modular bounded if $\sup_{m(x)\leqq1}|\overline{x}(x)|<\infty$.

25) m is said to be uniformly increasing if

$$\sup_{\xi>0}\inf_{\|x\|\geqq1}\frac{m(\xi x)}{\xi}=\infty.$$

26) See H. Nakano [3].

we can see conversely by the similar way that (UI) on $\overline{m}$ implies (F) on m.

For any $\overline{x}\in\overline{R}$, by (F), we have

$$(\sharp)\qquad \overline{m}(\overline{x})\leqq\sup_{x\in R}\left\{\overline{x}(x)-\frac{1}{\gamma}m(2x)+\frac{1}{\gamma}m([x]c)\right\}$$

$$\leqq\frac{1}{\gamma}\sup_{x\in R}\left\{\frac{\gamma}{2}\overline{x}(2[\overline{x}]^{R\,27)}x)-m(2[\overline{x}]x)+m([[\overline{x}]^{R}x]c)\right\}$$

$$\leqq\frac{1}{\gamma}\overline{m}\left(\frac{\gamma}{2}\overline{x}\right)+\frac{1}{\gamma}m([\overline{x}]^{R}c)\,.$$

Therefore, the proof is complete if we prove that for above $c>0$ there exist $\overline{c}\in\overline{R}$ with $\overline{m}(\overline{x})<\infty$ such that for some $[N_c]$

$$m([p]c)=\overline{m}(\overline{c}[p])\qquad\text{for all } [p]\leqq[N_c]$$

$$\overline{x}[N_c^{\perp}]\leqq\overline{c}[N_c^{\perp}]\qquad\text{for all } \overline{m}(\overline{x})<\infty\,.$$

Because, for any $\overline{x}$ with $\overline{m}\left(\frac{\gamma}{2}\overline{x}\right)<\infty$ we have

$$\gamma\overline{m}(\overline{x})=\gamma\overline{m}(\overline{x}[N_c])+\gamma\overline{m}(\overline{x}[N_c^{\perp}])$$

$$\leqq\overline{m}\left(\frac{\gamma}{2}\overline{x}[N_c]\right)+m([\overline{x}]^{R}[N_c]c)+\gamma\overline{m}\left(\frac{2}{\gamma}\overline{c}[N_c^{\perp}][\overline{x}]^{R}\right)$$

$$\leqq\overline{m}\left(\frac{\gamma}{2}\overline{x}\right)+2\overline{m}(\overline{c}[\overline{x}]^{R}[N_c])+2\overline{m}(\overline{c}[\overline{x}]^{R}[N_c^{\perp}])$$

$$=\overline{m}\left(\frac{\gamma}{2}\overline{x}\right)+2\overline{m}([\overline{x}]\overline{c})$$

by (♯). We can show the existence of such $\overline{c}$ by generalizing the method considered in H. Nakano [3; §62]. Q.E.D.

Corollary. *Let R be non-atomic. m is uniformly increasing if and only if m has the type* (UI).

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27) $[\overline{x}]^{R}=[\{x;\,|\overline{x}|(|x|)=0\}^{\perp}]$.

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