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PARTIALLY ORDERED ABELIAN SEMIGROUPS. III

ON THE REVERSIBLE PARTIAL ORDER DEFINED ON AN ABELIAN SEMIGROUP

By

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In their paper,¹⁾ Ben Dushnik and E. W. Miller introduced the concept of the reversible partial order and expressed the theorem about this concept. In this Part III, I shall show that the same one is held in the partially ordered abelian semigroup by adding the certain condition.

Definition 1. A set S is said to be a *partially ordered abelian semigroup* (p.o. semigroup), when S is (I) an abelian semigroup (not necessarily contains the unit element), (II) a partially ordered set, and satisfies (III) the *homogeneity*: $a \geq b$ implies $ac \geq bc$ for any c of S .

A partial order which satisfies the condition (III) is called a *partial order defined on an abelian semigroup*.

Moreover, if a partial order defined on an abelian semigroup S is a linear order, then S is said to be a *linearly ordered abelian semigroup* (l.o. semigroup). (Definition 1, O.I.)

Definition 2. Let $\mathfrak{S} = \{P_\alpha\}$ be any set of partial orders, each defined on the same abelian semigroup S . We define the new partial order P on S as follows: For any two elements a, b , we put $a \geq b$ in P if and only if $a \geq b$ in every P_α of the set $\mathfrak{S}$. Indeed, P is again a partial order defined on S . This partial order P is said to be the *product* of the partial orders P_α or to be *realized* by the set $\mathfrak{S}$ of partial orders P_α . (Definition 9, O.I.)

By the *dimension* of a partial order P defined on an abelian semigroup S is meant the smallest cardinal number m such that P is realized by m linear orders defined on S .

Definition 3. Let P and Q be two partial orders defined on the same

Partially ordered abelian semigroup. I. On the extension of the strong partial order defined on abelian semigroups. Journ. Fac. Sci., Hokkaido University, Series I, vol. XI (1951), pp. 181-189; this is referred to hereafter as "O.I."

1) Ben Dushnik and E. W. Miller: Partially ordered sets, Amer. Math. Journ. vol. 63 (1941), pp. 600-610.

abelian semigroup S , and suppose that *any* two distinct elements of S are comparable in *just one* of these partial orders; in such a case we shall say that P and Q are *conjugate* partial orders. A partial order will be called *reversible* if and only if it has a conjugate.²⁾

If P is a partial order defined on S , then the partial order obtained from P by inverting the sense of all ordered pairs will be called a *dual* order, which is denoted by P^* .

Theorem 1. *Let P and Q are conjugate partial orders defined on an abelian semigroup S . Then we can define a linear order L_1 on S such that $a > b$ in L_1 if and only if $a > b$ in either P or Q ; denoted by $L_1 = P + Q$. Similary $L_2 = P + Q^*$ is a linear order defined on S .*

Proof. We shall prove only the transivity of L_1 .

From $a > b$, $b > c$ in L_1 , we can consider the following four cases: (i) $a > b$, $b > c$ in P , (ii) $a > b$, $b > c$ in Q , (iii) $a > b$ in P , $b > c$ in Q , (iv) $a > b$ in Q , $b > c$ in P .

In cases (i) and (ii), $a > c$ in L_1 is cleary. In case (iii), if $c > a$ in P or Q , then $c > b$ in P or $b > a$ in Q respectively, which is absured, therefore $a > c$ in P or Q and hence in L_1 . Similary, in case (iv) $a > c$ in L_1 is held.

Theorem 2. *The following two properties of a partial order P defined on an abelian semigroup S are equivalent to each other:*

- (1) P is reversible.
- (2) The dimension of P is 2.

Proof. We shall show first that (1) implies (2). Suppose that the partial order P defined on S is reversible, and let Q be a partial order defined on S conjugate to P and Q^* be the dual order of Q . Then by Theorem 1, $L_1 = P + Q$ and $L_2 = P + Q^*$ are linear extensions of P and it is obvious that P is realized by linear orders L_1 and L_2 .

Next we show that (2) implies (1). Let L_1 and L_2 be any two linear orders defined on S which together realize P . We define the other order Q as follows: $a > b$ in Q if and only if a and b are non-comparable in P and $a > b$ in L_1 (likewise $a > b$ in L_1 and $b > a$ in L_2). Then $a > b$ and $b > a$ in Q are contradictory. If $a > b$ and $b > c$ in Q , then we have $a > b > c$ in L_1 and $c > b > a$ in L_2 , hence $a > c$ in Q . $a > b$ in Q implies that $ac \geq bc$ in L_1 and $bc \geq ac$ in L_2 , i.e. $ac \geq bc$ in Q . Therefore Q is a partial order defined on S . Evidently P and Q are conjugate.

Definieion 4. A linear extension L of a partial order P defined on

2) Cf. Ben Dushnik and E. W. Miller: l.c.

an abelian semigroup S will be called *separating* if and only if there exist three elements a, b and c in S such that $a > c$ in P , and b is not comparable with either a or c in P , while in L we have $a > b > c$.

Theorem 3.³⁾ *Let P be a partial order defined on an abelian semigroup S which satisfies the condition (E).⁴⁾ Then the following three properties of a partial order P are equivalent to each other:*

- (1) P is reversible.
- (2) The dimension of P is 2.
- (3) There exists a linear extension of P which is non-separating.

Proof. (1) and (2) are equivalent by Theorem 2.

We show now that (2) implies (3) without the condition (E). Let L_1 and L_2 be any two linear orders defined on S which together realize $P: P = L_1 \times L_2$. If L_1 is separating, then there exist three elements a, b and c such that $a > c$ in P , $a > b > c$ in L_1 and b is not comparable with either a or c in P . Hence we have $c > b > a$ in L_2 which is impossible.

To show that (3) implies (1) we shall suppose that L is a non-separating linear extension of P . We define the other order Q as follows: $a > b$ in Q if and only if a and b are non-comparable in P and $a > b$ in L . Then clearly $a > b$ and $b > a$ in Q are contradictory. If $a > b$ and $b > c$ in Q , then we have $a > b > c$ in L and a and c are non-comparable in P , for otherwise $a > c$ in P would imply that L is separating contrary to the assumption, hence we have $a > c$ in Q . $a > b$ in Q implies that $ac \geq bc$ in L . If $ac > bc$ in P , then by the condition (E) $a > b$ in P which is impossible. Hence $ac = bc$ or ac and bc are non-comparable in P , and hence $ac \geq bc$ in Q . Therefore Q is a partial order defined on S . Clearly P and Q are conjugate.

Definition 5. Let S be a p.o. semigroup and P be the partial order defined on S . For any element a of S , we denote the set of all elements x such that $x \leq a$ in P by $\bar{a}$. Then the correspondence $a \leftrightarrow \bar{a}$ is one-to-one. We put $\bar{a} \geq \bar{b}$ if and only if $\bar{b}$ is a subset of $\bar{a}$, likewise $a \geq b$ in P , then the family $\bar{S} = \{\bar{a}\}$ is become a partially ordered set. Next we define the product $\bar{a} \cdot \bar{b} = \overline{ab}$, then the family $\bar{S}$ is a commutative semigroup, moreover $\bar{S}$ become a p.o. semigroup. Clearly S and $\bar{S}$ are order-isomorphic.⁵⁾

More generally, if there exists a one-to-one correspondence between

3) Ben Dushnik and E. W. Miller: l.c. Theorem 3.61.

4) Condition (E) (order cancellation law):

$ac > bc$ in P implies $a > b$ in P .

5) See Definition 3, O.I.

the elements of the p.o. semigroup S and the family $\mathfrak{R}$ of subsets of the certain set R (a subset of R which corresponds with an element a of S , denote by $s(a)$), and $a \geq b$ in P if and only if $s(a) \supseteq s(b)$ (in the sense of set-inclusion), then by the defining the product $s(a) \cdot s(b) = s(ab)$, two p.o. semigroups S and $\mathfrak{R}$ are order-isomorphic.

Any family $\mathfrak{R}$ of the subsets of the set R which has the above properties will be called a *representation* of P .

Theorem 4. *Let P be a partial order defined on an abelian semigroup S which satisfies the condition (E). Then the following two properties are equivalent to each other:*

(1) P is reversible.

(4) *There exists a representation of P by means of a family $\mathfrak{R} = \{I_a\}$ of closed intervals on some l.o. semigroup R , and let $I_a = [\alpha_1, \alpha_2]$, $I_b = [\beta_1, \beta_2]$, $I_{ac} = [\gamma_1, \gamma_2]$, $I_{bc} = [\delta_1, \delta_2]$, and if a and b are non-comparable in P , then $\alpha_1 < \beta_1$ (and $\alpha_2 < \beta_2$) implies $\gamma_1 \leq \delta_1$ (and $\gamma_2 \leq \delta_2$) or its dual.*

Proof. We shall show (1) implies (4). Let P be reversible, and hence the dimension of P is 2. Let A and B be any two linear orders defined on S which together realize P .

Let S' be a l.o. semigroup which is anti-order-isomorphic to the l.o. semigroup S in the linear order B , where the set S' is disjoint from S , and the linear order defined on S' is denoted by B' .

Let R be the union of S , S' and the new element 0 which belongs to neither S nor S' .

We define the multiplication in R as follows:

$$0 \cdot 0 = 0,$$

$$x \cdot 0 = 0 \cdot x = 0 \quad \text{for any } x \text{ in } S \text{ or } S',$$

$$a \cdot a' = a' \cdot a = 0 \quad \text{for any } a \text{ in } S \text{ and } a' \text{ in } S',$$

and for any two elements x and y of $S(S')$ the product is the same as in $S(S')$.

Thus R becomes the abelian semigroup under the multiplication introduced above.

Let us now define the order-relation L in R as follows:

$$x > y \text{ in } L (x, y \in S) \quad \text{if and only if } x > y \text{ in } A,$$

$$x > y \text{ in } L (x, y \in S') \quad \text{if and only if } x > y \text{ in } B',$$

and we put

$$a > 0 > a' \text{ in } L (a \in S, a' \in S').$$

Then R becomes a l.o. semigroup.

For each a in S denote by a' the image of a in S' , and denote by I_a the closed interval $[a', a]$ of R .

We will show that the family $\mathfrak{R}=\{I_a\}$ of all such intervals is a representation of P . Suppose first that $a>b$ in P . Then $a>b$ in A and $a'<b'$ in B' , so that we have $a'<b'<b<a$ in L . This means that I_b is a proper subset of I_a .

Let $I_a=[a', a]$, $I_b=[b', b]$, $I_{ac}=[a'c', ac]$, $I_{bc}=[b'c', bc]$. If a and b are non-comparable in P , then from $a>b$ ($a'>b'$) in L we have $ac\geq bc$ ($a'c'\geq b'c'$) in L or its dual.

We prove that (4) implies (1). Suppose that P is a partial order which is represented by a family $\mathfrak{R}$ of intervals taken from some l.o. semigroup R , whose linear order is denoted by L . For each a in S , denote by I_a the interval of the family $\mathfrak{R}$ which corresponds to a . We notice first that if a and b are distinct elements of S which are not comparable in P , then I_a and I_b cannot have the same left (right)-hand end-point.

Suppose that $I_a=[\alpha_1, \alpha_2]$, $I_b=[\beta_1, \beta_2]$, $I_c=[\gamma_1, \gamma_2], \dots$.

We define a new partial order Q defined on S as follows:

(i) a and b are not comparable in P ,

(ii) $\alpha_1<\beta_1$ (and $\alpha_2<\beta_2$) in L .

It is easy to see that Q is the partial order defined on the set S . We shall now prove the homogeneity. Let $a>b$ in Q and $I_{ac}=[\lambda_1, \lambda_2]$, $I_{bc}=[\mu_1, \mu_2]$. If ac and bc are distinct and comparable in P , then by the condition (E) a and b are comparable in P which is impossible. If ac and bc are non-comparable in P , then $\lambda_1<\mu_1$, $\lambda_2<\mu_2$ and hence $ac>bc$ in Q .

Example 1. Let S_1 be an abelian semigroup generated by two elements a and b with the relation

$$a^m b^n = ab^n \quad \text{for any positive integers } m \text{ and } n.$$

By putting the order-relation

$$P: \quad \begin{cases} a^{m+1} > a^m \\ b^n > ab^n \end{cases} \quad \text{for any positive integer } n$$

S_1 becomes a p.o. semigroup, and the partial order P is reversible. Its conjugate order Q is as follows:

$$Q: \quad \begin{cases} a^m > b^n > b^{n+1} > ab^{n+1} \\ a^m > ab^n > b^{n+1} \end{cases} \quad \text{for any positive integers } m \text{ and } n.$$

The linear orders which together realize P are

$$a^{m+1} > a^m > b^n > ab^n > b^{n+1} > ab^{n+1}$$

and

$$b^{n+1} > ab^{n+1} > b^n > ab^n > a^{m+1} > a^m$$

for any positive integers m and n .

Example 2. Let S_2 be an abelian semigroup generated by two elements a and b with the relation

$$a^m b^n = b^n \quad \text{for any positive integers } m \text{ and } n.$$

By putting the two order-relations A and B

$$A: \quad a^{m+1} > a^m > b^n > b^{n+1}$$

$$B: \quad b^{n+1} > b^n > a^{m+1} > a^m$$

for any positive integers m and n ,

S_2 becomes a l.o. semigroup in the orders A and B respectively. Let P be the partial order which is the product of A and B , that is

$$P: \quad a^{m+1} > a^m.$$

Then P has the conjugate order Q such that

$$Q: \quad a^m > b^n > b^{n+1}.$$

Example 3. Let S'_3 and S''_3 be free abelian semigroups generated by elements a and b respectively. And by defining the order-relations

$$a^{m+1} > a^m \quad (m \geq 1), \quad b^{n+1} > b^n \quad (n > 1),$$

S'_3 and S''_3 becomes a l.o. semigroup and a p.o. semigroup respectively. Let S_3 be the direct product of S'_3 and S''_3 . Then S_3 becomes a p.o. semigroup by introducing the following order-relation P :

$$a^i b^j > a^m b^n$$

if and only if

$$a^i > a^m \quad \text{or} \quad a^i = a^m \quad \text{and} \quad b^j > b^n.$$

Since ab^2 and ab are non-comparable in P in spite of $(ab^2)(ab) = a^2b^3 > a^2b^2 = (ab)(ab)$, P does not satisfy the condition (E).

Now, in S''_3 we define the another order-relation:

$$b^{n+1} > b^n \quad (n \geq 1),$$

then we get the non-separating linear extension of P .

But we cannot realize the partial order P by two linear orders.

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