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# PARTIALLY ORDERED ABELIAN SEMIGROUPS. III

## ON THE REVERSIBLE PARTIAL ORDER DEFINED ON AN ABELIAN SEMIGROUP

By

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In their paper,[1] Ben Dushnik and E. W. Miller introduced the concept of the reversible partial order and expressed the theorem about this concept. In this Part III, I shall show that the same one is held in the partially ordered abelian semigroup by adding the certain condition.

**Definition 1.** A set $S$ is said to be a *partially ordered abelian semigroup* (p.o. semigroup), when $S$ is (I) an abelian semigroup (not necessarily contains the unit element), (II) a partially ordered set, and satisfies (III) the *homogeneity*: $a \geqq b$ implies $ac \geqq bc$ for any $c$ of $S$.

A partial order which satisfies the condition (III) is called a *partial order defined on an abelian semigroup.*

Moreover, if a partial order defined on an abelian semigroup $S$ is a linear order, then $S$ is said to be a *linearly ordered abelian semigroup* (l.o. semigroup). (Definition 1, O.I.)

**Definition 2.** Let $\mathfrak{S}=\{P_\alpha\}$ be any set of partial orders, each defined on the same abelian semigroup $S$. We define the new partial order $P$ on $S$ as follows: For any two elements $a, b$, we put $a \geqq b$ in $P$ if and only if $a \geqq b$ in every $P_\alpha$ of the set $\mathfrak{S}$. Indeed, $P$ is again a partial order defined on $S$. This partial order $P$ is said to be the *product* of the partial orders $P_\alpha$ or to be *realized* by the set $\mathfrak{S}$ of partial orders $P_\alpha$. (Definition 9, O.I.)

By the *dimension* of a partial order $P$ defided on an abelian semigroup $S$ is meant the smallest cardinal number $\mathfrak{m}$ such that $P$ is realized by $\mathfrak{m}$ linear orders defined on $S$.

**Definition 3.** Let $P$ and $Q$ be two partial orders defined on the same

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Partially ordered abelian semigroup. I. On the extension of the strong partial order defined on abelian semigroups. Journ. Fac. Sci., Hokkaido University, Series I, vol. XI (1951), pp. 181–189; this is referred to hereafter as "O.I."

1) Ben Dushnik and E. W. Miller: Partially ordered sets, Amer. Math. Journ. vol. 63 (1941), pp. 600–610.

abelian semigroup $S$, and suppose that *any* two distinct elements of $S$ are comparable in *just one* of these partial orders; in such a case we shall say that $P$ and $Q$ are *conjugate* partial orders. A partial order will be called *reversible* if and only if it has a conjugate.[2)]

If $P$ is a partial order defined on $S$, then the partial order obtained from $P$ by inverting the sense of all ordered pairs will be called a *dual* order, which is denoted by $P^*$.

**Theorem 1.** *Let $P$ and $Q$ are conjugate partial orders defined on an abelian semigroup $S$. Then we can define a linear order $L_1$ on $S$ such that $a>b$ in $L_1$ if and only if $a>b$ in either $P$ or $Q$; denoted by $L_1=P+Q$. Similary $L_2=P+Q^*$ is a linear order defined on $S$.*

*Proof.* We shall prove only the transivity of $L_1$.

From $a>b$, $b>c$ in $L_1$, we can consider the following four cases: (i) $a>b$, $b>c$ in $P$, (ii) $a>b$, $b>c$ in $Q$, (iii) $a>b$ in $P$, $b>c$ in $Q$, (iv) $a>b$ in $Q$, $b>c$ in $P$.

In cases (i) and (ii), $a>c$ in $L_1$ is cleary. In case (iii), if $c>a$ in $P$ or $Q$, then $c>b$ in $P$ or $b>a$ in $Q$ respectively, which is absured, therefore $a>c$ in $P$ or $Q$ and hence in $L_1$. Similary, in case (iv) $a>c$ in $L_1$ is held.

**Theorem 2.** *The following two properties of a partial order $P$ defined on an abelian semigroup $S$ are equivalent to each other:*

(1) *$P$ is reversible.*

(2) *The dimension of $P$ is 2.*

*Proof.* We shall show first that (1) implies (2). Suppose that the partial order $P$ defined on $S$ is reversible, and let $Q$ be a partial order defined on $S$ conjugate to $P$ and $Q^*$ be the dual order of $Q$. Then by Theorem 1, $L_1=P+Q$ and $L_2=P+Q^*$ are linear extensions of $P$ and it is obvious that $P$ is realized by linear orders $L_1$ and $L_2$.

Next we show that (2) implies (1). Let $L_1$ and $L_2$ be any two linear orders defined on $S$ which together realize $P$. We define the other order $Q$ as follows: $a>b$ in $Q$ if and only if $a$ and $b$ are non-comparable in $P$ and $a>b$ in $L_1$ (likewise $a>b$ in $L_1$ and $b>a$ in $L_2$). Then $a>b$ and $b>a$ in $Q$ are contradictory. If $a>b$ and $b>c$ in $Q$, then we have $a>b>c$ in $L_1$ and $c>b>a$ in $L_2$, hence $a>c$ in $Q$. $a>b$ in $Q$ implies that $ac\geqq bc$ in $L_1$ and $bc\geqq ac$ in $L_2$, i.e. $ac\geqq bc$ in $Q$. Therefore $Q$ is a partial order defined on $S$. Evidently $P$ and $Q$ are conjugate.

**Definieion 4.** A *linear* extension $L$ of a partial order $P$ defined on

2) Cf. Ben Dushnik and E. W. Miller: l.c.

an abelian semigroup $S$ will be called *separating* if and only if there exist three elements $a$, $b$ and $c$ in $S$ such that $a>c$ in $P$, and $b$ is not comparable with either $a$ or $c$ in $P$, while in $L$ we have $a>b>c$.

**Theorem 3.**[3] *Let $P$ be a partial order defined on an abelian semigroup $S$ which satisfies the condition $(E)$.*[4] *Then the following three properties of a partial order $P$ are equivalent to each other:*

(1) *$P$ is reversible.*

(2) *The dimension of $P$ is 2.*

(3) *There exists a linear extension of $P$ which is non-separating.*

*Proof.* (1) and (2) are equivalent by Theorem 2.

We show now that (2) implies (3) without the condition $(E)$. Let $L_1$ and $L_2$ be any two linear orders defined on $S$ which together realize $P: P=L_1\times L_2$. If $L_1$ is separating, then there exist three elements $a, b$ and $c$ such that $a>c$ in $P$, $a>b>c$ in $L_1$ and $b$ is not comparable with either $a$ or $c$ in $P$. Hence we have $c>b>a$ in $L_2$ which is impossible.

To show that (3) implies (1) we shall suppose that $L$ is a non-separating linear extension of $P$. We define the other order $Q$ as follows: $a>b$ in $Q$ if and only if $a$ and $b$ are non-comparable in $P$ and $a>b$ in $L$. Then clearly $a>b$ and $b>a$ in $Q$ are contradictory. If $a>b$ and $b>c$ in $Q$, then we have $a>b>c$ in $L$ and $a$ and $c$ are non-comparable in $P$, for otherwise $a>c$ in $P$ would imply that $L$ is separating contrary to the assumption, hence we have $a>c$ in $Q$. $a>b$ in $Q$ implies that $ac\geqq bc$ in $L$. If $ac>bc$ in $P$, then by the condition $(E)$ $a>b$ in $P$ which is impossible. Hence $ac=bc$ or $ac$ and $bc$ are non-comparable in $P$, and hence $ac\geqq bc$ in $Q$. Therefore $Q$ is a partial order defined on $S$. Clearly $P$ and $Q$ are conjugate.

**Definition 5.** Let $S$ be a p.o. semigroup and $P$ be the partial order defined on $S$. For any element $a$ of $S$, we denote the set of all elements $x$ such that $x\leqq a$ in $P$ by $\bar{a}$. Then the correspondence $a\leftrightarrow\bar{a}$ is one-to-one. We put $\bar{a}\geqq\bar{b}$ if and only if $\bar{b}$ is a subset of $\bar{a}$, likewise $a\geqq b$ in $P$, then the family $\bar{S}=\{\bar{a}\}$ is become a partially ordered set. Next we define the product $\bar{a}\cdot\bar{b}=\overline{ab}$, then the family $\bar{S}$ is a commutative semigroup, moreover $\bar{S}$ become a p.o. semigroup. Clearly $S$ and $\bar{S}$ are order-isomorphic.[5]

More generally, if there exists a one-to-one correspondence between

3) Ben Dushnik and E. W. Miller: l.c. Theorem 3.61.

4) Condition $(E)$ (order cancellation law):
$ac>bc$ in $P$ implies $a>b$ in $P$.

5) See Definition 3, O.I.

the elements of the p.o. semigroup $S$ and the family $\mathfrak{R}$ of subsets of the certain set $R$ (a subset of $R$ which corresponds with an element $a$ of $S$, denote by $s(a)$), and $a \geqq b$ in $P$ if and only if $s(a) \supseteq s(b)$ (in the sense of set-inclusion), then by the defining the product $s(a) \cdot s(b) = s(ab)$, two p.o. semigroups $S$ and $\mathfrak{R}$ are order-isomorphic.

Any family $\mathfrak{R}$ of the subsets of the set $R$ which has the above properties will be called a *representation* of $P$.

**Theorem 4.** *Let $P$ be a partial order defined on an abelian semigroup $S$ which satisfies the condition (E). Then the following two properties are equivalent to each other:*

(1) *$P$ is reversible.*

(4) *There exists a representation of $P$ by means of a family $\mathfrak{R} = \{I_a\}$ of closed intervals on some l.o. semigroup $R$, and let $I_a = [\alpha_1, \alpha_2]$, $I_b = [\beta_1, \beta_2]$, $I_{ac} = [\gamma_1, \gamma_2]$, $I_{bc} = [\delta_1, \delta_2]$, and if $a$ and $b$ are non-comparable in $P$, then $a_1 < \beta_1$ (and $\alpha_2 < \beta_2$) implies $\gamma_1 \leqq \delta_1$ (and $\gamma_2 \leqq \delta_2$) or its dual.*

*Proof.* We shall show (1) implies (4). Let $P$ be reversible, and hence the dimension of $P$ is 2. Let $A$ and $B$ be any two linear orders defined on $S$ which together realize $P$.

Let $S'$ be a l.o. semigroup which is anti-order-isomorphic to the l.o. semigroup $S$ in the linear order $B$, where the set $S'$ is disjoint from $S$, and the linear order defined on $S'$ is denoted by $B'$.

Let $R$ be the union of $S$, $S'$ and the new element 0 which belongs to neigther $S$ nor $S'$.

We define the multiplication in $R$ as follows:

$$0 \cdot 0 = 0,$$
$$x \cdot 0 = 0 \cdot x = 0 \qquad \text{for any } x \text{ in } S \text{ or } S',$$
$$a \cdot a' = a' \cdot a = 0 \qquad \text{for any } a \text{ in } S \text{ and } a' \text{ in } S',$$

and for any two elements $x$ and $y$ of $S(S')$ the product is the same as in $S(S')$.

Thus $R$ becomes the abelian semigroup under the multiplication introduced above.

Let us now define the order-relation $L$ in $R$ as follows:

$$x > y \text{ in } L \ (x, y \in S) \quad \text{if and only if } x > y \text{ in } A,$$
$$x > y \text{ in } L \ (x, y \in S') \quad \text{if and only if } x > y \text{ in } B',$$

and we put

$$a > 0 > a' \text{ in } L \ (a \in S, \ a' \in S').$$

Then $R$ becomes a l.o. semigroup.

For each $a$ in $S$ denote by $a'$ the image of $a$ in $S'$, and denote by $I_a$ the closed interval $[a', a]$ of $R$.

We will show that the family $\mathfrak{R}=\{I_a\}$ of all such intervals is a representation of $P$. Suppose first that $a>b$ in $P$. Then $a>b$ in $A$ and $a'<b'$ in $B'$, so that we have $a'<b'<b<a$ in $L$. This means that $I_b$ is a proper subset of $I_a$.

Let $I_a=[a', a]$, $I_b=[b', b]$, $I_{ac}=[a'c', ac]$, $I_{bc}=[b'c', bc]$. If $a$ and $b$ are non-comparable in $P$, then from $a>b$ $(a'>b')$ in $L$ we have $ac \geqq bc$ $(a'c' \geqq b'c')$ in $L$ or its dual.

We prove that (4) implies (1). Suppose that $P$ is a partial order which is represented by a family $\mathfrak{R}$ of intervals taken from some l.o. semigroup $R$, whose linear order is denoted by $L$. For each $a$ in $S$, denote by $I_a$ the interval of the family $\mathfrak{R}$ which corresponds to $a$. We notice first that if $a$ and $b$ are distinct elements of $S$ which are not comparable in $P$, then $I_a$ and $I_b$ cannot have the same left (right)-hand end-point.

Suppose that $I_a=[\alpha_1, \alpha_2]$, $I_b=[\beta_1, \beta_2]$, $I_c=[\gamma_1, \gamma_2], \cdots$ .

We define a new partial order $Q$ defined on $S$ as follows:

(i) $a$ and $b$ are not comparable in $P$,

(ii) $\alpha_1<\beta_1$ (and $\alpha_2<\beta_2$) in $L$.

It is easy to see that $Q$ is the partial order defined on the set $S$. We shall now prove the homogeneity. Let $a>b$ in $Q$ and $I_{ac}=[\lambda_1, \lambda_2]$, $I_{bc}=[\mu_1, \mu_2]$. If $ac$ and $bc$ are distinct and comparable in $P$, then by the condition $(E)$ $a$ and $b$ are comparable in $P$ which is impossible. If $ac$ and $bc$ are non-comparable in $P$, then $\lambda_1<\mu$, $\lambda_2<\mu_2$ and hence $ac>bc$ in $Q$.

**Example 1.** Let $S_1$ be an abelian semigroup generated by two elements $a$ and $b$ with the relation

$$a^m b^n = ab^n \qquad \text{for any positive integers } m \text{ and } n.$$

By putting the order-relation

$$P: \quad \begin{cases} a^{m+1} > a^m \\ b^n > ab^n \end{cases} \qquad \text{for any positive integer } n$$

$S_1$ becomes a p.o. semigroup, and the partial order $P$ is reversible. Its conjugate order $Q$ is as follows:

$$Q: \quad \begin{cases} a^m > b^n > b^{n+1} > ab^{n+1} \\ a^m > ab^n > b^{n+1} \end{cases} \qquad \text{for any positive integers } m \text{ and } n.$$

The linear orders which together realize $P$ are

$$a^{m+1}>a^m>b^n>ab^n>b^{n+1}>ab^{n+1}$$

and

$$b^{n+1}>ab^{n+1}>b^n>ab^n>a^{m+1}>a^m$$

for any positive integers $m$ and $n$.

**Example 2.** Let $S_2$ be an abelian semigroup generated by two elements $a$ and $b$ with the relation

$$a^mb^n=b^n \quad \text{for any positive integers } m \text{ and } n.$$

By putting the two order-relations $A$ and $B$

$A$: $a^{m+1}>a^m>b^n>b^{n+1}$

$B$: $b^{n+1}>b^n>a^{m+1}>a^m$

for any positive integers $m$ and $n$,

$S_2$ becomes a l.o. semigroup in the orders $A$ and $B$ respectively. Let $P$ be the partial order which is the product of $A$ and $B$, that is

$P$: $a^{m+1}>a^m$.

Then $P$ has the conjugate order $Q$ such that

$Q$: $a^m>b^n>b^{n+1}$.

**Example 3.** Let $S_3'$ and $S_3''$ be free abelian semigroups generated by elements $a$ and $b$ respectively. And by defining the order-relations

$$a^{m+1}>a^m \quad (m\geqq 1), \qquad b^{n+1}>b^n \quad (n>1),$$

$S_3'$ and $S_3''$ becomes a l.o. semigroup and a p.o. semigroup respectively. Let $S_3$ be the direct product of $S_3'$ and $S_3''$. Then $S_3$ becomes a p.o. semigroup by introducing the following order-relation $P$:

$$a^ib^j>a^mb^n$$

if and only if

$$a^i>a^m \quad \text{or} \quad a^i=a^m \quad \text{and} \quad b^j>b^n.$$

Since $ab^2$ and $ab$ are non-comparable in $P$ in spite of $(ab^2)(ab)=a^2b^3>a^2b^2=(ab)(ab)$, $P$ does not satisfy the condition $(E)$.

Now, in $S_3''$ we define the another order-relation:

$$b^{n+1}>b^n \quad (n\geqq 1),$$

then we get the non-separating linear extension of $P$.

But we cannot realize the partial order $P$ by two linear orders.

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