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Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 15(1-2), 131-133
Issue Date	1960
Doc URL	https://hdl.handle.net/2115/56013
Type	departmental bulletin paper
File Information	JFSHIU_15_N1-2_131-133.pdf



ON *-MODULAR RIGHT IDEALS OF AN ALTERNATIVE RING

By

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It is well known that, in any alternative ring A , the Smiley radical $SR(A)$ is contained in every modular maximal right ideal M . E. Kleinfeld has shown that every primitive alternative, non-associative ring is a Cayley-Dickson algebra.

Now we introduce the notion of $*$ -modularity as follows: a right ideal I of an alternative ring A is called $*$ -modular if there exist two elements $a, u \in A$ such that

$$(1) \quad x + ax + (a, x, u) \in I$$

for all $x \in A$, where (a, x, u) denotes the associator $ax \cdot u - a \cdot xu$ of a, x, u , and in this case we call a a *left $*$ -modulo unit of I* . Clearly, modularity implies $*$ -modularity.

In this note, we shall show that the above results are also true if we replace modular ideals by $*$ -modular ideals.

If a ring A is assumed to be alternative, then (a, b, c) becomes a skew-symmetric function of its three variables.

The Smiley radical $SR(A)$ of an alternative ring A is defined as the totality of elements $z \in A$ for which each element of $(z)_r$ is right quasi-regular.

In the next lemma we develop an important property of $*$ -modular right ideals.

Lemma 1. *Let I^* be a $*$ -modular right ideal of an alternative ring A , and suppose that a left $*$ -modulo unit a of I^* is right quasi-regular. Then $I^* = A$.*

Proof. Let b be a right quasi-inverse of a :

$$(2) \quad a + b + ab = 0.$$

Since a is a left $*$ -modulo unit of I^* and since $(a, a, u) = 0$, we have $a + a^2 \in I^*$ by putting $x = a$ in (1), while $(a + a^2)b - (a, b, u) = ab + a^2b - (a, a, u) - (a, b, u) = ab + a^2b - (a, a + b, u) = ab + a^2b + (a, ab, u) \in I^*$ by (2). Hence it follows that $(a, b, u) \in I^*$. On the other hand, if we put $x = b$ in (1), we

have $b+ab+(a, b, u) \in I^*$, whence $b+ab \in I^*$. Thus $a \in I^*$, which implies together with (1) that every $x \in A$ is in I^* , that is, $I^* = A$.

As any modular maximal right ideal is a member of a set of *-modular maximal right ideals, the intersection of all the modular maximal right ideals is contained in the intersection of modular maximal right ideals.

Now, we show a connection between the intersection of all the *-modular maximal right ideals and the radical $SR(A)$ in an alternative ring A .

Theorem 1. *Let A be an alternative ring. Then the Smiley radical $SR(A)$ is contained in the intersection of all the *-modular maximal right ideals M^* :*

$$SR(A) \subseteq \bigcap M^*.$$

Proof. Let $z \in A$ be an element not contained in the intersection $\bigcap M^*$. Then there exists a *-modular maximal right ideal M^* does not contain z . And we have $A = M^* + (z)_r$. Let a, u be elements such that $x+ax+(a, x, u) \in M^*$ for all $x \in A$, and let m^* and z' be elements of M^* and $(z)_r$ respectively such that $a = m^* + z'$. Then $x+z'x+(z', x, u) = x + (a - m^*)x + (a - m^*, x, u) = x + ax + (a, x, u) - m^*x + (m^*, x, u) \in M^*$ for all $x \in A$. Thus z' is also a left *-modulo unit of M^* . But, since $M^* \neq A$, z' is not right quasi-regular by Lemma 1, and so z' is not in $SR(A)$. This proves our theorem.

Next we refer to the structure of *-primitive alternative ring.

An alternative ring is defined to be *-primitive in case it contains a *-modular maximal right ideal whose quotient is zero.

Lemma 2. *The quotient $(I^*: A) = \{x \in A; Ax \subseteq I^*\}$ of a *-modular right ideal I^* is an ideal of A .*

Proof. The *-modularity of I^* assures the existence of $a, u \in A$ with the property that $x+ax+(a, x, u) \in I^*$ for every $x \in A$. Since $ax \in I^*$ for $x \in (I^*: A)$, we have $x+(a, x, u) = x + a \cdot ux - au \cdot x \in I^*$. And further $ux + a \cdot ux + (a, ux, u) \in I^*$. Combining this with $ux \in I^*$, we obtain $a \cdot ux \in I^*$ and eventually $x \in I^*$. Hence, for any $y \in A$, $A \cdot xy$ and $A \cdot yx$ are both in I^* .

By the light of this lemma, it is clear that A is *-primitive if and only if A has a *-modular maximal right ideal which contains no nonzero two-sided ideals.

An alternative ring is called simple if it has no nonzero proper two-sided ideals and is not a nil ring.

The following lemma is due to E. Kleinfeld [1].

Lemma 3. *A simple alternative ring is either a Cayley-Dickson algebra or associative.*

Most results in primitive alternative rings which were stated in [2] are also true in our *-primitive case under a slight modification of the modularity.

We obtain the following:

Theorem 2. *Every *-primitive, alternative, non-associative ring A is a Cayley-Dickson algebra.*

Proof. We may prove, with the help of the proof in primitive case [2], that every *-modular maximal right ideal M^* of A is zero. And so, it is enough only to show that A is a simple alternative ring. For any left *-modulo unit a of M^* , $a+a^2=0$ and then $a^n=\pm a$ for every integer $n\geq 1$. It shows that a is not nilpotent, and hence A is simple by Lemma 3. Therefore, A is a Cayley-Dickson algebra.

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