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VECTOR FIELDS AND SPACE FORMS

By

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The surface of rotation R in an $(n+1)$-dimensional Euclidean space has some remarkable intrinsic properties. Among them, the following two are the most typical:

Property I. Let G be the Lie group consisting of all isometries. Then there exists a subgroup H of G of the type:

$$\exists x \in R\,[\dim H(x) = n-1],$$

provided that $H(x)$ denotes the orbit of x.

Property II. There exists a vector field V which either is parallel or satisfies these three conditions:

(i) In case of the movement in the direction orthogonal to V, the end point of V is always fixed (intrinsically, with respect to Levi-Civita parallelism of R).

(ii) The trajectories of V are geodesics in regard to the induced metric of R.

(iii) V admits a family of transversal hypersurfaces.

Remarks. It is worth noting that from the global point of view the above mentioned vector field V may generally have certain singularities,[1] particularly in case of R being a closed hypersurface.

Property I and II give rise conversely to the interesting questions of determing the global nature of Riemannian spaces possessing either Property I or Property II respectively. Each of these questions propounds quite a different problem than the other and the methods by means of which these problems can be solved must differ very much from each other. In either case, however, results to be obtained will show that the spaces in question have remarkable similarity to the surface of rotation. Conversely speaking, to solve these problems is in a sense nothing but to make clear this similarity which such spaces have.

In fact it is from this point of view that P. Mostert has dealt with the former problem and determined it to a large extent [1]. It must

1) The word singularity means that V may have not only 0-points, but also a kind of discontinuity.

be added that T. Nagano has contributed to further investigation of this problem and gained excellent and much sharper results [2].

But the latter problem has attracted little attention and hardly any paper has been written from this point of view. One of the main objectives of the present paper is to solve the latter problem (in a form enlarged to a Finsler case).

On the other hand, the present author has proved that the vector field described in Property II is reduced to a special kind of the torse-forming vector field discovered by K. Yano (see [21]), where the torse-forming vector field means on which describes a torse when developed along a curve by the Levi-Civita parallelism [3] or [4].

Then our problem can be generalized in such a way as this: To determine the global nature of Finsler spaces admitting a torse-forming vector field with certain singularities.

Anyway our guide is the vector field admitted by the surface of rotation; then "the torse-forming vector field in the large" should be defined in such a fashion that it contains the vector field of the surface of rotation seen in the light of the global theory.

Thus we have the following definition.

Definition: *Let M be a Finsler space, that is to say, a space whose metric is given by $dx=L(x, dx)$ $(x\in M)$, where $L(x, dx)$ is a positively homogeneous function of degree 1 with respect to dx. Let*

$$g_{ij}(x, x')=\frac{1}{2}\frac{\partial^2 L(x, x')}{\partial x'^i \partial x'^j}, \quad A_{ijk}(x, x')=\frac{\partial g_{ij}(x, x')}{\partial x'^k}.$$

The torse-forming vector field in the large $\boldsymbol{V}$ is one satisfying the following postulates:

(i) *$\boldsymbol{V}$ is a single-valued and C'-differentiable vector field over K, provided that K means a dense open set in M, that is to say: $K^0=K$ and $K^-=M$.*

(ii) *There exist two point-functions defined over M such that, taking a suitable coordinate neighborhood of x, we have in it*

$$V_{i;j}=Ag_{ij}+BV_iV_j, \tag{0.1}$$

$$A_{ijk}V^k=0 \tag{0.2}$$

[2]

(iii) *For every $y\in K^c$ we can find a neighborhood U of y in which there exists a torse-forming vector fields in the local sense $\boldsymbol{W}$ and an appropriate point-function F joined together by a relation:*

2) This expression is obviouly unnecessary for a Riemannian case.

(0.3) $$\boldsymbol{W} = F\boldsymbol{V} \quad in \quad U \cap K,$$

provided that a torse-formin vector field in the local sense means one satisfying a differential equation: $W_{i;j}=Cg_{ij}+DW_iW_j$ *with a condition:* $A_{ijk}W^k=0$ *for suitable point-functions C and D defined over U.*

In addition to these postulates, one more postulate may be used in order to prove some of the theorems in the present paper, namely:

(iv) *A never vanishes at any 0-point of* $\boldsymbol{V}$.

We shall see that this postulate guarantees the mutual isolation of 0-points.

Let x be a non-0-point and $W(x)$ a maximal transversal hypersurface passing through x. Moreover, denote the number of the 0-point of $\boldsymbol{V}$ by $N(\boldsymbol{V})$. Then the complete answer to our generalized problem is as follws, provided that M is assumed to be complete.

Case I: $N(\boldsymbol{V})=1$.

In this case M is homemorphic to the n-dimensional Euclidean space or to the n-dimensional projective space.

Case II: $N(\boldsymbol{V})=2$.

In this case M is homeomorphic to the n-dimensional sphere.

Case III: $N(\boldsymbol{V})=0$.

Let g_0 be an arbitrary translation on a trajectory T.

Then there exists a diffeomorphism g on M whose restriction to T is g_0 and which satisfies

(0.4) $$g\{W(x)\}=W\{g_0(x)\} \quad (x\in T).$$

Furthermore if we assume Postulate (iv), then $N(\boldsymbol{V})$ is always smaller than or equal to 2. Therefore the above three cases include all the possibilities. Moreover, in Case I and II, the metric of M is completely determined and M becomes Riemannian (see § 5); and all the maximal transversal submanifolds are conformal to a $(n-1)$-dimensional sphere.

The present paper is divided into two parts, the first one of which is a preliminary on differential equations and the second one of which is assigned to the proof of the above theorem and some others including the ones on a generalization of certain theorems of W. Rinow [5] and on the characterization of the spaces of constant curvature by the following torse-forming vector field (in the large):

(0.5) $$V_{i;j}=c_1(g_{ij}+V_iV_j);\ A_{ijk}V^k=0$$

in the case of positive constant curvature.

(0.6) $$V_{i;j}=c_2(g_{ij}-V_iV_j);\ A_{ijk}V^k=0$$

in the case of negative constant curvature.

(0.7) $$V_{i;j}=c_3 g_{ij};\ A_{ijk}V^k=0$$ [3]

in the case of a Euclidean space.[4]

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Part I

Some Theorems on certain Types of Differential Equation

§ 1. The following conventions will be used throughout the present paper.

Conventions. We denote the arc length of a curve by s, a vector field by a bold letter, the inner product of $\boldsymbol{A}$ and $\boldsymbol{B}$ by $\langle \boldsymbol{A}, \boldsymbol{B} \rangle$, the openner of K by K^0, the closure of K by K^-, the complement of K by K^c, neighborhoods of x by $U(x)$, $V(x)$, etc., and the distance with respect to a manifold W between x and $y (x, y \in W)$ by dis $W(x, y)$.

The following lemma is obvious from an intuitive point of view and the strict proof is also given in a straight-forward way.

Lemma 1.1. *Let J be an arbitrary set in an n-dimensional Euclidean space E_n and x_0 a limiting point of J. Then we can choose a suitable sequence $\{x_i\}_{1\leq i<+\infty}$ of J such that there is a curve $x(s)$ satisfying the following conditions, where $0\leq s\leq L$.*

(i) *A monotone decreasing sequence $\{s_i\}_{1\leq i<+\infty}$ can be taken in such a way as this:*

(1.1) $$x_0=x(0) \quad \textit{and} \quad x_i=x(s_i)\ (1=1, 2, \cdots),$$

(ii) *$x(s)$ is differentiable on $[0, L]$ (including 0).*

The lemma stated below also is necessary for the proof of Theorem 1.1.

Lemma 1.2. *Let $\gamma : y=y(s)$ $(0\leq s<+\infty)$ be a bounded differentiable curve in E_n having infinite length. Let S_r^{n-1} be a concentric sphere of center y_0 and radius r and let $\theta(s)$ be the angle between γ and S_r^{n-1} if*

3) This case has been proved by S. Sasaki and M. Goto [6] as to the Riemannian space, but our results include that the spaces concerned become simply-connected ones.

4) c_1, c_2, and c_3 mean constants; c_1 and c_2 are intimately connected with the scalar curvatures of the spaces and uniquely determined by these.

$y(s)\in S_r^{n-1}$. *Assume* $y_0\notin\gamma$. *Then there exists a sequence* $\{s_i\}_{1\leq i<+\infty}$ *such that*

$$\lim_{i\to+\infty}\theta(s_i)\equiv 0 \pmod{\pi} \text{ and } \lim_{i\to+\infty}s_i=+\infty. \tag{1.2}$$

Proof. Suppose that the lemma is false. Then we can find a positive number ε such that

$$\varepsilon<\underline{\lim_{s\to+\infty}}\,\theta(s)\leq\overline{\lim_{s\to+\infty}}\,\theta(s)<\pi-\varepsilon \pmod{\pi}.$$

On the other hand

$$ds=\frac{|dr|}{|\sin\theta(s)|} \quad \text{consequently} \quad \int ds=\frac{|dr|}{|\sin\theta|}\leq\frac{1}{\sin\varepsilon}\int|dr|.$$

Since $\int ds=+\infty$, $\int|dr|$ likewise is infinite. It follows that we can find sequences $\{s'_i\}_{1\leq i<+\infty}$ $(\nu=1,2)$ and $\{r_i\}_{1\leq i<+\infty}$ such that $s'_1<s'_2<\cdots\uparrow+\infty$ and $y(s'_{2i+1})$, $y(s'_{2i})\in S_{r_i}^{n-1}$ $(\nu=1,2;\ i=1,2,\cdots)$, because of γ being bounded. Since a sphere is closed, we can choose a segment of γ $(s'_{2i-1}\leq s\leq s'_{2i})$ in such a way as its end points are on $S_{r_i}^{n-1}$ and all the points other than these are inside $S_{r_i}^{n-1}$. We denote it by γ_i and set: $d_i=\min\{\mathrm{dis}\,(y,y_0)|\ y\in\gamma_i\}$. Since γ_i is compact, $\exists x_i\in\gamma_i\,[\mathrm{dis}\,(x_i,y_0)=d_i]$. Let $x(s_i)=x_i$. Then we can easily see that $\theta(s_i)\equiv 0 \pmod{\pi}$. This is contrary to the assumption that the lemma is false.

Noting that y_0 has no restriction, we have the

Corollary. *If* $\lim_{s\downarrow+\infty}\dfrac{dy}{ds}$ *exists for* γ *in Lemma* 1.2, *then* $\lim_{s\to+\infty}\dfrac{dy}{ds}=0$.

Now consider a partial differential equation of the type:

$$\frac{\partial \boldsymbol{E}}{\partial x^i}=\boldsymbol{F}_i(x,\boldsymbol{E})\ (i=1,2,\cdots,n). \tag{1.3}$$

where the vector-valued functions $\boldsymbol{F}_i(x,\boldsymbol{E})$ $(i=1,2,\cdots,n)$ are assumed to be continuous with respect to x and $\boldsymbol{E}$ on the domain concerned.

Next we prove the following theorem which concerns itself with the behavior of the solution of differential equations (1.3). This theorem is of vital importance to our whole argument.

Theorem 1.1. *Suppose that* (1.3) *has a solution with* x_0 *as its isolated* 0*-pointed. If* $\boldsymbol{F}_i(x_0,0)$ $(i=1,2,\cdots,n)$ *are linearly independent, then*

$$\exists U(x_0)\forall x\in U(x_0)\ [x\sim x_0],$$

provided that $x\sim x_0$ *means that* x *and* x_0 *can be joined together by some* 2*-differentiable trajectory of the solution.*

Proof. The verification is divided into three parts. First it is shown that every trajectory tends to x_0, secondly that it has finite length in a

suitable neighborhood of x_0, and finally that it is differentiable at x_0.

(I) There is a sphere S^{n-1} of center x_0 such that for an arbitrary point x_1 belonging to its interior we can find a trajectory $x(s)$ $(0 \leq s < L$, where L is a positive number or $+\infty)$ satisfying

$$x(0)=x_1 \quad \text{and} \quad \lim_{s \to L} x(s)=x_0 . \tag{1.4}$$

Assume that this is false. Then no matter how small positive number r is, there exists always a trajectory γ_r for which one of the following two cases arises namely:

(i) If we denote the curves obtained by dividing γ_r into halves at a suitable point by γ_r^1 and γ_r^2 respectively, then

(1°) either γ_s^1 or γ_r^2 is entirely inside S_r^{n-1} and has at least two limiting points (that is: vibrates) as a tends to $+\infty$ or $-\infty$.

or

(2°) either γ_r^1 or γ_r^2 has a limit different from x_0.

(ii) γ_r has at least two points in common with S_r^{n-1}.

Among these cases, (1°) of (i) never arises. This is easily seen from the corollary of Lemma 1.2. In Case of (1°) of (i), we can find a sequence $\{x_i'\}_{1 \leq i < +\infty}$ such that $\dfrac{\langle \boldsymbol{E}(\boldsymbol{x}_i'), \boldsymbol{x}_i' \rangle}{\| \boldsymbol{E}(x_i) \| \, \| \boldsymbol{x}_i' \|}$ tends to 0 as $i \to +\infty$ and $\lim_{i \to +\infty} x_i = x_r$ (x_r is a point inside S_r^{n-1}), where $\boldsymbol{x}_i'$ means the position vector of x_i'.[5] If $x_r \neq x_0$, then we have

$$\langle \boldsymbol{E}(x_r'), \boldsymbol{x}_r \rangle = 0 . \tag{1.3}$$

In Case (ii), we can also find a point x_r satisfying (1.3).

After all, since r is an arbitrary number, we have a sequence $\{x_i\}_{1 \leq i < +\infty}$ such that

$$\lim_{i \to +\infty} \frac{\langle \boldsymbol{E}(x_i'), \boldsymbol{x}_i \rangle}{\| \boldsymbol{E}(x_i) \| \, \| \boldsymbol{x}_i \|} = 0 . \tag{1.4}$$

and

$$\lim_{i \leftarrow +\infty} x_i = x_0 . \tag{1.5}$$

(1.4) is rewritten in the form

$$\left\langle \boldsymbol{E}(x_i), \frac{\boldsymbol{x}_i}{\| \boldsymbol{x}_i \|} \right\rangle = o\{\mathrm{dis}\,(x_i, x_0)\} , \tag{1.6}$$

where o means a infinitesimal number of higher order than dis (x_i, x_0).

Due to Lemma 1.1 we can make a differentiable curve $x^*(\tau)$ such that

5) This kind of symbols will be used throughout the present paper.

$$x^*(0)=x_0 \qquad x_{i_k}=x^*(\tau_k)\ (k=1, 2, \cdots)$$

for some monotone decreasing sequence $\{\tau_k\}_{1\leq k<+\infty}$ tending to 0 provided that $\{x_{i_k}\}_{1\leq k<+\infty}$ means a suitable subsequence of $\{x_i\}_{1\leq i<+\infty}$.

For such a curve it follows from (1.6) that

$$0=\lim_{k\to+\infty}\left\langle \frac{\boldsymbol{E}(x_{i_k})}{\tau_{i_k}}, \frac{\boldsymbol{x}_{i_k}}{\|\boldsymbol{x}_{i_k}\|}\right\rangle=\left\langle \left(\frac{d\boldsymbol{E}}{d\tau}\right)_{\tau=0}, \left(\frac{dx^*}{d\tau}\right)_{\tau=0}\right\rangle, \tag{1.7}$$

and if we take $\boldsymbol{F}_i(x_0, 0)$ $(i=1, 2, \cdots, n)$ as coordinate axes, then it follows

$$0=\left\langle \left(\frac{d\mathrm{X}^*}{d\tau}\right)_{\tau=0}, \left(\frac{d\mathrm{X}^*}{d\tau}\right)_{\tau=0}\right\rangle=1\,.$$

This is a contradiction.

Thus it has been proved that there exists a neighborhood of x_0 in which a half of the trajectory passing through an arbitrary point tends to x_0 in the above-stated sense.

(II) If we assume that a trajectory tending to x_0 has infinite length, then by virtue of Lemma 1.2, there exists a sequence satisfying (1.4) and (1.5). Therefore we can not help falling to a contradiction from the same reason as above. Consequently all the trajectories are rectifiable around x_0.

(III) It still remain to prove that the trajectory is differentiable at x_0. Since it is rectifiable around x_0 as shown in (I) and (II), its length can be measured with x_0 as the starting point. Let S_u^{n-1} be the unit sphere of center x_0 and φ be the mapping which, to every point $x\in E_n$, assigns the radius vector of S_u^{n-1} passing through it.

We shall first prove

$$\lim_{s\to 0}\left[\varphi\{x(s)\}-\frac{\boldsymbol{E}\{x(s)\}}{\|\boldsymbol{E}\{x(s)\}\|}\right]=0\,. \tag{1.8}$$

In order to verify the above, assume it is false. Then we can find a positive number ε satisfying this:

$$\left\|\varphi\{x(s_i'')\}-\frac{\boldsymbol{E}\{x(s_i'')\}}{\|\boldsymbol{E}\{x(s_i'')\}\|}\right\|\geq\varepsilon \quad (i=1, 2, \cdots), \tag{1.9}$$

provided that $\{s_i''\}_{1\leq i<+\infty}$ is a suitable sequence tending to 0. The identity

$$\left\|\varphi\{x(s_i'')\}-\frac{\boldsymbol{E}\{x(s_i'')\}}{\|\boldsymbol{E}\{x(s_i'')\}\|}\right\|^2=2-2\left\langle\varphi\{x(s_i'')\}, \frac{\boldsymbol{E}\{x(s_i'')\}}{\|\boldsymbol{E}\{x(s_i'')\}\|}\right\rangle$$

implies

$$1-\frac{\varepsilon^2}{2}\geq\left\langle\varphi\{x(s_i'')\}, \frac{\boldsymbol{E}\{x(s_i'')\}}{\|\boldsymbol{E}\{x(s_i'')\}\|}\right\rangle. \tag{1.10}$$

On the other hand, by the use of Lemma 1.1, we can find a differentiable curve $x^*(\tau)$ $(0\leq\tau\leq 1)$ such that $x^*(\tau_k)=x(s''_{i_k})$ $(k=1, 2, \cdots)$ and $x^*(0)=x_0$ for an appropriate subsequence $\{s''_{i_k}\}_{1\leq k<+\infty}$ of $\{s''_i\}_{1\leq i<+\infty}$, where τ denotes the arc length of x^*. Then from (1.10), we have

$$\left(1-\frac{\varepsilon^2}{2}\right)\frac{\|\boldsymbol{E}\{x^*(\tau_k)\}\|}{\tau_k}\geq\left\langle\varphi\{x^*(\tau_k)\},\ \frac{\boldsymbol{E}\{x^*(\tau_k)\}}{\tau_k}\right\rangle. \tag{1.11}$$

There is no loss of generality in assuming $\{\boldsymbol{F}_i(x_0, 0)\}_{1\leq i\leq n}$ forms the basis of E_n. Then the following holds,

$$\lim_{k\to\infty}\frac{\boldsymbol{E}\{x^*(\tau_k)\}}{\tau_k}=\left\{\frac{d\boldsymbol{E}\{x(\tau)\}}{d\tau}\right\}_{\tau=0}=\sum_{i=1}^{n}\boldsymbol{F}_i(x_0, 0)\frac{dx^i}{d\tau}=\left(\frac{dx^*}{d\tau}\right)_{\tau=0}.$$

By means of this and (1.11), we have

$$\begin{aligned}\left(1-\frac{\varepsilon^2}{2}\right)&=\left(1-\frac{\varepsilon^2}{2}\right)\left\|\frac{dx^*}{d\tau}\right\|=\left(1-\frac{\varepsilon^2}{2}\right)\left\|\lim_{k\to\infty}\frac{\boldsymbol{E}\{x^*(\tau_k)\}}{\tau_k}\right\|\\&\geqq\left\langle\lim_{k\to\infty}\varphi\{x^*(\tau_k)\},\ \lim_{k\to\infty}\frac{\boldsymbol{E}\{x^*(\tau_k)\}}{\tau_k}\right\rangle=\left\|\frac{dx^*}{d\tau}\right\|^2_{\tau=0}=1.\end{aligned}$$

This contradiction completes the proof of (1.8).

Now assume that the trajectory $x(s)$ is not differentiable at $s=0$. Then there exist two unit vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ such that

$$\lim_{i\to\infty}\varphi\{x(s_i)\}=\boldsymbol{r}_1\quad\text{and}\quad\lim_{i+\infty}\varphi\{x(s'_i)\}=\boldsymbol{r}_2, \tag{1.12}$$

for suitable sequences tending to 0 $\{s_i\}_{1\leq i<+\infty}$ and $\{s'_i\}_{1\leq i<+\infty}$. According to Lemma 1.1, there exist a differentiable curve $x'(\tau)$ $(0\leq\tau\leq 1)$ and a suitable sequence $\{\tau_k\}_{1\leq k<+\infty}$ tending to 0 such that

$$x'(\tau_{2k-1})=x(s_{i_k})\quad\text{and}\quad x'(\tau_{2k})=x(s_{i_k}), \tag{1.13}$$

where $\{s_{i_k}\}_{1\leq k<+\infty}$ and $\{s'_{i_k}\}_{1\leq k<+\infty}$ means appropriate subsequences of $\{s_i\}_{1\leq i<+\infty}$ and $\{s'_i\}_{1\leq i<+\infty}$ respectively, and τ denotes the arc length. Then given an arbitrary positive positive number ε, we can find an integer k_0 such that for $k\geq k_0$, $\left\|\left(\frac{dx'}{d\tau}\right)_{\tau=0}-\varphi\{(x'(\tau_k)\}\right\|<\frac{\varepsilon}{6}$. Hence for such k,

$$\|\varphi\{x'(\tau_{2k-1})\}-\varphi\{x'(\tau_{2k})\}\|<\frac{\varepsilon}{3}. \tag{1.14}$$

On the other hand, (1.12) implies

$$\|\varphi\{x(\tau_{2k-1})\}-\boldsymbol{r}_1\|<\frac{\varepsilon}{3}\quad\text{and}\quad\|\varphi\{(\tau_{2k})\}-\boldsymbol{r}_2\|<\frac{\varepsilon}{3} \tag{1.15}$$

for a sufficiently large integer k. From (1.14) and (1.15) we have

$$\|\boldsymbol{r}_1-\boldsymbol{r}_2\|\leq\|\varphi\{\tau_{2k-1})\}-\varphi\{x(\tau_{2k})\}\|+\|\varphi\{x(\tau_{2k-1})\}-\boldsymbol{r}_1\| \\ +\|\varphi\{x_{2k}(\tau_{2k})\}-\boldsymbol{r}_2\|<\varepsilon.$$

Since ε is arbitrary, $\boldsymbol{r}_1=\boldsymbol{r}_2$. This contradiction shows that the trajectory is differentiable at $s=0$.

As a corollary to Theorem 1.1, we have the

Theorem 1.2. *Let f be a function defined on a manifold M. If x_0 is an isolated non-degenerate critical point*[6] *of f, then every trajectory of* grad f *converges to x_0 and it is differentiable at x_0.*

Remark. Theorem 1.2 is well-known in the theory of variation. It is a direct result of the fact that a function of Class C^2 is expressed in a so-called normal coordinate system $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ in this form: $f=\pm\bar{x}_1^2\pm\bar{x}_2^2\pm\cdots\pm\bar{x}_n^2$. It is interesting to note that a torse-forming vector field is not necessarily able to be expressed as a gradient of a function in a domain including a 0-point. Hence it is clear that Theorem 1.2 does not serve the passing need of the present paper at all. Besides Theorem 1.1 will be useful, for example, in examining the space which has a non-linear connection admitting a parallel vector field [11].

§ 2. In this section we shall deal with some problems concerning a non-linear ordinary equation of the type: $\frac{dy}{dt}=A(t)+B(t)y^2$. We begin by proving the following theorem which means that a solution of this equation can be extended all over the whole t-line uniquely if it can take $\pm\infty$ also as its values.

Theorem 2.1. *Let $A(t)$, $B(t)$ be continuous functions defined on $-\infty<t<+\infty$, a non-linear ordinary differential equation*

$$\frac{dy}{dt}=A(t)+B(t)y^2 \tag{2.1}$$

has a solution $y(t)$ with values in an interval $[-\infty, +\infty]$ (containing $\pm\infty$) and defined on $-\infty<t<+\infty$. This solution is unique if its initial condition is given and it is continuous with respect to the non-separated topology[7] *introduced into $[-\infty, +\infty]$.*

Proof. We can find the unique solution with an arbitrary value $y(0)$ as its initial condition and defined on a certain interval $(-b, a)$. We may consider $(-b, a)$ as the maximum interval on which the solution

6) This means

$$\det\left(\frac{\partial_2 f}{\partial\bar{x}_i\partial\bar{x}_j}\right)_{\substack{i=1,2,\cdots,a\\ j=1,2,\cdots,a}}\neq 0 \text{ and } \operatorname{grad} f=0 \text{ at } x_0.$$

7) The non-separated topology means that which $+\infty$ and $-\infty$ have the same neighborhoods in respect to.

can be defined. Let us assume $a \neq +\infty$. It is easily seen that the solution $y(t)$ either vibrates between $\lim_{t\to a-0} y(t)$ and $\overline{\lim}_{t\to a-0} y(t)$ or converges to $\pm\infty$ as $t \to a-0$.

(1°) *The former case.* Let $[c, d]$ be a subinterval of $(\lim_{t\to a-0} y(t), \overline{\lim}_{t\to a-0} y(t))$. By use of the continuity of $y(t)$, there is a monotone increasing sequence $\{t_i\}_{1\leq i<+\infty}$ such that $\lim_{i\to+\infty} t_i = a$, $y(t_{2i-1}) = c$ and $y(t_{2i}) = d$. Due to the mean value theorem, we can find a sequence $\{t_i'\}_{1\leq i<+\infty}$ such that $t_{2i-1} < t_i' < t_{2i}$ and $\left\{\frac{d}{dt} y(t)\right\}_{t=t_i'} = \frac{d-c}{t_{2i}-t_{2i-1}}$. Moreover we easily see that another condition $c \leq y(t_i) \leq d$ may be added before-hand. For this sequence we have $\lim_{i\to+\infty}\left\{\frac{d}{dt} y(t)\right\}_{t=t_i'} = +\infty$. On the other hand, from (2.1)

$$\lim_{i\to+\infty}\left\{\frac{d}{dt} y(t)\right\}_{t=t_i'} \leqq |A(a)| + |B(a)| \overline{\lim_{i\to+\infty}} y(t_i')^2 = |A(a)| + |B(a)| \max(c^2, d^2).$$

This inequality is contrary to the preceding. This shows that $y(t)$ does not vibrate as t converges to a from the left hand.

(2°) *The latter case.* A differential equation

$$\frac{dz}{dt} = B(t) + A(t)z^2 \tag{2.2}$$

has a solution $z(t)$ with $z(a) = 0$ as its initial condition and defined as $a-\varepsilon \leq t < c$ where ε is a suitable positive number and c means the maximum value such that the solution can be obtained on $[a-\varepsilon, c)$. Because $y(t)$ may be assumed not to vanish on $a-\varepsilon \leq t < a$, consider a function $z^*(t) = -\frac{1}{y(t)}$ and replace $y(t)$ in (2.1) by $-\frac{1}{z^*(t)}$. Then we get exactly the same differential equation as (2.2). It follows that $z(t) = z^*(t) = -\frac{1}{y(t)}$ for $a-\varepsilon \leq t < a$. We define $y(t) = -\frac{1}{z^*(t)}$ for $a \leq t < c$.

The same problem as the above-stated occurs at c and the procedure in (2°) can be repeated. Now assume that we can not obtain $y(t)$ defined on $(-b, +\infty)$ by the repetition of the above-mentioned procedures. Then there is a monotone increasing sequence $\{a_i\}_{1\leq i<+\infty}$ such that (a_i, a_{i+1}) is an interval which appears through these procedures and $\lim_{i\to+\infty} a_i \neq +\infty$. So it is obvious that $y(a_{2i-1}) = 0$ and $y(a_{2i}) = \pm\infty$. We may suppose that $\lim_{t\to a_{2i}} y(t) = +\infty$ for an infinite number of i. Then we have a sequence

$\{a'_k\}$ such that $a_{2i_k-1}<a'_k<a_{2i_k}$, $0\leq y(a'_k)\leq\alpha$ and $\left\{\frac{d}{dt}y(t)\right\}_{t=a'_k}=\frac{\alpha}{a_{2i_k}-a_{2i_k-1}}$, where $\{a_{2i_k}\}_{1\leq k<+\infty}$ means a suitable subsequence of $\{a_i\}_{1\leq i<+\infty}$ and α an arbitrary positive number independent of the index k. This shows the same kind of contradiction as (1°).

It is obvious that the solution thus obtained is unique.

Theorem 2.2. *Let $A(t)$ and $B(t)$ be continuous functions defined on an interval $[0, h]$. In addition we assume $A(0)\neq 0$. Now consider a differentiable equation* (2.1) *and denote the solution with $y(0)=0$ as its initial condition by $y(t)$. Then $\frac{1}{t}e^{\int_a^t \frac{A(t)}{y(t)}dt}$ converges to a definite value different from 0 as $t\to+0$, where a means a sufficiently small positive number.*

Proof.[8] Since $A\neq 0$ in $[0, \varepsilon]$ for a small positive number ε, $y(t)$ is never vanishes.
From (2.1), we have

$$\frac{A(t)}{y(t)}=\frac{\frac{dy}{dt}}{y}-By$$

Consequently

$$\lim_{t\to+\infty}\frac{1}{t}e^{\int_a^t\frac{A(t)}{y(t)}dt}=\lim_{t\to+0}\frac{|y|}{t}\lim_{t\to+0}e^{-\int_a^t By\,dt}$$

$$=\left|\frac{dy}{dt}\right|_{t=0}e^{\int_0^a By\,dt}=A(0)\,e^{\int_0^a By\,dt}.$$

Part II

Proofs of the Main Theorems and its Applications

§ 3. The metric property will play hardly any rôle in our proving the theorems stated in §4, almost all of which are based only upon a non-metric property of a vector field in question. In other words, our theory can be considered as an application of a certain much wider theory which has no immediate connection with the metric property and can be used in a much more extensive field of differential geometry.

8) This proof was suggested by Dr. I. Amemiya and Dr. T. Shibata.

From this point of view it is more convenient to treat of much broader conservative force field than a torse-forming vector field in the large. As one of them we take a pseudo-concurrent vector field, the local theory of which substantially coincides with that of a torse-forming vector field [21], but the global theory of which is very different from that of a torse-forming vector field. A global example of that vector field is given by the so-called Appollonius' circles, whose radius vectors form that vector field [21]. But it can not be said that we succeed in dealing with this pseudo-concurrent vector field. In fact most of our theorems are only concerned with the torse-forming vector field in the large. First we define a pseudo-concurrent vector field as follows.

A pseudo-concurrent vector field in the large is a vector field given by replacing (0.1) by the following (3.1) and letting $\boldsymbol{W}$ in (0.3) mean a pseudo-concurrent vector field in the local sense, that is one satisfying (3.2).

$$V_{i;j}=Ag_{ij}+BV_iV_j+\sigma_{;i}V_j+\sigma_{;j}V_i\,, \tag{3.1}$$

$$W_{i;j}=Cg_{ij}+DW_iW_j+\rho_{;i}W_j+\rho_{;j}W_i\,, \tag{3.2}$$

where σ means a point function defined over the space and ρ one defined in U.

The following theorem orginally given by S. Sasaki and K. Yano is our starting point.

Theorem. *Let M be a Finsler space admitting a pseudo-concurrent vector field. Then for every point x but 0-points there exists a neighorhood $U(x)$ and a coordinate mapping α defined over $U(x)$ such that the image of $U(x)$ by α can be decomposed into the product of an open intervel and an $(n-1)$-dimensional cubic, each slice of the former being a segment of a trajectory and each slice of the latter being an integral manifold of* (3.1).

In the present paper we call such a neighborhood $U(x)$ what satisfies Condition ξ.

Let N be the set of all the 0-points. For $x\in K\frown N^c$ let us consider integral manifolds of $\boldsymbol{V}$ containing x as neighborhoods of x, for $x\in K^c\frown N^c$ integral manifolds $\boldsymbol{W}$ of containing x, and for $x\in N$ x itself. Then we can introduce a topology into M by using them as basis (see [8] or [7] p. 92~93). We denote the component of x with respect to this topology by $W(x)$. We can also define $T(x)$ in eactly the same way, which means a maximal trajectory. Later on a symbol $T^+(x)$ will be used. That

means the connected component of x in case that 0-points has been removed from $T(x)$. Besides $W(x)$ or $T(x)$ is sometimes replaced by W or T unless there is any danger of confusion. We call a parameter of T which is compatible with the decomposition of Condition ξ, canonical. Henceforth $\boldsymbol{V}$ means a pseudo-concurrent vector field in the large unless stated otherwise.

The following theorem is easily seen.

Theorem 3.1. *Let x_0 be an isolated 0-point. Then*

$$\exists U(x_0) \forall x \in U(x_0)\ [x \sim x_0],$$

Besides the segment of a trajectory joining x with x_0 can be assumed to be completely inside $U(x_0)$.

Furthermore we have the

Theorem 3.2. *If $\boldsymbol{V}$ satisfies Condision* (iv), *then all the 0-poits of $\boldsymbol{V}$ are isolated.*

Proof. Let x_0 be a limiting point of 0-points of $\boldsymbol{V}$. Then we can choose a differentiable curve $x(s)$ in such a way as this:

$$x_0 = x(0) \quad \text{and} \quad x(s_i) = 0\text{-point} \quad (i = 1, 2, \cdots), \tag{3.3}$$

where $\{s_i\}_{1 \leq i < +\infty}$ means a sequence tending to 0 (Lemma 1.1).

The following immediately follows from (3.3):

$$0 = \lim \frac{\boldsymbol{V}\{x(s_i)\}}{s_i} = \lim_{i \to +\infty} \left(\frac{d\boldsymbol{V}}{ds}\right)_{s=s_i} = \lim_{i \to +\infty} A\{x(s_i)\}\left(\frac{dx}{ds}\right)_{s=s_i} = A(x_0)\left(\frac{dx}{ds}\right)_{s=0}.$$

Hence $$A(x_0) = 0$$

This is contrary to the assumption.

§ 4. In this section we study the nature of $\boldsymbol{V}$ satisfying (i)–(iv) for the most part.

Lemma 4.1. *Let $x \neq a$ 0-point. Then*

$$x_2 \in W(x_1) \Rightarrow \| \boldsymbol{V}(x_2) \| = e^{\sigma(x_2) - \rho(x_1)} \| \boldsymbol{V}(x_1) \|. \tag{4.1}$$

Proof. In a coordinate neighborhood, we have

$$\| \boldsymbol{V}(x') \|_{;j} = \left\{ \frac{A(x')}{\| \boldsymbol{V}(x') \|} + B(x') \| \boldsymbol{V}(x') \| + \frac{\sigma(x')_{;k} V^k}{\| \boldsymbol{V}(x') \|} \right\} V_j + \| \boldsymbol{V}(x') \| \sigma(x')_{;j}.$$

It is easy to obtain (4.1) from this equation.

Corollary. *Let $\boldsymbol{V}$ be a torse-forming vector field in the large. Then $\| \boldsymbol{V} \|$ is constant over W.*

Theorem 4.1. *If M is complete, then W also is complete, provided*

that to be complete means that any Cauchy sequence converges to a definite point.

Proof. Let $\{x_i\}_{1\leq i<+\infty}\subset W$ be a Cauchy sequence of W with respect to the induced metric. That is:

$$(4.2)\qquad \forall\varepsilon>0\exists i_0\ [i,j\geq i_0\Rightarrow \operatorname{dis} W(x_i,x_j)<\varepsilon]\,.$$

Since $\operatorname{dis} W(x_i,x_j)\geq\operatorname{dis}(x_i,x_j)$, $\{x_i\}$ is a Cauchy sequence of M as well. Consequently $\exists x\in M$, $\lim_{i\to+\infty} x_i=x$ where this convergence means one with respect to the topology of M. On the other hand, from (4.1) we have

$$\|V(x)\|=\lim_{i\to+\infty} e^{\sigma(x_i)-\sigma(x_1)}\|V(x_1)\|=e^{\sigma(x)-\sigma(x_1)}\|V(x_1)\|\,.$$

This shows $V(x)\neq 0$. Hence there exists a neighborhood of x satisfying satisfying condition ξ. We denote it by $U(x)$. $U(x)$ is homeomorphic to $I\times R^{n-1}$, where I is the open interval $(-1,1)$ and R^{n-1} is a cubic of center 0 and length 2, where 0 means the origin of E_{n-1}. Describe a sphere of center 0 and radius $\frac{1}{2}$, and describe another sphere of the same center and radius $\frac{1}{4}$, in E_{n-1}, let them be denoted by S^{n-2} and S^{*n-2} respectively. Let the interval $(-\varepsilon,\varepsilon)$ be denoted by I_ε, where ε is an arbitrary positive number. Then $I_\varepsilon^-=[-\varepsilon,\varepsilon]$. Denote a coordinate mapping on $U(x)$ by α. Set $X_1=I_\varepsilon^-\times S^{n-2}$ and $X_2=I_\varepsilon^-\times S^{*n-2}$. Then $\alpha^{-1}(X_1)$ and $\alpha^{-1}(X_2)$ are closed and bounded.
Moreover $\alpha^{-1}(X_1)\frown\alpha^{-1}(X_2)\neq\phi$. Hence $\operatorname{dis}\{\alpha^{-1}(X_1),\ \alpha^{-1}(X_2)\}=d_0>0$.
On the other hand, $\exists i_0'[i,j\geq i_0'\Rightarrow \operatorname{dis} W(x_i,x_j)<d_0$ and $x_i,x_j\in\alpha^{-1}\{I_\varepsilon\times$ the interior of $S^{*n-2}\}]$. Denote the connected component of x_i,x_j with respect to the relative topology of $W\frown U(x)$ induced by the topology of W by W_i, W_j respectively. Let us choose t_i such that $W_i=\alpha^{-1}[\{t_i\}\times R^{n-1}]$ and $t_i\in I_\varepsilon$.

Now assume $W_i\neq W_j$. Then $W_i\frown W_j=\phi$. Therefore $W_j\subset$ the exterior of $\alpha^{-1}[\{t_i\}\times S^{n-2}]$ with respect to $W(x_i)$. Of course, $x_i\in$ the interior of $\alpha^{-1}[\{t_i\}\times S^{*n-2}]$ with respect to $W(x_i)$. It follows that an arbitrary curve γ which joins x_i with x_j can not help meeting $\alpha^{-1}[\{t_i\}\times S^{*n-2}]$ and $\alpha^{-1}[\{t_i\}\times S^{n-2}]$. Consequently, the length of $\gamma\geq$ the distance (with respect to M) between $\alpha^{-1}(X_1)$ and $\alpha^{-1}(X_2)$ (that is; d_0).

Therefore

$$(4.3)\qquad \operatorname{dis} W(x_i,x_j)\geq d_0\,.$$

This is contrary to (4.2).

Hence we have $W_i=W_j$. This means that if $i\geq i_0'$, x_i belongs to the common component $W_{i_0'}$. Then

$$x=\lim x_i \in W_{i_0'}^{-} \frown V(x)=W_{i_0'} \subset W$$

Namely, every Cauchy sequence of W has limit in W.

Theorem 4.2. *Let T be a trajectory passing through a 0-point $T(0)$ of V. Then*

$$\lim_{t\to 0} W_t = T(0), \tag{4.4}$$

where W_t means $W\{T(t)\}$. Strictly speaking, (4.4) means

$$\forall U \ni T(0) \exists \delta > 0 [\delta > |t| \Rightarrow W_t \subset U]. \tag{4.4'}$$

Proof. Let us suppose that (4.4) is false. Let $S_r^{*n-1}=[x \mid \mathrm{dis}\,(x, x_0)=r]$ for an arbitrary positive number r, let $\mathrm{In}\, S_r^{*n-1} \equiv [x \mid \mathrm{dis}\,\{x, T(0)\}<r]$, and let $\mathrm{Out}\, S_r^{*n-1} \equiv [x \mid \mathrm{dis}\,\{x, T(0)\}>r]$. Since (4.4') is false, for a suitable positive number r and a suitable sequence $\{t_i\}_{1\le i<+\infty}$, of positive terms we have

$$\lim t_k = 0 \text{ and } T(t_i) \in \mathrm{In}\, S_r^{*n-1} \text{ and } W_{t_i} \not\subset \mathrm{In}\, S_r^{*n-1},$$

where $i=1, 2, \cdots$. Then either of the following holds,

$$W_{t_i} \frown S_r^{*n-1} \neq \phi, \tag{4.4}$$

or

$$W_{t_i} \frown \mathrm{Out}\, S_r^{*n-1} \neq \phi. \qquad (i=1, 2, \cdots) \tag{4.5}$$

By means of the arc-connecticity of W_{t_i}, (4.5) can be reduced to (4.4). Then we can select an element x_i from $W_{t_i} \frown S_r^{*n-1}$ for $i=1, 2, \cdots$. The sequence $\{x_i\}_{1\le i<+\infty}$ has a limit x on S_r^{*n-1}.

There exist a subsequence $\{x_{i_k}\}_{1\le k<+\infty}$ of $\{x_i\}$ such that $\lim_{k\to\infty} x_{i_k}=x$. Consequently using Lemma 4.1, we have

$$\begin{aligned} \|V(x)\| &= \lim e^{\sigma(x_{i_k})-\sigma\{T(t_{i_k})\}} \|V\{T(t_{i_k})\}\| \\ &= e^{\sigma(x)-\sigma\{T(0)\}} \|V\{T(0)\}\| = 0. \end{aligned}$$

This shows a contradiction. Thus our theorem has been proved.

Corollary. *Let x_0 be a 0-point of a pseudo-concurrent vector field. Then*

$$\exists W [W \frown T^+ \neq \phi \Rightarrow T \ni x_0].$$

Proof. Let $U(x_0)$ be such a neighborhood as stated in Theorem 3.1. Take W contained in $U(x_0)$. The existence of such W is guaranteed by the above theorem. It is clear that this W is one to suit our purpose.

Lemma 4.2. *Let $W \in x_1, x_2$. Join x_1 to x_2 by a curve γ on W (γ: $x=x(\tau)$, $0\le\tau\le 1$). Then taking a suitalbe positive number, the following hold.*

(i) $t_0-\varepsilon\leq t\leq t_0+\varepsilon\Rightarrow T\{x(\tau)\}\frown W_t\neq\phi$

(ii) *There exists one and only one way of choosing* $x_t(\tau)$ *from* $T\{x(\tau)\}\frown W_t$ *in such a fashion that* $x_t(\tau)$ *describes a continuous curve as* τ *varies from* 0 *to* 1,
where t *means a canonical parameter of* $T(x_1)$ *and* W_t *means* $W\{T(x_1)(t)\}$.

Proof. Denote the neighborhood of Condition ξ by U_ξ. For an arbitrary point $x\in\gamma$, set $r_x=\sup\{\delta\,|\,\exists U(x)[\delta\geq\mathrm{dis}\,(x,y)\Rightarrow y\in U_\xi(x)]\}$. Assign a sphere $S^{n-1}(x)$ of center x and radius r_x to x.

Lemma 4.3. *Let* r *be* $\inf_{x\in\gamma} r_x$. *Then* $r\neq 0$.

Proof of Lemma 4.3. Assume $r=0$. Then there exists a sequence $\{r_{x_i}\}_{1\leq i<+\infty}$ such that $\lim_{i\to\infty} r_{x_i}=0$. $\{x_i\}_{1\leq i<+\infty}$ has a limiting point. Let it be denoted by x_3. Then there exists a subsequence $\{x_{i_k}\}_{1\leq k<+\infty}$ of $\{x_i\}$ converging to x_3. For this subsequence,

$$\exists k_0\left[k\geq k_0\Rightarrow \mathrm{dis}\,(x_3, x_{i_k})<\frac{r_{x_3}}{2}\right]. \tag{4.6}$$

Hence if $\mathrm{dis}\,(x_{i_k}, x)\leq\frac{r_{x_3}}{2}$ for $k\geq k_0$, then $\mathrm{dis}\,(x_3, x)<r_{x_3}$. Therefore $k\geq k_0$ there exists U_ξ such that if $\mathrm{dis}\,(x, x_{i_k})\leq\frac{r_{x_3}}{3}\Rightarrow U_\xi\ni x$. It follows that

$$r_{i_k}\geq\frac{r_3}{2}\quad\text{for } k\geq k_0\,. \tag{4.7}$$

(4.7) is contrary to (4.6).

Proof of Lemma 4.2 *continued.* Define $F=\{y\,|\,\exists x\in\gamma,\ \mathrm{dis}\,(x,y)\leq r\}$ and $\sigma_0=\inf_{x\in F}\sigma(x)$. Notice that $\sigma_0=-\infty$ because $\inf_{x\in F}=\min_{x\in F}$. Setting $T_\tau=T\{x(\tau)\}$, introduce a canonical parameter t_τ into T_τ in such a way as $T_\tau(0)=x(\tau)$. Then for a sufficiently small positive number ε, if $\varepsilon>t_1>0$, we have

$$\mathrm{dis}\,\{x_1, T_1(t_1)\}<r\quad\text{and}\quad\int_0^{t_1} e^\sigma ds<\frac{r}{2}e^{\sigma_0}\,. \tag{4.8}$$

Fix t_1, and denote $W\{T_1(t_1)\}$ by W^*. Consider a positive number δ satisfying the following condition:

$$0\leq\tau\leq\delta\Rightarrow$$

(i) $\exists x_\tau^*\in W^*\frown T_\tau$. (Define $x^*(\tau)=x_\tau^*$).

(ii) γ^*: $x=x^*(\tau)$ is a continuous curve.

(iii) Setting $x^*(\tau)=T_\tau(t_\tau)$

$$\int_{0}^{t_\tau} e^\sigma\, ds = \int_{0}^{t_1} e^\sigma\, ds\,.$$

(iv) $\operatorname{dis}\{x(\tau), T_\tau(u)\} < r$ for $0 \leq u \leq t_\tau$.

Lemma 4.4. *Let τ_0 be the supremum of such δ. Then $\tau_0 = 1$.*

Proof of Lemma 4.4. First assume that τ_0 does not have the above property of δ. Then $\tau_0 > 0$. For a sufficiently small positive number ε', if $\tau_0 - \varepsilon' < \tau < \tau_0$, τ has the property of δ and we may assume $\operatorname{dis}\{x(\tau_0), x(\tau)\} < \frac{r}{2}$. Hence, from (4.8),

$$e^{\sigma_0} \int_{0}^{t_\tau} ds \leq \int_{0}^{t_\tau} e^\sigma\, ds = \int_{0}^{t_1} e^\sigma\, ds \leq \frac{r}{2} e^{\sigma_0}\,,$$

Hence

$$\int_{0}^{t_\tau} ds \leq \frac{r}{2}\,.$$

Therefore if $0 \leq t_\tau^* \leq t_\tau$,

$$\begin{aligned}\operatorname{dis}\{x(\tau_0), T_\tau(t_\tau^*)\} &\leq \operatorname{dis}\{x(\tau_0), x(\tau)\} + \operatorname{dis}\{x(\tau), T_\tau(t_\tau^*)\} \\ &< \frac{r}{2} + \int_{0}^{t_\tau} ds < r\,.\end{aligned}$$

Consequently,

$$(4.9) \qquad \exists\, U_\xi\{x(\tau_0)\}[0 \leq t_\tau^* \leq t_\tau \Rightarrow T_\tau(t_\tau^*) \in U_\xi\{x(\tau_0)\}]\,.$$

Let W' be a component of $T_\tau(t_\tau)$ with respect to the relative topology of $W\{T_\tau(t_\tau)\} \frown U_\xi\{x(\tau_0)\}$ induced by the topology of $W\{T_\tau(t_\tau)\}$. Then $W' \subset W^*$.

Let $W' \frown T_{\tau_0} = T_{\tau_0}(t_{\tau_0})$. Then $T_{\tau_0}(t_{\tau_0}) \in W^* \frown T_{\tau_0}$ and $\lim_{\tau \to \tau_0} x^*(\tau) = T_{\tau_0}(t_{\tau_0})$ and moreover,

$$\int_{0}^{t_{\tau_0}} e^\sigma\, ds = \int_{0}^{t_\tau} ds = \int_{0}^{t_1} e^\sigma\, ds\,.$$

This shows that τ_0 has the property of δ. This is contrary to our assumption.

Next assume $\tau_0 < 1$. According to (4.4), $U_\xi\{x(\tau_0)\} \ni T_{\tau_0}(t_{\tau_0})$. It follows that if $\tau_0 < \tau \leq \tau_0 + \varepsilon''$ for a sufficiently small positive number ε'', then $T\{x(\tau)\}$ meets $W[T\{x(\tau_0)\}]$ and τ itself has the property of δ. This shows a contradiction. Therefore $\tau_0 = 1$.

Thus together with Lemma 4.2 has been proved.

Lemma 4.5. *Let M be a complete space. Let $\boldsymbol{V}$ be a torse-forming*

vector field. Let $T(x_0) \ni x_1$ *and* $W(x_0) \ni y_0$. *Join* x_0 *with* y_0 *by a curve* γ *on* $W(x_0)$ (γ: $x=x(\tau)$ $(0 \leq \tau \leq 1)$; $x_0=x(0)$, $y_0=x(1)$). *Denote* $T\{x(\tau)\}$ *by* T_τ *and* $W\{T(s)\}$ *by* W_s, *where* $x_0=T(0)$ *and* $x_1=T(s_0)$ *are assumed. Then assuming the arc length of* T_τ *is measured from* $W(x_0)$, *we have*

$$T_\tau(s) \in W_s \quad (0 \leq s \leq s_0;\ 0 \leq \tau \leq 1). \tag{4.10}$$

Especially

$$T_1(s_0) \in W_{s_0}. \tag{4.11}$$

Proof. Let δ_0 be the supremum of such a number δ that if s_0 is replaced by δ in the statement of Lemma 4.5, this lemma holds. Let $\{s_i\}_{1 \leq i < +\infty}$ be a monotone increasing sequence tending to δ_0 as $i \to +\infty$. Since $\operatorname{dis}\{T_\tau(s_i), T_\tau(s_j)\} \leq$ the length of the segment $T(s_i, s_j) = |s_j - s_i|$, $T(s_i, s_j) \to 0$ as $i, j \to 0$. Therefore $\{T_\tau(s_i)\}_{1 \leq i < +\infty}$ form a Cauchy sequence of M. Hence $\exists x_\tau \in M[\lim T_\tau(s_i) = x_\tau]$. Notice that $\boldsymbol{V}(x_\tau) \neq 0$. In fact $\|V\{T_\tau(s_i)\}\| \to \|V\{T(\delta_0)\}\| \neq 0$ as $i \to +\infty$. Thus we have found that $T_\tau(\delta_0)$ exists because of M being complete. It is obvious that this convergence is uniform. If follows that $T_\tau(\delta_0)$ is a continuous curve with respect to τ. Moreover, it is clear that this curve is on W_{δ_0}. It follows from Lemma 4.2 that $\delta_0 = s_0$ or δ_0 is not the supremum. Thus δ_0 must be equal to s_0.

Let $\mathfrak{M}'$ and $\mathfrak{M}''$ be partitions of topological spaces M' and M'' respectively. Let us suppose that for every $x' \in M'\mathfrak{M}'$ has a neighborhood $U(x')$ in M' with the condition that $U(x') \frown \pi'$ contains at most one element for every $\pi' \in \mathfrak{M}'$ and that for every $x'' \in M''$ $\mathfrak{M}''$ has an analogous neighborhood $U(x'')$ in M''.

Definition. *A mapping* φ *of* $\mathfrak{M}'$ *into* $\mathfrak{M}''$ *is called locally homeomorphic if it satisfies the following condition:*

If we take suitable neighborhoods $V(x') \subset U(x')$ *and* $V(x'') \subset U(x'')$, *the mapping* φ^* *which is defined by means of* $\varphi^* = P''^{-1} \circ \varphi \circ P'$ *is a topological mapping of* $V(x')$ *into* $V(x'')$, *where* P' *and* P'' *are the partition mappings of* $\mathfrak{M}'$ *and* $\mathfrak{M}''$.

We call φ a *local homeomorphism of* M' *into* M''.

Theorem 4.3. *Let* M *be a space. Let* $\boldsymbol{V}$ *be a torse-forming vector field in the large with isolated* 0*-points only. Then for arbitrary non-*0*-points* x_1 *and* x_2, *there exists a local homeomorphism of* $W(x_1)$ *onto* $W(x_2)$ *canonically.*

Proof. Denote the join of W for which there exists a certain T satisfying $T \frown W(x_1) \neq \phi$ and $T \frown W \neq \phi$ by M_1, and denote the join of W

not satisfying the above condition by M_2. Then $M-N=M_1 \smile M_2$, where N means the set of all 0-points of $\boldsymbol{V}$. It is abvious that M_1 and M_2 are open. On the other hand, N is an isolated set. Hence $M-N$ is connected. It follows that $M_1=\phi$ or $M_2=\phi$. Since $M_1=\phi$, $M_2=\phi$. Consequently no matter what x_2 is, there exists a certain T such that $T \frown W(x_1) \neq \phi$ and $T \frown W(x_2) \neq \phi$. Thus we can prove the theorem from Lemma 4.5.

Corollary. *Let x_1 and x_2 be elements of $T^+ \frown W$. Let x_1' be an element of $T'^+ \frown W'$. Then there exist $x_2' \in T'^+ \frown W'$ and $x_3 \in T^+ \frown W'$ such that*

$$\text{(4.12)} \quad \text{the length of } T'^+(x_1', x_2') = \begin{cases} \text{the length of } T^+(x_1, x_2). \\ \text{or} \\ \text{the length of } T^+(x_1, x_2) \pm 2 \text{ the length} \\ \text{of } T^+(x_1, x_3). \end{cases}$$

Proof. Due to Theorem 4.2, $T^+ \frown W' \neq \phi$. Let x_3 be its element. Then using Lemma 4.2, we have the length of $T^+(x_3, x_2)$=the length of $T'^+(x', x_3')$ for some $x_3' \in T'^+ \frown W$, and, the length of $T^+(x_1, x_3)$=the length of $T'^+(x_3', x_2')$ for some $x_2' \in T'^+ \frown W'$. Hence

the length of $T'^+(x_1', x_2')$=the length of $T'^+(x_1', x_3') \pm$the length of $T'^+(x_3', x_2')$=the length of $T^+(x_3, x_2) \pm$the length of $T^+(x_1, x_2)$

$$= \begin{cases} \text{the length of } T^+(x_1, x_2) \\ \text{or} \\ \text{the length of } T^+(x_1, x_2) \pm 2 \text{ the length of } T^+(x_1, x_3) \end{cases}$$

Theorem 4.4. *Let $\boldsymbol{V}$ be a torse-forming vector field satisfying Condition* (iv). *Let x_0 be a 0-point of $\boldsymbol{V}$. Then for an arbitrary trajectory T, $T \ni x_0$.*

Proof. Consider a neighborhood $U(x_0)$ mentioned in Theorem 4.2 (which is moreover assumed to be included in such a neighborhood as stated in Theorem 3.1.) By virtue of Theorem 4.3, $T \frown W(x) \neq \phi$. Consequently $T \ni x_0$ (Theorem 3.1).

Theorem 4.5.[9] *Let $\chi(W, T^+)$ be the intersection number of W and T^+. Then $\chi(W, T^+)$ does not depend on T^+,* provided that V has at least one 0-point.

Proof. Let $x_1 \in W \frown T^+$ and $x_1' \in W \frown T'^+$. Then to each $x \in W \frown T^+$ we can assign the point of $T'^+ \frown W$ which is determined through the procedure stated in Lemma 4.2. This mapping becomes a 1–1 correspondence between $T^+ \frown W$ and $T'^+ \frown W$. Thus the number of the elements of $T^+ \frown W$ coincides with that of $T'^+ \frown W$. Namely: $\chi(W, T^+)=\chi(W, T'^+)$.

Lemma 4.6. *In a neighborhood of an isolated 0-point all W are regularly imbedded.*

9) From this reason we write $\chi(W)$ instead of $\chi(W, T)$, when the vector field has 0-point.
10) The intersection number means the number of the points at which W and T^+ meet.

Proof. Otherwise there exists a coordinate neighborhood:
$$0<x^i<d \quad (d\text{: a positive number})$$
such that $x^n=\text{const.}\overset{\text{def}}{\equiv}x_k^n$ $(k=1, 2, \cdots)$ express portions of the same W, where we assume $\lim_{k\to\infty} x_k^n=x_1^n$. By virtue of Theorem 4.1, $\|\boldsymbol{V}\|=\text{const.}$, if $x\in W$. Hence $\dfrac{\partial\|\boldsymbol{V}\|}{\partial x^n}=\lim_{k\to\infty}\dfrac{\Delta\|\boldsymbol{V}\|}{x_k^n-x_1^n}=0$. Therefore $\left(\dfrac{\partial\|\boldsymbol{V}\|}{ds}\right)_{x=x_1}=0$, where x_1 means a point having x_1^n as its n-th coordinate. On the other hand, neighborhood of a 0-point, $\dfrac{d\|\boldsymbol{V}\|}{ds}\neq 0$. This is a contradiction.

Lemma 4.7. *In a neighborhood of a 0-point, all W contained in it are compact.*

Proof. This is evident, because a submanifold which is bounded, complete, and regularly imbedded is compact.

Lemma 4.8. *If a torse-forming vector field in the large satisfies Condition* (iv) *and has 0-points, then $\chi(W)$ is equal to 1 or 2 for W sufficiently close to a 0-point.*

Proof. Let $x_0=T(0)$ be a 0-point. Let $U(x_0)$ be the intersection of a neighborhood stated in Theorem 3.1 and one mentioned in Lemma 4.6. Now assume that this lemma is false. Then there exists a monotone decreasing sequence $\{s_i\}_{1\leq i<+\infty}$ such that $\chi[W\{T(s_i)\}]\geq 3$. Henceforth we denote $W\{T(s_i)\}$ by W_i. Due to Theorem 4.2, we may consider $W_i\subset U(x_0)$. Lemma 4.6 shows that $T^+\frown W_i$ is a finite set. Then there exist $\min\{s\mid T^+(s)\in W_i\}$ $(\overset{\text{def}}{\equiv}s_i^{(1)})$, $\min\{s\mid T^+(s)\in W_i, s>s_i^{(1)}\}$ $(\overset{\text{def}}{\equiv}s_i^{(2)})$, and $\max\{s\mid T^+(s)\in W_i\}$ $(\overset{\text{def}}{\equiv}s_i^{(3)})$. Let $T\{s_i^{(2)}-(s_i^{(1)}-s_{i+1}^{(1)})\}\in W_{i+1}$. Then it is easily seen from the corollary of Theorem 4.3 that $s_i^{(2)}-(s_i^{(1)}-s_{i+1}^{(1)})=s_{i+1}^{(2)}$. Hence $s_i^{(2)}-s_i^{(1)}=s_{i+1}^{(2)}-s_{i+1}^{(1)}$. Let $T^+\{s_i^{(2)}-(s_i^{(1)}-s_{i+1}^{(1)})\}\notin W_{i+1}$. Then by virue of the above mentioned corollary again, we have $T^+\{s_i^{(2)}+(s_i^{(1)}-s_{i+1}^{(1)})\}\in W_{i+1}$. In this case $s_i^{(2)}+(s_i^{(1)}-s_{i+1}^{(1)})=s_{i+1}^{(2)}$ and $s_{i+1}^{(2)}-s_{i+1}^{(1)}=s_i^{(2)}-s_i^{(1)}+2(s_i^{(1)}-s_{i+1}^{(1)})$. In either case we have $s_{i+1}^{(2)}-s_{i+1}^{(1)}\geq s_i^{(2)}-s_i^{(1)}$. Consequently we easily find $\sim\forall V(x_0)\exists i_0$ $[i>i_0\Rightarrow T^+(s_i^{(1)}, s_i^{(2)})\subset V(x_0)]$,[11] because T is rectifiable around x_0 (see Theorem 1.1). By use of this fact we have

$$\forall V(x_0)\exists i_0[i>i_0\Rightarrow T^+(s_i^{(2)}, s_i^{(3)})\subset V(x_0)]\,. \tag{4.13}$$

Otherwise there exists a sequence $\{\gamma_i\}_{1\leq i<+\infty}$ of segments of T such that γ_i is entirely contained inside S_i^{n-1} excepting for its end points exactly on S_i^{n-1}, provided that S_i^{n-1} $(i=1, 2, \cdots)$ are spheres of a radius tending to 0. This contradicts the same argument as stated in (I) of the proof of Theorem 1.1. Hence (4.13) holds. Set $s_i^{(4)}=\max\{s\mid T^+(s)\in W_i \text{ and } s<s_i^{(3)}\}$.

11) The symbol $\sim$ means the negation of what follows.

Then (4.13) still holds for the segment $T^+(s_i^{(4)}, s_i^{(3)})$. It is seen from exactly the same reason that $T^+(s)$ $(s>s_i^{(3)})$ straightforwardly tends to x_0.

Now let $m(s)=\max\{s' \mid T^+(s') \in W\{T^+(s)\}\}$. Assume that $m(s)$ is not monotone increasing. Then since $m(s)$ is a continuous 1–1 mapping, $m(s)$ is monotone-decreasing. On the other hand $T^+\{m(s)\}$ must tend to x_0 as $s \to 0$. This is a contradiction. Namely: $m(s)$ is monotone-increasing and $\{s_i^{(3)}\}$ likewise does so. Hence we can find, by precisely the same way, that $T^+(s_i^{(4)}, s_i^{(3)})$ has length not tending to 0. This is contrary to (4.13), because T is rectifiable around x_0.

Lemma 4.9. *Let $\boldsymbol{V}$ be a torse-forming vector field in the large satisfying Condition* (iv) *and having* 0*-points. Then $\chi(W)=1$ or 2.*

Proof. Let x_0 be one of the 0-points of $\boldsymbol{V}$. Suppose that the above lemma is not true. Then there exists a transversal submanifold W such that $\chi(W)\geq 3$. We may assume that $\chi(W)=3$. In this case an arbitrary T^+ meets W three times. Let $T^+(s_1)$, $T^+(s_2)$, and $T^+(s_3)$ be the points at which T^+ meets W, provided that the arc length s is measured from x_0. Let s_1' be a sufficiently small positive number. Then $\chi\{W(s_1')\}=1$ or 2 (Lemma 4.6), where $W(s_1')=W\{T^+(s_1')\}$. According to Lemma 4.5, we can find such two points $T^+(s_2')$, $T^+(s_3') \in W(s_1')$ as this:

$$|s_1'-s_1|=|s_2'-s_2|=|s_3'-s_3|. \tag{4.14}$$

Among these three points, two must coincide. For example assume $T(s_1')=T(s_2')$.
Then it follows from $s_1 \neq s_2$ that $s_2=s_1'-(s_1-s_1')$. Consequently $s_1' \geq s_1-s_1'$. This is contrary to the fact that s_1' is an arbitrary small positive number. In the case where $T^+(s_3')=T^+(s_2')$ etc., the same arguments hold.

Theorem 4.6. *Let $\boldsymbol{V}$ be a torse-forming vector field satisfying Condition* (iv) *and having* 0*-points. If there exists a W such that $\chi(W)\geq 2$, then*

(i) *there exists only one* 0*-point.*

(ii) *$\chi(W)=2$ for all W but one.*[12]

Proof. Let x_0 and s be the same symbols as used in the proof of Lemma 4.9, denote $W\{T^+(s)\}$ by $W(s)$ briefly, and if $\chi(W(s))=2$, define a function $d(s)$ as the arc length of T^+ between the two points at which T^+ meets $W(s)$. Furthermore let s_0 be the upper limit of such a value δ as the following conditions are fulfilled:

$$\delta>s>0 \Rightarrow \begin{cases} \chi\{W(s)\}=2, & (4.15) \\ d(s)=L-2s, & (4.16) \end{cases}$$

where L is the whole length of T^+.

12) Of course, for the exceptional W, $\chi(W)=1$.

Then the following lemma holds:

Lemma 4.10.

$$0<s_0\leq\frac{L}{2}. \tag{4.17}$$

Proof of Lemma 4.10. Let $\chi\{W(s_1)\}\geq 2$. Then $T^+(s_2)\in W(s_1)$ for some $s_2>s_1$. It is easily seen by considering s_1 and s_2 that $\chi(W(s))=2$[13] for a sufficiently small positive number s owing to Lemma 4.5 and Lemma 4.8. Besides (4.16) holds likewise. Hence $s_0>0$.

On the other hand, (4.15) and (4.16) are not compatible for $s=\frac{L}{2}$. Hence $s_0\leq\frac{L}{2}$. Thus we have (4.17).

Lemma 4.11. $\chi(W(s_0))=1$.

Proof of Lemma 4.11. Assume that $\chi(W(s_0))\geq 2$. Hence it follows that $\chi(W(s_0))=2$. It is evident that, if $s-s_0$ (>0) is sufficiently small, then $\chi(W(s))\geqq 2$. Consequently $\chi(W(s))=2$. If (4.16) does not hold for such s, then $d(s)=L-2s_0$ as we see easily from Lemma 4.2. On the other hand $T^+[L-\{s_0-(s-s_0)\}]\in W\{s_0-(s-s_0)\}$. Since $L-\{s_0-(s-s_0)\}=s+L-2s_0$, $T^+[L-\{s_0-(s-s_0)\}]\in W(s)$. This means that $W(s)=W\{s_0-(s-s_0)\}$ meets T^+ at least three times. This can not arise because of Lemma 4.9. Hence (4.16) likewise holds for s sufficiently close to s_0. But this is contrary to the definition of s_0. Thus the above lemma has been proved. What remains for us is to prove the

Lemma 4.12.

$$s_0=\frac{L}{2}. \tag{4.18}$$

Proof of Lemma 4.12. Let s be smaller than s_0. Then $W(s)\ni T^+(s')$ for some $s'>s$. According to Lemma 4.5, $T^+\{s'-(s_0-s)\}$ or $T^+\{s'+(s_0-s)\}\in W(s_0)$. Hence $s'=s_0+(s_0-s)$, because $W(s_0)$ and T^+ meet together only once. On the other hand $s'=L-s$ from (4.15) and (4.16). Consequently (4.18) can be derived.

Lemma 4.13. *Let $\boldsymbol{V}$ be a torse-forming vector field satisfying Condition* (iv). *Then the number of the* 0*-points of $\boldsymbol{V}$, $N(\boldsymbol{V})$, is at most* 2.

Proof. Assume that there are more than two 0-points and let $x_1, x_2, \cdots$ be those 0-points. Let $W_1, W_2, \cdots$ be maximal transversal hypersurfaces sufficiently close to $x_1, x_2, \cdots$ respectively. Let $U(x_2)$ be

13) This shows that L is finite.

such a neighborhood of the 0-point x_2 as stated in Theorem 3.1. We can assume $W_3 \subset U(x_2)^c$. Since $T^+ \frown W_1 \neq \phi$, $T^+ \frown W_2 \neq \phi, \cdots$, let $T^+(s_1) \in W_1$, $T^+ \in W_2, \cdots$, provided that we assume $s_1 < s_2 < \cdots$. Then after starting from $T^+(s_1)$ and reaching $T^+(s_2)$, T^+ must tend to x_2 without going beyond $U(x_2)$. This means that T^+ has a 0-point between $T^+(s_2)$ and $T^+(s_3)$. This is contrary to the definition of T^+ (see § 3).

By virtue of the above-stated theorems, we can easily see that the following theorems hold good. It seems to the present author that these theorems are a satisfactory answer to the topological aspect of the problem which is given rise to in the introduction of the present paper.

We shall deal with the metric aspect of the problem in § 5.

Theorem 4.7. *If a 2-differentiable complete Finsler space M admits a torse-forming vector field in the large $\boldsymbol{V}$ satisfying Condition* (iv). *then the number of the 0-points of $\boldsymbol{V}$ is at most 2 and the type of M is decomposed into the following four:*

I. *The case in which $N(\boldsymbol{V})=1$.*

This case is divided into the following two:

i) *All geodesics throgh the 0-point deverge to the infinity, and they cover the whole space one-foldly excepting for the 0-point.*

ii) *All geodesics throgh the 0-point are simply closed, and they cover the whole space one-foldly excepting for the 0-point.*

II. *The case in which $N(\boldsymbol{V})=2$.*

All geodesics through one of the 0-points reach the other 0-point, and they cover the whole space one-foldly excepting for the 0-points.

III. *The case in which $N(\boldsymbol{V})=0$.*

The geodesic conruence tangent to $\boldsymbol{V}$ covers the whole space exclusively. All geodesics of the congruence are homeomorphic to a straight-line or a torus.

Corollary. *If a 2-differentiable Finsler space M admits a torse-forming vector field in the large $\boldsymbol{V}$ satisfying Condition* (iv) *and having at least one 0-point, then the space form of M can be classified in the following way accoding to the number of the 0-points of $\boldsymbol{V}$.*

I. *M is homeomorphic to a Euclidean space or a projective space ($N(\boldsymbol{V})=1$).*

II. *M is homeomorphic to a sphere ($N(\boldsymbol{V})=2$).*

Corollary. *If a 2-differentiable complete Finsler space admits a torse-forming vector field in the large satisfying Condition* (iv) *and having at least one 0-point, then the space is simply connected.*

Theorem 4.8. *If a 2-differentiable complete Finsler space admits a torse-forming vector field having two 0-points x_0 and x_1 and if $A(x_0)$ $A(x_1)>0$, then there exists at least one W over which $||\boldsymbol{V}||=+\infty$.*

Corollary. *A 2-differentiable compact Finsler space admits a torse-forming vector field having one and only one 0-point, then there exists at least one W over which $||\boldsymbol{V}||=+\infty$.*

§ 5. The objective of this paragraph is to determine the metric of a Finsler space admitting a torse-forming vector field in the large to the fullest extent. Let M be a complete Finsler space which admits a torse-forming vector field in the large $\boldsymbol{V}$ satisfying Condition (iv). Then we set

$$H(x) = e^{\int_{T(x)} \frac{A(x)}{||\boldsymbol{V}||} ds} \tag{5.1}$$

and call it the characteristic function of M, provided that the integral of the right-hand member is made from a fixed W to x along $T(x)$. Hence $H(x)$ can not generally be a single-valued function.

Theorem 5.1. *In order that a point x_0 be a 0-point of $\boldsymbol{V}$, it is necessary and sufficient that*

$$\lim_{x \to x_0} H(x) = 0 . \tag{5.2}$$

Proof. Assume that (5.2) holds. Let T be a trajectory passing through x_0 and $T(0)$ be x_0. Then

$$\lim_{s \to 0} e^{\int_a^s \frac{A}{||\boldsymbol{V}||} ds} = 0 .$$

Accordingly

$$\lim_{s \to 0} \int_a^s \frac{A}{||\boldsymbol{V}||} ds = -\infty , \qquad \text{and} \qquad \lim_{s \to 0} ||\boldsymbol{V}|| = 0 ,$$

The converse is obvious.

In what follows, *we assume that $\boldsymbol{V}$ has at least one 0-point, and every trajectory has the arc length measured from a 0-point as its parameter.* We say that $\boldsymbol{V}$ *satisfies H-condition*, if $H(x)=H(x')$ for $x, x' \in W$. We write $W(s)$ instead of $W\{T(s)\}$. Then there exists a natural homeomorphism between $W(s)$ and $W(s')$ for $0<s, s'<L$, where L means the constant length of the trajectories, including $L=+\infty$. We denote it by $\Omega(s, s')$.

Now consider a mapping which assigns each element of $W(s)$ to the unit initial vector of $T(s)$, that is: $\left\{\frac{dT(s)}{ds}\right\}_{s=0}$. We denote it by $\Omega(s)$

briefly. $\Omega(s)$ is a diffeomorphism between $W(s)$ and the unit sphere S_u^{n-1} of center $T(0)$ (in the tagent space of $T(0)$). $\Omega(s)$ can be extended to a 1–1 bundle map from the tangent bundle with base space $W(s)$ to the tagent bundle with base space S_u^{n-1}. We denote it by $d\Omega(s)$. On the other hand, left $d\Omega(s, s')$ express the natural bundle map from the tangent bundle $\mathfrak{B}_1$ with base space $W(s)$ to the tangent bundle $\mathfrak{B}_2$ with base space $W(s')$ and let $d\Omega(s)$ be a bundle map defined in the same way as above. These tagent bundles $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are subbundles of the tangent bundle $\mathfrak{B}$ with base space X. We can consider a bundle map of $\mathfrak{B}$ onto $\mathfrak{B}$ called a scaler multiplication in the natural way and we denote it by the scaler itself. Let $\frac{1}{s'}d\Omega(s, s')$ mean the product map of $d\Omega(s, s')$ and a scaler multiplication $\frac{1}{s'}$ and let S_u^{n-1} be identified with a bundle with the base space consisting of a single point x_0. Notice that this bundle can be considered as a subbundle of $\mathfrak{B}$. Then we have the

Theorem 5.2.

$$\lim_{s'\to 0}\frac{1}{s'}d\Omega(s, s')=d\Omega(s),$$

namely

$$\frac{d}{ds'}d\Omega(s, s')=d\Omega(s) \quad at\ s'=0. \tag{5.3}$$

Proof. Describe a curve γ on $W(s)$ and let its equation be $x=x(\tau)$ $(0\leq\tau\leq 1)$. Next consider a function given by the equation: $f(s', \tau)=\Omega(s, s')x(\tau)$. Then

$$\frac{d}{ds'}d\Omega(s, s')\frac{dx}{d\tau}=\frac{d}{ds'}\frac{d}{d\tau}f(s', \tau)=\frac{d}{d\tau}\frac{d}{ds'}f(s', \tau)$$

$$=\frac{d}{d\tau}\Omega(s)x(\tau)=d\Omega(s)\cdot\frac{dx}{d\tau} \quad \text{at } s'=0.$$

Thus we have obtained (5.3).

By virtue of Theorem 2.2, we see that $\sqrt{H}$ [14] has a derivative in every direction at x_0 (more strictly speaking a derivative for every vector) and the derivative is different from 0. We denote the derivative for a vector $\boldsymbol{X}$ by $(d\sqrt{H})_0\boldsymbol{X}$. Then we have the

Theorem 5.3.

$$\|d\Omega(s)\boldsymbol{X}\|=\frac{(d\sqrt{H})_0\circ\Omega\{s(p(\boldsymbol{X}))\}p(\boldsymbol{X})}{\sqrt{H\{p(\boldsymbol{X})\}}}, \tag{5.4}$$

14) $\sqrt{H}$ can not be differentiable at x_0.

where p is the projection of $\mathfrak{B}$ and $s(x)$ expresses the distance along $T(x)$ from the 0-point and x.

Proof. It is readily seen that

$$||d\Omega(s, s')\boldsymbol{X}|| : ||\boldsymbol{X}|| = \sqrt{H\{\Omega(s, s')p(\boldsymbol{X})\}} : \sqrt{H\{p(\boldsymbol{X})\}}$$

$$\frac{||d\Omega(s, s')\boldsymbol{X}||}{s'} = \left\{ \frac{\sqrt{H\{\Omega(s, s')p(\boldsymbol{X})\}}}{s'} \Big/ \sqrt{H\{p(\boldsymbol{X})\}} \right\} ||\boldsymbol{X}|| .$$

By virtue of (5.3), we obtain

$$||d\Omega(s)\boldsymbol{X}|| = \lim_{s' \to 0} \left|\left| \frac{d\Omega(s, s')\boldsymbol{X}}{s'} \right|\right| = \{(d\sqrt{H})_0 \Omega(s) p(X) / \sqrt{H\{p(\boldsymbol{X})\}}\} ||\boldsymbol{X}|| .$$

Let γ be a curve on $W(s)$. Then $\Omega(s)\gamma$ is curve on S_u^{n-1}. Then we have a corollary of Theorem 5.3.

Corollary.

$$\textit{The length of } \gamma = \int_\gamma \frac{\sqrt{H(x)}}{(d\sqrt{H})_0 \Omega(s) x} ||d\Omega(s)\, dx|| = \int_{\Omega(s)\gamma} \frac{\sqrt{H\{\Omega(s)^{-1} z\}}}{(d\sqrt{H})_0 z} ||dz|| ,$$

where $||dz||$ means the norm with respect to the metric of S_u^{n-1}.

Theorem 5.4. *Every W is conformal to a sphere.*

Besides a conformal mapping between $S_u^{n-1}(x_0)$ and $W(s)$ is given by $\Omega(s)$.

Theorem 5.5. *In order that $\Omega(s)$ be homothetic, it is necessary and sufficient that $\boldsymbol{V}$ satisfies H-condition.*

If $\boldsymbol{V}$ satisfies H-condition, then $H(x)$ depends only upon s. That is the reason why we write $H(s)$ instead of $H(x)$ in this case.

Corollary. *If $\boldsymbol{V}$ satisfies H-condition, every $W(s)$ is a sphere and its radius is given by this:*

$$r(s) = \frac{\sqrt{H(s)}}{\left(\frac{d}{ds}\sqrt{H(s)}\right)_{s=0}} .$$

The following important theorem is likewise obvious from Theorem 5.3.

Theorem 5.6. *A complete Finsler space which admits a torse-forming vector field in the large with at least one 0-point is a Riemannian one.*

§6. In this section we shall make some applications of the preceding theorems. First we have the

Theorem 6.1. *Let M and M' be analytic Riemeannian spaces admitting torse-forming vector fields in the large V and V' respectively. Let V satisfy H-condition and have at least one 0-point x_0. Then in order that M be isometric to M', it is necessary and sufficient that there exists a mapping f of M into M' which preserves A, B, and $||V||$ that is:*

$$A(x)=A'\{f(x)\} \tag{6.1}$$

$$B(x)=B'\{f(x)\} \tag{6.2}$$

$$||V(x)||=||V'\{f(x)\}|| . \tag{6.3}$$

Proof. Let us restrict our consideration on a trajectory T and define a function $h(s)$ through a relation: $s'=h(s) \Longleftrightarrow f\{T(s)\} \in W'(s')$, where the primes indicate symbols with respect to M'. Then we have

$$\frac{d||V||}{ds}=\{A(s)+B(s)||V||^2\}, \tag{6.4}$$

and using (6.1) and (6.2)

$$\frac{d||V'||}{ds}=\{A(s)+B(s)||V'||^2\}\frac{dh}{ds} .$$

By (6.3)

$$\frac{d||V||}{ds}=\{A(s)+B(s)||V||^2\}\frac{dh}{ds} . \tag{6.5}$$

It follows from (6.4) and (6.5) that $\frac{dh}{ds}=1$ holds on $0\leq s\leq\varepsilon$ for some positive number ε or there exists a sequence $\{s_i\}_{1\leq i<+\infty}$ such that $\lim_{i\leftarrow+\infty} s_i=0$ and $A(s_i)+B(s_i)||V||^2=0$ $(i=1, 2, \cdots)$. In the latter case we get $A(0)=\lim_{i\to+\infty} A(s_i)=\lim\{-B(s_i)||V||^2\}=0$.
This contradicts our assumption $A(0)\neq 0$. Consequently only the former case arises. Therefore

$$h(s)=s \quad \text{for } 0\leq s\leq\varepsilon . \tag{6.6}$$

Since $$\left(\frac{d||V||}{ds}\right)_{s=0}=\left(\frac{d||V'||}{ds'}\right)_{s'=0}=A(0)\neq 0 ,$$

any transversal hypersurface to V or V' in a neighborhood of x_0 or $f(x_0)$ can be determined as equi-valued hypersurfaces of $||V||$ or $||V'||$. Thus we find that f preserves transversal hypersurfaces. It is obvious that

$$H(x)=e^{-\int_a^s \frac{A}{||V||}ds}=e^{-\int_a^s \frac{A'}{||V'||}ds}=H'\{f(x)\} . \tag{6.7}$$

With the help of Theorem 5.3, (5.6) and (5.7) show that an arbitrary isometry u between the tangent spaces of x_0 and $f(x_0)$ can be extended to an isometry v between a neighborhood of x_0 and that of $f(x_0)$ in the sense $(dv)_{x_0}=u$. Besides, M and M' are simply-connected by the corollary of Theorem 4.4. Therefore according to H. Hopf-W. Rinow's theorem ([19] or [2]), v can be extended to a global isometry between M and M'.

We would like to conjecture the following unproved theorem.

Theorem. *Let M and M' be spaces stated in the above theorem, and V and V' be vector fields done so. Then in order that M be isometric to M', it is necessary and sufficient that there exists a mapping f of M into M' which preserves A, B, and H.*

Definition. *We call a Riemannian space R a rotation element, if R admits a torse-forming vector field satisfying H-condition and having $x_0 \in R$ as an isolated 0-point.*[16]

Then we have the

Theorem 6.3. *In the 2-dimensional case, a rotation element of center x_0 is an equivalent of a Riemannian spaces admitting an isometry group I such that* $\mathrm{dis}(x, x_0)=\mathrm{dis}(y, x_0) \Rightarrow g(x)=y$ *for some* $g \in I(x_0)$[15] *(see* [5]).

In one of his noteworthy paper [5], W. Rinow pursued what kind of analytic rotation element (of 2 dimension) can be extended to a complete space, and his result is one to the effect:

Let us take a geodesic polar coordinate system with x_0 as center. In that coordinate system suppose that the line element is given by $ds^2=dr^2+g_{22}d\theta^2$. Set $f(r)=\sqrt{g_{22}}$.

I. The case where the analytic function $f(r)$ has an irregular point. In this case the extension is evidently impossible.

II. The case where $f(r)$ has no irregular point.

(i) If always $f(r)>0$ $(r>0)$, then the extension is possible and the extended space is homeomorphic to a Euclidean plane.

(ii) If $f(a)=0$ and $f(r)>0$ for $0<r<a$, then in order that the extension be possible, it is necessary and sufficient that $f(r)$ is a periodic function with period $2a$ such that $f'(a)=-1$. In this case, the extended space is homemorphic to a sphere.

Our next theorem is a generalized one of the above to a non-analytic space of arbitrary dimension.

Theorem 6.4. *Let $A(t)$ and $B(t)$ be continuous functions of one*

15) $I(x_0)$ means the isotropic group at x_0 of I.

16) By a torse-forming vector field, we mean one satisfying Condition (iv).

variable. Let R be a rotation element with respect to coefficient functions:

(6.8) $$A(x)=A\{\mathrm{dis}\,(x_0, x)\},\quad B(x)=B\{\mathrm{dis}\,(x_0, x)\}.$$

Let $y(t)$ be the solution with condition $y(0)=0$ of the differential equation (2.1). *Then in order that R can be extended to a complete rotation element*[1] *with respect to coefficient functions* (5.8), *it is necessary and sufficient that one of the following mutually incompatible cases arises:*

I. *$y(t)$ is always positive for $t>0$.*[16]

II. (i) $y(a)=0$, (ii) $y(t)>0$[17] *for* $0<t<a$, (iii) $A(a)\neq 0$, *and* (iv)[18] $e^{\int_0^s B(t)y(t)dt}$ *is bounded on* $[0, a]$

In case I, the extended space is homeomorphic to a Euclidean space, and in case II, that is homeomorphic to an n-dimensional sphere or an n-dimensional projective space. Furthermone the extended Riemannian space is uniquely determined in case the extension is possible and in case the space form is given.

In this theorem, $A(t)$ and $B(t)$ are assumed not to vanish constantly on any interval.

Proof. We treat only of Case II, because Case I can be dealt with in an analogous way. Let I be an interval $(0, a)$ and S_u^{n-1} the unit sphere. Let $J=S_u^{n-1}\times I$ and consider a product bundle $J\xrightarrow{p}S_u^{n-1}$, where p means a natural mapping. To every point of J we assign a vector which is tangent to the fibre and has an absolute value $y(t)$. The direction is assumed to be appropriately chosen so as to be continuous. Let $K'=\{t\mid y(t)=\pm\infty,\ t\in I\}$. It is obvious that K'^c is open. K' has no inner point because of $B(t)$ never vanishing on any interval (if $K'(t)$ has an inner point, then $B(t)=0$ in a neighborhood of that point). Hence $K'^{-0}=K'^{c-c}=M^c=\phi$. Namely K' is nowhere dense. Let $K=S_u^{n-1}\times K'$. Then K likewise is nowhere dense. Define a function $Y(s)$ by

$$Y(s)=\left\{\frac{d}{dt^*}e^{\int_{t^*}^{s}\frac{A}{|y(t)|}dt}\right\}_{t^*=0}=A(a)\,e^{\int_0^s By(t)dt}.$$

This function never vanishes for $0<s<a$. Let the line-element at $x\in J$ be given by this:

$$ds^2=\langle dp^*(dx),\ dp^*(dx)\rangle+Y\{p^*(x)\}\langle dp(dx),\ dp(dx)\rangle$$

where p^* is a natural mapping to the fibre and the metrics of the base and the fibre are the natural ones. It is readily seen that manifold J with

17) If $|y(t_0)|=+\infty$, then $y(t_0)$ is assumed to be $+\infty$.

18) This condition may perhaps be verified from the other conditions.

the above metric can be compactified by adding one point to either end of J, if condition (iii) is fulfilled. As a matter of fact,

$$\lim_{t\to a-0} e^{\frac{1}{a-t}\int_t^s \frac{A}{y(t)}dt} = A(a)e^{\int_0^s By(t)dt}.$$

The manifold obtained like this is what we are seeking for.

Theorem 6.5. *A (simply-connected) spherical form admits a torse-forming vector field in the large* (0.5), *a (simply-connected) hyperbolic form* (0.6), *and a Euclidean form* (0.7).

Proof. Take the example of a sphere. Differential equation (0.5) satisfies the integrability condition on the sphere. By giving an initial condition $V(x_0)=0$, we can make a vector field (0.5) defined in a certain region on the sphere. We easily find that the region is a hemi-sphere of x_0. Taking the antipodal point to x_0 as starting point, we can also define V over the other hemi-sphere. The equator is the common hyper-surface of singularity. On it, the absolute value of V diverges. In a sense, we may consider $||V||=+\infty$. It is easily seen that V defined in such a way make a torse-forming vector field in the large. The other cases are dealt with in an analogous way, but it is interesting to note that in the cases excepting for the spherical no discontinuity appears.

Theorem 6.6. *If a space admits such a type of torse-forming vector field in the large as* (0.5), (0.6), *or* (0.7), *then the space is a (simply-connected) spherical form, a (simply-connected) hyperbolic form, or a Euclidean form.*

Proof. For example, take (0.5). Along a trajectory we have

$$\frac{d||V||}{ds} = c_2(1+||V||^2). \tag{6.9}$$

In the case of the spherical form likewise, we have precisely the same equation as (6.9). Hence by virtue of Theorem 6.4, the space in question must be isometric to the spherical form.

Since the proof of Theorem 6.5 is also available for the spaces of constant curvature, we have the following well-known classical theorem.

Theorem 6.7. *A simply connected complete Riemannian space of constant curvature c is isometric to one of the spaces:*

(1) *A euclidean space, if* $c=0$; (2) *A hyperbolic space, if* $c<0$;
(3) *A spherical space, if* $c>0$ ([17] *or* [22]).

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