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ON AUTOMORPHISM GROUPS OF FINITE ORDER IN DIVISION RINGS

By

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It is a well-known theorem of E. Artin that if F is an algebraically closed (commutative) field of characteristic 0 then any automorphism of F of finite order is of order at most 2. Recently, in [1], H. Lenz drew out the essence of Artin's proof [1, Satz 3], and obtained several results concerned with automorphism groups of finite order. In the present note, we shall prove an extension of [1, Satz 3] to division rings and that of [1, Satz 5], whose proof is notably easy.

Let K be a division ring with the center Z , and $\mathfrak{G}$ an automorphism group of K . Then, $\mathfrak{G}$ induces an automorphism group $\bar{\mathfrak{G}}$ of the group K^*/Z^* , where K^* is the multiplicative group consisting of all non-zero elements of K . Particularly, if $\bar{\mathfrak{G}}$ coincides with the identity group then $\mathfrak{G}$ will be called an M -group. In case K is commutative, the notion of M -group is trivial of course. Now let p be a prime number, and k an element of K . If there exists a division subring K' of K such that $k^p \in K'$ and $k \notin K'$, then k will be called a p -th root of K . We consider here the following property of K :

(P) *For each p -th root k of K , the equation $x^p - k = 0$ has a solution in K .*

At first, we shall prove the following fundamental lemma.

Lemma 1. *Let K be strictly Galois¹⁾ with respect to an M -group $\mathfrak{P} = \{\sigma^i\}$ of order p , and possess the property (P). If Z contains a primitive p^2 -th root η of 1, then η is not contained in $L = J(\mathfrak{P}, K)$ (=the fixing subring of $\mathfrak{P}$).*

Proof. Evidently, $\zeta = \eta^p$ is a primitive p -th root of 1, and $[\Phi(\zeta) : \Phi] < p$, where Φ is the prime subfield of K . If $\zeta \sigma \neq \zeta$, then $\Phi(\zeta)$ being $\mathfrak{P}$ -normal, we have $[\Phi(\zeta) : \Phi] \geq p$. We see therefore ζ is contained in L . Accordingly, by [2, Corollary 2], there exists a non-zero element $x \in K$ such that $x\sigma = x\zeta$. As $x^p \in L$ and $K = L[x]$ consequently, the property (P)

1) Cf. [3].

secures the existence of $y \in K$ with $y^p = x$. Recalling here $\mathfrak{P}$ is an M -group, we have $y\sigma = yz$ for some $z \in Z$. $(y\sigma)^p = y^p\sigma = x\zeta = (y\eta)^p$ yields therefore $\{(y\sigma)y^{-1}\eta^{-1}\}^p = (z\eta^{-1})^p = 1$, whence we have $y\sigma = y\eta\zeta^\lambda$ where $0 \leq \lambda \leq p-1$. If $\eta \in L$, then $y\sigma^p = y\eta^p\zeta^{p\lambda} = y\zeta \neq y$. But this contradicts $\sigma^p = 1$.

The following lemma will be almost evident from the proof of [3, Lemma 5].

Lemma 2. *Let K be strictly Galois with respect to $\mathfrak{G}$. Then, for an arbitrary subgroup $\mathfrak{H}$ of $\mathfrak{G}$, K is strictly Galois with respect to $\mathfrak{H}$.*

Now we can extend [1, Satz 3] to division rings.

Theorem 1. *Let K be a division ring of characteristic 0 which is strictly Galois with respect to an M -group $\mathfrak{G}$ of order n . If K and Z possess the property (P), and Z contains a primitive $2p$ -th root of 1, then p does not divide n if $p > 2$, 4 does not divide n if $p = 2$.*

Proof. (I) $p = 2$. If 4 divides n , then $\mathfrak{G}$ contains a subgroup $\mathfrak{H}$ of order 4, and K is strictly Galois with respect to $\mathfrak{H}$ (Lemma 2). Let $\mathfrak{P}_1$ be a subgroup of $\mathfrak{H}$ of order 2, and set $L_1 = J(\mathfrak{P}_1, K)$. Then, $K = L_1[i]$ by Lemma 1, where i is an element of Z with $i^2 = -1$. (i is a primitive $4 (= 2p = p^2)$ -th root of 1.) And so, $i \notin L = J(\mathfrak{H}, K)$. If we set $L_2 = L[i]$, then $[L_2 : L] = 2$ and $\mathfrak{H}(L_2) \neq 1$.²⁾ For, if not, every σ in $\mathfrak{H}$ different from 1 moves i into $-i$. But this is impossible. This proves that K/L_2 is strictly Galois with respect to $\mathfrak{H}(L_2)$ of order 2. Hence, again by Lemma 1, $K = L_2[i] = L_2$. This contradiction shows that $4 \nmid n$.

(II) $p > 2$. If p divides n , then $\mathfrak{G}$ contains a subgroup $\mathfrak{P} = \{\sigma^i\}$ of order p , and K is strictly Galois with respect to $\mathfrak{P}$. Since Z possesses the property (P), it contains primitive p^j -th roots of 1 for all j . Since $E_{p, \infty}$, the subfield (of Z) generated by all the p^j -th roots of 1 ($j = 1, 2, \dots$), is $\mathfrak{P}$ -normal, Lemma 1 shows that σ induces an automorphism of $E_{p, \infty}$ which leaves invariant every primitive p -th root of 1 and moves really primitive p^2 -th roots of 1. But this contradicts [1, Satz 1].

Let L be a division subring of K . If, for every $k \in K$, there exists a finite number of division subrings $K_1, \dots, K_m$ such that $L(k) = K_1 \supseteq \dots \supseteq K_m = L$ and $[K_i : K_{i+1}]_i \leq n$ (fixed) for $i = 1, \dots, m-1$, then we say that K/L is n -accessible. Further if, for every intermediate division subring L' of K/L , K/L' is n -accessible, then K/L will be said to be completely n -accessible. To be easily seen, in case K is commutative, the notion of n -accessibility coincides with that of complete n -accessibility. Now, let K/L be completely n -accessible and strictly Galois with respect to $\mathfrak{G}$ of

2) $\mathfrak{H}(L_2) = \{\sigma \in \mathfrak{H} \mid x\sigma = x \text{ for all } x \in L_2\}$.

prime order p , where $\mathfrak{G}$ is an automorphism group of K which leaves invariant every element of L . Then, $[K:L'] = p$ where $L' = J(\mathfrak{G}, K)$, whence $K = L'[k]$ for some k . On the other hand, there exists a finite number of division subrings $K_1, \dots, K_m$ such that $K = K_1 \supseteq \dots \supseteq K_m = L'$ and $[K_i : K_{i+1}] = n_i \leq n$. And so, $p = [K:L'] = \prod_{i=1}^{m-1} n_i$. Thus, we have proved the following which contains [1, Satz 5].

Theorem 2. *Let K be a division ring, and completely n -accessible over a division subring L . If K is strictly Galois with respect to $\mathfrak{G}$ of prime order p and $J(\mathfrak{G}, K) \supseteq L$, then $p \leq n$.*

References

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