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A NOTE ON STRICTLY GALOIS EXTENSION OF PRIMARY RINGS

By

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Let $R \ni 1$ be a primary ring with minimum condition (for one-sided ideals). One of the present authors proved in [1]¹⁾ that if R is strictly Galois with respect to $\mathfrak{G}$ then R possesses a $\mathfrak{G}$ -normal basis element. The purpose of this note is to present a slight generalization of this fact.

In what follows, R be always a primary ring with minimum condition which is strictly Galois with respect to (an F -group) $\mathfrak{G}$ of order n , $N \ni 1$ a subring of R with minimum condition such that $N\mathfrak{G} = N$ and R possesses a linearly independent right N -basis consisting of t elements.²⁾ Further, we set $t = nq + r$, where $0 \leq r < n$. Under this situation, our theorem can be stated as follows:

Theorem. *There exist q elements $x_1, \dots, x_q \in R$ and a $\mathfrak{G}N_r$ -submodule M of R such that*

(1) *M is $\mathfrak{G}N_r$ -homomorphic to $\mathfrak{G}N_r$ and possesses a linearly independent right N -basis consisting of r elements,*

$$(2) \quad R = \sum_{i=1}^q \oplus \sum_{\sigma \in \mathfrak{G}} (x_i \sigma) N \oplus M.$$

Proof. As is shown in [1], $\text{Hom}_{S_t}(R, R) = \mathfrak{G}R_r = \sum_{\sigma \in \mathfrak{G}} \oplus \sigma R_r$, where $S = J(\mathfrak{G}, R)$. Since $[R : S]_t = n$, and so, since R is S -left regular, R is $\text{Hom}_{S_t}(R, R)$ -right regular too. In fact, $R^{(n)}$ is $\mathfrak{G}R_r$ -isomorphic to $\mathfrak{G}R_r$, where $R^{(n)}$ means the direct sum of n -copies of R as $\mathfrak{G}R_r$ -module. Accordingly, $R^{(n)}$ is $\mathfrak{G}N_r$ -isomorphic to $\mathfrak{G}R_r$ of course. Now let $R = u_1 N \oplus \dots \oplus u_t N$. Then, we have $\mathfrak{G}R_r = R_r \mathfrak{G} = \sum_{i=1}^t u_{ir} N_r \mathfrak{G} = \sum_{i=1}^t \oplus u_{ir} \mathfrak{G}N_r$. Hence, $\mathfrak{G}R_r$ is $\mathfrak{G}N_r$ -isomorphic to $(\mathfrak{G}N_r)^{(t)}$, and so we have eventually that $R^{(n)}$ is $\mathfrak{G}N_r$ -isomorphic to $(\mathfrak{G}N_r)^{(t)}$. Here let $p_1, \dots, p_s$ be all the non-isomorphic directly indecomposable direct summands of the $\mathfrak{G}N_r$ -module R (or $\mathfrak{G}N_r$ itself). And, in the Remak decompositions of $\mathfrak{G}N_r$ -modules R and $\mathfrak{G}N_r$, the re-

1) As to notations and terminologies used in this note, we follow [1]. And we will use freely the results cited in [1].

2) N does not necessarily contain the subring $S = J(\mathfrak{G}, R)$.

spective numbers of directly indecomposable components which are isomorphic to $\mathfrak{p}_i$ will be denoted by n_i and m_i . Then, our isomorphism mentioned above yields at once $n_i n = m_i t = m_i (nq + r)$, whence we have $m_i r = n k_i$ with some non-negative integer $k_i < m_i$. Consequently, it follows that $n_i = m_i q + k_i$ ($i = 1, \dots, s$). This proves clearly the existence of a $\mathbb{G}N_r$ -isomorphism φ of R onto $(\mathbb{G}N_r)^{(q)} \oplus T$, where $T = \sum_{i=1}^s \mathfrak{p}_i^{(k_i)}$. Recalling here $m_i > k_i$, we see that T is $\mathbb{G}N_r$ -homomorphic to $\mathbb{G}N_r$. Now, let $y_i = (0, \dots, 0, \overset{(i)}{1}, \dots, 0) \in (\mathbb{G}N_r)^{(q)}$. Then, one will easily verify that $x_i = \varphi^{-1}\{y_i\}$ ($i = 1, \dots, q$) and $M = \varphi^{-1}\{T\}$ are desired ones.

Our theorem may be considered as a generalization of [1, Theorem 1]. Moreover, in case R is a division ring we obtain the following which secures the existence of the so-called semi-normal basis.

Corollary. *Let R be a division ring which is strictly Galois with respect to $\mathbb{G}$ of order n , and N a division subring of R with $N\mathbb{G} = N$ and $[R:N]_r = t$. If $t = nq + r$ ($0 \leq r < n$) then there exist some $x_0, x_1, \dots, x_q \in R$ such that $R = \sum_{i=1}^q \oplus \sum_{\sigma \in \mathbb{G}} (x_i \sigma)N \oplus \sum_{\tau} \oplus (x_0 \tau)N$, where τ runs over a suitable subset of $\mathbb{G}$ consisting of r elements.*

References

- [1] H. TOMINAGA: A note on Galois theory of primary rings, Math. J. Okayama Univ., 8 (1958), 117-124.

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