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SOME REMARKS ON CARTAN-BRAUER-HUA THEOREM[1)]

By

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Concerning Cartan-Brauer-Hua theorem, T. Nagahara and H. Tominaga proved the following [3, Lemma 3.5]:

Let U be a ring with 1, *and B a two-sided simple subring of U containing* 1. *If A is a division subring of U containing* 1 *such that B is invariant relative to all inner automorphisms determined by non-zero elements of A, then either $A\subseteq B$ or $A\subseteq V_U(B)$.*

In what follows, by making use of the same method as in the proof of this fact, we shall present a slight generalization of [3, Lemma 3.5] (Theorem 1) and an extension of [2, Theorem 7.13.1 (2)] (Theorem 2). And finally, we shall prove that Theorem 2 is still valid for inner automorphisms provided A is a simple ring (Theorem 3). Throughout the present note, a ring will mean a ring with the identity element 1, and a subring one with this identity element.

Our first theorem containing [3, Lemma 3.5] can be stated as follows:

Theorem 1. *Let U be a ring, and A and B a subring of U satisfying minimum condition for right ideals and a two-sided simple subring of U respectively. If for each $a\in A$ and $b\in B$ there exists an element $b_1\in B$ such that $ab=b_1a$, then either $A\subseteq B$ or $A\subseteq V_U(B)$.*

Proof. To be easily seen from the proof of [3, Lemma 3.5], it suffices to prove that $A=(A\frown B)\smile V_A(B)$. Let a be an arbitrary element of A. If a and 1 are linearly left independent over B, then for each $b\in B$, $ab=b_1a$ and $(a+1)b=b_2(a+1)$ yield $(b_1-b_2)a+(b-b_2)=0$, whence it follows $b_1=b_2=b$. Consequently, we obtain $a\in V_A(B)$. If, on the other hand, a and 1 are linearly dependent, then there holds $d_1a=d_2$ for some non-zero $d_1\in B$. In case $d_2\neq 0$, since B is two-sided simple, we obtain $da=1$ for some $d\in B$. And so, recalling that A satisfies minimum condition for right ideals, one will readily see that a is a regular element of A. And then, $aB=Ba=B$ will yield at once $a\in B$. In case $d_2=0$ too, since $d_1(a+1)=d_1\neq 0$, we obtain $a+1\in B$. Thus, in either case, a is contained in B. We have proved tnerefore $A=(A\frown B)\smile V_A(B)$.

Combining our method with the one employed in the proof of [2,

1) The author wishes to express his gratitude to Prof. G. Azumaya for his kind guidance.

Theorem 7.13.1 (2)], we can obtain the following:

Theorem 2.[2] *Let U be a ring and B a two-sided simple subring of U. If B is not of characteristic 2 and A is a subring of U such that B is invariant relative to all inner derivations determined by elements of A, then either $A \subseteq B$ or $A \subseteq V_U(B)$.*

Proof. Let a be an arbitrary element of A. For any element $b \in B$, we set $[b, a]=ba-ab=b_1$, $[[b, a]a]=b_2$, $[b, a^2]=b_3$ where b_1, b_2 and b_3 are in B. Then, one will easily see that $2b_1a=2(ba^2-aba)=b_2+b_3 \in B$. And, if a and 1 are linearly left independent over B, we obtain $b_1=0$. This means obviously that $a \in V_A(B)$. On the other hand, if a and 1 are linearly dependent: $b^*a-b^{**}=0$ with non-zero $b^* \in B$, then noting that B is two-sided simple, it will be easy to see that $a \in A \frown B$. We have proved therefore $A=(A \frown B) \smile V_A(B)$. Now, the rest of the proof is the same with the latter half of the proof of [3, Lemma 3.5].

Finally, we shall present the following:

Theorem 3. *Let U be a ring and B a two-sided simple subring of U. If B is not of characteristic 2, and A a simple subring of U such that B is invariant relative to all inner automorphisms determined by regular elements of A, then either $A \subseteq B$ or $A \subseteq V_U(B)$.*

Proof. Let K be the prime field of A (which is evidently contained in the center of B), and let a be an arbitrary α-biregular element[3] of A $(0 \neq \alpha \in K)$. If a and 1 are linearly left independent over B, then for an arbitrary $b \in B$, $ab=b^*a$ and $(a-\alpha)b=b^{**}(a-\alpha)$ yield at once $(b^*-b^{**})a+\alpha(b^{**}-b)=0$, whence it follows $b^*=b^{**}=b$. Hence we obtain $a \in V_A(B)$. On the other hand, if a and 1 are linearly dependent, then it will be easy to see that $a \in B$. Since each element of A is a sum of biregular elements by [1], the fact proved above will show that B is invariant relative to all inner derivations determined by elements of A. Hence, our assertion is a direct consequence of Theorem 2.

References

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2) This theorem is essentially due to Dr. H. Tominaga who kindly permitted us to cite it here. We are indebted to him for his helpful suggestions and advices.

3) Cf. [1].