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Title	ON INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATION
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Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 16(1-2), 001-004
Issue Date	1962
Doc URL	https://hdl.handle.net/2115/56026
Type	departmental bulletin paper
File Information	JFSHIU_16_N1-2_001-004.pdf



ON INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATION

By

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§0. **Introduction.** Recently, T. Ôtsuki and Y. Tashiro [1]¹⁾ have studied holomorphically projective correspondences of Kählerian manifolds.

On the other hand, K. Yano and T. Nagano [2] and T. Sumitomo [3] have studied infinitesimal projective transformations in a Riemannian manifold and obtained valuable results. Further, S. Tachibana and S. Ishihara [4] have considered analogous problems concerning the infinitesimal holomorphically projective transformations, which will be briefly called an *HP-transformation*, and obtained that a Kählerian manifold satisfying $R_{ij;k}=0$, which admits a non-trivial analytic HP-transformation reduces to an Einstein one.

The purpose of the present paper is to generalize more the above result of S. Tachibana and S. Ishihara, that is, we shall give a theorem about a Ricci-recurrent Kählerian manifold in §1 and one in a Ricci-recurrent K-space in §2, which is one of the generalization of the theorem in §1.

The present author wishes to express his sincere thanks to Prof. A. Kawaguchi and Prof. Y. Katsurada for their constant guidances and criticisms, and also thanks to Mr. T. Nagai and Mr. T. Sumitomo who gave the author many valuable suggestions.

§1. An analytic HP-transformation on a Kählerian manifold.

A vector field v^i is called an HP-transformation, if it satisfies

$$(1.1) \quad \mathfrak{L}_{\{jk\}}^i = P_h(\delta_j^h \delta_k^i - \varphi_j^h \varphi_k^i) + P_h(\delta_k^h \delta_j^i - \varphi_k^h \varphi_j^i),$$

where P_h is a certain vector and φ_j^i is the complex structure, and semi-colon and $\mathfrak{L}$ denote the covariant differentiation with respect to v^i and Lie differentiation with respect to v^i , respectively. We shall call P_h in (1.1) the associated vector of the HP-transformation. Contracting (1.1)

1) Numbers in brackets refer to the references at the end of the paper.

with respect to i and k , we get $P_h = \frac{1}{n+2} v^i_{;i;h}$, which shows that P_h is gradient.

An infinitesimal affine transformation v^i is defined by

$$\mathfrak{L}_{\{jk\}}^i \equiv v^i_{;j;k} + R^i_{jkl} v^l = 0.$$

If $P_h = 0$, then the HP-transformation reduces to an affine one.

A vector field v^i is called analytic on a Kählerian manifold, if it satisfies

$$(1.2) \quad \mathfrak{L}_{\varphi_j^i} = -\varphi_j^k v^i_{;k} + \varphi_k^i v^k_{;j} = 0.$$

We shall give here preliminary formulas on Kählerian manifold. Let our manifold be a real $n(=2m>2)$ dimensional Kählerian manifold with local coordinates $\{x^i\}$. Then the Riemannian metric g_{ij} and the complex structure φ_i^j satisfy

$$\begin{aligned} \varphi_i^k \varphi_k^j &= -\delta_i^j, & g_{hk} \varphi_i^h \varphi_j^k &= g_{ij}, \\ \varphi_i^j_{;k} &= 0, & g_{ij;k} &= 0. \end{aligned}$$

Then the following equation holds:

$$(1.3) \quad R_{hk} \varphi_i^h \varphi_j^k = R_{ij},$$

where R^i_{jkl} is the Riemannian curvature tensor, and

$$R^i_{jki} = R_{jk}, \quad R^h_{jkl} g_{hi} = R_{ijkl}.$$

If P_h is the associated vector of an analytic HP-transformation, then we get

$$(1.4) \quad P_{h;k} \varphi_i^h \varphi_j^k = P_{i;j}.$$

Moreover, if v^i be an analytic HP-transformation, then we have

Lemma. *Let v^i be an analytic HP-transformation, then the following relation holds:*

$$(1.5) \quad (\mathfrak{L}_{g_{ik}}) R_j^k = (\mathfrak{L}_{g_{jk}}) R_i^k.$$

Proof. From the assumptions, it follows that

$$\mathfrak{L}_{\varphi_j^i} = 0,$$

$$\mathfrak{L}_{\{jk\}}^i \equiv v^i_{;j;k} + R^i_{jkl} v^l = P_h (\delta_j^h \delta_k^i - \varphi_j^h \varphi_k^i) + P_h (\delta_k^h \delta_j^i - \varphi_k^h \varphi_j^i).$$

Since R_{ijkl} is anti-symmetric with respect to i and j , we get

$$(1.6) \quad (\mathfrak{L}_{g_{ij}})_{;k} \equiv (v_{i;j} + v_{j;i})_{;k} = 2P_k g_{ij} + P_j g_{ik} + P_i g_{jk} - P_a \varphi_j^a \varphi_{ki} - P_a \varphi_i^a \varphi_{kj}.$$

The integrable condition of the above equation is that

$$\begin{aligned} (\mathfrak{L}_{g_{aj}}) R^a_{ikl} + (\mathfrak{L}_{g_{ia}}) R^a_{jkl} &= P_{j;k} g_{il} + P_{i;k} g_{jl} - P_{j;l} g_{ik} - P_{i;l} g_{jk} \\ &+ \varphi_j^a (P_{a;l} \varphi_{ki} - P_{a;k} \varphi_{li}) + \varphi_i^a (P_{a;l} \varphi_{kj} - P_{a;k} \varphi_{lj}). \end{aligned}$$

If we contract g^{jl} to this equation and take account of (1.4), then we have

$$(\mathfrak{L}g_{aj})R_{ik}^a - (\mathfrak{L}g_{ia})R_k^a = nP_{i;k} - P_a{}^a g_{ik}.$$

Since P_h is gradient and $(\mathfrak{L}g_{aj})R_{ik}^a$ is symmetric with respect to i and k , we obtain the conclusion.

Recently S. Tachibana and S. Ishihara [4] obtained the following

Theorem. *If a Kählerian manifold satisfying $R_{ij;k}=0$ admits an analytic non-affine HP-transformation, it is a Kähler-Einstein manifold.*

We shall now consider a Ricci-recurrent Kählerian manifold, i.e., a Kählerian manifold such that $R_{ij;k}=R_{ij}v_k$, and we obtain the following

Theorem. *If a Kählerian manifold satisfying $R_{ij;k}=R_{ij}v_k$ admits an analytic non-affine HP-transformation, it is a Kähler-Einstein manifold.*

Proof. Covariantly differentiating (1.5) with respect to x^l and making use of (1.5), we find

$$(\mathfrak{L}g_{ia})_{;l}R_k^a = (\mathfrak{L}g_{ka})_{;l}R_i^a.$$

Substituting (1.6) into the last equation, we have easily

$$\begin{aligned} & (P_a g_{il} + P_i g_{al} + 2P_l g_{ia} - \varphi_a^b \varphi_{li} P_b - \varphi_i^b \varphi_{la} P_b) R_k^a \\ & = (P_a g_{kl} + P_k g_{al} + 2P_l g_{ka} - \varphi_a^b \varphi_{lk} P_b - \varphi_k^b \varphi_{la} P_b) R_i^a. \end{aligned}$$

Contracting this equation with g^{il} and R^{il} , and taking account of (1.3) and (1.4), we have

$$\begin{aligned} nP_a R_k^a &= R P_k, \\ R R_k^a P_a &= R_{ij} R^{ij} P_k. \end{aligned}$$

From the above equations, we get

$$\left(R_{ij} R^{ij} - \frac{R^2}{n} \right) P_k = 0.$$

Since $P_k \neq 0$, we must have

$$R_{ij} R^{ij} - \frac{R^2}{n} = 0.$$

On the other hand, according to the theorem obtained by T. Sumitomo [3], a Riemannian manifold satisfying the relation $R_{ij} R^{ij} = \frac{R^2}{n}$ is an Einstein manifold. Therefore, we get the conclusion.

§2. An analytic HP-transformation in a K-space.

In this section, we shall consider only a K-space, which is another generalization of a Kählerian manifold.

If φ_{ij} ($\varphi_{ij} \stackrel{\text{def}}{=} \varphi_i^k g_{kj}$) is a Killing tensor, i.e., it satisfies the equation

$$\varphi_{ij;k} + \varphi_{ik;j} = 0,$$

an almost-Hermitian space is called a K-space. After some calculations we get also the following identities in a K-space:

$$(2.1) \quad R_{hk} \varphi_i^h \varphi_j^k = R_{ij},$$

$$(2.2) \quad P_{h;k} \varphi_i^h \varphi_j^k = P_{i;j}.$$

Thus, by virtue of (2.1), (2.2), and Lemma, we have the following

Theorem. *If a K-space satisfying $R_{ij;k} = R_{ij} v_k$ admits an analytic non-affine HP-transformation, it is an Einstein K-space.*

The method of the proof is analogous to that in Kählerian manifold.

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(Received June 2, 1961)