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A NOTE ON HOCHSCHILD COHOMOLOGY GROUPS

By

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Let K be a commutative ring with identity element, and let A be a K -algebra, that is, an algebra over K with identity element 1. We denote by A^* the opposite algebra of A , which is in an opposite-isomorphism $\lambda \rightarrow \lambda^*$ with A . Any right A -module is converted into a left A^* -module (and conversely) by setting the left multiplication of λ^* to be the right multiplication of λ . Furthermore, every two-sided A -module is, and in particular A is, regarded as a left module for the enveloping algebra $A^e = A \otimes A^*$ ¹⁾ in the natural manner.

Let A be a two-sided A -module. Hochschild defined in [2] the cohomology groups of A for A as the homology groups $H^n(A, A)$ of the complex whose components are the cochain groups $C^n(A, A)$, i. e., the module of all K -multilinear mappings $f = f(\lambda_1, \dots, \lambda_n)$ of A into A , and whose differentiation operators $\delta^n : C^n(A, A) \rightarrow C^{n+1}(A, A)$ are defined by

$$\begin{aligned} (\delta^n f)(\lambda_1, \dots, \lambda_{n+1}) &= \lambda_1 f(\lambda_2, \dots, \lambda_{n+1}) + \\ &\sum_{i=1}^n (-1)^i f(\lambda_1, \dots, \lambda_i \lambda_{i+1}, \dots, \lambda_{n+1}) + (-1)^{n+1} f(\lambda_1, \dots, \lambda_n) \lambda_{n+1}. \end{aligned}$$

Here, the two-sided A -module A needs not be assumed to be unital (i. e., the identity element 1 of A does not necessarily act on A as the identity-operator on both left and right hands), but we may replace A by the unital module $1A1$ to obtain the same cohomology groups according to Hochschild [3], Th. 1: $H^n(A, A) \cong H^n(A, 1A1)$. On the other hand, Cartan and Eilenberg gave in [1] another definition of cohomology groups for a unital two-sided A -module A ; namely, they called $\text{Ext}_{A^e}^n(A, A)$ the n -th cohomology group of A for A . The defined two groups $H^n(A, A)$ and $\text{Ext}_{A^e}^n(A, A)$, for unital A , coincide (that is, are naturally equivalent as functors of A) always for $n=0, 1$ ([1], Chap. IX, Prop. 4.1.). But this is not the case in general for $n > 1$. It is shown in [1], Chap. IX, §6 that the both groups coincide if A is K -projective. In this note, we shall however generalize this to prove that the both groups coincide whenever A^e is projective as a right A -module. In this connection, it may be of some interest to give in Proposition 2 below a homological significance of the Hochschild two-sided A -module $\text{Hom}_K(A, A)$ introduced in [2].

1) We shall mean by the mere \otimes the tensor product over K ; thus, $A \otimes A^* = A \otimes_K A^*$.

Now, consider three K -algebras Λ, Γ, Σ . Let A, B, C be respectively a unital two-sided Λ - Γ -, Γ - Σ -, Λ - Σ -module; they are also regarded respectively as a left $\Lambda \otimes \Gamma^*$ -, $\Gamma \otimes \Sigma^*$ -, $\Lambda \otimes \Sigma^*$ -module in the natural manner. Then there is a well-known natural isomorphism ([1], Chap. IX, Prop. 2.2)

$$(1) \quad \text{Hom}_{\Lambda \otimes \Gamma^*}(A, \text{Hom}_{\Sigma}(B, C)) \cong \text{Hom}_{\Lambda \otimes \Sigma^*}(A \otimes_{\Gamma} B, C).$$

On the basis of this isomorphism, we have the following analogy of [1], Chap. IX, Th. 2.8a :

Proposition 1. *In the situation $(\Lambda A_{\Gamma}, {}_{\Gamma} B_{\Sigma}, \Lambda C_{\Sigma})$ assume that A is Γ -projective, B is Σ -projective, and $\Lambda \otimes \Gamma^*$ is (left) Γ^* -projective²⁾. Then there is a natural isomorphism*

$$\text{Ext}_{\Lambda \otimes \Gamma^*}^n(A, \text{Hom}_{\Sigma}(B, C)) \cong \text{Ext}_{\Lambda \otimes \Sigma^*}^n(A \otimes_{\Gamma} B, C).$$

Proof. Let X be a $\Lambda \otimes \Gamma^*$ -projective resolution of A . Then, since $\Lambda \otimes \Gamma^*$ is Γ^* -projective, X is also a (Γ^* - or Γ -projective resolution of A . Hence the homology group of $X \otimes_{\Gamma} B$ is $\text{Tor}^r(A, B)$. But, since A is Γ -projective, $\text{Tor}_n^r(A, B) = 0$ for $n > 0$. Further, since B is Σ -projective, $X \otimes_{\Gamma} B$ is $\Lambda \otimes \Sigma^*$ -projective and so a $\Lambda \otimes \Sigma^*$ -projective resolution of $A \otimes_{\Gamma} B$ by [1], Chap. IX, Prop. 2.3. Thus, replacing in (1) A by X and taking the homology group, we have the desired isomorphism.

As the particular case where $\Lambda = \Gamma = \Sigma$ and $A = \Lambda$, we obtain.

Corollary. *Let B and C be unital two-sided Λ -modules, and assume that B is right Λ - (or left Λ^* -) projective and Λ^e is left Λ^* -projective. Then there is a natural isomorphism*

$$\text{Ext}_{\Lambda^e}^n(\Lambda, \text{Hom}_{\Lambda^*}(B, C)) \cong \text{Ext}_{\Lambda^e}^n(B, C).$$

We next consider a Λ^e -epimorphism $\rho : \Lambda^e \rightarrow \Lambda$ which is defined by $\rho(\lambda \otimes \mu^*) = (\lambda \otimes \mu^*)1 = \lambda\mu$. Let J be the kernel of ρ . Then J is a left ideal of Λ^e , and we have an exact sequence of left Λ^e -modules

$$(2) \quad 0 \longrightarrow J \xrightarrow{\iota} \Lambda^e \xrightarrow{\rho} \Lambda \longrightarrow 0,$$

where ι means the imbedding mapping. Hence, to every unital two-sided Λ -module A , there corresponds, for each $n > 1$, an exact sequence

$$\text{Ext}_{\Lambda^e}^{n-1}(\Lambda^e, A) \longrightarrow \text{Ext}_{\Lambda^e}^{n-1}(J, A) \longrightarrow \text{Ext}_{\Lambda^e}^n(\Lambda, A) \longrightarrow \text{Ext}_{\Lambda^e}^n(\Lambda^e, A).$$

But, since Λ^e is Λ^e -projective, the first and the last terms = 0, and thus we have a natural isomorphism

(2) We regard $\Lambda \otimes \Gamma^*$ as a left Γ^* -module by setting the left multiplication of r^* as the left multiplication of $1 \otimes r^*$.

$$(3) \quad \text{Ext}_{A^e}^{n-1}(J, A) \cong \text{Ext}_{A^e}^n(A, A), \quad n > 1.$$

On the other hand, since (2) is exact as left A^* -modules too and since A is A^* -projective, (2) must split, i. e., $A^e = J \oplus A^*$. It follows therefore that if A^e is A^* -projective then J is also A^* -projective. Thus, there is, by the above corollary, a natural isomorphism

$$\text{Ext}_{A^e}^n(A, \text{Hom}_{A^*}(J, A)) \cong \text{Ext}_{A^e}^n(J, A),$$

if A^e is A^* -projective. Now, replacing here n by $n-1$ and comparing with (3), we obtain the following reduction theorem:

Theorem 1. *Let A be a unital two-sided A -module, and assume that A^e is left A^* -projective. Then there is a natural isomorphism*

$$\text{Ext}_{A^e}^{n-1}(A, \text{Hom}_{A^*}(J, A)) \cong \text{Ext}_{A^e}^n(A, A), \quad n > 1.$$

Now, let A be any two-sided A -module. Hochschild converted $\text{Hom}_K(A, A)$ into a two-sided A -module by setting, for any $f \in \text{Hom}_K(A, A)$ and $\lambda \in A$, λf and $f\lambda$ as the mappings $\mu \rightarrow \lambda f(\mu)$ and $\mu \rightarrow f(\lambda\mu) - f(\lambda)\mu$, $\mu \in A$, respectively ([2], §1). Since A is K -unital we have $1f(\mu) = f(\mu) = f(\mu)1$ for every $f \in \text{Hom}_K(A, A)$ and $\mu \in A$, and this implies first that $\text{Hom}_K(A, A)$ is unital as a left A -module. However, it is not necessarily unital as a right A -module (even if A is unital), and in fact $\text{Hom}_K(A, A) \cdot 1$ consists of those f in $\text{Hom}_K(A, A)$ which satisfy $f(1) = 0$, because if $f = g1$ with some $g \in \text{Hom}_K(A, A)$ then $f(1) = g(1) - g(1)1 = 0$ and conversely if $f(1) = 0$ then $f(\mu) - f(1)\mu = f(\mu)$ for all $\mu \in A$, showing $f1 = f$.

Proposition 2. *Let A be a two-sided A -module and $\text{Hom}_K(A, A)$ the Hochschild two-sided A -module. Let $\varphi: A \rightarrow J$ be the K -homomorphism defined by $\varphi(\lambda) = \lambda \otimes 1 - 1 \otimes \lambda^*$. Then, by associating with each $h \in \text{Hom}_{A^*}(J, A)$ the product mapping $h \cdot \varphi \in \text{Hom}_K(A, A)$, we obtain a natural isomorphism $\text{Hom}_{A^*}(J, A) \cong \text{Hom}_K(A, A) \cdot 1$ as two-sided A -modules.*

Proof. Let $h \in \text{Hom}_{A^*}(J, A)$ and put $f = h \cdot \varphi$. Then, since $\varphi(1) = 0$, it follows that $f(1) = h(\varphi(1)) = h(0) = 0$. Further, for any λ, μ in A , we have $\lambda f(\mu) = \lambda h(\mu \otimes 1 - 1 \otimes \mu^*)$, i. e., $\lambda f = (\lambda h) \cdot \varphi$, while $f(\lambda\mu) - f(\lambda)\mu = f(\lambda\mu) - \mu^* f(\lambda) = h((\lambda\mu) \otimes 1 - 1 \otimes (\lambda\mu)^*) - h((1 \otimes \mu^*)(\lambda \otimes 1 - 1 \otimes \lambda^*)) = h((\lambda\mu) \otimes 1 - \lambda \otimes \mu^*) = h((\lambda \otimes 1)(\mu \otimes 1 - 1 \otimes \mu^*))$, i. e., $f\lambda = (h\lambda) \cdot \varphi$. These together show that the mapping $h \rightarrow h \cdot \varphi$ gives a homomorphism $\text{Hom}_{A^*}(J, A) \rightarrow \text{Hom}_K(A, A) \cdot 1$ as two-sided A -modules. Now J is, as a left A^* -module, generated by the elements $\varphi(\lambda)$, $\lambda \in A$, because $\sum \lambda_i \otimes \mu_i^* \in J$ means $\sum \lambda_i \mu_i = 0$ whence $\sum \lambda_i \otimes \mu_i^* = \sum \lambda_i \otimes \mu_i^* - 1 \otimes (\sum \lambda_i \mu_i)^* = \sum (\lambda_i \otimes \mu_i^* - 1 \otimes (\mu_i^* \lambda_i^*)) = \sum (1 \otimes \mu_i^*)(\lambda_i \otimes 1 - 1 \otimes \lambda_i^*)$ (cf. [1], Chap. IX, Prop. 3.1), which implies that the mapping $h \rightarrow h \cdot \varphi$ is one-to-one. In order to see that the mapping is moreover an epimorphism, take any f from $\text{Hom}_K(A, A) \cdot 1$. As is easily seen,

there exists a K -homomorphism $\bar{h}: A^e \rightarrow A$ such that $\bar{h}(\lambda \otimes \mu^*) = (f(\lambda) \mu =) \mu^* f(\lambda)$, and \bar{h} is necessarily a A^* -homomorphism because $\bar{h}((1 \otimes \nu^*)(\lambda \otimes \mu^*)) = \bar{h}(\lambda \otimes (\mu\nu)^*) = (\mu\nu)^* f(\lambda) = \nu^* \mu^* f(\lambda) = \nu^* \bar{h}(\lambda \otimes \mu^*)$. Let h be the restriction of \bar{h} to J . Then $h \in \text{Hom}_{A^*}(J, A)$ and $h(\lambda \otimes 1 - 1 \otimes \lambda^*) = \bar{h}(\lambda \otimes 1) - \bar{h}(1 \otimes \lambda^*) = f(\lambda) 1 - \lambda^* f(1) = f(\lambda)$, i. e., $h \cdot \varphi = f$. This completes our proof.

We can now prove

Theorem 2. *Let A be a unital two-sided A -module and $H^n(A, A)$ the n -th Hochschild cohomology group of A for A . Suppose that $A^e = A \otimes A^*$ is left A^* -projective (or equivalently, right A -projective)³⁾. Then there is a natural isomorphism $H^n(A, A) \cong \text{Ext}_{A^e}^n(A, A)$.*

Proof. The theorem is true for $n=0, 1$ by [1], Chap. IX, Prop. 4.1 (even without assuming the A^* -projectivity of A^e). So we may assume $n > 1$, and we suppose the theorem is true for $n-1$ instead of n and for all (unital) A . Then in particular

$$(4) \quad H^{n-1}(A, \text{Hom}_{A^*}(J, A)) \cong \text{Ext}_{A^e}^{n-1}(A, \text{Hom}_{A^*}(J, A)).$$

By Theorem 1 the right side of (4) is naturally isomorphic to $\text{Ext}_{A^e}^n(A, A)$. On the other hand, the left side of (4) is by Proposition 2, whence, by [3], Th. 1, naturally isomorphic to $H^{n-1}(A, \text{Hom}_K(A, A))$, and moreover this is isomorphic to $H^n(A, A)$ according to [2], Th. 3.1. Thus, $H^n(A, A) \cong \text{Ext}_{A^e}^n(A, A)$, and by induction this proves our theorem completely.

References

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3) The mapping $\lambda \otimes \mu^* \rightarrow \mu \otimes \lambda^*$ defines an involution of A^e , and by this mapping $\nu^*(\lambda \otimes \mu^*)$ is mapped on $(\mu \otimes \lambda^*) \nu = (\mu\nu) \otimes \lambda^*$. This shows that A^e is left A^* -projective if and only if it is right A -projective.