CORRESPONDENCE OF BOUNDARIES OF RIEMANN SURFACFS

By

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Let R be a Riemann surface with positive boundary and let R_n $(n=0,1,2,\cdots)$ be its exhaustion with compact relative boundary ∂R_n. Let $N(z,p)$ be a positive harmonic function in $R-R_0-p:p\in R-R_0$ such that $N(z,p)=0$ on ∂R_0, $N(z,p)$ has a logarithmic singularity at p and $N(z,p)$ has minimal D^*irichlet integral over $R-R_0$, where D^*irichlet integral is taken with respect to $N(z,p)+\log|z-p|$ in a neighbourhood of p. We call such $N(z,p)$ the N-Green's function of $R-R_0$ with pole at p. Consider now a sequence of points $\{p_i\}$ of $R-R_0$ having no points of accumulation in $R-R_0+\partial R_0$. Since the functions $N(z,p_i)$ $(i=1,2,\cdots)$ forms, from some i on, a bounded sequence of harmonic functions—thus a normal family. A sequence of these functions, therefore is convergent in every compact part of $R-R_0$ to a positive harmonic function. A sequence $\{p_i\}$ of $R-R_0$ having no point of accumulation in $R-R_0+\partial R_0$, for which the corresponding $N(z,p_i)$'s have the property just mentioned, that is, $\{N(z,p_i)\}$ coverges to a harmonic function—will be called fundamental. If two fundamental sequences determine the same limit function $N(z,p)$, we say that they are equivalent. Two fundamental sequences equivalent to a given one determine an ideal boundary point of R. The set of all the ideal boundary points of R will be denoted by B and the set $R-R_0+B$ by $\bar{R}-R_0$. The domain of definition of $N(z,p)$ may now be extended by writing $N(z,p)=\lim_i N(z,p_i)$ $(z\in R-R_0, p\in B)$, where $\{p_i\}$ is any fundamental sequence determining p. The function $N(z,p)$ is characteristic of the point p of their corresponding $N(z,p)$ as a function of z. The function $\delta(p_1,p_2)$ of two points p_1 and p_2 in $\bar{R}-R_0$ is defined as

$$\delta(p_1,p_2)=\sup_{z\in R_1-R_0}\left|\frac{N(z,p_1)}{1+N(z,p_1)}-\frac{N(z,p_2)}{1+N(z,p_2)}\right|.$$

Evidently, $\delta(p_1,p_2)=0$ is equivalent to $N(z,p_1)=N(z,p_2)$ for all points z in $\bar{R}_1-R_0$. Therefore we have $N(z,p_1)=N(z,p_2)$ in $\bar{R}-R_0$, i. e. $\delta(p_1,p_2)=0$ implies $p_1=p_2$ and it is clear that $\delta(p_1,p_2)$ satisfies the axioms of distance. Therefore $\delta(p_1,p_2)$ can be considered as the distance between p_1 and p_2 of $\bar{R}-R_0$.

The topology (we call N-Martin's topology) induced by this metric is homeomorphic to the original topology, when it is restricted in $R-R_0$ and we continue this topology into R_0. Since $\{N(z,p_i)\}:p_i\in\bar{R}-R_0$ is also a normal family, both $\bar{R}-R_1+\partial R_1+B$ and B are closed and compact. For a fixed point z, $N(z,p)$ is continuous with respect to this metric as a function of p in $\bar{R}-R_0$ except at p. Other topologies can be introduced on any Riemann surfaces (with null or positive boundary). For example K-Martin's topology, Green's topology, and Stoilow's topology. We proved that N-Martin's, and Stoilow's and Green's topologies are Dirichlet-separative[1]. We define the ideal boundary B of R by the completion of R. Then $\bar{R}+B$ is compact with respect to N-Martin's and Stoilow's topology.

We use in the present paper the notions of capacity, superharmonic functions, N-minimal points and others which are referred to the papers[2]. Suppose N-Martin's topology is introduced on $\bar{R}=R+B$. Let G be a domain and ${}_{CG}N(z,p)$ be the least positive superharmonic function in $R-R_0$ larger than $N(z,p)$ on CG (complementary set of G), where $p\in B_1^N$ (B_1^N is the set of N-minimal boundary points). If $N(z,p)>{}_{CG}N(z,p)$, we say that G contains p N-approximately and denote it by $G\overset{N}{\ni}p$.

Let $w=f(z)$: $z\in R$, $w\in \underline{R}$ be an analytic function whose values fall on a basic surface $\underline{R}$, where $\underline{R}$ is a Riemann surface with positive or null-boundary and a topology is defind on $\underline{R}+\underline{B}$. Put

$$M(p)=\bigcap_{\tau}\overline{f(G_\tau)}\quad\text{for } p\in B_1^N,$$

where $\{G_\tau\}$ means the all sets G_τ such that $G_\tau\overset{N}{\ni}p$.

1). For any point $p\in\underline{R}$, there exists an open disc $C(p)$ in $\underline{R}$ such that the area of R over $C(p)$ is finite.

2). There exists a number n_0 such that $n(w)\leqq M<\infty$ in $\underline{R}-\underline{R}_{n_0}$, where $n(w)$ is the number of points of R over p and $\{\underline{R}_n\}$ is an exhaustion of $\underline{R}$ with compact relative boundary $\partial\underline{R}_n$.

If the above two conditions are satisfied, we say that $w=f(z)$ is of *almost finitely sheeted.* Then we proved

Theorem 1[3]**.** *Let R be a covering surface with positive boundary and*

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1) Z. KURAMOCHI: On the behaviour of analytic functions on the ideal boundary. III. Proc. Japan Acad., 38 (1962).

2) Z. KURAMOCHI: Potentials on Riemann surfaces. Journ. Fac. Sci. Hokkaido Univ., XVI (1962). In this paper we denote it by P.

3) Z. KURAMOCHI: On the behaviour of analytic functions on the boundary. I. II. III and IV and Correction to the paper "On the behaviour—", Proc. Japan Acad., 39 (1963).

with N-Martin's topology over $\underline{R}$ *with* D. S. (Dirichlet separative[4]) *topology and suppose* $\underline{R}+\underline{B}$ *is compact with respect to the topology on* $\underline{R}$. *If* R *is a covering surface of almost finitely sheeted*: $w=f(z)$, $z\in R$ *and* $w\in\underline{R}$. *Then* $M(p)$ *is defined in* B *except an* F_σ *set of capacitly zero and* $S=E[p\in B_1^N$: diameter of $M(p)>0]$ *is a* $G_{\delta\sigma}$ *set of inner capacity zero, where* B_1^N *is the set of N-minimal boundary points of* R.

This is an extension of Beurling's theorem[5].

Let $w=f(z)$ be an analytic function in $|z|<1$ and let R be the Riemann surface, generated by $w=f(z)$ over the w-sphere K. Let a be any point of K and R_r be the part of R, which lies over a disc $C_r(a)$ of radius r with a as its centre and $A(r)$ be the sepherical area of R_r. If

$$\varliminf_{r\to 0}\frac{A(r)}{r^2}<\infty,$$

then a is called *an ordinary value of* $f(z)$ *in Beurling's sense.* Then

Theorem 2[6] **(A. Beurling).** *Let* $w=f(z)$ *be an analytic function in* $|z|<1$: $w\in w$*-sphere. Suppose the spherical area of the covering surface generated by* $w=f(z)$ *is finite. Let* a *be an ordinary value of* $f(z)$ *in Beurling's sense. Then the set* E *of* $e^{i\theta}$ *such that* $\lim_{r\to 1} f(re^{i\theta})=a$ *is a set of capacity zero.*

M. Tsuji[6] proved the above theorem under the condition that $f(z)$ takes three values α, β, γ only a finite number of times in $|z|<1$ instead of the condition that the spherical area of the image is finite.

The purpose of the present paper is to extend Theorem 2 to Riemann surfaces and to study the correspondence of boundaries of Riemann surfaces.

1). Properties of domains[7] **containing** $p\in B_1^N$ ***N*-approximately.** We shall study the properties of a domain $G\overset{N}{\ni} p: p\in B_1^N$, some of which are useful in the following. We proved the following

Lemma[8]**.** a). *For* $p\in B_1^N$ *we have* 1°). *If* $G_i\overset{N}{\ni}p$, $\bigcap_{i=1}^{l} G_i\overset{N}{\ni}p$. 2°). *If* $G\overset{N}{\ni}p$, (int CG) $\overset{N}{\not\ni}$ p. 3°). $E\left[z\in\bar{R}: \mathrm{dist}(z,p)<\frac{1}{n}\right]=v_n(p)\overset{N}{\ni}p$. *Since* $\underline{R}+\underline{B}$ *is closed,*

4). See 1).

5) A. BEURLING: Ensembles exceptionnelles. Act Math., 72 (1940)

6) See 5) also M. TSUJI: Beurling's theorem on exceptional sets. Tohoku Math. Journ., 2 (1950).

7) In the present paper we suppose for any domain G in Riemann surfaces R that ∂G consists of at most enumerably infinite number of analytic curves clustering nowhere in R.

8) See 1) of 3) and 2).

by 1°) *we have* $M(p)=\bigcap_{\tau}\overline{f(G_\tau)}\neq 0$, *and since* $\underline{R}+\underline{B}$ *is separable with respect to the given topology on* $\underline{R}+\underline{B}$, *we can find a sequence* $\{G_i\}: i=1,2,3,\cdots$ *from* $\{G_\tau\}$ *such that* $M(p)=\bigcap\overline{f(G_i)}$. *We see by* 2°) *that there exists only one component* G' *of* G *such that* $G'\overset{N}{\in}p$. 4°). *Let* G *and* G' *be domains such that* $G\supset G'$ *and* $D(\omega(G'\cap B,z,G))>0$. *Then there exists at least one point* $p\in B_1^N$ *such that* $G\overset{N}{\ni}p$, *where* $\omega(G'\cap B,z,G)=\lim_n\lim_i\omega_{n,n+i}(z)$ *and* $\omega_{n,n+i}(z)$ *is a harmonic function in* $(G\cap R_{n+i})-(G'\cap(R_{n+i}-R_n))-R_0$ *such that* $\omega_{n,n+i}(z)=0$ *on* ∂G, $\omega_{n,n+i}(z)=1$ *on* $\partial(G'\cap(R_{n+i}-R_n))$, $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ *on* $\partial R_{n+i}-G'$ *and* $D(\omega_{n,n+i}(z))\leqq M<\infty$ *for a certain* n_0 *and for every* i.

b). *Let* G *be a domain in* R. *Then the set of points* $\in B_1^N$ *not contained in* G *N-approximately is a* G_δ *set, i.e. the set of points* $\in B_1^N$ *contained in* G *is an* F_σ *set.*

It is known[9] *that for any* $V_M(p)=E[z\in R:N(z,p)>M]:M<M^*=\sup N(z,p)$, $p\in B_1^N$, *there exists a* $v_n(p)$ *such that* $(R\cap v_n(p))\subset V_M(p)$. *Hence we have by* 1°) *of* a)

c). *For any* $V_M(p):M<M^*$ *and for any domain* $G\overset{N}{\ni}p$ *we can find a domain* $G'(=G\cap v_n(p))$ *such that* $p\in G'\subset v_n(p)\subset V_M(p)$.

d). *If* $\omega(p,z)=\lim_n\omega(v_n(p),z)>0:p\in B^N$ (*this is equivalent to* $\sup N(z,p)<\infty$), *we call* p *a singular point. Let* B_s^N *be the set of singular points. Then* B_s^N *is an* F_σ *set and* $B_s^N\subset B_1^N$. *Then*

$$\text{for any } v_n(p)\ \lim_{M=\infty}\ \underset{V_M(p)\cap Cv_n(p)}{N}(z,p)=0 \text{ and } \lim_{M=\infty}\int_{\partial V_M(p)\cap Cv_n(p)}\frac{\partial}{\partial n}N(z,p)ds=0 \text{ for}$$

$p\in B_1^N-B_s^N$.

Since $\min(M,N(z,p))=M\ \omega(V_M(p),z))$ *and* $\int_{\partial V_M(p)}\frac{\partial}{\partial n}N(z,p)ds=2\pi$ *for almost* $M<M^*$, d) *means that the system* $\{V_{M_i}(p)\}$ $(M_1<M_2<M_3\cdots)$ *is almost equivalent to the neighbourhood system* $\{v_n(p)\}$. *By* c) *we know* $\{G_\tau\}:G_\tau\overset{N}{\ni}p$ *is finer than* $\{v_n(p)\}$. We shall prove the following theorem to show that G has the similar properties as $\{v_n(p)\}$.

Theorem 3. *Suppose* $G\overset{N}{\ni}p:p\in B_1^N$. *Then*

1). ${}_p(_{CG}N(z,p))=\lim_{n=\infty}{}_{v_n(p)}(_{CG}N(z,p))=0$ *for* $p\in B_1^N-B_s^N$.

9) See p. 42 of P. and see 2).

2). ${}_{p\cap CG}N(z,p)=\lim_n {}_{v_n(p)\cap CG}N(z,p)=0$ *and* ${}_{G\cap p}N(z,p)={}_GN(z,p)=N(z,p)$ *for* $p\in B_1^N$.

3). $\lim_{M\to M^*} {}_{V_M(p)\cap CG}N(z,p)=0$ *and* $\lim_{M\to M^*}\int_{\partial V_M(p)\cap CG}\frac{\partial}{\partial n}N(z,p)ds=0$ *for* $p\in B_1^N$, *where* $M^*=\sup_N N(z,p)$.

By $v_n(p)\ni p$ 3) *implies* d) *of the Lemma.*

Proof of 1). Assume ${}_p({}_{CG}N(z,p))>0$. Since ${}_p({}_{CG}N(z,p))$ has mass only at p, ${}_p({}_{CG}N(z,p))=a_1 N(z,p)$, where $1\geqq a_1>0$, by $N(z,p)\geqq{}_{CG}N(z,p)$. Now p is a set of capacity zero and $0\leqq U_1(z)={}_{CG}N(z,p)-{}_p({}_{CG}N(z,p))$ is also superharmonic[10]. Hence similarly as before ${}_pU_1(z)=a_2 N(z,p)$ and $U_1(z)=a_2 N(z,p)+U_2(z)$ and $U_2(z)$ is superharmonic, whence

$$\begin{aligned}&{}_p(U_n(z))=a_{n+1}N(z,p)\quad\text{and}\\ &U_n(z)=a_{n+1}N(z,p)+U_{n+1}(z),\qquad n=1,2,3,\cdots.\end{aligned}$$

Then $a_1\geqq a_2\cdots$ and $\Sigma\ a_n\leqq 1$, $\lim_n a_n=0$ and $U_n(z)\downarrow$. Put $U^*(z)=\lim U_n(z)$. Then $U^*(z)$ is also superharmonic[11] and $U^*(z)\leqq a_nN(z,p)$ for any n. Thus $U^*(z)=0$ and $N(z,p)=(\Sigma a_n)N(z,p)$. On the other hand, $N(z,p)={}_{CG}N(z,p)$ on CG. Whence $\Sigma\ a_n=1$ and ${}_{CG}N(z,p)=N(z,p)$. This contradicts $G\overset{N}{\ni}p$. Hence ${}_p({}_{CG}N(z,p))=0$.

Proof of 2). Case 1. $p\in B_1^N-B_s^N$. In this case ${}_{v_n(p)\cap CG}N(z,p)=N(z,p)={}_{CG}N(z,p)={}_{v_n(p)}({}_{CG}N(z,p))$ on $v_n(p)\cap CG$. Now since ${}_AU(z)\geqq{}_BU(z)$ for a superharmonic function $U(z)$ for $A\supseteqq B$,

${}_{v_n(p)\cap CG}N(z,p)={}_{v_n(p)\cap CG}({}_{CG}N(z,p))\leqq{}_{v_n(p)}({}_{CG}N(z,p))$ by $v_n(p)\supset(v_n(p)\cap CG)$. Let $n\to\infty$. Then by 1)

$$ {}_{p\cap CG}N(z,p)\leqq{}_p({}_{CG}N(z,p))=0\,. \tag{1}$$

Case 2. $p\in B_s^N$ in this case it was proved that $\omega(p\cap CG,z)=0$[12]. Whence

$$ {}_{p\cap CG}N(z,p)\leqq a\ \omega(p\cap CG,z)=0\,, \tag{1'}$$

by $\omega(p,z)\leqq 1$ and $N(z,p)=a\ \omega(p,z)$, where $a=2\pi\Big/\int_{\partial R_0}\frac{\partial}{\partial n}\omega(p,z)\,ds$. We shall

10) p. 32 of P (see 2)).

11) We can prove $D(\min(N,U^*(z))\leqq D(\min(M,N(z,p))\leqq 2\pi M$. Then by p. 23 of P $U(z)$ is superharmonic.

12) Z. KURAMOCHI: Singular point of Riemann surfaces. Journ. Fac. Sci. Hokkaido Univ., XVI (1962) and see p. 97 of the above paper.

prove the latter part of 2). ${}_{p\cap CG}N(z,p)+{}_{p\cap G}N(z,p)\geqq {}_{p}N(z,p)$. Hence by (1) and (1′) we have ${}_{p\cap CG}N(z,p)=0$ and ${}_{G}N(z,p)\geqq {}_{p\cap G}N(z,p)\geqq {}_{p}N(z,p)=N(z,p)$.

Proof of 3). Case 1. $p\in B_1^N-B_s^N$. Put $S(z)=\lim\limits_{M=\infty}\ {}_{CG\cap V_M(p)}N(z,p)$. Assume $S(z)>0$. Then $S(z)$ is superharmonic and since $\lim\limits_{M=\infty}\mathrm{Cap}(CG\cap V_M(p))\leqq \lim\limits_{M=\infty}\mathrm{Cap}(V_M(p))=0$, $N(z,p)-S(z)$ is also superhamonic[10]. On the other hand, $N(z,p)$ is N-minimal, whence $S(z)$ must be a multiple of $N(z,p)$, i. e. $S(z)=a\,N(z,p):a>0$. Clearly ${}_{CG}N(z,p)\geqq {}_{CG\cap V_M(p)}N(z,p)\geqq S(z)=a\,N(z,p)$. Whence ${}_{p}({}_{CG}N(z,p))\geqq {}_{p}(a\,N(z,p))=a\ N(z,p)>0$. This contradicts 1). Hence $\lim\limits_{M=\infty}{}_{V_M(p)\cap CG}N(z,p)=0$.

Case 2. $p\in B_s^N$. In this case it was proved[13] that $\lim\limits_{M\to 1}\omega\,(V_M(p)\cap CG,z)=0$ to show $\lim\limits_{M\to 1}\omega(Cv_n(p)\cap V_M(p),z)=0$. But ${}_{V_M(p)\cap CG}N(z,p)=a\,\omega(V_M(p)\cap CG,z)$. Hence $\lim\limits_{M\to M^*}\ {}_{V_M(p)\cap CG}N(z,p)=0$.

We show $\lim\limits_{M\to M^*}\displaystyle\int_{\partial V_M(p)\cap CG}\frac{\partial}{\partial n}N(z,p)\,ds=0$ for $p\in B_1^N$. $M\omega(V_M(p)\cap CG,z)$ is M on $V_M(p)\cap CG$ and superharmonic in $\bar{R}$. Hence $M\omega(V_M(p)\cap CG,z)\leqq {}_{V_M(p)\cap CG}N(z,p)\leqq N(z,p)$. Since $N(z,p)=M$ on $\partial V_M(p)$,

$$\int_{\partial R_0}\frac{\partial}{\partial n}M\,\omega(V_M(p)\cap CG,z)\,ds\geqq\int_{\partial(V_M(p)\cap CG)}\frac{\partial}{\partial n}M\,\omega(V_M(p)\cap CG,z)ds$$
$$\geqq\int_{\partial V_M(p)\cap CG}\frac{\partial}{\partial n}\omega(V_M(p)\cap CG,z)\,ds\geqq\int_{\partial V_M(p)\cap CG}\frac{\partial}{\partial n}N(z,p)\,ds\,.$$

Assume $\overline{\lim\limits_{M\to M^*}}\displaystyle\int_{\partial V_M(p)\cap CG}\frac{\partial}{\partial n}N(z,p)\,ds\geqq\delta>0$. Then there exists a sequence $M_1<M_2\cdots.\ M^*$ such that $0<\lim\limits_{i}M\omega(V_{M_i}(p)\cap CG,z)$ by $\displaystyle\int_{\partial R_0}\frac{\partial}{\partial n}M_i\omega(V_{M_i}(p)\cap CG,z)\,ds>\frac{\delta}{2}$. On the other hand, $\lim\limits_{M\to M^*}M\omega(V_M(p)\cap CG,z)=\lim\limits_{M\to M^*}\ {}_{V_M(p)\cap CG}N(z,p)$. This contradicts $\lim\limits_{M\to M^*}{}_{V_M(p)\cap CG}N(z,p)=0$. Hence $\lim\limits_{M\to M^*}\displaystyle\int_{\partial V_M(p)\cap CG}\frac{\partial}{\partial n}N(z,p)\,ds=0$ for $p\in B_1^N$. We consider the domain $\overset{N}{G}\ni p$ and we shall show that $N'(z,p)\,(=N(z,p)-{}_{CG}N(z,p)>0)$ in G has the similar properties as those of $N(z,p)$ in $R-R_0$.

Theorem 4. *Let G be a domain in $R-R_0$ such that $\overset{N}{G}\ni p\,(p\in B_1^N)$.*

13) See p. 97 of 12) and p. 21) of P.

Then

1. a). ${}_{p\cap V'_M(p)}N(z,\ p) = {}_{V'_M(p)}N(z,\ p) = N(z,\ p)$ *and* $\lim\limits_{M=\infty} {}_{p\cap V'_M(p)}N(z,\ p) =$ $N(z, p)$ *for* $p\in B_1^N - B_s^N$, *where* $V'_M(p) = E[z\in G : N'(z,p) > M]$.

1. b). $\omega(p, z) = {}_{p\cap V'_M(p)}\omega(p, z) : M < 1 - M_0 = M^{**}$ *for* $p\in B_s^N$, *where* M_0 *is the infinimum of numbers* M_t *such that* $\omega(p\cap V^C_{M_t}, z) = 0$, $M_0 < 1$[12] *and* $V'_M(p)$ $= E[z\in G : \omega(p, z) - {}_{CG}\omega(p, z) > M]$, $V^c_{M_t} = E[z\in R : {}_{CG}\omega(p, z) > M_t]$.

1. c). $D_G(\min(M, N'(z,p)) \leqq 2\pi M$ *for* $p\in B_1^N - B_s^N$.

$D_G(N'(z,p)) < \infty$ *for* $p\in B_s^N$.

2. a). *Let* ${}^{*}_{G'}N^L(z,p)$ *be a function in* G *such that* ${}^{*}_{G'}N^L(z,p) = \min(L,$ $N'(z,p))$ *on* $G' + \partial G$ *and* ${}^{*}_{G'}N^L(z,p)$ *has* M. D. I. ($\leqq 2\pi L$ by 1. c)) *over* $G - G'$. *Put* ${}^{*}_{G'}N'(z,p) = \lim\limits_{L=\infty} {}^{*}_{G'}N^L(z,p)$. *Then* ${}^{*}_{G'}N'(z,p) \leqq N'(z,p)$ *for any domain* G' *in* G. *Thus* $N'(z,p)$ *is superharmonic in* G.

2. b). *As a special case we have*

$\min(M, N'(z,p)) = M\ \omega(V'_M(p), z, G)$ *for* $M < \infty$ *and* $p\in B_1^N - B_s^N$,

$\min(M,\ \omega'(p, z)) = M\ \omega(V'_M(p), z, G)$ *for* $M < M^{**}$ *for* $p\in B_s^N$,

where $\omega'(p, z) = \omega(p, z) - {}_{CG}\omega(p, z)$ *and* $\omega(V'_M(p), z, G)$ *is C. P. of* $V'_M(p)$ *relative to* G.

And

$$V'_M(p) \ni p.$$

By 2. a) *and* 2). b) *it is seen that* $V'_M(p)$ *has similar properties as* $V_M(p)$ *in* $R - R_0$,

where $V'_M(p) = E[z\in G :\ N'(z,p) > M]$ *for* $p\ \in B_1^N - B_s^N$ *and* $V'_M(p) = E[z\in G :$ $\omega(p, z) > M]$ *for* $p\in B_s^N$.

3). *Let* G' *be a domain in* G *such that* ${}_{G'}N(z,p) = N(z,p)$. *Then* ${}^{*}_{G'}N'(z,p) = N'(z,p)$. *As a special case by* $v_n(p) \overset{N}{\ni} p$ *we have*

$$N'(z,p) = {}^{*}_{p}N'(z,p) = \lim_{n=\infty} {}_{v_n(p)\cap G}^{*}N'(z,p) = {}_{v_n(p)}^{*}N'(z,p) \text{ for } p\in B_1^N .$$

Now $v_n(p) \overset{N}{\ni} p$, *hence by considering* $v_n(p)$ *a domain* G *we have by* 2). b) *the following fact: for any* $v_n(p)$ *there exists a domain* $G' (= V'_M(p))$ *such that* $v_n(p) \supset G' \overset{N}{\ni} p$. *Hence for any* $V_M(p)$ *there exist* $v_n(p)$ *and* $V'_M(p)$ *such that* $V_M(p) \supset v_n(p) \supset V'_M(p) \overset{N}{\ni} p$. *This means* $\{V_M(p)\}$, $\{v_n(p)\}$ *and* $\{V'_M(p)\}$ *are equivalent.*

Proof of 1). a). Now $p\in B_1^N - B_s^N$. In this case $N(z,p) \leqq {}_{CG}N(z,p) + M$

on $CV'_M(p)$. Hence

$_{CV'_M(p)\cap p}N(z, p) \leqq {}_{CV'_M(p)\cap p}({}_{CG}N(z, p)) + {}_{CV'_M(p)\cap p}M = {}_{CV'_M(p)\cap p}({}_{CG}N(z, p)) + M\omega(p\cap CV'_M(p), z) \leqq {}_p({}_{CG}(N(z, p)) = 0$ by $\omega(p, z)=0$ and by 2) of Theorem 3. On the other hand,

$$_{CV'_M(p)\cap p}N(z, p) + {}_{V'_M(p)\cap p}N(z, p) \geqq {}_pN(z, p) = N(z, p).$$ Whence
$$N(z, p) \geqq {}_{V'_M(p)}N(z, p) \geqq {}_{V'_M(p)\cap p}N(z, p) \geqq N(z, p).$$ Hence
$$\lim_{M'=\infty} {}_{V'_M(p)\cap p}N(z, p) = N(z, p).$$

Proof of 1). b). Since $p\in B_s^N$, $N(z, p)=a\ \omega(p, z): a>0$. Let $1>M'>M_0>0$. Then $\omega(p\cap V_{M'}^C, z)=0$, where $V_{M'}^C=E[z\in R: {}_{CG}\omega(p, z)>M']$. Further we have

$$\omega(\tilde{V}_{1-\varepsilon}\cap p, z) = \omega(p, z)^{12)},$$

where $\tilde{V}_{1-\varepsilon} = CV_{M'}^C \cap V_{1-\varepsilon}(p) = E[z\in G: \omega(p, z) - {}_{CG}\omega(p, z) > 1-\varepsilon-M']$. Since $\omega(p, z) > 1-\varepsilon$ on $\tilde{V}_{1-\varepsilon}$, ${}_{p\cap\tilde{V}_{1-\varepsilon}}\omega(p, z) \geqq {}_{p\cap\tilde{V}_{1-\varepsilon}}(1-\varepsilon) = (1-\varepsilon)\omega(p\cap\tilde{V}_{1-\varepsilon}, z)$. We have by letting $\varepsilon\to 0$ $\lim_{\varepsilon\to 0} {}_{p\cap\tilde{V}_{1-\varepsilon}}\omega(p, z) = \omega(p, z)$.
Because $\lim_{\varepsilon\to 0}\omega(p\cap\tilde{V}_{1-\varepsilon}, z)$ has mass only at p and $\sup\omega(p\cap\tilde{V}_{1-\varepsilon}, z)=1$.
Clearly $V'_M(p)\supset\tilde{V}_{1-\varepsilon}$ for $M<1-\varepsilon-M'$ and
$\omega(p, z) \geqq {}_{p\cap V'_M(p)}\omega(p, z) \geqq \lim_{\varepsilon\to 0} {}_{\tilde{V}_{1-\varepsilon}\cap p}\omega(p, z) = \omega(p, z)$. Thus

$$\omega(p, z) = {}_{p\cap V'_M(p)}\omega(p, z).$$

Proof of 1). c). Case 1. $p\in B_1^N - B_s^N$. By 1). a) ${}_{V'_M(p)}N(z, p) = N(z, p)$ and since ${}_{V_M(p)\cap R_m}N(z, p) \uparrow {}_{V'_M(p)}N(z, p) = N(z, p)$ as $m\to\infty$, there exists a number m_0 such that

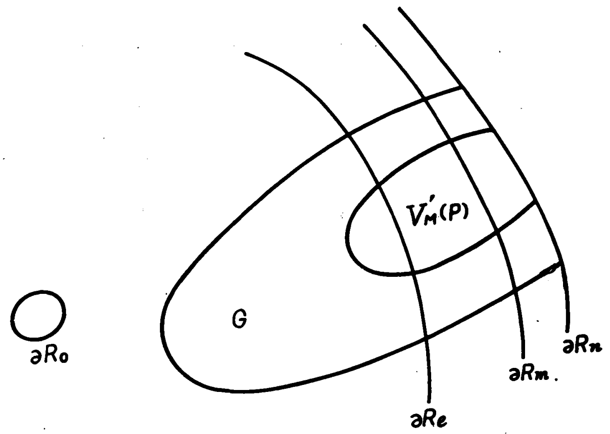

Fig. 1.

$$0 \leqq N(z, p) - {}_{V'_M(p) \cap R_m}N(z, p) > \varepsilon \text{ on } R_l \text{ for } m \geqq m_0, \quad (2)$$

for any given R_l (R_l is an exhaustion of R) and $\varepsilon > 0$. Let $N_{m,n}(z, p)$ be a harmonic function in $R_n - R_0$ such that $N_{m,n}(z, p) = 0$ on ∂R_0, $N_{m,n}(z, p) = N(z, p)$ on $R_m \cap V'_M(p)$ and $\frac{\partial}{\partial n} N_{m,n}(z, p) = 0$ on ∂R_n. Then

$$N_{m,n}(z, p) \Rightarrow {}_{V'_M(p) \cap R_m}N(z, p) \; (\leqq {}_{V'_M(p)}N(z, p)) \text{ as } n \to \infty,$$

where $\Rightarrow$ means convergence in mean. Hence there exists a number $n_1(m)$ such that

$$|N_{m,n}(z, p) - {}_{V'_M(p)}N(z, p)| < \varepsilon \text{ on } R_l \text{ for } n \geqq n_1(m) \text{ and } n > m. \quad (3)$$

Let $U_{l,m,n}(z)$ be a harmonic function in $R_n - (R_l \cap CG)$ such that $U_{l,m,n}(z) = N_{m,n}(z, p)$ on $\partial(R_l \cap CG)$, $\frac{\partial}{\partial n} U_{l,m,n}(z) = 0$ on ∂R_n. Then by (2) and (3)

$$|U_{l,m,n}(z) - N(z, p)| < 2\varepsilon \text{ on } \partial(R_l \cap CG) \text{ for } n > n_1(m) \text{ and } n > m. \quad (4)$$

On the other hand, let $U'_{l,m,n}(z)$ be a harmonic function in $R_n - (R_l \cap CG)$ such that $U'_{l,m,n}(z) = N(z, p)$ on $\partial(R_l \cap CG)$, $\frac{\partial}{\partial n} U'_{l,m,n}(z) = 0$ on ∂R_n. Then by the maximum principle

$$|U_{l,m,n}(z) - U'_{l,m,n}(z)| < 2\varepsilon \text{ in } R_n - (R_l \cap CG). \quad (5)$$

Let $n \to \infty$. Then by $U'_{l,m,n}(z) \Rightarrow {}_{CG \cap R_l}N(z, p)$

$$|\overset{*}{\lim_n} U_{l,m,n}(z) - {}_{CG \cap R_l}N(z, p| < 2\varepsilon \text{ in } R_n - (R_l \cap CG), \quad (6)$$

where $\overset{*}{\lim_n} U_{l,m,n}(z)$ means a limit by a converging subsequence of $\{U_{l,m,n}(z)\}$. Hence there exists a number $n_2 > n_1$ such that

$$|U_{l,m,n}(z) - {}_{CG \cap R_l}N(z, p)| < 3\varepsilon \text{ for } n > n_2 \text{ in } (R_m \cap G) \supset (R_m \cap V'_M(p)). \quad (7)$$

Now by ${}_{CG \cap R_l}N(z, p) \uparrow {}_{CG}N(z, p)$ as $l \to \infty$ we have $U_{l,m,n}(z) < {}_{CG}N(z, p) + 3\varepsilon$ on $R_m \cap V'_M(p)$. On the other hand, $M \leqq N(z, p) - {}_{CG}N(z, p) \leqq N(z, p) - U_{l,m,n}(z) + 3\varepsilon$ on $R_m \cap V'_M(p)$. Since $N_{m,n}(z) = U_{l,m,n}(z)$ on $\partial(R_l \cap CG)$ and ${}_{CG}N(z, p) = N(z, p)$ on $R_m \cap V'_M(p)$,

$V^* = E[z \in R_n - (R_l \cap CG): \; N_{m,n}(z, p) - U_{l,m,n}(z) > M - 3\varepsilon] \supset E[z \in R_m: N(z, p) - {}_{CG}N(z, p) > M] = V'_M(p) \cap R_m$.

Whence $N_{m,n}(z, p) - U_{l,m,n}(z)$ is harmonic in $R_n - (R_l \cap CG) - V^*$. It follows that

$$0 \leqq \int_{\partial V^* \cap R_n} \frac{\partial}{\partial n} N_{m,n}(z,p)\, ds = \int_{\partial R_0} \frac{\partial}{\partial n} N_{m,n}(z,p)\, ds \uparrow \int_{\partial R_0} \frac{\partial}{\partial n}\, {}_{V'_M(p)\cap R_m}N(z,p)\, ds$$

$$\leqq \int_{\partial R_0} \frac{\partial}{\partial n} N(z,p)\, ds = 2\pi \text{ as } n \to \infty .$$

On the other hand, $\displaystyle\int_{\partial V^* \cap R_n} \frac{\partial}{\partial n} U_{l,m,n}(z)\, ds = \int_{\partial R_n \cap V^*} \frac{\partial}{\partial n} U_{l,m,n}(z) ds = 0$. Hence

$\displaystyle\int_{\partial V^* \cap R_n} \frac{\partial}{\partial n} (N_{m,n}(z,p) - U_{l,m,n}(z))\, ds \leqq 2\pi + \varepsilon$ for $n > n_3(m, l)$, whence

$$\underset{R_n-(CG\cap R_l)-V^*}{D}(N_{m,n}(z,p) - U_{l,m,n}(z)) \leqq (2\pi+\varepsilon)\,(M-3\varepsilon)\,. \qquad (8)$$

Let $\mathfrak{G}$ be a compact domain completely contained in $G - V'_M(p)$. Then since $(N_{m,n}(z,p) - U_{l,m,n}(z)) \to ({}_{V'_M(p)\cap R_n}N(z,p) - {}_{CG\cap R_l}N(z,p) + \delta(z)$ as $n\to\infty$, (where $|\delta(z)| < 2\varepsilon$) by (6) and since $CG\cap R_l$ is compact, $({}_{V'_M(p)\cap R_m}N(z,p) - {}_{CG\cap R_l}N(z,p) + \delta(z) \to ({}_{V'_M(p)}N)(z,p) - {}_{CG\cap R_l}N(z,p) + \delta(z)$ as $m \to \infty$. Further ${}_{V'_M(p)}N(z,p) - {}_{CG\cap R_l}N(z,p) + \delta(z) \to N(z,p) - {}_{CG}N(z,p)$ as $l\to\infty$ by ${}_{V'_M(p)}N(z,p) = N(z,p)$. Hence there exist numbers l, m, n such that $\mathfrak{G} \subset (R_n - (CG\cap R_l) - V^*)$ for any $\varepsilon > 0$ by (8) and

$$\underset{\mathfrak{G}}{D}\,(N_{m,n}(z,p) - U_{l,m,n}(z)) \leqq (2\pi+\varepsilon)\,(M-3\varepsilon)\,.$$

Let $n\to\infty$. Then $\underset{\mathfrak{G}}{D}\,(N_m(z,p) - U_{l,m}(z)) \leqq \underline{\lim}_{n}\, \underset{\mathfrak{G}}{D}\,(N_{m,n}(z,p) - U_{l,m,n}(z))$. Similarly let $m\to\infty$ and then $l\to\infty$ and then $\varepsilon\to 0$ and then $\mathfrak{G} \to (G - V'_M(p))$. Then

$$\underset{G-V'_M(p)}{D}\,(N(z,p) - {}_{CG}N(z,p)) = D(\min(M, N(z,p))) \leqq 2\pi M\,.$$

Case 2. $p \in B^N_s$. In this case it is evident that $D(\omega(p,z)) < \infty$ and $D({}_{CG}\omega(p,z)) < \infty$ and we have at once $D(\omega(p,z) - {}_{CG}\omega(p,z)) < \infty$.

Proof of 2). a). Let ${}_DN^T(z,p)$ be a function such that ${}_DN^T(z,p) = \min(T, N(z,p))$ on D and ${}_DN^T(z,p)$ has M. D. I. over $R - R_0 - D$ and let ${}^*_{G'}N'^T(z,p)$: $G' \subset G$ be a function in $G - G'$ such that ${}^*_{G'}N'^T(z,p) = 0$ on ∂G, ${}^*_{G'}N'^T(z,p) = \min(T, N'(z,p))$ on G' and ${}^*_{G'}N'^T(z,p)$ has M. D. I. over $G - G'$ (this can be defined by 1). c)). Then by $N(z,p) = N'(z,p) + {}_{CG}N(z,p)$ we hare ${}_{CG+G'}N^{2L}(z,p) \geqq {}^{L}_{CG+G'}({}_{CG}N(z,p)) + {}_{CG+G'}(N'^L(z,p))$ on $\partial G + \partial G'$ and by the maximum principle*[13] ${}_{CG+G'}N^{2L}(z,p) \geqq {}_{CG+G'}({}_{CG}N^L(z,p) + N'^L(z,p))$. Let $L\to\infty$. Then

$${}_{CG+G}\,N(z,p) \geqq {}_{CG+G'}({}_{CG}N(z,p) + N'(z,p)).$$

Next similarly we have ${}_{CG+G'}N^{L}(z,p)\leqq{}^{L}_{CG+G'}({}_{CG}N(z,p)+N'(z,p))$ and ${}_{CG+G'}N(z,p)$ $\leqq{}_{CG+G'}({}_{CG}N(z,p)+N'(z,p))$. On the other hand, by $CG+G'\supset CG$ we have ${}_{CG+G'}({}_{CG}N(z,p))={}_{CG}N(z,p)$. Hence by ${}_{CG+G'}(N'(z,p))={}^{*}_{G'}N'(z,p)$ we have

$${}_{CG}N(z,p)+N'(z,p)=N(z,p)\geqq{}_{CG+G'}N(z,p)={}_{CG}N(z,p)+{}^{*}_{G'}N'(z,p) \quad (9)$$

and $N'(z,p)\geqq{}^{*}_{G'}N'(z,p)$. Thus $N'(z,p)$ is $\overline{\text{superharmonic}}$ in G.

Proof of 2). b). By 1). a) and 1. b) ${}_{V'_M(p)}N(z,p)=N(z,p)$ whence

$N(z,p)\geqq{}_{CGV+V'_M(p)}N(z,p)\geqq{}_{V'_M(p)}N(z,p)=N(z,p)$ for $M<\infty$ for $p\in B^N_1-B^N_s$,

${}_{CG+V'_M(p)}N(z,p)={}_{V'_M(p)}N(z,p)=N(z,p)$ for $M<M^*$:

$N(z,p)=a\ \omega(p,z)$ for $p\in B^N_s$.

Hence ${}_{CG}N(z,p)+N'(z,p)=N(z,p)={}_{CG+V'_M(p)}N(z,p)={}_{CG+V'_M(p)}({}_{CG}N(z,p)+N'(z,p))$ $={}_{CG}N(z,p)+{}^{*}_{V'_M(p)}N'(z,p)$,

because ${}_{CG+V'_M(p)}({}_{CG}N(z,p))={}_{CG}N(z,p)$ by $(CG+V'_M(p))\supset CG$ and ${}_{CG+V'_M(p)}N'(z,p)$ $={}^{*}_{V'_M(p)}N'(z,p)$. Hence $N'(z,p)={}^{*}_{V'_M(p)}N'(z,p)$. Clearly by definition ${}^{*}_{V'_M(p)}N'(z,p)$ $=M\ \omega(V'_M(p),z,G)$. Hence

$$N'(z,p)=M\ \omega(V'_M(p),z,G) \text{ in } G-V'_M(p).$$

We show $V'_M(p)\overset{N}{\ni}p$. Case 1. $p\ni B^N_1-B^N_s$. Assume $V'_M(p)\overset{N}{\not\ni}p$. Then ${}_{CV'_M(p)}N(z,p)=N(z,p)$ and ${}_{CG}N(z,p)+N'(z,p)=N(z,p)={}_{CV'_M(p)}N(z,p)=$ ${}_{CV'_M(p)}({}_{CG}N(z,p))+{}_{CV'_M(p)}N'(z,p)\leqq{}_{CG}N(z,p)+M$. Whence $N(z,p)={}_{p}N(z,p)\leqq$ ${}_{p}({}_{CG}N(z,p)+M\omega(p,z)$. But by a) of Theorem 3 ${}_{p}({}_{CG}N(z,p))=0$ and $\omega(p,z)=0$ by $p\in B^N_1-B^N_s$. This implies $N(z,p)=0$. This is a contradiction. Hence $V'_M(p)\overset{N}{\ni}p$.

Case 2. $p\in B^N_s$. It is sufficient to prove for $\omega(p,z)$. Now $\omega'(p,z)=$ $\omega(p,z)-{}_{CG}\omega(p,z)$ and

$${}_{CG+CV'_M(p)}\omega(p,z)={}_{CG+CV'_M(p)}({}_{CG}\omega(p,z))+{}_{CV'_M(p)}^{*}(\omega'(p,z)). \quad (10)$$

Assume $V'_M(p)\overset{N}{\notin}p$. Then ${}_{CV'_M(p)}\omega(p,z)=\omega(p,z)$ and $\omega(p,z)\geqq{}_{CG+V'_M(p)}\omega(p,z)\geqq$ ${}_{CV'_M(p)}\omega(p,z)=\omega(p,z)$. On theo ther hand, ${}_{V'_M(p)}^{*}\omega'(p,z)=M\ \omega(V'_M(p),z,G)$ for $M<M^{**}$. Hence $\sup\limits_{z\in G}\omega'(p,z)\geqq M^{**}$ and $\omega'(p,z)>{}_{CV'_M(p)}^{*}\omega'(p,z)(\leqq M)$. Hence by (10) and by ${}_{CG+CV'_M(p)}({}_{CG}\omega(p,z))={}_{CG}\omega(p,z)$ we have

$\omega(p,z)={}_{CG+CV'_M(p)}\omega(p,z)={}_{CG}\omega(p,z)+{}_{CV'_M(p)}^{*}\omega'(p,z)<{}_{CG}\omega(p,z)+\omega'(p,z)=\omega(p,z)$.

This is a contradiction. Hence $V'_M(p)\overset{N}{\ni}p$ for $M<M^{**}$.

Proof of 3). By ${}_{G'}N(z, p) \doteq N(z, p)$ we have ${}_{CG+G'}N(z, p) = N(z, p)$. This implies ${}_{CG}N(z, p) + N'(z, p) = N(z, p) \leqq {}_{CG+G'}N(z, p) = {}_{CG+G'}({}_{CG}N(z, p)) + {}^{*}_{G'}N'(z, p)$ $= {}_{CG}N(z, p) + N'(z, p) \leqq N(z, p)$. Hence ${}^{*}_{G'}N'(z, p) = N'(z, p)$.

2. On capacitary potentials. Let $\{G_n\}$ $(n = 1, 2, 3, \cdots)$ be a decreasing sequence of domains in $R - R_0$. Let G be a domain in $R - R_0$ such that $G \supset G_1$ and let Γ be a compact arc on ∂G. Let $\omega^G_{n,n+i}(z)$ be a harmonic function in $(R_{n+i} \cap G) - G_n$ such that $\omega^G_{n,n+i}(z) = 0$ on ∂G, $\omega^G_{n,n+i}(z) = 1$ on G_n and $\frac{\partial}{\partial n}\omega^G_{n,n+i}(z) = 0$ on $\partial R_{n+i} - G_n$. Then if $D(\omega^G_{n,n+i}(z)) < M < \infty$ for a number n_0 and for every i, then $\omega^G_{n,n+i}(z) \Rightarrow \omega^G_n(z)$[14] as $i \to \infty$ and $\omega^G_n(z) \Rightarrow \omega(\{G_n\}, z, G)$ called C.P. (capacitary potential) of the set determined by $\{G_n\}$ as $n \to \infty$. If $G = R - R_0$, we denote $\omega^G_{n,n+i}(z)$ by $\omega_{n,n+i}(z)$. Then $D(\omega_{n,n+i}(z)) < M < \infty$ for any and i by the fact that dist $(\partial R_0, G_1) > 0$. We denote $\omega(\{G_n\}, z, R - R_0)$ simply by $\omega(\{G_n\}, z)$. Let $\omega^{\mathfrak{N}}_{n,n+i}(z)$ be a harmonic function in $(R_{n+i} \cap G) - G_n$ such that $\omega^{\mathfrak{N}}_{n,n+i}(z) = 1$ on G_n, $\omega^{\mathfrak{N}}_{n,n+i}(z) = 0$ on $\partial\Gamma$, $\frac{\partial}{\partial n}\omega^{\mathfrak{N}}_{n,n+i}(z) = 0$ on $(\partial G - \Gamma) + (\partial R_{n+i} - G_n)$. Then since Γ is compact and by dist $(\Gamma, G_n) > 0$, $D(\omega^{\mathfrak{N}}_{n,n+i}(z) < M < \infty$ for $n \geqq n_0$ and every i. Then $\omega^{\mathfrak{N}}_{n,n+i}(z) \Rightarrow \omega^{\mathfrak{N}}_n(z)$ and $\omega^{\mathfrak{N}}_n(z) \Rightarrow \omega(\{G_n\}, z, G, \Gamma)$. It can be proved easily if $\omega(\{G_n\}, z, R - R_0) > 0$, $\omega(\{G_n\}, z, R - R'_0) > 0$ for any compact set R'_0 such that $R'_0 \cap G_{n_0} = 0$ (n_0 is a certain number).

Theorem 5. *If the condition* $D(\omega^G_{n,n+i}(z)) < M < \infty$ *for a number* n_0 *and for any number* i *is satisfied, then*

1). $\omega(\{G_n\}, z) > 0$ *if and only if* $\omega(\{G_n\}, z, G) > 0$,

2). $\omega(\{G_n\}, z, G) > 0$ *if and only if* $\omega(\{G_n\}, z, G, \Gamma) > 0$,

3). If $\omega(\{G_n\}, z, G, \Gamma) > 0$, $\omega(\{G_n\}, z) > 0$ *without the above condition.*

Proof. Suppose $\omega(\{G_n\}, z) > 0$. Then by the Dirichlet principle $D(\omega_{n,n+i}(z)) \leqq D(\omega^G_{n,n+i}(z))$. By the mean covergency of $\omega_{n,n+i}(z)$ and $\omega^G_{n,n+i}(z)$ we have

$$D(\omega(\{G_n\}, z, G)) \geqq D(\omega(\{G_n\}, z)) > 0. \quad \text{Hence } \omega(\{G_n\}, z, G) > 0.$$

Next suppose $\omega(\{G_n\}, z, G) > 0$. Then by the maximum principle $\omega_{n,n+i}(z)$ $\geqq \omega^G_{n,n+i}(z)$. Hence

$$\omega(\{G_n\}, z) > 0\,.$$

14) See p. 21 of P.

By $\omega(\{G_n\}, z, G) \leqq \omega(\{G_n\}, z, G, \Gamma)$ and by $D(\omega(\{G_n\}, z, G)) \geqq D(\omega(\{G_n\}, z, G, \Gamma))$ we have 2. Also by the Dirichlet principle $D(\omega(\{G_n\}, z, G)) \geqq D(\omega(\{G_n\}, z, G, \Gamma))$ and we have 3).

It is known that $\omega(\{G_n\}, z, G)$ (or $\omega(\{G_n\}, z, G, \Gamma)$) attains 1 almost on $\{G_n\}$ in the following sense: $\omega(\{G_n \cap V^c_{1-\varepsilon}\}, z, G)=0$ (or $\omega(\{G_n \cap V^c_{1-\varepsilon}\}, z, G, \Gamma)=0$), where $V^c_{1-\varepsilon}=E[z \in G: \omega(\{G_n\}, z, G)<1-\varepsilon]$ (or $V^c_{1-\varepsilon}=E[z \in G: \omega(\{G_n\}, z, G, \Gamma)<1-\varepsilon]$) for any $\varepsilon>0$. Conversely we shall prove. Let G_1 and G_2 be two domains (or closed domains), if there exists at least one C_1[15]-function $U(z)$ such that $U(z)=0$ on G_1, $U(z)=1$ on G_2 and $D(U(z))<\infty$, we say that G_1 and G_2 are *Dirichlet-disjoint*.

Lemma 1. 1) *If there exist numbers i_0 and n_0 such that CG_{n_0} and $G_{n_0+i_0}$ are Dirichlet-disjoint,*

$$\omega(CG_{n_0} \cap V_{1-\varepsilon}, z, G-G_{n_0+i_0}) \Rightarrow 0 \text{ as } \varepsilon \to 0, \text{ and}$$
$$\omega(CG_{n_0} \cap V_{1-\varepsilon}, z, G) \Rightarrow 0 \text{ as } \varepsilon \to 0,$$

where $V_{1-\varepsilon}=E[z \in G: \omega(\{G_n\}, z, G)>1-\varepsilon]$.

This fact means that $\omega(\{G_n\}, z, G)<1$ almost everywhere outside of G_n.

1'). *If there exist numbers i_0 and n_0 such that CG_{n_0} and $G_{n_0+i_0}$ are Dirichlet-disjoint,*

$$\omega(CG_{n_0} \cap V_{1-\varepsilon}, z, G, \Gamma) \Rightarrow 0 \text{ as } \varepsilon \to 0,$$

where $V_{1-\varepsilon}=E[z \in G: \omega(\{G_n\}, z, G, \Gamma)>1-\varepsilon]$.

2). *Under the condition of* 1)

$$\int_{\partial V_{1-\varepsilon} \cap CG_{n_0}} \frac{\partial}{\partial n} \omega(\{G_n\}, z, G) \downarrow 0 \text{ as } \varepsilon \to 0.$$

2'). *Under the condition of* 1')

$$\int_{\partial V_{1-\varepsilon} \cap CG_{n_0}} \frac{\partial}{\partial n} \omega(\{G_n\}, z, G, \Gamma) \downarrow 0 \text{ as } \varepsilon \to 0.$$

Proof. Since $\omega(V_{1-\varepsilon}, z, G)$ has M.D.I. over $G-V_{1-\varepsilon}$, $D(\omega(V_{1-\varepsilon}, z, G)) \leqq D\left(\frac{1}{1-\varepsilon}\omega(\{G_n\}, z, G)\right)=M'<\infty$, because $\omega(\{G_n\}, z, G) \geqq (1-\varepsilon)$ on $V_{1-\varepsilon}$ and $=0$ on ∂G. Now CG_{n_0} and $G_{n_0+i_0}$ are Dirichlet-disjoint, whence there exists a C_1-function $U(z)$ such that $U(z)=0$ on $CG_{n_0+i_0}$, $U(z)=1$ on G_{n_0} and $D(U(z))<M''<\infty$. Put $S(z)=\min(\omega(V_{1-\varepsilon}, z, G), U(z))$. Then $S(z)=0$ on $\partial G+\partial G_{n_0+i_0}$, $S(z) \geqq 1$ on $V_{1-\varepsilon} \cap CG_{n_0}$ and $D(S(z)) \leqq M'+M''<\infty$. Hence $D(\omega(V_{1-\varepsilon} \cap CG_{n_0}, z, G-G_{n_0+i_0})) \leqq D(S(z))<\infty$ and $\lim_{\varepsilon \to 0} \omega(V_{1-\varepsilon} \cap CG_{n_0}, z, G-G_{n_0+i_0})$ exists. By

15) If Uz has partial derivatives almost everywhere, we call $U(z)$ a C_1-function.

$D(\omega(V_{1-\varepsilon}\cap CG_{n_0}, z, G))\leqq D(\omega(V_{1-\varepsilon}\cap CG_{n_0}, z, G-G_{n_0+i_0})$ $\lim_{\varepsilon\to 0}\omega(V_{1-\varepsilon}\cap CG_{n_0}, z, G)$ exists. Assume $\omega(z)=\lim_{\varepsilon\to 0}\omega(V_{1-\varepsilon}\cap CG_{n_0}, z, G-G_{n_0+i_0})>0$. Let $C_r=E[z:\omega(z)=r]$. Then

$$\int_{C_r}\frac{\partial}{\partial n}\omega(z)\,ds=D(\omega(z))\quad\text{for almost } r,$$

such C_r is called *a regular niveau curve*[16].

Now $\omega(\{G_n\}, z, G)$ is a harmonic function in $G-G_{n_0+i_0}$ having M. D. I. over $G-G_{n_0+i_0}$ among all functions with the same value on $\partial G+\partial G_{n_0+i_0}$[17], whence

$$A_m(z)\Rightarrow\omega(\{G_n\}, z, G)\text{ as } m\to\infty,$$

where $A_m(z)$ is a harmonic function in $R_m\cap(G-G_{n_0+i_0})$ such that $A(z)=\omega(\{G_n\}, z, G)$ on $(\partial G+\partial G_{n_0+i_0})\cap R_m$ and $\frac{\partial}{\partial n}A_m(z)=0$ on $(G-G_{n_0+i_0})\cap R_m$. Also $\omega(z)$ has M. D. I. over $G-G_{n_0+i_0}-V_{1-\varepsilon}$, whence $\omega_m(z)\Rightarrow\omega(z)$, where $\omega_m(z)$ is a harmonic function in $R_m\cap(G-G_{n_0+i_0}-\tilde{V}_{1-\varepsilon})$ such that $\omega_m(z)=\omega(z)$ on $\partial\tilde{V}_{1-\varepsilon}$, $\frac{\partial\omega_m}{\partial n}(z)=0$ on $\partial R_m\cap(G-G_{n_0+i_0}-\tilde{V}_{1-\varepsilon})$, where $\tilde{V}_{1-\varepsilon}=E[z\in G-G_{n_0+i_0}:\omega(z)>1-\varepsilon]$. Then by Green's formula

$$\int_{C_{1-\varepsilon}\cap R_m}A_m(z)\frac{\partial}{\partial n}\omega_m(z)ds=\int_{C_\delta\cap R_m}A_m(z)\frac{\partial\omega_m}{\partial n}(z)\,ds,$$

where $C_{1-\varepsilon}$ and C_δ are regular niveau curves of $\omega(z)$. Then by letting $m\to\infty$

$$\int_{C_{1-\varepsilon}}\omega(\{G_n\}, z, G)\frac{\partial}{\partial n}\omega(z)\,ds=\int_{C_\delta}\omega(\{G_n\}, z, G)\frac{\partial}{\partial n}\omega(z)\,ds.$$

By $\omega(z)\leqq\omega(\{G_n\}, z, G)$ we have

$$(1-\varepsilon)D(\omega(z))=\int_{C_{1-\varepsilon}}\omega(z)\frac{\partial\omega}{\partial n}(z)\,ds\leqq\int_{C_{1-\varepsilon}}\omega(\{G_n\}, z, G)\frac{\partial}{\partial n}\omega(z)\,ds$$
$$=\int_{C_\delta}\omega(\{G_n\}, z, G)\frac{\partial}{\partial n}\omega(z)\,ds. \tag{11}$$

On the other hand, $C_\delta\subset G$, $\omega(\{G_n\}, z, G)<1$ in G and there exists a number $\varepsilon_0>0$ such that

16) See theorem 2 of P.
17) See See 15)

$$\int_{C_\delta} \omega(\{G_n\}, z, G)\frac{\partial}{\partial n}\omega(z)\,ds < (1-\varepsilon_0)\int_{C_{1-\varepsilon}} \frac{\partial}{\partial n}\omega(z)\,ds = (1-\varepsilon_0)D(\omega(z)).$$

Take $\varepsilon < \frac{\varepsilon_0}{2}$. Then (11) will be a contradiction. Hence

$$\omega(z) = \lim_{\varepsilon\to 0} \omega(V_{1-\varepsilon}\cap CG_{n_0}, z, G-G_{n_0+i_0}) = 0.$$

Also by the Dirichlet principle we have $\lim\limits_{\varepsilon\to 0} \omega(V_{1-\varepsilon}\cap CG, z, G)=0$.

Proof of 2). Assume $\overline{\lim\limits_{r\to 1}} \int\limits_{CG_{n_0}\cap\partial V_r} \frac{\partial}{\partial n}\omega(\{G_n\}, z, G)\,ds > \delta > 0$. Then by $r\,\omega(V_r\cap CG_{n_0}, z, G-G_{n_0+i_0}) \leqq \omega(\{G_n\}, z, G)$ and $r\,\omega(V_r\cap CG_{n_0}, z, G-G_{n_0+i_0}) = \omega(\{G_n\}, z, G) = r$ on ∂V_r and we have

$$r\delta \leqq r\int\limits_{C_r\cap CG_{n_0}} \frac{\partial}{\partial n}\omega(\{G_n\}, z, G)\,ds \leqq r\int\limits_{C_r\cap CG_{n_0}} \frac{\partial}{\partial n}\omega(V_r\cap CG_{n_0}, z, G-G_{n_0+i_0})\,ds \leqq$$

$$\int\limits_{\partial(V_r\cap CG_{n_0})} r\frac{\partial}{\partial n}\omega(V_r\cap CG_{n_0}, z, G)\,ds \leqq r\, D(\omega(V_r\cap CG_{n_0}, z, G-G_{n_0+i_0}).$$

Let $r\to 1$. Then by $r\,\omega(V_r\cap CG_{n_0}, z, G-G_{n_0+i_0}) \Rightarrow \omega(z)$ we have $D(\omega(z))>\delta$ and $\omega(z)>0$. This is a contradiction. Hence

$$\lim_{r\to 1}\int\limits_{CG_{n_0}\cap\partial V_r} \frac{\partial}{\partial n}\omega(\{G_n\}, z, G)\,ds = 0.$$

The other part can be proved similarly.

Angular domain $\Delta(p)$ of $p\in B_1^N$. Let $\Delta(p)$ be a domain. If

$$\lim_{n\to\infty} {}_{\Delta(p)\cap v_n(p)}N(z, p) > 0,$$

we call $\Delta(p)$ an angular domain with vertex at p. Then

Theorem 6. 1). *Let G be a domain. If* $\overline{\lim\limits_{M\to M^*}}\int\limits_{G\cap C_M}\frac{\partial}{\partial n}N(z,p)\,ds>0$, *G is an angular domain, where* $C_M = E[z\in R : N(z,p) = M]$: $M<M^* = \sup\limits_{z\in R} N(z,p)$.

2). *If* $G\overset{N}{\ni} p$, *G is an angular domain.*

3). *Let $\Delta(p)$ be an angular domain. If* $G\overset{N}{\ni} p$, *then* $G\cap\Delta(p)$ *is also an angular domain.*

Proof of 1. Suppose $\overline{\lim\limits_{M\to M^*}}\int\limits_{C_M\cap G}\frac{\partial}{\partial n}N(z,p)\,ds>\delta>0$. Then there exists a sequence $M_1, M_2, \cdots$ and $M_i\uparrow M^*$ such that $\int\limits_{C_{M_i}\cap G}\frac{\partial}{\partial n}N(z,p)\,ds>\frac{2\delta}{3}$ for $i\geqq 1$.

Now for any given $v_n(p)$ $\lim_{M\to M^*}\int_{\partial V_M \cap Cv_n(p)} \frac{\partial}{\partial n} N(z,p)\,ds=0$ by Theorem 3.) 3).
Hence by $(\partial V_M(p)\cap v_n(p)\cap G)+(\partial V_M(p)\cap Cv_n(p)\cap G)=\partial V_M(p)\cap G$, there exists a number $i_0(n)$ such that $\int_{\partial V_{M_i}(p)\cap v_n(p)\cap G} \frac{\partial}{\partial n} N(z,p)\,ds \geqq \frac{\delta}{2}$ for $i>i_0(n)$. Now $M_i\ \omega(V_{M_i}(p)\cap v_n(p)\cap G,\ z)\leqq {}_{V_{M_i}(p)}N(z,p)\leqq N(z,p)$ and $M_i\ \omega(V_{M_i}(p)\cap v_n(p)\cap G, z)=M_i=N(z,p)$ on $\partial V_{M_i}(p)$, whence $\int_{\partial R_0} \frac{\partial}{\partial n} M_i\ \omega(V_{M_i}(p)\cup v(p)\cap G,z)\,ds\geqq$

$$\int_{\partial V_{M_i}(p)\cap v_n(p)\cap G} \frac{\partial}{\partial n} M_i\ \omega(V_{M_i}(p)\cap v_n(p)\cap G, z)\,ds \geqq \int_{\partial V_{M_i}(p)\cap v_n(p)\cap G} \frac{\partial}{\partial n} N(z,p)\,ds \geqq \frac{\delta}{2}.$$

On the other hand, by $M_i\ \omega(V_{M_i}(p)\cap v_n(p)\cap G, z)\leqq {}_{V_{M_i}(p)\cap v_n(p)\cap G}N(z,p)$ we have $\int_{\partial R_0} \frac{\partial}{\partial n}\, {}_{V_{M_i}(p)\cap v_n(p)\cap G}N(z,p)\,ds\geqq\frac{\delta}{2}$. Let $n\to\infty$. Then $V_M(p)\cap v_n(p)\cap G\to p\cap G$ for any given $M_i<M^*$ by $V_M(p)\supset v_{n'}(p): n'=n'(M)$ and $\lim_{n=8}\ {}_{v_n(p)\cap G\cap V_{M_i}(p)}N(z,p)$ has its mass only at p and $\lim_{n=\infty}\ {}_{v_p(u)\cap G\cap V_{M_i}(p)}N(z,p)\geqq\frac{\frac{\delta}{2}}{2\pi}N(z,p)>0$. Thus G is an angular domain.

Proof of 2). If $G\overset{N}{\ni}p$, $\int_{\partial V_M(p)\cap CG}\frac{\partial}{\partial n} N(z,p)\,ds\downarrow 0$ as $M\uparrow M^*$ by Theorem 3). 3). But $\int_{M\partial V(p)}\frac{\partial}{\partial n}N(z,p)\,ds=2\pi$ for almost M. Hence $\overline{\lim}_{M\to M^*}\int_{\partial V_M(p)\cap G}\frac{\partial}{\partial n}N(z,p)ds$ $=2\pi$ and by 1) G is an angular domain.

Proo of 3). By $v_n(p)\cap\varDelta(p)=(v_n(p)\cap\varDelta(p)\cap CG)+(v_n(p)\cap\varDelta(p)\cap G)$ and by $\lim_n\ {}_{v_n(p)\cap CG}N(z,p)={}_{p\cap CG}N(z,p)=0$ by Theorem 3). 2) we have $\lim_n\ {}_{v_n(p)\cap\varDelta(p)}N(z,p)\leqq\lim_n\ {}_{v_n(p)\cap\varDelta(p)\cap CG}N(z,p)+\lim_n\ {}_{v_n(p)\cap\varDelta(p)\cap G}N(z,p)$ and $\lim_n\ {}_{v_n(p)\cap\varDelta(p)\cap G}N(z,p)>0$. Hence $G\cap\varDelta(p)$ is also an angular domain.

Let R be a disc: $|z|<1$ and R_0 be $|z|<\frac{1}{2}$. Then $N(z,p)=U(z)+\log\frac{1}{|z-p|}: p=e^{i\theta}$, where $U(z)$ is a bounded harmonic function in $E\left[z: \frac{1}{2}<|z|<2\right]$. Hence it is easily verified that $E\left[z: \left|\arg\frac{z-e^{i\theta}}{e^{i\theta}}\right|<\frac{\pi}{2}\right]=A(e^{i\theta})$

satisfies the condition of Theorem 6). 1) and $A(e^{i\theta})$ is an angular domain in our sense.

3. Capacitary potentials of the boundary determined by a domain G containing an end part $\Delta(p)$ or containing p N-approximately.

Theorem 7). 1). *Let F be a closed set (with respect to N-Martin's topology) on B of positive capacity. Let G be a domain such that $G\supset(\Delta(p)\cap v_n(p))$ for $p\in(B_1^N\cap F)$, where n depends on p and $\Delta(p)$ is an angular domain of p. Then*

$$\lim_n \omega(G\cap F_n, z)>0,$$

where $F_n=E\left[z\in\bar{R}:\ \mathrm{dist}(z,F)\leqq\frac{1}{n}\right]$.

2). *Let G be a domain such that $G\supset G_n(n=1,2,\cdots)$, where $\{G_n\}$ is a decreasing sequence of domains and $G_n\overset{N}{\ni}p$ for any $p\in(B_1^N\cap F)$. Let Γ be a compact arc on G. Then*

$$\omega(\{G_n\}, z, G, \Gamma)>0.$$

Further if CG and G_{n_0} (n_0 is a certain number) are Dirichlet-disjoint, then

$$\omega(\{G_n\}, z)>0 \quad and \quad \omega(\{G_n\}, z, G)>0.$$

Proof of 1). Since $\mathrm{Cap}(B_0^N)=0$, we can suppose that $\mathrm{Cap}(F\cap B_1^N)>0$. By the condition $G\supset(\Delta(p)\cap v_n(p))$ for $n>n(p)$ and $F_n\supset v_n(p): p\in F\cap B_1^N$, we have

$${}_{G\cap F}N(z,p)=\lim_n {}_{G\cap F_n}N(z,p)\geqq\lim_n {}_{G\cap\Delta(p)\cap v_n(p)}N(z,p)>0. \quad (11)$$

Now $N(z,p)$ is continuous for $z\in R$ with respect to p and $\int_{\partial R_0}\frac{\partial}{\partial n}\,{}_{G\cap R_m\cap F_n}N(z,p)\,ds$ is also continuous with respect to p and ${}_{G\cap R_m\cap F_n}N(z,p)\uparrow{}_{G\cap F_n}N(z,p)$ as $m\to\infty$, ${}_{G\cap F_n}N(z,p)\downarrow{}_{G\cap F}N(z,p)$ as $n\to\infty$. Hence the set $A_l=E\left[z\in F\cap B_1^N:\ \int_{\partial R_0}\frac{\partial}{\partial n}\,{}_{G\cap F}N(z,p)\,ds>\frac{1}{l}\right]$ is a Borel set and

$$\sum_{l=1}^{\infty}A_l=F\cap B_1^N.$$

Capacity of Borel sets in B is the supremum of total mass of positive canonical distributions $\{\mu\}$ such that the potential $U(z)$ of μ satisfies the condition that $U(z)\leqq 1$ (this is equivalent to that the infinimum of energy integrals $I(\mu)$

of canonical distributions of unit masses $=1/\mathrm{Cap}$). Hence $\mathrm{Cap}(A_l)\uparrow\mathrm{Cap}(F)$ as $l\to\infty$, because μ is Borel measurable. Therefore we can find a set $A_l\subset F$ such that $\mathrm{Cap}(A_l)>\frac{1}{2}\mathrm{Cap}(F)$. Since $\omega(F_n,z)\downarrow\omega(F,z)$, for any given number $\left(\frac{1}{2l}>\right)\varepsilon>0$ there exists a number $n(\varepsilon)$ such that

$$\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F,z)ds\geqq\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F_n,z)ds-\varepsilon\left(\text{clearly}\geqq\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F_n\cap R_m,z)ds-\varepsilon\right). \tag{12}$$

Since ${}_{F_n\cap R_m}N(z,p)\uparrow{}_{F_n}N(z,p)\geqq{}_FN(z,p)$ as $m\to\infty$,

$$\sum_m A_{n,m}=A_l\,, \tag{13}$$

where $A_{n,m}=E\left[p\in A_l\cap B_1^N:\int_{\partial R_0}\frac{\partial}{\partial n}{}_{F_n\cap R_m}N(z,p)\,ds>\frac{1}{2l}\right]$.

Hence there exists a number $m(n(\varepsilon))$ for any number $n(\varepsilon)$ such that

$$\mathrm{Cap}(A_{n,m})\geqq\frac{1}{2}\mathrm{Cap}(A_l)\geqq\frac{1}{4}\mathrm{Cap}(F)\ \text{ for } m>m(n(\varepsilon))\,.$$

By the definitions of $\mathrm{Cap}(A_{n,m})$, there exists a positive canonical mass distribution μ on $A_{n,m}$ such that $\int d\mu=\mathfrak{M}>0$ and its potential $U(z)$ satisfies $U(z)\leqq1$ in R. Since $N(z,p)$ is continuous with respect to p, we can find a sequence $\{U_i(z)\}$ of the form $U_i(z)=\sum_{j=1}^{i}c_j\,N(z,p_j):p_j\in A_{n,m}:c_j>0$, $\sum c_j=\mathfrak{M}<\frac{1}{4}\mathrm{Cap}(F)$ such that $U_i(z)\to U(z)$ for any compact set R_m of R. Hence we can suppose $U_i(z)<1+\varepsilon$ on $R_{m(n(\varepsilon))}$, whence $(1+\varepsilon)\omega(F_n\cap R_{m(n(\varepsilon))},z)\geqq U_i(z)$ on $R_{m(n(\varepsilon))}$. Hence by (12) and (13) we have by $\omega(F_n\cap R,z)\geqq\omega(F_n\cap R_m,z)$

$$\int_{\partial R_0}(1+\varepsilon)\frac{\partial}{\partial n}\omega(F,z)ds+\varepsilon\geqq\int_{\partial R_0}(1+\varepsilon)\frac{\partial}{\partial n}\omega(F_n\cap R_{m(n)\varepsilon)}),z)ds\geqq$$

$$\int_{\partial R_0}(1+\varepsilon)U(z)\frac{\partial}{\partial n}{}_{F_n\cap R_{m(n(\varepsilon))}}U_i(z)ds=\int_{\partial R_0}\frac{\partial}{\partial n}\sum c_j\;N(z,p_j)ds\geqq\frac{\mathfrak{M}}{2l}>0\,.$$

Thus by letting $\varepsilon\to0$ $\qquad\omega(F,z)>0$.

Proof of 2). By $G\overset{N}{\supset}G_n\ni p(p\in F\cap B_1^N)$ we have by Theorem 3). 2) ${}_{G_n}N(z,p)=N(z,p)$ and by Theorem 4). 3) ${}^*_{G_n}N'(z,p)=N'(z,p)>0$ for any n and

$\int_{\Gamma}\frac{\partial}{\partial n}N'(z,p)\,ds>0$, where $N'(z,p)=N(z,p)-{}_{CG}N(z,p)$. Let $A_{l,n}=E\Big[p\in F\cap B_1^N$: $\int_{\Gamma}\frac{\partial}{\partial n}\,{}_{G_n}^{\ *}N'(z,p)\,ds>\frac{1}{l}\Big]$ and $A_l=E\Big[p\in F\cap B_1^N:\int_{\Gamma}\frac{\partial}{\partial n}N'(z,p)\,ds>\frac{1}{l}\Big]$. Then $A_{l,n}$ and A_l are measurable, $A_{l,n}=A_l$ and $\sum_l A_{l,n}=F\cap B_1^N$. Hence there exists a set $A_{l,n}$ such that $\mathrm{Cap}(A_{l,n})\geqq\frac{1}{2}\mathrm{Cap}(F)$ for any n. Now ${}_D^{\ *}N'(z,p)=\lim_{L=\infty}{}^*N'^L(z,p)=\lim_{m=\infty}{}_{D\cap R_m}^{\ *}N'(z,p)$[18] for any domain D in G can be proved as superharmonic functions in $R-R_0$, where ${}_{D\cap R_m}^{\ *}N'(z,p)$ is a function in G such that ${}^*N'(z,p)=N(z,p)$ on $((D\cap R_m)\cap G)+\partial G$ and ${}_{D\cap R_m}^{\ *}N'(z,p)$ has M. D. I. over $G-(D\cap R_m)$. Hence $\sum_m^{\infty} A_{l,n,m}=A_{l,n}=A_l$, where

$$A_{l,m,n}=E\Big[p\in A_l:\int_{\Gamma}\frac{\partial}{\partial n}\,{}_{G_n\cap R_m}^{\ \ *}N'(z,p)\,ds>\frac{1}{2l}\Big].$$

There exists a number $m(n)$ such that $\mathrm{Cap}(A_{l,n,m})>\frac{1}{4}\mathrm{Cap}(F)=\mathfrak{M}$ for $m>m(n)$. By the definition of $\mathrm{Cap}(A_{l,m,n})$, there exists a positive canonical mass distribution μ on $A_{l,m,n}$ such that $\frac{\mathfrak{M}}{2}=\int d\mu$ and the potential $U(z)$ of $\mu<1$ in $R-R_0$. Since $G\cap R_m$ is compact, $U(z)$ can be approximated on $G\cap R_m$ by a sequence $U_i(z)=\sum_{j=1}^{i}c_j\,N(z,p_j):p_j\in A_{l,m,n}$, $\sum c_j=\frac{\mathfrak{M}}{2}$. Hence for any given number $\varepsilon>0$ $U_i(z)<1+\varepsilon$ on $G\cap R_m$ for $i>i(\varepsilon)$. Whence

$$(1+\varepsilon)\,\omega(G_n\cap R_m,z,G,\Gamma)\geqq U_i(z)=\sum_{j=1}^{i}c_j\,N(z,p_j)\ \text{ on } G\cap R_m\,.$$

On the other hand, clearly $N(z,p)\geqq N'(z,p)$ in G and $\sum_j^{i}c_j\,N'(z,p_j)=0\leqq\omega(G_n\cap R_m,z,G,\Gamma)$ on ∂G. Whence by the maximum principle

$$(1+\varepsilon)\,\omega(G_n\cap R_m,z,G,\Gamma)\geqq\sum_j^{i}c_j\,N'(z,p_j)\ \text{ and}$$

$$\int_{\Gamma}(1+\varepsilon)\frac{\partial}{\partial n}\omega(G_n\cap R_m,z,G,\Gamma)\,ds\geqq\int_{\Gamma}\frac{\partial}{\partial n}\Big(\sum c_j\,N'(z,p_j)\Big)ds\geqq\frac{\mathrm{Cap}(F)}{8}.$$

Let $i\to\infty$, then $m\to\infty$, then $n\to\infty$ and then $\varepsilon\to 0$. Then

18) See theorem 3 of P.

$$\int_{\Gamma} \frac{\partial}{\partial n} \omega(\{G_n\}, z, G, \Gamma) ds \geqq \frac{\mathrm{Cap}(F)}{8}. \quad \text{Thus } \omega(\{G_n\}, z, G, \Gamma) > 0.$$

By Theorem 5). 2) we have at once $\omega(\{G_n\}, z, G) > 0$ from $\omega(\{G_n\}, z, G, \Gamma) > 0$. Theorem 7) is an extension of the following

Theorem[19]. *Let F be a closed set of positive logarithmic capacity in the z-plane. Let $S(p)$ be a sector with vertex at p and with aperture θ: $2\pi > \theta > 0$. Let G be a domain (its connectivity may be ∞). Suppose G contain an endpart of $S(p)$ for any $p \in F$. Then $\omega(F, z, G, \Gamma) > 0$, where Γ is a compact arc on ∂G.*

4. Correspondence of sets on boundaries of Riemann surfaces by analytic functions.

Let R be a covering surface over $\underline{R}$. We suppose R is a surface with positive boundary and N-Martin's topology is defined on $R + B = \bar{R}$ ($N(z, p)$ is defined in $R - R_0$) and suppose that $\underline{R} + \underline{B}$ ($\underline{B}$ is the boundary of $\underline{R}$ and $\underline{R}$ is of with null or positive boundary) is compact with respect to the topology defined by a metric in $\underline{R}$. Let $\mathfrak{F}$ be a closed set in $\underline{R} + \underline{B}$ and let $\underline{D}$ be a compact disc in $\underline{R}$ such that dist $(\mathfrak{F}, \underline{D}) > 0$. We define the capacity $\int_{\partial \underline{D}} \frac{\partial}{\partial n} \omega(\mathfrak{F}, w, \underline{R} - \underline{D}) ds$ of $\mathfrak{F}$, where $\omega(\mathfrak{F}, w, \underline{R} - \underline{D})$ is C. P. of $\mathfrak{F}$. Clearly if $\mathfrak{F}$ is contained in a local parameter disc $C(p)$ of $p \in \underline{R}$, *$\mathfrak{F}$ is of capacity zero if and only if $\mathfrak{F}$ is of logarithmic capacity zero.*

Put $\mathfrak{F}_n = E\left[z \in \bar{R} : \mathrm{dist}(w, \mathfrak{F}) \leqq \frac{1}{n}\right]$. If for any given n, there exists a number $i(n)$ such that $\mathfrak{F}_n$ and $C\mathfrak{F}_{n+i}$ are Dirichlet-disjoint, we call $\mathfrak{F}$ a *D-regularly closed* set. Of course, if $\mathfrak{F}$ is a compact set in $\underline{R}$, we can construct a domains D_1 and D_2 such that $D_2 \subset D_1$, $D_2 \supset \mathfrak{F}_{n+i}$, $D \subset \mathfrak{F}_n$, $\mathrm{dist}(\partial D_1, \partial D_2) > 0$, $\partial D_i (i = 1, 2)$ is composed of a finite number of analytic curves and there exists a harmonic function in $D_1 - D_2$ such that $U(w) = 1$ on ∂D_1, $U(w) = 0$ on ∂D_2 and $D(U(w)) < \infty$. Put $U(w) = 1$ in $\underline{R} - D_1$, $U(w) = 0$ in D_2. Thus $\mathfrak{F}$ is D-regularly closed.

Cluster sets at $p \in B_1^N$. Let $w = f(z)$ be an analytic function: $z \in R$, $w \in \underline{R}$. Put $M(f(p)) = \bigcap_{\tau} \overline{f(G_\tau)}$, where G_τ runs over all domains $G_\tau \overset{N}{\ni} p$ and

$$f_\Delta(p) = \bigcap_n \overline{f(\Delta(p) \cap v_n(p))},$$

19) Z. KURAMOCHI: Correspondence of of sets on the boundaries of Riemann surfaces. Proc. Japan Acad., 36 (1960).

Then we have the following

Lemma. *Let $\mathfrak{F}$ be a closed set on $\underline{R}+\underline{B}$. Then*

1). *Suppose ${}_{\varDelta}f(p)\in\mathfrak{F}$. Then $f^{-1}(\mathfrak{F}_n)$ contains an endpart of $\varDelta(p)$.*

2). *Suppose $M(f(p))\in\mathfrak{F}$. Then $f^{-1}(\mathfrak{F}_n)\overset{N}{\ni}p$ for any n.*

In fact, clearly ${}_{\varDelta}f(p)$ is closed and non void. For any given number n there exists a number m such that $f(\varDelta(p)\cap v_m(p))\subset\mathfrak{F}_n$. Hence $f^{-1}(\mathfrak{F}_n)\supset(\varDelta(p)\cap v_n(p))$. Next $M(f(p))$ is closed. Since $\underline{R}+\underline{B}$ is a metric space and separable, we can find a sequence $G_1, G_2, \cdots$ from $\{G_\tau\}: G_\tau\overset{N}{\ni}p$ such that $\bigcap_{i=1}^{\infty}\overline{f(G_i)}=M(f(p))$. Hence for any given n there exists a number m such that $\bigcap_{j=1}^{m}\overline{f(G_j)}\supset\mathfrak{F}_n$. Put $G'_m=G_1\cap G_2\cdots\cap G_m$. Then $G'_m\overset{N}{\ni}p$. Whence by $f^{-1}(\mathfrak{F}_n)\supset G'_m$ we have $f^{-1}(\mathfrak{F}_n)\overset{N}{\ni}p$ for any n.

Let $n(w)$ be the number of points lying over $w\in\underline{R}$. Put $\bar{n}(w)=\lim_{n=\infty}(\sup_{w\in v_n(w)} n(w)): w\in\underline{R}+\underline{B}$. If $\bar{n}(w)<\infty$ for any point p of $\mathfrak{F}$, we can find a number M such that $n(w)<M<\infty$ for $w\in(v_n(w)\cap\underline{R})$. We shall prove the following

Theorem 8). *Let $w=f(z)$ be an analytic function from R into $\underline{R}$. Then*

1). *Let $\mathfrak{F}\subset(\underline{R}+\underline{B})$ be a D-regularly closed set of capacity zero and suppose $\bar{n}(w)<\infty$ for $w\in\mathfrak{F}$. Then the set $A=E[p\in B_1^N: {}_{\varDelta}f(p)\subset\mathfrak{F}]$ is of capacity zero.*

2). *Let $\mathfrak{F}\subset(\underline{R}+\underline{B})$ be a (not necessarily D-regularly) closed set of capacity zero and suppose $\bar{n}(w)<\infty$ for $w\in\mathfrak{F}$. Then the set $A=E[p\in B_1^N: M(f(p))\in\mathfrak{F}]$ is of capacity zero.*

Proof of 1). By the condition of 1) we can find a number n' such that $n(w)\leqq M<\infty$ in $\mathfrak{F}_{n'}$ and a number $n(>n')$ such that $\underline{R}-f^{-1}(\mathfrak{F}_n)$ has a compact disc $\underline{D}$. In fact, let G be a sufficiently small disc. Then f(interior of G) is an open set and $f(G)\cap\mathfrak{F}_n\neq0$ for $n\geqq n'$ (n' is a certain number). Hence there exists at least one point w_0 such that $w_0 \in f(G)$ and $w_0\in\mathfrak{F}_n$. $f^{-1}(\mathfrak{F}_n\cap R)$ is closed and $f^{-1}(\mathfrak{F}_n)\not\ni p_0(\in f^{-1}(w_0))$. Hence there exists a neighbourhood $v(p_0)$ such that $v(p_0)\cap f^{-1}(\mathfrak{F}_n)=0$ $(n>n')$. Take $v(p_0)$ as $\underline{D}$. Then $\underline{D}$ is the required domain.

Aussume A is of positive capacity. Then there exists a positive canonical mass distribution μ whose $I(\mu)<\infty$. In this case we can find also a closed set F in A of positive capacity. Now a closed set F is of positive capacity if and only if F is of function theoretic capacity positive[20], whence $\omega(F,z)$ $(=\omega(F,z,R-R_0))>0$. Let D be a compact disc in R such that $f(D)\subset\underline{D}$.

20) See p. 73 of P.

Then $\omega(F, z)>0$ if and only if $\omega(F, z, R-D)>0$. Similarly by $\mathrm{Cap}(\mathfrak{F})>0$ we have $\omega(\mathfrak{F}, w, R-D)>0$. Let n'' be a number such that $\underline{D}\subset\underline{R}-\mathfrak{F}_{n''}$ $(n''>n')$ and fix n'' at present.

Since ${}_{\Delta}f(p)\subset\mathfrak{F}$, by lemma $f^{-1}(\mathfrak{F}_m)$ contains an endpart of $\Delta(p)$ for any p and m. Then by Theorem 7). 1) by putting $G_m=f^{-1}(\mathfrak{F}_m)$

$$\lim_{m=\infty}\omega(f^{-1}(\mathfrak{F}_m), z, R-D)\geqq\lim_{m=\infty}\omega(F\cap f^{-1}(\mathfrak{F}_m), z, R-D)>0\,. \quad (14)$$

Now $\mathfrak{F}$ is D-regularly closed and there exists a C_1-function $U(w)$ in $\underline{R}$ such that $U(w)=0$ on $\partial\mathfrak{F}_{n''}$, $U(w)=1$ on $\mathfrak{F}_{n''+i_0}$ and $D(U(w))\leqq L<\infty$. Hence $D(\omega(\mathfrak{F}_m, w, \mathfrak{F}_{n''}))\leqq L$ for $m>n''+i_0$ and since $\mathfrak{F}$ is of capacity zero

$$D(\omega(\mathfrak{F}_m, w, \mathfrak{F}_{n''}))\downarrow 0 \quad \text{as } m\to\infty\,. \quad (15)$$

Consider $U(z)=U(f^{-1}(w))$ in $f^{-1}(\mathfrak{F}_{n''})-f^{-1}(\mathfrak{F}_m)$. Then by $n(w)\leqq M$ in $\mathfrak{F}_{n''}$ $D(\omega(f^{-1}(\mathfrak{F}_m), z, f^{-1}(\mathfrak{F}_{n''})))\leqq MD(U(w))\leqq ML<\infty$ and

$$D(\omega(f^{-1}(\mathfrak{F}_m), z, f^{-1}(\mathfrak{F}_{n''})))\leqq MD(\omega(\mathfrak{F}_m, w, \mathfrak{F}_{n''}))\,. \quad (16)$$

Let $m\to\infty$. Then by (15) and (16)

$$D(\omega(f^{-1}(\mathfrak{F}_m), z, f^{-1}(\mathfrak{F}_{n''}))\downarrow 0 \quad \text{as } m\to\infty\,. \quad (17)$$

On the other hand, by the Dirichlet principle and by (14) $D(\omega(f^{-1}(\mathfrak{F}_m)\cap F, f^{-1}(\mathfrak{F}_{n''}))\geqq D(\omega(f^{-1}(\mathfrak{F}_m)\cap F, z, R-D))>0$. This contradicts (17). Hence A is a set of capacity zero.

Proof of 2). As (1) we can find a number n' such that $n'(w)\leqq M<\infty$ in $\mathfrak{F}_{n'}$ and $R-f^{-1}(\mathfrak{F}_{n'})$ contains a compact disc $\underline{D}$. Fix n' at present. Put $G=R\cap f^{-1}(\mathfrak{F}_{n'})$. Then G may consist of an enumerably infinite number of domains (components) $G^1, G^2, \cdots$. Since $M(f(p))\subset\mathfrak{F}$, $G_n=f^{-1}(\mathfrak{F}_n)$ contains any point of $A\cap B_1^N$ by Lemma 4). Let A_i be the set of points contained N-approximately in G^i. Then A_i is a Borel set by Lemma b) and $\sum_i A_i=A$. Hence there exists at least one component G' of G such that $G'\ni p$, $p\in A'$ and $\mathrm{Cap}(A')>0$, where A' is the set of points contained in G'. Hence we can find a closed set F of positive capacity in A'. Since $\mathfrak{F}$ is a set of capacity zero, we have by Theorem 5). 3) $0=\omega(\mathfrak{F}, w, \underline{R}-\underline{D})$ implies

$$D(\omega(\mathfrak{F}_n, w, \mathfrak{F}_{n'}, \Gamma_w))\to 0 \quad \text{as } n\to\infty\,, \quad (18)$$

where Γ_w is a compact arc on $\partial\mathfrak{F}_{n'}$.

Let Γ_z be a compact arc on $\partial G'$ such that $f(\Gamma_z)\subset\Gamma_w$. Since $G_n\overset{N}{\ni}p:p\in(B_1^N\cap F)$, we have by Theorem 7). 2)

$$\omega(\{F\cap G_n\}, z, G', \Gamma_z)=\lim\,\omega(F\cap G_n, z, G', \Gamma_n)>0\,. \quad (19)$$

Now $U(z)=f^{-1}(\omega(\mathfrak{F}_n, w, \mathfrak{F}_{n'}, \Gamma_w))$ is a harmonic function in $G'-(f^{-1}(\mathfrak{F}_n)\cap G')$ such that $U(z)=1$ on $f^{-1}(\mathfrak{F}_n)=G_n\supset(G_n\cap F)$ and $U(z)=0$ on $\Gamma_z\subset f^{-1}(\Gamma_w)$ and since $\omega(G_n, z, G', \Gamma_z)$ has M. D. I.,

$$D(\omega(G_n, z, G', \Gamma_z))\leqq MD(\omega(\mathfrak{F}_n, w, \mathfrak{F}_{n'}, \Gamma_w))\downarrow 0 \quad \text{as } n\to\infty . \qquad (20)$$

But $\omega(G_n, z, G', \Gamma_z)\geqq\omega(G_n\cap F, z, G', \Gamma_z)$. This contradicts (20). Thus A is a set of capacity zero.

Similarly as (1) we can prove the following

Theorem 7). 3). Suppose $n(w)\leqq M<\infty$ outside of a compact set $\underline{D}$ of $\underline{R}$. Then 1) is valid even if $\mathfrak{F}$ is not D-regularly closed.

In fact, in the proof of 1) take $\underline{R}-\underline{D}'$ instead of $\mathfrak{F}_{n'}$, where $\underline{D}'$ is a compact domain such that $\underline{D}'\supset\underline{D}$, $\underline{D}'\cap\mathfrak{F}_{n_0}=0$ (n_0 is a certain number such that $n_0>1$) and dist $(\partial\underline{D}, \partial\underline{D}')>0$. Then $D(\omega(f^{-1}(\mathfrak{F}_n), z, R-f^{-1}(D')))\leqq MD(\omega(\mathfrak{F}_n, w, \underline{R}-\underline{D}'))\leqq L<\infty$ for $n>n_{n_0}$ and $\omega(\{G_n\}, z, R-f^{-1}(D'))$ can be defined, where $G_n=f^{-1}(\mathfrak{F}_n)$. We used the D-regularity of $\mathfrak{F}_n$ only to define $\omega(\{G_n\}, z, f^{-1}(\mathfrak{F}_{n''}))$ in the proof of 1). Hence the above theorem can be proved similarly as 1).

Remark. The assumption of 1) on $\mathfrak{F}$ is stronger than that of 2) but the assumption of 1) on the cluster set at p is weaker than that of 2). The assumption of 3) on the cluster set is that of 1) but the assumption of 3) on $n(w)$ is strongest.

We shall consider an extension of Beurling's theorem. Let $\mathfrak{F}$ be a compact set in $\underline{R}$ and suppose $\mathfrak{F}$ is contained in a local parameter disc Ω : Ω can be thought equivalent to $|z|<1$ and suppose $\mathfrak{F}$ is contained in $E\left[z : |z|<\frac{1}{2}\right]$. In this case we suppose that the distance between two points in Ω means the enclidean distance. Let $A(r)$ be the area of R over $\mathfrak{F}_r=E\left[z\in\Omega : \text{dist } (z, \mathfrak{F})\leqq r : r<\frac{1}{2}\right]$. Then we have the following

Theorem 9. *Let $w=f(z)$ be an analytic function from R (with positive boundary and with N-Martin's topology) into $\underline{R}$. Let $\mathfrak{F}$ be a closed set contained in a local parameter disc $\Omega=E[|z|<1]$ and $\mathfrak{F}$ in $E\left[|z|<\frac{1}{2}\right]$. Suppose*

$$\varlimsup_{r\to 0}\frac{A(r)}{r^2}<\infty .$$

Then $A=E[z\in B_1^N : M(f(p))\subset\mathfrak{F}]$ is a set of capacity zero.

Proof. We can find a sequence $r_1, r_2, \cdots$ such that $\frac{A(r_i)}{r_i^2} < K < \infty$. Put $G_i = f^{-1}(\mathfrak{F}_i)$, where $\mathfrak{F}_i = E[w \in \Omega : \mathrm{dist}(w, \mathfrak{F}) \leqq r_i]$. Then we see for sufficiently large number i $R - f^{-1}(\mathfrak{F}_i) \neq 0$. This implies that $R - f^{-1}(\mathfrak{F}_i)$ contains a compact disc. Hence we can suppose without loss of generality that $R - f^{-1}(\mathfrak{F}_i) \supset R_0(\{R_n\}$ $(n = 0, 1, 2, \cdots)$ is an exhaustion of $R)$ for $i \geqq 1$ and $\frac{A(r_i)}{r_i^2} < K$ for $i \geqq 1$. Now $G_1 = f^{-1}(\mathfrak{F}_1)$ consists of enumerably infinite number of components $G^1, G^2, \cdots : \sum_j G^j = G_1$. By Lemma b) $A^j = E[p \in A : p \overset{N}{\in} G^j]$ and $E[p \in A : p \overset{N}{\in} G]$ are measurable. Since $G_1 \overset{N}{\ni} p$ implies that there exists only one component G^j of G_1 containing p N-approximately. Hence

$$\sum_j A^j = A .$$

Assume A is a set of positive capacity. Then there exists at least one G^j such that $\mathrm{Cap}(A^j) > 0$. We fix G^j at present and denote it by G. We can find a closed set F of positive capacity in A^j. Put $G_i = G \cap f^{-1}(\mathfrak{F}_i)$. Then also $G_i \overset{N}{\ni} p$ for any p of A^j and G_i is clearly non compact. Hence by Theorem 7). 2) $\omega(F \cap \{G_i\}, z, G, \Gamma) > 0$ and by Theorem 5). 3) $\omega(\{G_i\}, z, R - R_0) > 0$, where Γ is a compact arc on ∂G. Put $\omega(\{G_i\}, z, R - R_0) = U(z) > 0$. Then $U(z) = 0$ on ∂R_0, $\int_{\partial R_0} \frac{\partial}{\partial n} U(z) ds = D(U(z)) = \int_{\partial V_M} \frac{\partial}{\partial n} U(z) ds$ for almost M: $0 < M < 1$, where $V_M = E[z \in R : U(z) > M]$ and such ∂V_M is called a regular niveau curve.

We shall show that G_i and $CG_{i+j} : G_{i+j} = f^{-1}(\mathfrak{F}_{i+j}) \cap G$ are Dirichlet-disjoint. In fact, $\mathrm{dist}(\partial \mathfrak{F}_i, \partial \mathfrak{F}_{i+j}) > 0$ and $\partial \mathfrak{F}_i$ and $\partial \mathfrak{F}_{i+j}$ are compact. We can construct compact domains Ω_1 and Ω_2 such that $\mathfrak{F}_i \supset \Omega_1 \supset \Omega_2 \supset \mathfrak{F}_{i+j}$, $\mathrm{dist}(\partial\Omega_1, \partial\Omega_2) > 0$ and $\partial\Omega_i$ $(i = 1, 2)$ is composed of a finite number of components of which each one is consists of only one analytic curve. Hence we can define a C_1-function $V(w)$ in $\underline{R}$ such that $V(w)$ is harmonic in $\Omega_1 - \Omega_2$, $V(w) = 1$ in Ω_1, $V(w) = 0$ on $C\Omega_2$ and $\left|\frac{\partial}{\partial u} V(w)\right| < L$, $\left|\frac{\partial}{\partial v} V(w)\right| < L$: $L < \infty$, where $w = u + iv$. Put $V(z) = V(f^{-1}(w))$. Then $D(V(z)) \leqq KL^2 r_i^2$. Thus G_i and G_{i+j} are Dirichlet-disjoint. Hence by Lemma 1). 2)

$$\int_{\partial V_M - G_i} \frac{\partial}{\partial n} U(z) ds \downarrow 0 \quad \text{as } M \uparrow 1 \text{ for any given } G_i . \tag{21}$$

Put $\alpha = D(U(z))$. Then $\int_{\partial V_M} \frac{\partial}{\partial n} U(z) ds = \alpha$ for a regular niveau curve ∂V_M. Let

$U_n^M(z)$ be a harmonic function in $R_n - R_0 - V_M$ such that $U_n^M(z)=0$ on ∂R_0, $U_n^M(z)=M$ on $\partial V_M - R_n$, $\frac{\partial}{\partial n} U_n^M(z) = 0$ on $\partial R_n - V_M$ and put $\alpha_n^M = D(U_n^M(z)) = \int_{\partial R_0} \frac{\partial}{\partial n} U_n^M(z)\,ds \left(= \int_{\partial V_M \cap R_n} \frac{\partial}{\partial n} U_n^M(z)\,ds\right.$, since $R_n - R_0 - V_M$ is compact$\left.\right)$. Then

$$U_n^M(z) \Rightarrow U(z) \quad \text{and} \quad \alpha_n^M \to \alpha, \text{ as } M \to 1 .$$

Suppose ∂V_M is regular. Then by $\alpha_n^M \to \alpha$ and by $\varliminf_n \int_{\partial V_M \cap G_i} \frac{\partial}{\partial n} U_n^M(z)\,ds \geqq \int_{\partial V_M \cap G_i} \frac{\partial}{\partial n} U(z)\,ds$ we can find a numbers M and $n_1(\varepsilon, i, M)$ for any given numbers ε and i such that

$$\int_{(\partial V_M \cap R_n) \cap G_i} \frac{\partial}{\partial n} U_n^M(z)\,ds \geqq \alpha - 2\varepsilon \quad \text{for } n > n_1(\varepsilon, i, M) . \tag{22}$$

Put
$$D(\mathfrak{F}_k) = \iint_{f^{-1}(\mathfrak{F}_k)} |f'(z)| \left\{\left(\frac{\partial}{\partial x} U(z)\right)^2 + \left(\frac{\partial}{\partial y} U(z)\right)^2\right\}^{\frac{1}{2}} dxdy \text{ and}$$

$$D(\mathfrak{F}_k, CV_M \cap R_n) = \iint_{f^{-1}(\mathfrak{F}_k) \cap CV_M \cap R_n} |f'(z)| \left\{\left(\frac{\partial}{\partial x} U_n^M(z)\right)^2 + \left(\frac{\partial}{\partial y} U_n^M(z)\right)^2\right\}^{\frac{1}{2}} dxdy.$$

Clearly $\varliminf_{r \to 0} \frac{A(r)}{r^2} < \infty$ implies that area of $f^{-1}(\mathfrak{F}_k) \to 0$ as $k \to \infty$. Since $f(z)$ is analytic in R and for any R_n area of $f^{-1}(\mathfrak{F}_k)$ in $R_n \to 0$ as $k \to \infty$, $D(U(z))$ over $R_n \cap f^{-1}(\mathfrak{F}_k) \to 0$ as $k \to \infty$. Further $D(U(z)) < \infty$ and $\underset{R-R_n}{D}(U(z)) \to 0$ as $n \to \infty$ and $\underset{f^{-1}(\mathfrak{F}_k)}{D}(U(z)) \leqq \underset{f^{-1}(\mathfrak{F}_k) \cap R_n}{D}(U(z)) + \underset{R-R_n}{D}(U(z))$. Hence for any given number ε we can find a number $k(\varepsilon)$ such that

$$\underset{f^{-1}(\mathfrak{F}_k)}{D}(U(z)) < \varepsilon \quad \text{for } k > k_1(\varepsilon) .$$

Hence $D(\mathfrak{F}_k)^2 \leqq \iint_{f^{-1}(\mathfrak{F}_k)} |f'(z)|^2\, dxdy \underset{f^{-1}(\mathfrak{F}_k)}{D}(U(z)) \leqq A(r_k)\varepsilon \leqq K r_k^2 \varepsilon$ for $k > k_1$ and

$$D(\mathfrak{F}_k) \leqq \sqrt{K\varepsilon}\, r_k \qquad \text{for } k > k_1 . \tag{23}$$

Put $D(\mathfrak{F}_k, CV_M) = \iint_{CV_M \cap f^{-1}(\mathfrak{F}_k)} |f'(z)| \left\{\left(\frac{\partial}{\partial x} U(z)\right)^2 + \left(\frac{\partial}{\partial y} U(z)\right)^2\right\}^{\frac{1}{2}} dxdy.$ Then by $U_n^M(z) \Rightarrow U(z)$ $D(\mathfrak{F}_k, CV_M \cap R_n) \to D(\mathfrak{F}_k, CV_M)$ as $n \to \infty$. Hence for any ε we can find a number $n_2(\varepsilon, k, M)$ such that

$$D(\mathfrak{F}_k, CV_M \cap R_n) \leqq D(\mathfrak{F}_k, CV_M) + \varepsilon \leqq D(\mathfrak{F}_k) + \varepsilon \text{ for } n \geqq n_2 .$$

Next by (22) for any given $\mathfrak{F}_k$ and $\mathfrak{F}_i$ $(i > k)$ and ε we can find M' and n_3 such that

$$\int_{G_i \cap R_n \cap \partial V_{M'}} \frac{\partial}{\partial n} U_n^{M'}(z) ds > \alpha - 2\varepsilon \text{ and } D(\mathfrak{F}_k, CV_{M'} \cap R_n) \leqq \sqrt{K\varepsilon}\ r_k + \sqrt{\varepsilon} \tag{24}$$

for $n > n_3 > \max(n_1, n_2)$.

We consider $D(\mathfrak{F}_k, CV_{M'} \cap R_n)$ in the following. Put $\mathfrak{z} = \exp(U_n^{M'}(z) + iV_n^{M'}(z)) = re^{i\theta}$, where $V_n^M(z)$ is the conjugate function of $U_n^M(z)$. Then $|\mathfrak{z}| = e^{M'}$ on $\partial V_{M'} \cap R_n$ and by (22)

$$\int_{\partial V_{M'} \cap R_n \cap G_i} d\theta = \int_{\partial V_{M'} \cap R_n \cap G_i} \frac{\partial}{\partial n} U_n^{M'}(z) ds \geqq \alpha - 2\varepsilon . \tag{25}$$

Now

$$D(\mathfrak{F}_i, CV_M \cap R_n) = \iint_{G_i \cap R_n \cap CV_{M'}} |f'(z)| \left\{ \left(\frac{\partial}{\partial r} U_n^{M'}(z) \right)^2 + \frac{1}{r^2} \left(\frac{\partial}{\partial \theta} U_n^{M'}(z) \right)^2 \right\}^{\frac{1}{2}} r\, dr\, d\theta$$

$$\geqq \iint_{G_i \cap R_n \cap CV_{M'}} \frac{1}{r} |f'(re^{i\theta})| \left| \frac{\partial}{\partial r} U_n^{M'}(re^{i\theta}) \right| r\, dr\, d\theta \text{ and } U_n^{M'}(z)$$

is harmonic in a compact domain $R_n - R_0 - V'_M$. Hence $U_n^{M'}(z)$ has a finite number of branch points and we can trace the trajectries T_θ along which θ $=$const from $r=1$ $(U_n^{M'}(z) = 0$ on $\partial R_0)$ to $r = e^{M'}$ $(U_n^M(z) = M'$ on $\partial V_{M'})$. Let Θ be the set of θ such that T_θ intersects G_i when z goes from R_0 to $\partial V_{M'}$ along T_θ. Then by (25) $\Theta > \alpha - 2\varepsilon$. If $\theta \in \Theta$, image $f(T_\theta)$ of T_θ must intersect G_k and G_i, whence $\int_{T_\theta} |f'(re^{i\theta})|\ dr \geqq r_k - r_i$ and

$$D(\mathfrak{F}_k) + \varepsilon \geqq D(\mathfrak{F}_k, CV_M \cap R_n) \geqq (\alpha - 2\varepsilon)(r_k - r_i) . \tag{26}$$

Let $r_i \to 0$ and then $\varepsilon \to 0$. Then (26) contradicts (23). Hence A is a set of capacity zero.

As an application of Theorem 9 we shall prove

Theorem 10. *Suppose R is a covering surface almost finitely sheeted over $\underline{R}$ and $\underline{\bar{R}} = \underline{R} + \underline{B}$ is compact. Then the set A of point $p \in B_1^N$ such* *that* ${}_{\Delta}f(p)\subset\mathfrak{F}$ *is a set of capacity zero, where* $\mathfrak{F}$ *satisfies the condition*

$$\varlimsup_{r\to 0}\frac{A(r)}{r^2}<\infty,$$

and $A(r)$ *is the area of* R *over* $\mathfrak{F}_r=E[w:$ euclidean dist $(w,\mathfrak{F})\leqq r]$.

Proof. By Theorem 1. $M(f(p))=$ one point q for any point p of B_1^N except at most a set of capacity zero. Suppose $M(f(p))=q$. We can find a sequence $G_i(i=1,2,\cdots)$ such that $G_i\ni p$ and $\bigcap_i\overline{f(G_i)}=q$ from $\{G_\tau\}:G_\tau\overset{N}{\ni}p$. Put $G_i'=G_i\cap v_i(p)$. Then also $G_i'\overset{N}{\ni}p$ and by the Theorem 6).3) $\Delta(p)\cap G_i'$ is an angular domain, whence $f(\Delta(p)\cap G^i)\neq 0$ for every i. On the other hand, $\bigcap_i\overline{f(\Delta(p)\cap G_i')}\subset\bigcap_i\overline{f(\Delta(p)\cap v_i(p))}={}_{\Delta}f(p)\subset\bigcap_i\overline{f(G_i)}=q$, hence ${}_{\Delta}f(p))=M(f(p))=q\in\mathfrak{F}$. Thus by Theorem 9) we have Theorem 10.

In the previous paper[21] we constructed a covering surface R over the w-plane: $w=f(z):z\in R$ with finite area such that R has a singular point $p\in B_s^N$ with the following properties: 1) $\omega(p,z)>0$ and 2) $\bigcap_n\overline{f(v_n(p))}=$ one point. This example shows that even when $\mathfrak{F}=$ one point, some condition (for instance $\varlimsup_{r\to 0}\frac{A(r)}{r^2}<\infty$) is necessary for the validity of Theorems 9) or 10.)

(*Remark to the condition in Theorem* 9). The condition $\varlimsup_{r\to 0}\frac{A(r)}{r^2}<\infty$ in Theorem 9 imposed on a closed set $\mathfrak{F}$ in $\underline{R}$ does not necessarily mean that $\mathfrak{F}$ is so small as Cap$(\mathfrak{F})=0$ but means the part of R over $\mathfrak{F}_r$ is so small as $\omega(\{\mathfrak{F}_r\},z)\to 0$ as $r\to 0$. In reality the part of R over $\mathfrak{F}_r$ may be so small even when $\mathfrak{F}$ is a segment.

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21) The last example of the paper "Examples of Singular points: Journ. Fac. Sci. Hokkaido Univ., XVI (1962).